Nonlinear Theory of the Electromagnetic Field

By

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INTRODUCTION

The natural development of any scientific research is characterized by an intimate collaboration between experiment and theory. In certain epochs, theory takes the lead and may foreshadow new and unknown effects; we have seen an extraordinary example of such a foreshadowing, by de Broglie, of the wavelike character of electronic motion. On the contrary, in other epochs − like the present one, for example − the discoveries of new facts achieve such a rhythm that theory lags the experimental developments quite considerably. Nevertheless, one may say that the observation of facts and their interpretations are always on a par.

    The theory that I shall discuss here does not have the character that one would call normal. It sometimes happens that a well-established experimental fact remains unexplained for a long time, and completely isolated from other known observations. A celebrated example is provided for us by the phenomenon - which was already known by Newton – of the equality between mass (a measure of inertia) and weight (a measure of gravitation); in other words, by the confirmation that all bodies fall with the same acceleration in a vacuum. The researchers who usually employed this fact forgot this, and regarded it as an unsolved problem; 200 years passed before Einstein began to wonder about it again, and discovered, in a theory which has grown lengthy and commonplace, the foundations of the theory of general relativity.

    The situation that we are placed in here is not that different from the preceding one, in the sense that the problem that we are concerned with is neither too old nor completely forgotten; nevertheless, 50 years have already passed since I first posed it. The problem is that of the difficulties that are raised by the existence of an infinite proper energy for point-like electric charges.

    J. J. Thomson, who was the first to discover the existence of free electrons and measure their specific charge, has likewise proposed the fundamental hypothesis that their mass is of electromagnetic origin. I will cite a phrase from an autobiography that was recently published [1] ([1]), in which he described his first researches (1891) on the consequences of Maxwell's theory:

([1]) The numbers between brackets ,[ ], refer to the bibliography that one finds at the end of this work.

"If one adopts the hypothesis of an electric constitution for matter then it is permissible to assume that mass has an electric origin, and that, as a consequence, it does not arise in the atoms themselves, but in the space that surrounds their charges."

J. J. Thomson admits that this "space surrounding a charge e" is the space that is external to a small sphere of radius, a, and proves that the mass that is actually calculated must be proportional to the electrostatic energy, which is e^2/a. This expression must be infinite as a τ 0; the idea of a mass of electromagnetic origin therefore necessarily leads to speculations concerning the structure of the electron. This problem did not receive a satisfactory solution during the "classical" period of physics, and has not disappeared even with the creation of the modern theory of quanta. At the present time, it constitutes a "black hole" in our theoretical description of natural phenomena, and it is only a meager consolation that in the course of time other black holes have likewise obscured our horizon.

The latter difficulties have their origin in the discovery of new elementary particles, which were unknown up till now and made the electron give up its privileged position as a unique fundamental particle.

At present, there exists several particles: the electron, the positron, the proton, the neutron, and probably the neutrino, and we are certain that any correct theory of these particles must embrace the whole set and not just be content to describe them one at a time; indeed, one already knows a certain number of phenomena in which these particles are produced or destroyed, or ultimately transformed into each other.

Nonlinear field theory is a residue of the epoch during which the electron was considered to be the premier element of physics. It provides an elegant form solution to the problem of the infinite proper energy and the electromagnetic mass; in truth, this solution came much too late. The existence of the neutron clearly shows that mass is not indissolubly linked with charge. At the same time, experimental research concerning nuclear phenomena has shown the existence of forces between particles – charged or not – whose nature is completely different from that of electromagnetic forces. Moreover, it has not been possible to realize a complete agreement between nonlinear electrodynamics and the principle of quantum theory. Finally, we have not succeeded in ridding ourselves of the infinite terms that are characteristic of quantum field theory.

Despite these objections, I permit myself to develop this theory for your consideration, and for the following reasons: in the history of scientific research, one confirms, in a completely general fashion, that often one of the reasons for progress in a given direction is the radical elimination of all parasitic hypotheses that are hidden in the theory. The linear electrodynamics of Maxwell contains such parasitic hypotheses. They are not only useless, but also give rise to considerable difficulties, particularly as far as proper energy is concerned.

Now, it is possible to develop a general theory that avoids these difficulties. Similarly, if this theory does not immediately solve the grand problem of elementary particles then it nevertheless merits the distinction of providing the means to grasp the essential points; one already confirms the existence of a certain number of connecting points that link nonlinear electrodynamics with the considerations of a completely different genre, like Dirac's theory of positrons, for example. Last, but not least, the generalized field theory forms a harmonious mathematical edifice, whose great beauty

might constitute a point of interest for those who appreciate the elegance of analytical methods.

For this last reason, the mode of exposition that is adopted here will have a very pronounced mathematical character. I will begin with some considerations concerning the principles of variation that apply to the motion of continuous media, and then pass to the general case, in which I conclude with the case of the electromagnetic field ([2])

([2]) For this exposition, I have made ample usage of the thesis that was presented to Cambridge Univeristy by my old student and collaborator, M. P. Weiss, whose has completely studied the mathematical basis for the field theory in the context of the calculus of variations and functional calculus; part of this thesis was published in [2].

CHAPTER I.

CLASSICAL THEORY.

1. Variational principle for a continuous medium [1]. – Consider an n-dimensional space, with coordinates $x^1, x^2, \ldots, x^n$.

We define a closed $n-1$-dimensional hypersurface S in this space, by means of the equations:

(1.1) $$x^i = x^i(u^1, \ldots, u^{n-1}) \qquad (i = 1, 2, \ldots, n),$$

in which the u^k ($k = 1, 2, \ldots, n-1$) are $n-1$ parameters. Call the domain that they refer to D and define ν functions in this domain $z^\alpha(x^1, \ldots, x^n)$, which we write, to abbreviate:

$$z^\alpha(x^i) \qquad\qquad (\alpha = 1, 2, \ldots, \nu);$$

let:

(1.2) $$z_k^\alpha(x^i) = \frac{\partial z^\alpha}{\partial x^k},$$

be their derivatives (^{3}).

Suppose that one is given a function of these various variables, a function that we call the "Lagrangian" of the system:

(1.3) $$\mathfrak{L}(x, z^\alpha, z_k^\alpha).$$

We form the integral:

(1.4) $$I = \int_D \mathfrak{L}[x^i, z^\alpha(x^i), z_k^\alpha(x^i)]dx,$$

in which dx signifies $dx^1, \ldots, dx^n$. I is a functional that depends on the choice of surface S that is given by (1.1), as well as the values of the ν functions z^α in D and on S.

In order to obtain a useful mechanical interpretation, it is necessary to make the number of dimensions n equal to 4 - three space dimensions and one time dimension.

The functions $z^\alpha(x^i)$ describe an arbitrary property of a medium that fills the space (elastic deformations, electromagnetic potentials, etc.). We postulate the *equations of motion* that are obtained by annulling the variation δI of I for independent variations of these arguments, i.e., of S and the z^α on S and in D.

It is essential to also consider the variation of the boundary, as well as that of dependent variables on the boundary if one would also like to understand the behavior of singularities. We treat the latter problem in section 9.

In order to perform the variation, we introduce a family $S(\varepsilon)$ of hypersurfaces S with one parameter ε, which are situated in a neighborhood of the given hypersurface, and for which:

(1.5) $$\overline{x}^i = x^i + \varepsilon \xi^i.$$

(^{3}) Latin indices always refer to the x's, and the Greek ones, to the z's. As usual, one must sum over any index that appears twice. We have taken pains to choose the notation in such a manner that the one index will appear on top and the other, on the bottom.

