"Anwendung dualer Quaternionen auf Kinematik," Annales Academiae Scientiarum Fennicae (1958), 1-13; Gesammelte Werke, v. 2.

Applications of dual quaternions to kinematics

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In the following, we shall speak of a classical situation that goes back to L. EULER in 1748 for its basic ideas, namely, the application of quaternions to the kinematics of rigid bodies, a situation that is only loosely connected with the theory of functions, and which was further developed by W. K. CLIFFORD, J. HJELMSLEV, and E. STUDY.

§ 1. Representation of rotations by quaternions.

Quaternions are expressions of the form:

(1) $$\mathbf{Q} = q_0\, e_0 + q_1\, e_1 + q_2\, e_2 + q_3\, e_3 ,$$

whose sums and products will be described by:

(2) $$\mathbf{Q} + \mathbf{Q}' = \sum (q_j + q'_j)\, e_j , \qquad \mathbf{QQ}' = \sum q_j q'_k e_j e_k ,$$

with the *product rules* for the units e:

(3) $$\begin{aligned} e_0\, e_j = e_j\, e_0 &= e_j , \\ e_k\, e_k &= -\, e_0 ; \qquad && k = 1, 2, 3, \\ e_j\, e_k = -\, e_k\, e_j &= e_l ; \qquad && j, k, l = 1, 2, 3;\ 2, 3, 1;\ 3, 1, 2, \end{aligned}$$

such that we can also set:

(4) $$e_0 = 1.$$

First, let the q_j be real numbers. One then has the associativity law:

(5) $$\mathbf{Q}(\mathbf{Q}'\,\mathbf{Q}) = (\mathbf{Q}\,\mathbf{Q}')\,\mathbf{Q}''.$$

The *conjugate quaternion* to (1) will be defined by:

(6) $$\tilde{\mathbf{Q}} = q_0\, e_0 - q_1\, e_1 - q_2\, e_2 - q_3\, e_3 ,$$

and one has:

(7) $$\mathbf{Q}\,\tilde{\mathbf{Q}} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = \langle \mathbf{Q}\,\mathbf{Q} \rangle$$

and

$$\widetilde{\mathbf{Q}\mathbf{Q}'} = \tilde{\mathbf{Q}}'\cdot\tilde{\mathbf{Q}}\,. \tag{8}$$

A quaternion with $q_0 = 0$ – so $\mathbf{q} = q_1\, e_1 + q_2\, e_2 + q_3\, e_3$ – is called a *vector*, and the product rule is given in vector notation as:

$$(q_0 + \mathbf{q})\,(q_0' + \mathbf{q}') = q_0 q_0' + q_0 q' + q_0' q - <\mathbf{q}\,\mathbf{q}'> + (\mathbf{q}\times\mathbf{q}'), \tag{9}$$

in which:

$$<\mathbf{q}\,\mathbf{q}'> = q_1 q_1' + q_2 q_2' + q_3 q_3' \tag{10}$$

is the *scalar product* and:

$$\mathbf{q}\times\mathbf{q}' = (q_2 q_3' - q_3 q_2')\,e_1 + (q_3 q_1' - q_1 q_3')\,e_2 + (q_1 q_2' - q_2 q_1')\,e_3 \tag{11}$$

means the *vector product*, with:

$$2(\mathbf{q}\times\mathbf{q}') = \mathbf{q}\,\mathbf{q}' - \mathbf{q}'\,\mathbf{q}. \tag{12}$$

Now, if:

$$\mathbf{x} = x_1\, e_1 + x_2\, e_2 + x_3\, e_3\,, \qquad \mathbf{x}' = x_1' e_1 + x_2' e_2 + x_3' e_3\,, \tag{13}$$

and if $\mathbf{Q}$ means a *normed* quaternion then, from EULER:

$$\mathbf{x} = \tilde{\mathbf{Q}}\mathbf{x}'\mathbf{Q} \tag{15}$$

represents a ternary orthogonal substitution $\mathbf{x}' \to \mathbf{x}$, and therefore, a *rotation* of the space of x_k around its origin. We set:

$$\mathbf{Q} = \cos\omega + \mathbf{a}\sin\omega, \qquad <\mathbf{a}\,\mathbf{a}> = 1, \tag{16}$$

then $\mathbf{a}$ means the rotational axis and 2ω is the rotation angle, whose sign is linked with the latter object.

If we interpret the q_j as homogeneous pointers in a projective space P_3 and we normalize the q_j by the requirement:

$$\mathbf{Q}\,\tilde{\mathbf{Q}} = q_0\, q_0 + q_1\, q_1 + q_2\, q_2 + q_3\, q_3 = 1, \tag{17}$$

then we can introduce the *separation* of two points $\mathbf{Q}$, $\mathbf{Q}'$ in P_3 by way of:

$$\cos\varphi = q_0 q_0' + q_1 q_1' + q_2 q_2' + q_3 q_3', \tag{18}$$

or:

$$\cos\varphi = \tfrac{1}{2}(\mathbf{Q}\tilde{\mathbf{Q}}' + \mathbf{Q}'\tilde{\mathbf{Q}}) = <\mathbf{Q}\,\mathbf{Q}'>. \tag{19}$$

This metric makes our P_3 into an *elliptic space* E_3. The particular points for which:

(20) $$\varphi = \frac{\pi}{2}, \qquad \langle \mathbf{Q}\,\mathbf{Q}' \rangle = 0$$

are called *conjugate in* E_3. E_3 is the *group space* for the rotation $\pm\,\mathbf{Q}$.

§ 2. Line map of E_3.

We can establish a (directed) line in E_3 by two of its conjugate points $\mathbf{Q}$, $\mathbf{Q}'$, such that one has:

(1) $$\langle \mathbf{Q}\,\mathbf{Q} \rangle = 1, \qquad \langle \mathbf{Q}\,\mathbf{Q}' \rangle = 0, \qquad \langle \mathbf{Q}'\,\mathbf{Q}' \rangle = 1.$$

For this, we compute the unit vectors:

(2) $$\mathbf{r} = \tilde{\mathbf{Q}}\mathbf{Q}', \qquad \mathbf{r}' = \mathbf{Q}'\tilde{\mathbf{Q}}$$

with:

(3) $$\mathbf{r} + \tilde{\mathbf{r}} = \mathbf{r}' + \tilde{\mathbf{r}}' = 0, \qquad \mathbf{r}\,\mathbf{r} = \mathbf{r}'\,\mathbf{r}' = -1.$$

If we replace $\mathbf{Q}$, $\mathbf{Q}'$ with another pair of the same kind of lines:

(4) $$\mathbf{Q}^* = \mathbf{Q}\cos\omega - \mathbf{Q}'\sin\omega, \qquad \mathbf{Q}'^* = \mathbf{Q}\sin\omega + \mathbf{Q}'\cos\omega$$

then $\mathbf{r}$ and $\mathbf{r}'$ remain the same:

(5) $$\mathbf{r} = \mathbf{r}^*, \qquad \mathbf{r}' = \mathbf{r}'^*.$$

Under a *motion of* E_3 – i.e., under a quaternary orthogonal substitution $\mathbf{Q} \to \mathbf{Q}^*$ – namely:

(6) $$\mathbf{Q}^* = \tilde{\mathbf{R}}'\,\mathbf{Q}\,\mathbf{R}; \qquad \mathbf{R}\tilde{\mathbf{R}} = \mathbf{R}'\tilde{\mathbf{R}}' = 1,$$

the image vectors $\mathbf{r}$, $\mathbf{r}'$ of our lines behave like:

(7) $$\mathbf{r}^* = \tilde{\mathbf{R}}'\,\mathbf{r}\,\mathbf{R}, \qquad \mathbf{r}'^* = \tilde{\mathbf{R}}'\,\mathbf{r}'\,\mathbf{R}',$$

so the two *image spheres* (3) will be rotated. From (2):

(8) $$\mathbf{Q}\,\mathbf{r} - \mathbf{r}'\,\mathbf{Q} = 0$$

is the condition for the point $\mathbf{Q}$ to be *united* with our line.