Similarly, we subject the functions z^α in D to a linear variation by replacing $z^\alpha(x^i)$ with:

(1.6) $$\overline{z}^\alpha(x^i) = z^\alpha(x^i) + \varepsilon\eta^\alpha(x^i) \qquad \text{in } D.$$

The values of these functions on the deformed boundary $S(\varepsilon) = \overline{S}$ will be:

(1.7) $$\overline{z}^\alpha(\overline{x}^i) = z^\alpha(\overline{x}^i) + \varepsilon\eta^\alpha(\overline{x}^i) = z^\alpha(x^i) + \varepsilon(z_i^\alpha\xi^i + \eta^\alpha) \qquad \text{on } S.$$

The total variation of z^α on the boundary may be written εz^α, and is given by:

(1.8) $$\overline{z}^\alpha(\overline{x}^i) = z^\alpha(x^i) + \varepsilon\zeta^\alpha.$$

By comparing this with (1.7), one sees that:

(1.9) $$\eta^\alpha = z^\alpha - z_i^\alpha\xi^i \qquad \text{on S.}$$

The first variation of the integral I is:

(1.10) $$\delta I = \left(\frac{dI(\varepsilon)}{d\varepsilon}\right)_0 = \int_D \left(\frac{d\mathfrak{L}}{d\varepsilon}\right)_0 dx + \left(\frac{d}{d\varepsilon}\int_{D+\varepsilon\Delta}(\mathfrak{L})_0\, dx\right)_0$$

in which the index 0 indicates that $\varepsilon \to 0$, and $D + \varepsilon\Delta$ represents the domain bounded by the deformed boundary $\overline{S}$.

In order to simplify this integral, we introduce a geometric language and speak of *vectors* and *tensors*. Nevertheless, one must note that in our case this is not the ordinary tensor calculus since, in our new form, we do not postulate any relation between, on the one hand, the transformation laws and independent variables, and, on the other, those of the dependent variables. We separately transform the space of x by introducing new variables:

$$\overline{x}^i = \varphi^i(x^1, x^2, \cdots, x^n)$$

and the space of x:

$$\overline{z}^\alpha = \psi^\alpha(x^1, x^2, \cdots, x^n),$$

but we make no hypothesis concerning the metric on these spaces. With these conventions, dx^i is a contravariant vector in the space of x and dz^α is a contravariant vector in the space of z; however, z^α, for example, is a scalar in the space of x, whereas is a covariant vector in the same space, but is contravariant in the space of z. The Lagrangian $\mathfrak{L}$ is a scalar in the space of x as well as of z. The transformation properties are deduced immediately for no particular quantity from the number and position of the Greek or Latin indices.

Much later, we shall establish the usual characteristic relations of ordinary tensor calculus between the transformations of the x and z spaces; at the moment, we shall

distinguish between *tensors, properly speaking,* and *tensor densities,* a distinction that is superfluous for the instant.

The n derivatives $\dfrac{dx^1}{du^k}$, $\dfrac{dx^2}{du^k}$, …, $\dfrac{dx^n}{du^k}$ are the components of contravariant vector that is tangent to S. There are $n-1$ vectors of this type that one may obtain by successively taking $k = 1, 2, …, n-1$; one may assume that they are linearly independent, or else the choice of parameters u^k on S would not satisfy the required conditions.

Consider an arbitrary contravariant vector a^k in the space of x and define a covariant vector N_k in the same space by relation:

$$(1.11) \qquad N_k\, a^k = \begin{vmatrix} a^1 & a^2 & \cdots & a^n \\[2mm] \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^2}{\partial u^1} & \cdots & \dfrac{\partial x^n}{\partial u^1} \\[2mm] \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^2}{\partial u^2} & \cdots & \dfrac{\partial x^n}{\partial u^2} \\[2mm] \cdots & \cdots & \cdots & \cdots \\[2mm] \dfrac{\partial x^1}{\partial u^{n-1}} & \dfrac{\partial x^2}{\partial u^{n-1}} & \cdots & \dfrac{\partial x^n}{\partial u^{n-1}} \end{vmatrix}.$$

N_k is therefore the minor that one obtains by starting with the matrix of $n-1$ rows and n columns:

$$\left(\left(\frac{\partial x^k}{\partial u^r}\right)\right)$$

by suppressing the column k, taken with the convenient sign.

The vector N_k is normal to S; indeed, by replacing a^k with $\dfrac{\partial x^k}{\partial u^r}$ in (1.11) one obtains:

$$(1.12) \qquad N_k \frac{\partial x^k}{\partial u^r} = 0 \qquad\qquad (r = 1, 2, …, n-1).$$

This being the case, first consider the second term in the variation δI in (1.10):

$$(1.13) \qquad \frac{d}{d\varepsilon}\int_{D+\varepsilon\Delta}(\mathcal{L})_0\,dx = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\int_{D+\varepsilon\Delta}(\mathcal{L})_0\,dx - \int_{D}(\mathcal{L})_0\,dx\right\}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\varepsilon\Delta}(\mathcal{L})_0\,dx.$$

This integral must be taken over the domain $\varepsilon\Delta$ that lies between the surfaces $\overline{S}$ and S, which one may imagine are decomposed into cylinders that have their axes normal to S. The volume of one of these cylinders is:

$$\begin{vmatrix} a^1 & a^2 & \cdots & a^n \\ \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^2}{\partial u^1} & \cdots & \dfrac{\partial x^n}{\partial u^1} \\ \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^2}{\partial u^2} & \cdots & \dfrac{\partial x^n}{\partial u^2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x^1}{\partial u^{n-1}} & \dfrac{\partial x^2}{\partial u^{n-1}} & \cdots & \dfrac{\partial x^n}{\partial u^{n-1}} \end{vmatrix} \varepsilon \, \Delta u^1 \, \Delta u^2 \, \dots \, \Delta u^{n-1},$$

which, by virtue of (1.5) and (1.11), is equal to:

$$\varepsilon \, N_k \, \xi^k \, \Delta u^1 \, \Delta u^2 \, \dots \, \Delta u^{n-1}.$$

From (1.13), one thus obtains:

(1.14)
$$\left(\frac{d}{d\varepsilon} \int_{D+\varepsilon\Delta} (\mathcal{L})_0 \, dx \right)_0 = \int_S \mathcal{L} N_k \xi^k \, du ,$$

in which du signifies $du^1 \dots du^{n-1}$.

The first term in (1.10) is:

(1.15)
$$\int_D \left[\frac{d}{d\varepsilon} \mathcal{L}(x^i, z^\alpha + \varepsilon \eta^\alpha, z_k^\alpha + \varepsilon \eta_k^\alpha) \right] dx = \int_D (\mathcal{L}_{z^\alpha} \eta^\alpha + \mathcal{L}_{z_k^\alpha} \eta_k^\alpha) N_k \xi^k \, dx ,$$

in which the indices that affect $\mathcal{L}$ are indicated by partial derivatives.

We introduce the following abbreviations:

(1.16)
$$\mathcal{L}_{z^\alpha} = q_\alpha, \qquad\qquad \mathcal{L}_{z_k^\alpha} = p_k^\alpha ;$$

p_k^α is a contravariant vector in the space of x. Since η^α is a scalar in the space of x, $p_\alpha^k \eta^\alpha$ is also a contravariant vector in the same space. Now, for any vector a^k of this type one has Gauss's theorem:

(1.17)
$$\int_D \frac{\partial a^k}{\partial x^k} \, dx = \int_S a^k N_k \, du .$$

We write:

$$\mathcal{L}_{z_k^\alpha} \eta_k^\alpha = p_\alpha^k \frac{\partial \eta^\alpha}{\partial x^k} = \frac{\partial}{\partial x^k} (p_\alpha^k \eta^\alpha) - \eta^\alpha \frac{\partial p_\alpha^k}{\partial x^k} ,$$

and apply (1.17); in this case, (1.15) takes the form:

(1.18)
$$\int_D \left(q_\alpha - \frac{\partial p_\alpha^k}{\partial x^k} \right) \eta^\alpha \, dx + \int_S p_\alpha^k N_k \eta^\alpha \, du \,.$$

We substitute expression (1.9) for η^α in the surface integral.