If the points $\mathbf{Q}_j$; $j = 0, 1, 2, 3$ satisfy the conditions:

(9) $$\langle \mathbf{Q}_j\,\mathbf{Q}_k \rangle = \delta_{jk}$$

with positive determinant:

(10) $$|\,\mathbf{Q}_0\,\mathbf{Q}_1\,\mathbf{Q}_2\,\mathbf{Q}_3\,| = +1$$

then we find that the image vectors $\mathbf{r}_{jk}$, $\mathbf{r}'_{jk}$ of the six edges of the *polar tetrahedron* of $\mathbf{Q}_j$, namely:

(11) $$\mathbf{r}_{jk} = \tilde{\mathbf{Q}}_j \mathbf{Q}_k = -\mathbf{Q}_k \tilde{\mathbf{Q}}_j, \qquad \mathbf{r}'_{jk} = \mathbf{Q}_k \tilde{\mathbf{Q}}_j = -\tilde{\mathbf{Q}}_j \mathbf{Q}_k,$$

satisfy the relations:

(12) $$\begin{aligned} \mathbf{r}_{01} + \mathbf{r}_{23} &= 0, & \mathbf{r}_{02} + \mathbf{r}_{31} &= 0, & \mathbf{r}_{03} + \mathbf{r}_{12} &= 0, \\ \mathbf{r}'_{01} - \mathbf{r}'_{23} &= 0, & \mathbf{r}'_{02} - \mathbf{r}'_{31} &= 0, & \mathbf{r}'_{03} - \mathbf{r}'_{12} &= 0, \end{aligned}$$

with:

(13) $$\langle \mathbf{r}_j \, \mathbf{r}_k \rangle = \langle \mathbf{r}'_j \, \mathbf{r}'_k \rangle = \delta_{jk}, \qquad [\mathbf{r}_{01} \, \mathbf{r}_{02} \, \mathbf{r}_{03}] = [\mathbf{r}'_{01} \, \mathbf{r}'_{02} \, \mathbf{r}'_{03}] = +1.$$

Once again, the square brackets in these equations mean the determinant. One can confirm (12) and (13) in the special case $\mathbf{Q}_j = e_j$, which is attainable by means of a motion (6) without altering these relations. The map (2) of the lines of E_3 onto the point-pairs of two spheres goes back to CLIFFORD, and was examined more closely around 1900 by HJELMSLEV and STUDY.

§ 3. Continual rotational processes.

In our formula (1.15), the normalized quaternion $\mathbf{Q}$ may now depend upon a real parameter – viz., time t – so what arises is a *one-parameter* or *continual rotational process:*

(1) $$\mathbf{x}(t) = \tilde{\mathbf{Q}}(t) \; \mathbf{x}' \, \mathbf{Q}(t); \qquad \mathbf{Q} \, \tilde{\mathbf{Q}} = 1.$$

The point $\mathbf{Q}(t) = \mathbf{Q}_0(t)$ of the elliptic space E_3 correspondingly describes a line C. In order to see this, we introduce a *comoving polar tetrahedron* such that the derived equations are valid:

(2) $$\begin{array}{lcccc} d\mathbf{Q}_0 = & * & +\mathbf{Q}_0\rho & * & * \\ d\mathbf{Q}_1 = & -\mathbf{Q}_0\rho & * & +\mathbf{Q}_2\sigma & * \\ d\mathbf{Q}_2 = & * & -\mathbf{Q}_1\sigma & * & +\mathbf{Q}_3\tau \\ d\mathbf{Q}_3 = & * & * & -\mathbf{Q}_2\tau & * \end{array}$$

with:

(3) $$\langle \mathbf{Q}_j \, \mathbf{Q}_k \rangle = \delta_{jk}, \qquad |\, \mathbf{Q}_0 \, \mathbf{Q}_1 \, \mathbf{Q}_2 \, \mathbf{Q}_3 \,| = +1.$$

Furthermore, as in (2.11), we define the two right-angled *canonical axis crosses:*

(4) $$\mathbf{r}_j = \mathbf{r}_{0j} = \tilde{\mathbf{Q}}_0 \, \mathbf{Q}_j, \qquad \mathbf{r}'_j = \mathbf{r}'_{0j} = \mathbf{Q}_j \tilde{\mathbf{Q}}_0.$$

One again has for the derived equations:

(5)
$$\begin{aligned} d\mathbf{r}_1 &= \quad * \quad +\mathbf{r}_2\lambda \quad * \\ d\mathbf{r}_2 &= -\mathbf{r}_1\lambda \quad * \quad +\mathbf{r}_3\mu \\ d\mathbf{r}_3 &= \quad * \quad -\mathbf{r}_2\mu \quad * \end{aligned} \qquad \begin{aligned} d\mathbf{r}'_1 &= \quad * \quad +\mathbf{r}'_2\lambda \quad * \\ d\mathbf{r}'_2 &= -\mathbf{r}'_1\lambda' \quad * \quad +\mathbf{r}'_3\mu' \\ d\mathbf{r}'_3 &= \quad * \quad -\mathbf{r}'_2\mu' \quad * \end{aligned}$$

as one finds by differentiating (1), (4), and the relations:

(6)
$$\lambda = \lambda' = \sigma, \qquad \mu = \tau - \rho, \qquad \mu' = \tau + \rho$$

exist between the differentials $\rho, \sigma, \tau; \lambda, \mu; \lambda', \mu'$.

The geometric interpretation for the canonical axes is the following: $\mathbf{r}_1(t)$ is the momentary rotational axis, $\mathbf{r}_3(t)$ is the common normal to the neighboring rotational axes $\mathbf{r}_1$ and $\mathbf{r}_1 + d\mathbf{r}_1$ in the moving body (e.g., a top). $\mathbf{r}'_1$ and $\mathbf{r}'_3$ have the corresponding meaning in the rest system. The relation $\lambda = \lambda'$ then means that the "moving cone" $\mathbf{r}_1(t)$ rolls without slipping on the "rest cone" $\mathbf{r}'_1(t)$.

If we denote a point $\mathbf{x}$ on the canonical axis by the Ansatz:

(7)
$$\mathbf{x} = x_1\,\mathbf{r}_1 + x_2\,\mathbf{r}_2 + x_3\,\mathbf{r}_3$$

this yields the *guiding condition*, as one says, that $\mathbf{x}$ is established in the moving system (i.e., the top) by:

(8)
$$\begin{aligned} dx_1 &= \quad * \quad +x_2\lambda \quad * \\ dx_2 &= -x_1\lambda \quad * \quad +x_3\mu' \\ dx_3 &= \quad * \quad -x_2\mu' \quad * \end{aligned}$$

Correspondingly, one finds the *rest conditions* for the point:

(9)
$$\mathbf{x}' = x'_1\mathbf{r}'_1 + x'_2\mathbf{r}'_2 + x'_3\mathbf{r}'_3$$

which indicate that $\mathbf{x}'$ is at rest:

(10)
$$\begin{aligned} dx'_1 &= \quad * \quad +x'_2\lambda \quad * \\ dx'_2 &= -x'_1\lambda \quad * \quad +x'_3\mu' \\ dx'_3 &= \quad * \quad -x'_2\mu' \quad * \end{aligned}$$

The kinematics of tops is included in these equations (8), (10). For the simplest rotational process, namely, rotation around an axis at rest, the associated line C is a line in E_3, and that is the kinematic interpretation of the contents of § 2.

In order to now transplant the formulas of the kinematics of tops up to now in the general kinematics of rigid bodies in space, we employ a "conversion principle."

§ 4. Conversion principle.

Now, let a line $\underline{\mathbf{r}}$ of Euclidian R_3 be given by a unit vector $\mathbf{r}$ that lies on it and a point $\mathbf{x}$ that lies on it. If $\mathbf{x}$ then means the vector that points from the origin $\mathbf{o}$ to the point $\mathbf{x}$ then we define the vector product:

$$\underline{\mathbf{r}} = \mathbf{x} \times \mathbf{r}, \tag{1}$$

and remark that $\overline{\mathbf{r}}$ does not depend upon the choice of $\mathbf{x}$ on $\underline{\mathbf{r}}$, since:

$$(\mathbf{x} + h\mathbf{r}) \times \mathbf{r} = \mathbf{x} \times \mathbf{r}. \tag{2}$$

(1) is then the condition for the point $\mathbf{x}$ to be united with the line $\underline{\mathbf{r}}$ that is given by the vector pair $\mathbf{r}$, $\overline{\mathbf{r}}$. This pair fulfills the conditions:

$$\langle \mathbf{r}\,\mathbf{r} \rangle = 1, \qquad \langle \mathbf{r}\,\overline{\mathbf{r}} \rangle = 0. \tag{3}$$

One can formally combine them into a single entity by introducing the *dual vectors:*

$$\underline{\mathbf{r}} = \mathbf{r} + \varepsilon\overline{\mathbf{r}}, \tag{4}$$

in which the ε obeys the rule of computation:

$$\varepsilon^2 = 0. \tag{5}$$

Thus, the calculations with such "dual numbers:"

$$\underline{a} = a + \varepsilon\,\overline{a} \tag{6}$$

are afflicted with exceptions, since the division by null components of the form $\varepsilon\,\overline{a}$ is inadmissible. In the sequel, dual numbers and vectors are always emphasized by underlines. From (4), eq. (5) gives an expression that is equivalent to (3):

$$\langle \underline{\mathbf{r}}\,\underline{\mathbf{r}}' \rangle = 0, \tag{7}$$

or, more precisely:

$$\langle \mathbf{r}\,\mathbf{r}' \rangle = 0, \qquad \langle \mathbf{r}\,\overline{\mathbf{r}}' \rangle + \langle \mathbf{r}'\,\overline{\mathbf{r}} \rangle = 0, \tag{9}$$

which imply the perpendicular intersection of the lines $\underline{\mathbf{r}}$ and $\underline{\mathbf{r}}'$.

If the function f has the derivative f' then we set:

$$f(\varphi + \varepsilon\,\overline{\varphi}) = f(\varphi) + \varepsilon\overline{\varphi}\, f'(\varphi). \tag{10}$$

One then has for two lines $\underline{\mathbf{r}}$, $\underline{\mathbf{r}}'$:

(11) $$<\underline{\mathbf{r}}\,\underline{\mathbf{r}}'> = \cos(\varphi + \varepsilon\bar{\varphi}),$$

if φ means the angle and $\bar{\varphi}$ is the shortest distance from $\underline{\mathbf{r}}$ to $\underline{\mathbf{r}}'$. In more detail, it follows from (11) that, in fact:

(12) $$<\mathbf{r}\,\mathbf{r}'> = \cos\varphi, \qquad <\mathbf{r}\,\bar{\mathbf{r}}'> + <\mathbf{r}'\,\bar{\mathbf{r}}> = -\varepsilon\,\bar{\varphi}\,\sin\varphi.$$

Thus, the signs of φ and $\bar{\varphi}$ are coupled to each other.

If we now take a *dual rotation* $\underline{\mathbf{r}}' \to \underline{\mathbf{r}}$ of the unit sphere:

(13) $$\underline{\mathbf{r}} = \tilde{\underline{\mathbf{Q}}}\underline{\mathbf{r}}'\underline{\mathbf{Q}},$$

with:

(14) $$\mathbf{Q} = \mathbf{Q} + \varepsilon\underline{\mathbf{Q}}, \qquad \underline{\mathbf{Q}}\tilde{\underline{\mathbf{Q}}} = \mathbf{Q}\tilde{\mathbf{Q}} + \varepsilon(\mathbf{Q}\tilde{\bar{\mathbf{Q}}} + \bar{\mathbf{Q}}\tilde{\mathbf{Q}}) = 1,$$

then the line $\underline{\mathbf{r}}$ obtains its interpretation in the space R_3 as a motion of R_3. If we set:

(15) $$\underline{\mathbf{Q}} = \cos\underline{\omega} + \underline{\mathbf{a}}\sin\underline{\omega},$$

with:

(16) $$\underline{\omega} = \omega + \varepsilon\underline{\omega}$$

and

(17) $$\underline{\mathbf{a}} = \mathbf{a} + \varepsilon\,\bar{\mathbf{a}}, \quad <\underline{\mathbf{a}}\,\underline{\mathbf{a}}> = 1$$

then $\underline{\mathbf{a}}$ becomes the screw axis of the motion $\underline{\mathbf{Q}}$, 2ω is its rotation angle around $\underline{\mathbf{a}}$, and $2\bar{\omega}$ is its displacement in the direction $\mathbf{a}$.

Therefore, it is possible to interpret the geometry of the sphere, when extended to dual space, in line space. The employment of dual numbers in geometry goes back to W. K. CLIFFORD (1845-70) and the conversion principle goes back to A. P. KOTJELNIKOFF (1865-1944) and E. STUDY (1862-1930). The contents of § 4 can also be obtained from those of § 2 by passing to the limit.