By adding (1.14) and (1.18), one obtains the total variation:

(1.19)
$$\delta I = \int_D [\mathfrak{L}]_\alpha \eta^\alpha \, dx + \int_S (X_k \xi^k + Z_\alpha \zeta^\alpha) du \,,$$

in which, to abbreviate, we have used the following notations:

(1.20)
$$[\mathfrak{L}]_\alpha = \mathfrak{L}_{z^\alpha} - \frac{\partial}{\partial x^k} \mathfrak{L}_{z_k^\alpha} = q_\alpha - \frac{\partial p_\alpha^k}{\partial x^k} \,.$$

This operation is called the *Euler variational derivative.* The q_α correspond to the "external forces" and the q_α^k to the "internal tensions." Therefore:

(1.21)
$$\begin{cases} X_k = U_k^i N_i, \\ Z_\alpha = p_\alpha^i N_i, \end{cases}$$

in which:

(1.22)
$$U_k^i = \mathfrak{L} \delta_k^i - p_\alpha^i z_k^\alpha \qquad \left(\delta_k^i = \begin{cases} 1 & for \quad i = k \\ 0 & for \quad i \neq k \end{cases} \right),$$

is a mixed tensor of second rank in the *x*-space. In the language of functional calculus, such as was developed by Volterra and other authors, $[\mathfrak{L}]_\alpha$ is called the derivative of *I* with respect to z^α, taken at the point x^i of the domain *D*, whereas X_k and Z_α are the derivatives of *I* with respect to x^k and z^α at the point u^k of the boundary *S*.

2. Euler equations and conservation laws. – From the fundamental lemma of the calculus of variations, since the functions η^α are arbitrary, the variation δI may be annulled only if:

(2.1)
$$[\mathfrak{L}]_\alpha = = q_\alpha - \frac{\partial p_\alpha^k}{\partial x^k} = 0, \qquad\qquad (\alpha = 1, 2, \ldots, \nu).$$

These equations are the *Euler differential equations,* which constitute the equations of equilibrium or motion (field equations) in dynamical applications. When they are satisfied, the first variation of *I* reduces "limiting form:"

(2.2)
$$\delta I = \int_S (X_k \xi^k + Z_\alpha \zeta^\alpha) du \,.$$

When one likewise annuls this integral, one restricts the number and nature of the possible boundary conditions for the functions z^α on S. The simplest case is the one in which the z^α take given values on a given surface, S; in this case, $\xi^k = 0$, $\zeta^\alpha = 0$, and, as a consequence, $\delta I = 0$. Nevertheless, when one imposes no "artificial" condition of this nature, the equation $\delta I = 0$ itself provides the "natural" boundary condition. For example, if the z^α may take any given value (ζ^α arbitrary) on a given surface ($\xi^k = 0$) then the derivatives z_k^α may not be completely arbitrary on S, but must take satisfy the following "dynamical" boundary condition:

(2.3) $$Z_\alpha = p_\alpha^i N_i = 0,$$

on S.

In the applications to electrodynamics, the value of the variation δI on the boundary will be used to determine the behavior of the field in the neighborhood of the singularities (charges).

It is often useful to consider the $n\nu$ quantities z_k^α no longer as the derivatives of the functions z^α, but as independent quantities. On the other hand, if a certain number $n\nu$ of similar functions $z_k^\alpha(x^i)$ may be essentially considered to be derivatives $\dfrac{\partial z^\alpha}{\partial x^k}$ then they must satisfy the following $\nu \dfrac{n(n-1)}{2}$ "integrability conditions:"

(2.4) $$\frac{\partial z_k^\alpha}{\partial x^i} - \frac{\partial z_i^\alpha}{\partial x^k} = 0.$$

We shall now deduce an identity that leads to the conservation law for dynamics.

To that effect, a celebrated method, due to Klein [1], uses the notion of infinitesimal transformation. In our case, we are particularly concerned with a displacement of all space that does not alter the functions, z^α; we must therefore take $\xi^k = $ const., and determine the variations η^α and z^α in such a fashion that this displacement does not vary the z^α. We therefore write:

$$z^\alpha(x^i + \varepsilon\xi^i) + z^\alpha = (x^i),$$

from which, it follows that:

$$z_i^\alpha \xi^i + \eta^\alpha = 0 \qquad \text{or} \qquad \zeta^\alpha = 0.$$

This displacement therefore provokes a variation of I that is equal to:

(2.5) $$\delta I = \left[\frac{d}{d\varepsilon} \int_D \mathfrak{L}(x^k + \varepsilon\xi^k ; z^\alpha, z_k^\alpha) dx \right]_0 = \xi^k \int_D \mathfrak{L}_{x^k} dx.$$

Now, our general formula (1.19) gives $\xi^k = $ const., $\zeta^\alpha = 0$, $\eta^\alpha = -z_k^\alpha \xi^k$ for this:

(2.6) $$\delta I = \xi^k \left[-\int_D [\mathfrak{L}]_\alpha z_k^\alpha \, dx + \int_S X_k \, du \right].$$

By equating (2.5) and (2.6) and using (1.21), one obtains:

(2.7) $$\int_S U_k^i N_i \, du = \int_D ([\mathfrak{L}]_{x^k} + [\mathfrak{L}]_\alpha z_k^\alpha) \, dx.$$

This *integral identity* may be transformed into another that contains only the derivatives by using Gauss's theorem to transform the surface integral into a volume integral.

One obtains:

(2.8) $$\frac{\partial U_k^i}{\partial x^i} = [\mathfrak{L}]_{x^k} + [\mathfrak{L}]_\alpha z_k^\alpha.$$

One may obtain a more general identity by directly differentiating relation (1.22). We admit that neither the Euler equations (2.1) nor the integrability conditions (2.4) are satisfied, and consider $\mathfrak{L}$ and $p_\alpha^i = \mathfrak{L}_{z_i^\alpha}$ as functions of x^i, $z^\alpha(x^i)$. This being the case, the derivative of $\mathfrak{L}$ with respect to x^k (since z^α and z_k^α are considered to be functions of x^k) is:

$$\frac{\partial \mathfrak{L}}{\partial x^i} = \mathfrak{L}_{x^k} + \mathfrak{L}_{z^\beta} \frac{\partial z^\beta}{\partial x^k} + \mathfrak{L}_{z_i^\beta} \frac{\partial z_i^\beta}{\partial x^k}.$$

From (1.22), it results that:

$$\frac{\partial U_k^i}{\partial x^i} = \frac{\partial \mathfrak{L}}{\partial x^k} - \frac{\partial p_\beta^i}{\partial x^k} z_k^\beta - p_\beta^i \frac{\partial z_k^\beta}{\partial x^k},$$

from which:

(2.9) $$\frac{\partial U_k^i}{\partial x^i} = \mathfrak{L}_{x^k} + \left(q_\beta \frac{\partial z_k^\beta}{\partial x^k} - \frac{\partial p_\beta^i}{\partial x^k} z_k^\beta \right) + p_\beta^i \left(\frac{\partial z_i^\beta}{\partial x^k} - \frac{\partial z_k^\beta}{\partial x^i} \right).$$

This identity is more general than (2.8); nonetheless, if the integrability conditions (2.4) are satisfied then the last term disappears, and one may set $z_k^\beta = \dfrac{\partial z^\beta}{\partial x^k}$; in this case, (2.9) reduces to (2.8).

If, moreover, the Euler equation (2.1) is satisfied then one will have:

(2.10) $$\frac{\partial U_k^i}{\partial x^i} = \mathfrak{L}_{x^k} \qquad \text{or} \qquad \int_S U_k^i N_i \, du = \int_G \mathfrak{L}_{x^k} \, dx.$$

Now consider "conservative" system, in which any phenomenon is independent of the absolute position of the system in space, i.e., for which $\mathfrak{L}$ does not depend upon the x^k (space and time) explicitly:

$$\mathfrak{L}_{x^k} = 0.$$

In this case, we have the *conservation laws*:

$$(2.11) \qquad \frac{\partial U_k^i}{\partial x^i} = 0 \qquad \text{or} \qquad \int_S U_k^i N_i \, du = 0.$$

At the end of this section, we mention a particular type of system that proves to be important in electrodynamics, namely, the ones for which the Euler equations (2.1) and the integrability conditions (2.4) are interchangeable; one may call these systems *self-dual* or *auto-conjugate.*

An obvious first condition for realizing self-duality is $q_a = \mathfrak{L}_{z^\alpha} = 0$, i.e., the "force" must be null. A second condition is obviously the equality of the number of equations (2.1), on the one hand, and (2.4), on the other, namely:

$$(2.12) \qquad \nu \frac{n(n-1)}{2} = \nu,$$

whose only positive solution is $n = 2$, ν arbitrary.

The self-dual systems are therefore possible only in a two-dimensional x-space, and must have a null "force." However, this result is valid only for the *most general* system, i.e., for the ones that contain the largest possible number of Euler equations and "independent" integrability conditions; this may not be the case for certain categories of particular systems and we shall see later on that electrodynamics fall into one of these categories precisely.

One may easily convince oneself that in the general case the two necessary conditions that we indicated above are likewise satisfied. Indeed, it suffices to introduce new quantities $\mathfrak{H}, z_\alpha^{*i}, p_i^{*\alpha}$ that are defined by:

$$(2.13) \qquad \begin{cases} z_\alpha^{*1} = z_2^\alpha & z_\alpha^{*2} = -z_1^\alpha, \\ p_1^{*\alpha} = -p_\alpha^2 & p_2^{*\alpha} = p_\alpha^1, \\ \mathfrak{H} = \mathfrak{L} - p_\beta^l z_l^\beta & = \mathfrak{L} + z_\beta^{*l} p_l^{*\beta}, \end{cases}$$

and to consider a new integral to vary:

$$(2.14) \qquad I^* = \int_G \mathfrak{H}(x^l, p_l^{*\alpha}) \, dx,$$

with the integrability conditions:

$$(2.15) \qquad \frac{\partial p_2^{*\alpha}}{\partial x^1} - \frac{\partial p_1^{*\alpha}}{\partial x^2} = 0.$$

In this last relation, one concludes the existence of ν functions $p^{*\alpha}$ such that:

(2.16) $$p_i^{*\alpha} = \frac{\partial p^{*\alpha}}{\partial x^i},$$

and one obtains:

(2.17) $$\mathfrak{H}_{p^{*\alpha}} = 0, \qquad\qquad z_\alpha^{*i} = \mathfrak{H}_{p_i^{*\alpha}}.$$

The Euler equations are:

(2.18) $$\frac{\partial z_\alpha^{*1}}{\partial x^1} + \frac{\partial z_\alpha^{*2}}{\partial x^2} = 0.$$

One immediately sees that this "dual" theory is identical with the one that we have studied.

As examples of "self-dual" theories with $\nu = 1$, one may cite the following (one sets $z_i = \frac{\partial z}{\partial x_i}$):

1. The Cauchy-Riemann conditions that appear in the theory of analytic functions may be considered as integrability conditions and Euler equations for a system that has the Lagrangian:

$$\mathfrak{L} = \tfrac{1}{2}[(z_1)^2 + (z_2)^2].$$

2. The theory of minimal surfaces is derived from the Lagrangian:

$$\mathfrak{L} = \sqrt{1 + (z_1)^2 + (z_2)^2}.$$

3. A simple form for nonlinear electrodynamics was studied by Pryce, in which one completely determine the solutions that correspond to point charges, which follow from :

$$\mathfrak{L} = \sqrt{1 - (z_1)^2 - (z_2)^2}.$$

3. Mie's electrodynamics [1]. – The theory that we have described up till now describes the properties of all types of continuous dynamical systems – elasticity, hydrodynamics, electromagnetic, or gravitational fields. The treatment is the same; the only differences are the physical significance of the variables z^a and the choice of the function $\mathfrak{L}$.

More particularly, we concern ourselves with the electromagnetic field here; nevertheless, in order to make out theory absolutely invariant, we are obliged to incorporate gravitation, which, following Einstein, is related to the geometric structure of space – i.e., to its metric.

We therefore have two groups of functions z^α: the one describes the electromagnetic field, and the other, the gravitational field, or metric. Likewise, if we are given that:

$$\sqrt{-g}\, s_i\, g^{ik} = \sqrt{-g}\, s^k = \mathfrak{s}^k\,,$$

then the rule for the displacement of indices is the same for tensors as it is for tensor densities.

Now examine the properly physical problem of the respective influence of the gravitational and electromagnetic forces on the phenomena that interest us here, namely, the ones that concern the behavior of elementary material particles.

Obviously, we are completely disposed to make a place for the phenomenon of inertia in out theory, whose point of departure has been the problem of electromagnetic mass, precisely. Nevertheless, there exist very strong arguments that permit us to neglect the gravitational forces with respect to the electromagnetic forces, especially "in the interior" of particles. Consider two identical material particles of mass m and charge e and let r be the distance between them; the ratio between the forces of electric attraction (Coulomb) and gravity is equal to $\dfrac{e^2}{Km^2}$, where $K = 6.66\ \%\ 10^{-8}$ C.G.S., and the gravitational constant. By introducing the numerical values into this expression that correspond to the electron, $e/m = 5.3 \times 10^{17}$ C.G.S., one thus obtains the value for the ratio, which demonstrates the insignificance of the role that gravitation plays in the structure of electrons.

In spite of this result, there exists a large number of works that have the objective of explaining the cohesion of the electron by gravitation (i.e., the fact that it resists the action of electric forces that tend to dissipate it into space). This idea, which was introduced by Einstein [3], has given rise to numerous attempts to synthesize the laws of electromagnetism and gravitation. One has developed elegant mathematical theories that generalize the initial idea of Einstein of describing the natural laws by means of Riemannian geometry. For example, today we possess an affine geometry of the universe and a projective one, which constitute some theories that unite the laws of gravitation and electromagnetism into the same mathematical formalism. Meanwhile, for my own part, I estimate that these theories are really quite remote from the physics that I will actually describe here, and which constitute, in my opinion, one of the necessary stages that one must pass through before reaching a satisfactory theory of elementary particles.