§ 5. Spatial motions in line space.

By means of the conversion principle of § 4, we can now apply the dual extensions of the formulas of § 3 to the kinematics of continual processes of motion when we let these motions act on lines. In place of (3.1), what enters in is:

(1) $$\underline{\mathbf{r}}(t) = \tilde{\underline{\mathbf{Q}}}(t)\underline{\mathbf{r}}'\underline{\mathbf{Q}}(t), \qquad \underline{\mathbf{Q}}\tilde{\underline{\mathbf{Q}}} = 1$$

with real t, and in place of (3.2), one has:

$$(2)\qquad \begin{array}{llllll}
d\underline{\mathbf{Q}}_0 = & * & +\underline{\mathbf{Q}}_1\rho & * & * \\
d\underline{\mathbf{Q}}_1 = & -\underline{\mathbf{Q}}_1\rho & * & +\underline{\mathbf{Q}}_1\sigma & * \\
d\underline{\mathbf{Q}}_2 = & * & -\underline{\mathbf{Q}}_1\sigma & * & +\underline{\mathbf{Q}}_3\tau \\
d\underline{\mathbf{Q}}_3 = & * & * & -\underline{\mathbf{Q}}_2\tau & *
\end{array}$$

More precisely, along with (3.2), the following formulas appear:

$$(3)\qquad \begin{array}{llllllllll}
d\bar{\mathbf{Q}}_0 = & * & +\mathbf{Q}_1\bar{\rho} & * & * & * & +\bar{\mathbf{Q}}_1\rho & * & * \\
d\bar{\mathbf{Q}}_1 = & -\mathbf{Q}_1\bar{\rho} & * & +\mathbf{Q}_2\bar{\sigma} & * & +\bar{\mathbf{Q}}_0\rho & * & +\bar{\mathbf{Q}}_2\sigma & * \\
d\bar{\mathbf{Q}}_2 = & * & -\mathbf{Q}_1\bar{\sigma} & * & +\mathbf{Q}_3\bar{\tau} & * & -\bar{\mathbf{Q}}_1\sigma & * & +\bar{\mathbf{Q}}_3\tau \\
d\bar{\mathbf{Q}}_3 = & * & * & -\mathbf{Q}_2\bar{\tau} & * & * & * & -\bar{\mathbf{Q}}_2\tau & *
\end{array}$$

with:

(4) $\underline{\rho} = \rho + \varepsilon\,\bar{\rho}$, $\qquad \underline{\sigma} = \sigma + \varepsilon\,\bar{\sigma}$, $\qquad \underline{\tau} = \tau + \varepsilon\,\bar{\tau}$.

Correspondingly, along with (3.5), one has the further formulas:

$$(5)\qquad \begin{array}{lllllll}
d\bar{\mathbf{r}}_1 = & * & +\mathbf{r}_2\bar{\lambda} & * & * & +\bar{\mathbf{r}}_2\lambda & * \\
d\bar{\mathbf{r}}_2 = & -\mathbf{r}_1\bar{\lambda} & * & +\mathbf{r}_3\bar{\mu} & -\bar{\mathbf{r}}_1\lambda & * & +\bar{\mathbf{r}}_3\mu \\
d\bar{\mathbf{r}}_3 = & * & -\mathbf{r}_2\bar{\mu} & * & * & -\bar{\mathbf{r}}_3\mu & *
\end{array}$$

and

$$(6)\qquad \begin{array}{lllllll}
d\bar{\mathbf{r}}_1' = & * & +\mathbf{r}_2'\bar{\lambda}' & * & * & +\bar{\mathbf{r}}_2'\lambda' & * \\
d\bar{\mathbf{r}}_2' = & -\mathbf{r}_1'\bar{\lambda}' & * & +\mathbf{r}_3'\bar{\mu}' & -\bar{\mathbf{r}}_1'\lambda' & * & +\bar{\mathbf{r}}_3'\mu' \\
d\bar{\mathbf{r}}_3' = & * & -\mathbf{r}_2'\bar{\mu}' & * & * & -\bar{\mathbf{r}}_3'\mu' & *
\end{array}$$

for the canonical axes. Thus, along with (3.6), one gets the relations:

(7) $\bar{\lambda} = \bar{\sigma}$, $\qquad \bar{\lambda}' = \bar{\sigma}'$, $\qquad \bar{\mu} = \bar{\tau} - \bar{\rho}$, $\qquad \bar{\mu}' = \bar{\tau} + \bar{\rho}$.