In order for us to perfectly account for the approximation that shall introduce, we shall take a variational principle for our point of departure that does not disregard gravitation *a priori*. The most general principal will a Lagrangian $\mathfrak{L}$ for its point of departure, which is tensorial density that is a function of $g_{ik}, \dfrac{\partial g_{ik}}{\partial x^l}, \dfrac{\partial^2 g_{ik}}{\partial x^l \partial x^m}$, as well as φ_i, $\dfrac{\partial \varphi_i}{\partial x^l} = \varphi_{l,i}$. In the absence of more precise information we are nevertheless reduced to assuming that $\mathfrak{L}$ is the sum of a term that corresponds to Einstein's theory and another term that represents the electromagnetic action:

(3.4) $$I = \int \tfrac{1}{2} R\sqrt{-g}\, dx + K \int \mathfrak{L}\, dx\,,$$

in which K is the gravitational constant (Einstein's notation), and $\mathfrak{L}$ depends on φ_i and $\varphi_{l,i}$ $= \dfrac{\partial \varphi_i}{\partial x^l}$; as a result of the invariance postulate, $\mathfrak{L}$ will also depend on the g_{ik}, but we assume that it is independent of the $\dfrac{\partial g_{ik}}{\partial x^l}$ (we shall return to this point). By varying the g_{ik}, one obtains the *equations of gravitation*:

$$(3.5) \qquad\qquad R_{ik} - \tfrac{1}{2} g_{ik} R = - K T_{ik},$$

in which R_{ik} is the Riemann curvature tensor and T_{ik} is a second rank tensor whose density is given by:

$$(3.6) \qquad \sqrt{-g}\; T_{ik} = \mathfrak{T}_{ik} = \mathfrak{T}^{rs} g_{ri} g_{sk}, \qquad \mathfrak{T}^{rs} = \frac{\partial \mathfrak{L}}{\partial g_{rs}} + \frac{\partial \mathfrak{L}}{\partial g_{sr}} \qquad (^4).$$

Our fundamental approximation consists of neglecting the term in K in (3.5), which expresses the coupling between the electromagnetic field and the gravitational field. By replacing the left-hand side of (3.5) with 0, one solution of this system is a quasi-euclidean space, which one assume to be reducible, by a convenient choice of coordinate axes, to the normal form:

$$(3.7) \qquad\qquad g_{ik} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

There is no doubt that this hypothesis may be reasonably applied to all of space, except perhaps at the center of an electron itself, at which one may find deviations if the fields become infinite. Nevertheless, the object of our theory is precisely to avoid the appearance of infinitely large quantities, at least as far as energy is concerned; with these conditions, it seems that there is no risk in assuming that space still remains Euclidean at the center of the particles considered. We shall return to this question later on, when we indicate the solution of a particular case of combined gravitational and electromagnetic fields. At that point in time we shall discuss the problem of the influence of the gravitational forces on the mass of the particle.

We may assume that the g_{ik} have the values, (3.7), which are valid in a Lorentz coordinate system. We meanwhile prefer to abstain from doing this, in order to avoid the possibility of sign difficulties that originate in the signature of the matrix (g_{ik}), and to have the possibility of introducing any coordinate system in spacetime that suits us.

(^{4}) In the sequel, we shall employ the symbol ∂ with two meanings: 1. derivatives with respect to φ_i, $\varphi_{i,k}$, g_{ik}, etc.; 2. "*total*" derivatives with respect to x^k. One must be careful to observe the distinction between these two meanings.

If one takes the indicated fundamental tensor then one obtains the general electrodynamics that was developed by Mie in 1913, but with two new hypotheses:

1. The Lagrangian depends on φ_k, $\varphi_{l,k} = \dfrac{\partial \varphi_k}{\partial x^l}$, and g_{kl}, but is independent of the derivatives $\dfrac{\partial g_{rl}}{\partial x^m}$.

The integral:

(3.8)
$$I = \int \mathfrak{L}(\varphi_k, \varphi_{l,k}, g_{kl})\, dx$$

is invariant under all coordinate transformations. The interpretation of the second postulate is the following: the value of I must be *the same* in any coordinate system if $\mathfrak{L}$ is *the same function* of the preceding variables, when taken in the system considered.

Instead of 2, Mie himself naturally postulated the invariance under Lorentz transformations. In place of 1, he introduced the hypothesis that $\mathfrak{L}$ essentially depends on the derivatives $\varphi_{l,k}$, but only by the intermediary of antisymmetric combinations:

(3.9)
$$f_{kl} = \varphi_{k,l} - \varphi_{l,k} = \frac{\partial \varphi_l}{\partial x^k} - \frac{\partial \varphi_k}{\partial x^l} = -f_{lk},$$

which represent the *components of the field.* Now, one may prove that this necessarily stems from 1 and 2 if we remark that all of the invariants of the $\varphi_{k,l}$ that are distinct from the invariants of the f_{rl} necessarily contain the derivatives $\dfrac{\partial g_{kl}}{\partial x^m}$. For example, one may form the invariant that corresponds to the Lorentz invariant div $\varphi = \dfrac{\partial \varphi_k}{\partial x^k} = \varphi_{k,k}$; in this case, we must employ the covariant derivative. The quantity:

$$\text{div } \varphi = \frac{i}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, \varphi_l\, g^{kl})}{\partial x^k}$$

is invariant, but contains $\dfrac{\partial g^{kl}}{\partial x^m}$. We do not insist upon that here because we give a simple proof of this theorem in the following section (sec. 4).

For the moment, we assume that $\mathfrak{L}$ is a function of φ_k, f_{kl}, g_{kl}.

The variation of $\mathfrak{L}$ is given by:

(3.10)
$$d\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial \varphi_{kl}} d\varphi_k + \frac{\partial \mathfrak{L}}{\partial f_{kl}} df_{kl} + \frac{\partial \mathfrak{L}}{\partial g_{kl}} dg_{kl}.$$

The partial derivative of $\mathfrak{L}$ with respect to one of the variables f_{kl} or g_{kl} must be taken without accounting for the symmetry properties of the latter; therefore, one must, for example, take the derivatives with respect to g_{kl} or g_{lk} separately.

We now introduce the following notation:

$$(3.10a) \qquad \mathfrak{s}^k = \frac{\partial \mathfrak{L}}{\partial \varphi_k}, \qquad \mathfrak{p}^{kl} = \frac{\partial \mathfrak{L}}{\partial f_{kl}} - \frac{\partial \mathfrak{L}}{\partial f_{lk}}, \qquad \mathfrak{T}^{kl} = \frac{\partial \mathfrak{L}}{\partial g_{kl}} + \frac{\partial \mathfrak{L}}{\partial g_{lk}}.$$

The derivatives of $\mathfrak{L}$ with respect to f_{kl} and g_{kl} have actual symmetry property; by contrast, $\mathfrak{p}^{kl}$ and $\mathfrak{T}^{kl}$ are antisymmetric and symmetric, respectively:

$$(3.11) \qquad\qquad \mathfrak{p}^{kl} = -\,\mathfrak{p}^{kl}, \qquad\qquad \mathfrak{T}^{kl} = \mathfrak{T}^{kl}.$$

One may now write (3.10) in the form:

$$d\mathfrak{L} = \mathfrak{s}^k\,d\varphi_k + \tfrac{1}{2}\left(\frac{\partial \mathfrak{L}}{\partial f_{kl}}\,df_{kl} + \frac{\partial \mathfrak{L}}{\partial f_{lk}}\,df_{lk} \right) + \tfrac{1}{2}\left(\frac{\partial \mathfrak{L}}{\partial g_{kl}}\,dg_{kl} + \frac{\partial \mathfrak{L}}{\partial g_{lk}}\,dg_{lk} \right),$$

and since $f_{kl} = -f_{lk}$, $g_{kl} = g_{lk}$:

$$(3.10b) \qquad\qquad d\mathfrak{L} = \mathfrak{s}^k\,d\varphi_k + \tfrac{1}{2}\mathfrak{p}^{kl}\,df_{kl} + \tfrac{1}{2}\mathfrak{T}^{kl}\,dg_{kl}.$$

The *invariance postulate* has another important consequence, which is expressed by the relation:

$$(3.12) \qquad\qquad \mathfrak{L}\delta_k^l = \mathfrak{s}^k\varphi_k + \mathfrak{p}^{kl}f_{kl} + \mathfrak{T}^{kl}g_{kl},$$

which we prove in the following section (sec. 4).

From our notion of derivative, we have:

$$\mathfrak{p}^{k,l} = \frac{\partial \mathfrak{L}}{\partial \varphi_{k,l}} = \frac{\partial \mathfrak{L}}{\partial f_{ij}} \frac{\partial f_{ij}}{\partial \varphi_{k,l}} = \frac{\partial \mathfrak{L}}{\partial f_{kl}} - \frac{\partial \mathfrak{L}}{\partial f_{lk}} = \mathfrak{p}^{kl}.$$

This being the case, the Euler equations that come from the variation of the integral (3.8) − in other words, the *field equations* – have the form:

$$(3.13) \qquad\qquad \frac{\partial \mathfrak{p}^{lk}}{\partial x^l} = \mathfrak{s}^k$$

(for the translation into ordinary vector notation, see sec. 5). They show that the $\mathfrak{p}^{lk}$ constitute the *other category of field components,* and that the $\mathfrak{s}^k$ represent the *spatial density of charge and current.* From (3.13), one deduces the *equation of continuity*:

$$(3.14) \qquad\qquad \frac{\partial \mathfrak{s}^k}{\partial x^l} = 0 \,.$$

The *integrability conditions* (2.4) of the general theory lose any significance if one takes the f_{kl} instead of the $\varphi_{k,l}$ as variables that $\mathfrak{L}$ depends upon; they are replaced by the following conditions:

$$(3.15) \qquad\qquad \frac{\partial f_{ik}}{\partial x^l} + \frac{\partial f_{kl}}{\partial x^i} + \frac{\partial f_{li}}{\partial x^k} = 0,$$

which are equivalent to (3.9), and which are necessary and sufficient for us to conclude the existence of potentials φ_i when one starts with a given set of f_{kl}.

In an *n*-dimensional space, their number is equal to:

$$\binom{n}{3} = \tfrac{1}{6} n(n-1)(n-2) \,,$$

which gives 4 for $n = 4$. They are not all independent; there are only 3 independent ones for $n = 4$. If one introduces the ordinary vector notation, one sees that (3.15) represents the *second group of field equations*.

We now examine the problem of *conservation laws*. For this, we transcribe the expression, (1.22), for U_k^l for our particular case:

$$(3.16) \qquad \mathfrak{U}_k^l = \mathfrak{L}\delta_k^l - \mathfrak{p}^{l,j}\varphi_{k,j} \qquad\qquad \left(\mathfrak{p}^{l,j} = \frac{\partial \mathfrak{L}}{\partial \varphi_{l,j}} \right).$$

We consider the g_{kl} to be given functions of the x^k in appearance, so it therefore does not seem to be a conservation law, properly speaking, but simply the relation (2.10), which may be written here:

$$(3.17) \qquad\qquad \frac{\partial \mathfrak{U}_k^l}{\partial x^l} = \tfrac{1}{2}\mathfrak{T}^{rs}\frac{\partial g_{rs}}{\partial x^k} \,.$$

Another difficulty is the fact that, from (3.16), $\mathfrak{U}_k^l$ depends on the $\varphi_{k,l}$ explicitly. Meanwhile, if one substitutes the expression (3.12) for $\mathfrak{L}\delta_k^l$ then, due to the fact that $\mathfrak{p}^{k,l} = \mathfrak{p}^{kl}$, one obtains:

$$(3.18) \qquad\qquad \mathfrak{U}_k^l = \mathfrak{T}_k^l + \mathfrak{s}^l\varphi_k + \mathfrak{p}^{jl}\varphi_{j,k} \,,$$

in which we have made use of the usual rule for lowering the index k of $\mathfrak{T}^{kl}$. With the aid of this equations, one may eliminate the $\mathfrak{U}_k^l$ from (3.17), and one obtains a relation for $\mathfrak{T}^{kl}$ that may be interpreted as a truly covariant conservation law.

In regard to (3.13), one has:

$$(3.19) \quad \begin{cases} \mathfrak{s}^l \varphi_k + \mathfrak{p}^{jl} \varphi_{j,k} = \dfrac{\partial \mathfrak{p}^{jl}}{\partial x^j} \varphi_k + \mathfrak{p}^{jl} \dfrac{\partial \varphi_k}{\partial x^j} = \dfrac{\partial}{\partial x^j} (\mathfrak{p}^{jl} \varphi_k), \\[2mm] \dfrac{\partial}{\partial x^j} (\mathfrak{s}^l \varphi_k + \mathfrak{p}^{jl} \varphi_{j,k}) = \dfrac{\partial^2}{\partial x^l \partial x^j} (\mathfrak{p}^{jl} \varphi_k) = 0, \end{cases}$$

since $\mathfrak{p}^{jl}$ is antisymmetric. This shows that $\mathfrak{T}_k^l$ and $\mathfrak{U}_k^l$, which are related by (3.18), differ only by a quantity with vanishing divergence; hence:

$$(3.20) \quad \frac{\partial \mathfrak{U}_k^l}{\partial x^l} = \frac{\partial \mathfrak{T}_k^l}{\partial x^l},$$

and (3.17) becomes:

$$(3.21) \quad \frac{\partial \mathfrak{T}_k^l}{\partial x^l} - \tfrac{1}{2} \mathfrak{T}^{rs} \frac{\partial g_{rs}}{\partial x^l} = 0.$$

The expression in the left-hand side is the *covariant divergence* of the tensorial density $\mathfrak{T}_k^l$, [4]; the latter satisfies a covariant conservation law, as was intended.

The preceding argument directly exhibits the relation between Mie's electrodynamics and the general case that was treated in sec. 1, by assuming that relation (3.12) is known. We indicate this in another way by starting with the law (3.21).

From the definition (3.10*b*) one has:

$$\frac{\partial}{\partial x^l} (\mathfrak{L} \delta_k^l) = \frac{\partial \mathfrak{L}}{\partial x^l} = \mathfrak{s}^k \frac{\partial \varphi_k}{\partial x^l} + \tfrac{1}{2} \mathfrak{p}^{kl} \frac{\partial f_{kl}}{\partial x^l} + \tfrac{1}{2} \mathfrak{T}^{kl} \frac{\partial g_{kl}}{\partial x^l}.$$

As a result, by virtue of (3.14):

$$\frac{\partial (\mathfrak{s}^k \varphi_k)}{\partial x^l} = \mathfrak{s}^l \frac{\partial \varphi_k}{\partial x^l} + \varphi_k \frac{\partial \mathfrak{s}^l}{\partial x^l} = \mathfrak{s}^l \frac{\partial \varphi_k}{\partial x^l},$$

and, by using (3.13):

$$\frac{\partial (\mathfrak{p}^{lj} f_{kj})}{\partial x^l} = \mathfrak{p}^{lj} \frac{\partial f_{kj}}{\partial x^l} + \frac{\partial \mathfrak{p}^{lj}}{\partial x^l} f_{kj}$$

$$= \mathfrak{p}^{lj} \frac{\partial f_{kj}}{\partial x^l} + \mathfrak{s}^j f_{kj} = \tfrac{1}{2} \mathfrak{p}^{lj} \left(\frac{\partial f_{kj}}{\partial x^l} + \frac{\partial f_{jk}}{\partial x^l} \right) + \mathfrak{s}^l f_{kl}.$$

By adding these equations and taking (3.15) into account, one obtains:

$$(3.22) \quad \frac{\partial}{\partial x^j} [\mathfrak{L} \delta_k^l - \mathfrak{s}_l \varphi_k - \mathfrak{p}^{lj} f_{kj}] = \tfrac{1}{2} \mathfrak{T}^{rs} \frac{\partial g_{rs}}{\partial x^j}.$$

This anticipates the proof of formula (3.12), which will be established in section 4 by invariance considerations; the expression between parentheses is found to be $\mathfrak{T}^{lm}g_{lm} = \mathfrak{T}^l_k$, and (3.22) becomes (3.21).