If we refer a line $\underline{\mathbf{r}}$ to the canonical axes by the Ansatz:

(8) $$\underline{\mathbf{r}} = \underline{x}_1\underline{\mathbf{r}}_1 + \underline{x}_2\underline{\mathbf{r}}_2 + \underline{x}_3\underline{\mathbf{r}}_3$$

then this yields the *guiding conditions:*

$$(9)\qquad \begin{array}{llll}
d\underline{x}_1 = & * & +\underline{x}_2\underline{\lambda} & * \\
d\underline{x}_2 = & -\underline{x}_1\underline{\lambda} & * & +\underline{x}_3\underline{\mu}' \\
d\underline{x}_3 = & * & -\underline{x}_2\underline{\mu}' & *
\end{array}$$

from (3.8). Likewise, for the line:

(10) $$\underline{\mathbf{r}}' = \underline{x}_1'\underline{\mathbf{r}}_1' + \underline{x}_2'\underline{\mathbf{r}}_2' + \underline{x}_3'\underline{\mathbf{r}}_3'$$

one obtains the *rest conditions:*

(11) $$\begin{aligned} d\underline{x}_1' &= \quad * \quad + \underline{x}_2'\underline{\lambda} \quad * \\ d\underline{x}_2' &= -\underline{x}_1'\underline{\lambda} \quad * \quad + \underline{x}_3'\underline{\mu} \\ d\underline{x}_3' &= \quad * \quad - \underline{x}_2'\underline{\mu} \quad * \end{aligned}$$

from (3.10).

§ 6. Motion of points and planes.

In order to transfer the formulas of § 5 to the motion of points and planes, one proceeds as follows: For a dual quaternion:

(1) $$\underline{\mathbf{Q}} = (q_0 + \varepsilon\, \bar{q}_0) + (\mathbf{q} + \varepsilon\, \bar{\mathbf{q}}),$$

the conjugate is:

(2) $$\tilde{\underline{\mathbf{Q}}} = (q_0 + \varepsilon\, \bar{q}_0) - (\mathbf{q} + \varepsilon\, \bar{\mathbf{q}}).$$

We now also introduce the quaternion that arises from $\underline{\mathbf{Q}}$ by changing the sign of ε:

(3) $$\underline{\mathbf{Q}}_\varepsilon = (q_0 - \varepsilon\, \bar{q}_0) + (\mathbf{q} - \varepsilon\, \bar{\mathbf{q}}).$$

We would like to associate the point **X** with the Cartesian pointers x_j with not only the vector:

(4) $$\mathbf{x} = x_1\, e_1 + x_2\, e_2 + x_3\, e_3\,,$$

but also the quaternion:

(5) $$\underline{\mathbf{X}} = 1 + \varepsilon\, \mathbf{x},$$

such that one has:

(6) $$\underline{\mathbf{X}}_\varepsilon - \tilde{\underline{\mathbf{X}}} = 0, \qquad \underline{\mathbf{X}}\tilde{\underline{\mathbf{X}}} = 1.$$

A plane $\underline{\mathbf{U}}$ with the equation:

(7) $$u_0 + u_1\, x_1 + u_2\, x_2 + u_3\, x_3 = 0$$

or:

(8) $$u_0 + \langle \mathbf{u}\ \mathbf{x} \rangle = 0, \qquad \langle \mathbf{u}\ \mathbf{u} \rangle = 1$$

with the normal vector:

(9) $$\mathbf{u} = u_1\, e_1 + u_2\, e_2 + u_3\, e_3\,,$$

might correspond to the quaternion:

(10) $$\underline{\mathbf{U}} = \mathbf{u} + \varepsilon\, u_0\,,$$

with:

(11) $$\underline{\mathbf{U}}_\varepsilon + \tilde{\underline{\mathbf{U}}} = 0, \qquad \underline{\mathbf{U}}\tilde{\underline{\mathbf{U}}} = 1.$$