When the field equations are satisfied, the variation of I reduces to its value on the boundary, which we write as:

(3.23)
$$\delta I = \int_S (\mathfrak{X}_k \, \delta x^k + \mathfrak{P}^k \, \delta \varphi_k) \, du$$

with

(3.24)
$$\mathfrak{X}_k = \mathfrak{U}^l_k N_l, \qquad\qquad \mathfrak{P}^k = \mathfrak{p}^{lk} N_l.$$

Since and differ only by terms with null divergence, one may replace the first equation with:

(3.24a)
$$\mathfrak{X}_k = \mathfrak{T}^l_k N_l.$$

The importance of this surface integral will appear much later, when we concern ourselves with the motion of point-like charges [5].

4. Generalized invariance. – Consider an infinitesimal coordinate transformation that is given by:

(4.1)
$$\bar{x}^k = x^k + \varepsilon \xi^k(\bar{x}^i),$$

in which the ξ^k are arbitrary continuous functions of the x^i. The inverse transformation is:

(4.2)
$$x^k = \bar{x}^k - \varepsilon \xi^k(\bar{x}^i).$$

We introduce that expression into (3.3), which represents the transformation of φ_i and g_{ik} that is induced by the preceding; one easily obtains:

(4.3)
$$\begin{cases} \bar{\varphi}_k = \varphi_k - \varepsilon \dfrac{\partial \xi^l}{\partial x^k} \varphi_l, \\[2ex] \bar{g}_{kl} = g_{kl} - \varepsilon \left(g_{ik} \dfrac{\partial \xi^i}{\partial x^l} + g_{ij} \dfrac{\partial \xi^j}{\partial x^l} \right), \end{cases}$$

and, upon differentiating $\bar{\varphi}_k$:

$$\bar{\varphi}_{l,k} = \frac{\partial \bar{\varphi}_k}{\partial \bar{x}^l} = \frac{\partial}{\partial x^m}\left(\varphi_k - \varepsilon \frac{\partial \xi^j}{\partial x^k} \varphi_j \right) \frac{\partial x^m}{\partial \bar{x}^l},$$

or

(4.4)
$$\bar{\varphi}_{l,k} = \varphi_{l,k} - \varepsilon \left(\varphi_{j,k} \frac{\partial \xi^j}{\partial x^l} + \varphi_{l,j} \frac{\partial \xi^j}{\partial x^k} + \frac{\partial \xi^j}{\partial x^k \partial x^l} \varphi_j \right).$$

In (4.3) and (4.4), the right-hand sides must be considered as functions of x^k and the left-hand sides as functions of $\overline{x}^k$, which are related to the x^k by the transformation (4.1).

We call the term in ε the "variation" $\delta\varphi_k$ of φ_k, the variation δg_{kl} of g_{kl}, the variation $\delta\varphi_{k,l}$ of $\varphi_{k,l}$, etc.; one has:

$$\delta\varphi_k = \overline{\varphi}_k(\overline{x}_l) - \varphi_k(x_l), \cdots$$

Now write:

(4.5)
$$\overline{\mathcal{L}} = \mathcal{L}(\overline{\varphi}_k, \overline{g}_{kl}, \overline{\varphi}_{l,k}) = \mathcal{L}(\varphi_k, g_{kl}, \varphi_{l,k}) + \delta\mathcal{L},$$

and use the notations:

(4.6)
$$\mathfrak{s}^k = \frac{\partial\mathcal{L}}{\partial\varphi_k}, \qquad \mathfrak{p}^{k,j} = \frac{\partial\mathcal{L}}{\partial\varphi_{k,j}}, \qquad \mathfrak{T}^{kj} = \frac{\partial\mathcal{L}}{\partial g^{kj}} + \frac{\partial\mathcal{L}}{\partial g^{jk}},$$

one has:

(4.7)
$$\delta\mathcal{L} = \mathfrak{s}^k \, \delta\varphi_k + \mathfrak{p}^{k,j} \, \delta\varphi_{k,j} + \tfrac{1}{2} \, \mathfrak{T}^{kj} \, \delta g_{kj}.$$

Upon substituting the value of the variations given by (4.3) and (4.4), and conveniently modifying the indices, one obtains:

(4.8)
$$\delta\mathcal{L} = -\varepsilon(\mathfrak{s}^k \varphi_k + \mathfrak{p}^{k,j} \varphi_{k,j} + \mathfrak{T}^{kj} g_{kj}) \frac{\partial\xi^k}{\partial x^l} + \mathfrak{p}^{k,j} \, \varphi_j \frac{\partial^2\xi^j}{\partial x^k \partial x^l}.$$

The Jacobian of the transformation, (4.1), is:

(4.9)
$$\frac{\partial(\overline{x}^1, \overline{x}^2, \overline{x}^3, \overline{x}^4)}{\partial(x^1, x^2, x^3, x^4)} = 1 + \varepsilon \frac{\partial\xi^k}{\partial x^k} + \cdots$$

With the notations that we introduced, the invariance postulate 2 of section 3 is expressed by:

(4.10)
$$\int_{\overline{D}} \overline{\mathcal{L}} d\overline{x} = \int_D \mathcal{L} dx,$$

in which $\overline{D}$ is the domain of the $\overline{x}^k$ that corresponds to the domain D of the x^k by the transformation (4.1). By virtue of (4.5) and (4.9), one obtains:

(4.11)
$$\int_D \left[(\mathcal{L} + \delta\mathcal{L})\left(1 + \varepsilon \frac{\partial\xi^k}{\partial x^k}\right) - \mathcal{L} \right] dx = 0.$$

Since this relation is valid for any domain D it must be the quantity under the $\int$ sign that is annulled.

By replacing $d\mathcal{L}$ with its value (4.8) one obtains an equation of the following form:

(4.12)
$$\mathfrak{D}_k^l \frac{\partial \xi^k}{\partial x^l} + \mathfrak{p}^{k,j} \varphi_j \frac{\partial^2 \xi^j}{\partial x^k \partial x^l} = 0,$$

in which

(4.13)
$$\mathfrak{D}_k^l = \mathfrak{L} \delta_k^l - (\mathfrak{s}^k \varphi_k + \mathfrak{p}^{k,j} \varphi_{k,j} + \mathfrak{p}^{j,l} \varphi_{j,k} + \mathfrak{T}^{kj} g_{kj}).$$

In (4.12), the values of the first two derivatives of x^k are arbitrary at any given point; since $\dfrac{\partial^2 \xi^j}{\partial x^k \partial x^l}$ is symmetric in $k, l,$ one has:

(4.14)
$$\mathfrak{p}^{k,l} + \mathfrak{p}^{l,k} = 0, \qquad\qquad \mathfrak{D}_k^l = 0.$$

The first of these equations shows that $\mathfrak{L}$ *depends only upon the combinations $f_{kl} = \varphi_{k,l} - \varphi_{l,k}$.* Because of (4.13), the second one shows [1] the relation (3.12):

(4.15)
$$\mathfrak{L} \delta_k^l = \mathfrak{s}^k \varphi_k + \mathfrak{p}^{k,j} f_{kj} + \mathfrak{T}^{kj} g_{kj}.$$

Now consider the problem of expressing the function $\mathfrak{L}$, or rather, the corresponding invariant L that is defined by $\mathfrak{L} = L\sqrt{-g}$ by means of the *elementary invariants* of φ_k, f_{kl}, g_{kl}.