This yields the condition for the line to be united with the point:

(12) $$\underline{\mathbf{r}}\underline{\mathbf{X}} - \underline{\mathbf{X}}\underline{\mathbf{r}} = 0,$$

for a line to be united with a plane:

(13) $$\underline{\mathbf{r}}\underline{\mathbf{U}} + \underline{\mathbf{U}}\underline{\mathbf{r}} = 0,$$

and for a point to be united with a plane:

(14) $$\underline{\mathbf{X}}\underline{\mathbf{U}} - \underline{\mathbf{U}}_\varepsilon\underline{\mathbf{X}}_\varepsilon = 0.$$

Thus, one confirms that the action of the same motion $\underline{\mathbf{Q}}$ on lines, points, and planes can be written thus:

(15) $$\underline{\mathbf{r}} = \tilde{\underline{\mathbf{Q}}}\underline{\mathbf{r}}'\underline{\mathbf{Q}}, \quad \underline{\mathbf{X}} = \tilde{\underline{\mathbf{Q}}}\underline{\mathbf{X}}'\underline{\mathbf{Q}}_\varepsilon, \quad \underline{\mathbf{U}} = \tilde{\underline{\mathbf{Q}}}_\varepsilon\underline{\mathbf{U}}'\underline{\mathbf{Q}}, \quad \underline{\mathbf{Q}}\tilde{\underline{\mathbf{Q}}} = 1.$$

Let **k** be the origin of the canonical axis cross of the $\mathbf{r}_j$. If we then refer a point **x** to the canonical cross by the Ansatz:

(16) $$\mathbf{x} = \mathbf{k} + \mathbf{r}_1 x_1 + \mathbf{r}_2 x_2 + \mathbf{r}_3 x_3$$

then this yields the *guiding condition:*

(17) $$\begin{aligned} dx_1 &= -\bar{\mu}' \quad * \quad +x_2\lambda \quad * \\ dx_2 &= \quad * \quad -x_1\lambda \quad * \quad +x_3\mu' \\ dx_3 &= -\lambda \quad * \quad -x_2\mu' \quad * \end{aligned}$$

Correspondingly, for:

(18) $$\mathbf{x}' = \mathbf{k}' + \mathbf{r}_1'x_1' + \mathbf{r}_2'x_2' + \mathbf{r}_3'x_3'$$

one has the *rest condition:*

(18) $$\begin{aligned} dx_1' &= -\bar{\mu} \quad * \quad +x_2'\lambda \quad * \\ dx_2' &= \quad * \quad -x_1'\lambda \quad * \quad +x_3'\mu \\ dx_3' &= -\bar{\lambda} \quad * \quad -x_2'\mu \quad * \end{aligned}$$

For the plane with the canonical equation:

(20) $$u_0 + u_1 u_1 + u_2 u_2 + u_3 u_3 = 0,$$

we have the *guiding equations:*

(21)
$$\begin{aligned} du_0 &= +u_1\bar{\mu}' \quad * \quad +u_3\bar{\lambda} \\ du_1 &= \quad * \quad +u_2\lambda \quad * \\ du_2 &= -u_1\lambda \quad * \quad +u_3\mu' \\ du_3 &= \quad * \quad -u_2\mu' \quad * \end{aligned}$$

and for the plane with the canonical equation:

(22)
$$u_0' + u_1'x_1' + u_2'x_2' + u_3'x_3' = 0$$

we have the *rest conditions:*

(23)
$$\begin{aligned} du_0' &= +u_1'\bar{\mu} \quad * \quad +u_3'\bar{\lambda} \\ du_1' &= \quad * \quad +u_2'\lambda \quad * \\ du_2' &= -u_1'\lambda \quad * \quad +u_3'\mu \\ du_3' &= \quad * \quad -u_2'\mu \quad * \end{aligned}$$

All of the formulas of spatial kinematics follow from this in a self-evident way.

I hope that, together with my colleague H. R. MÜLLER (Berlin), I can give a more thorough representation of this situation, in which multi-parameter processes of motion and integral formulas will be treated, as a continuation of our book "Ebene Kinematik" (Munich, 1956). A brief presentation of spatial kinematics shall appear in the publications of the University of Buenos Aires.

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