The invariants of the tensors f_{kl} and g_{kl} may be determined as a linear combination of the two matrices $g = (g_{ik})$, $f = (f_{ik})$ in a linear family $a(\lambda) = [a_{ik}(\lambda)]$ defined by:

(4.16)
$$g + \lambda f = (1 + \lambda f g^{-1})g = a(\lambda)g.$$

Since $g^{-1} = [g_{ik}]$, the general element of fg^{-1} is $f_{ij} g^{jk} = f_i^{\,k}$.

The element of the unit matrix δ_i^k is likewise that of a mixed tensor, and one has:

$$a_i^k(\lambda) = \delta_i^k + \lambda f_i^{\,k}.$$

It follows from this that the simultaneous invarinats of f and g are the coefficients of the characteristic polynomial, or of the determinant of $a(\lambda)$:

(4.17)
$$| a(\lambda) | = | 1 + fg^{-1} | = 1 + F_1 \lambda + F_2 \lambda^2 + \ldots + F_n \lambda^n.$$

Since f is anti-symmetric and g is symmetric, the transposed matrix to $g + \lambda f$ is $g - \lambda f = a(-\lambda)g$. Since the determinant of the transposed matrix is identical to that of the initial matrix, one has $| a(\lambda) | = | a(-\lambda) |$; therefore, the development (4.17) of $a(\lambda)$ contains only even powers. For $n = 4$, one will have:

(4.18)
$$| a(\lambda) | = 1 + F \lambda^2 - G^2 \lambda^4,$$

with the two invariants:

$$(4.19) \quad \begin{cases} F = \frac{1}{2} f_k^i f_i^k = \frac{1}{2} f_{ik} f^{ik} = \frac{1}{2} f_{im} f_{kn} g^{ik} g^{mn}, \\ G^2 = - \mid fg^{-1} \mid = \dfrac{\mid f \mid}{- \mid g \mid} = \dfrac{1}{- \mid g \mid} (f_{23} f_{14} + f_{31} f_{24} + f_{12} f_{34})^2. \end{cases}$$

The square root of the second invariant is not itself an invariant with respect to *all* of the transformations, but only with respect to the ones that have a positive Jacobian. *G* may be written in a simple form with the aid of the *dual tensor* that was introduced by Maxwell.

By slightly modifying the definition of the latter, we introduce the *contravariant dual tensorial density* that corresponds to the *covariant anti-symmetric* tensor f_{ik} the defining formulas will be:

$$(4.20) \quad \begin{cases} \mathfrak{f}^{*23} = f_{14}, \quad \mathfrak{f}^{*31} = f_{24}, \quad \mathfrak{f}^{*12} = f_{34}, \\ \mathfrak{f}^{*14} = f_{23}, \quad \mathfrak{f}^{*24} = f_{31}, \quad \mathfrak{f}^{*34} = f_{12}. \end{cases}$$

Here, the usual law for the displacement of indices in a Lorentz system (namely: change the sign for indices 1, 2, 3, but not 4) gives:

$$(4.21) \quad \begin{cases} \overset{*}{\mathfrak{f}}_{23} = -f^{14}, \quad \overset{*}{\mathfrak{f}}_{34} = -f^{24}, \quad \overset{*}{\mathfrak{f}}_{12} = -f^{34}, \\ \overset{*}{\mathfrak{f}}_{14} = -f^{23}, \quad \overset{*}{\mathfrak{f}}_{24} = -f^{31}, \quad \overset{*}{\mathfrak{f}}_{34} = -f^{12}. \end{cases}$$

$\overset{*}{\mathfrak{f}}_{kl}$ is a tensorial density; one may see this [2] by means of the *completely anti-symmetric tensorial density* δ^{iklm}, which is defined in the following manner. We develop the determinant of the tensor a_{kl}:

$$(4.22) \quad \mid a \mid = \frac{1}{4!} a_{k\kappa} a_{l\lambda} a_{m\mu} a_{n\nu} \delta^{klmn} \delta^{\kappa\lambda\mu\nu}.$$

The coefficients are:

$$(4.23) \quad \delta^{iklm} = \begin{cases} +1 & \textit{if iklm is an even permutation of } 1,2,3,4; \\ -1 & \textit{if iklm is an odd permutation of } 1,2,3,4; \\ 0 & \textit{in any other case.} \end{cases}$$

This being the case, we apply (4.22) to the tensor g_{kl}; if we divide by $- \mid g \mid$. One obtains the invariant -1 in the left-hand side; therefore, $\dfrac{\delta^{klmn}}{\sqrt{- \mid g \mid}}$ is a fourth order contravariant tensor and δ^{klmn} is the corresponding tensorial density. As usual, define the covariant tensorial density:

$$(4.24) \quad \mathfrak{d}_{klmn} = \mathfrak{d}^{\kappa\lambda\mu\nu} g_{k\kappa} g_{l\lambda} g_{m\mu} g_{n\nu},$$

and multiply by $\mathfrak{d}^{klmn}$; by virtue of (4.22), one obtains:

$$\eth_{klmn} \, \eth^{klmn} = 4! \mid g \mid.$$

By replacing a_{kl} with the unit tensor in (4.22), one sees that:

$$\eth^{klmn} \, \eth^{klmn} = 4! \, ;$$

therefore, from the preceding:

(4.25) $$\eth_{klmn} = \mid g \mid \eth^{klmn}.$$

We may now write:

(4.26) $$\mathfrak{f}^{*kl} = \tfrac{1}{2} \eth^{klmn} f_{mn}, \qquad \mathfrak{f}^*_{kl} = \tfrac{1}{2} \, \eth_{klmn} f^{mn},$$

which shows precisely that $\mathfrak{f}^{*kl}$ and $\mathfrak{f}^*_{kl}$ are tensorial densities.

The square root of the second equation (4.19) gives:

(4.27) $$G = \frac{1}{\sqrt{-g}} \, \tfrac{1}{8} \eth^{klmn} f_{kl} f_{mn} = \frac{1}{\sqrt{-g}} \, \tfrac{1}{4} f_{kl} \, \mathfrak{f}^{*mn} = \tfrac{1}{4} f_{kl} f^{*mn},$$

in which f^{kl} is the tensor that corresponds to the density $\mathfrak{f}^{kl}$.

Now consider the simultaneous invariants of φ_k and f_{kl} with respect to the metric tensor g_{kl}. We proceed in the following manner: we first form the vectors:

(4.28) $$\psi_k = f_{kl} \varphi^l, \qquad \chi_k = f^*_{kl} \, \varphi^l,$$

and then the invariants of the three vectors φ_k, ψ_k, and χ_k, namely:

the squares of the length:

(4.29) $$\begin{cases} A = \varphi_k \varphi^k = \varphi_k \varphi_l g^{kl}, \\ B = \psi_k \psi^k = f_{kl} f^{km} \varphi^l \varphi_m, \\ C = \chi_k \chi^k = f^*_{kl} f^{*km} \varphi^l \varphi_m, \end{cases}$$

the scalar products:

(4.30) $$\begin{cases} \alpha = \psi_k \varphi^k = f_{kl} \varphi^k \varphi^l, \\ \beta = \chi_k \varphi^k = f^*_{kl} \varphi^l \varphi_m, \\ \gamma = \psi_k \chi^k = f_{kl} f^{*km} \varphi^l \varphi^m. \end{cases}$$

since f_{kl} is anti-symmetric, one has:

(4.31) $$\alpha = 0, \quad \beta = 0;$$

moreover, one has the identities:

(4.32) $B - C = FA,$ $\gamma = GA,$

which are a consequence of the relations:

(4.33) $\begin{cases} f_{il} f^{kl} - f_{il}^* f^{*kl} = F \delta_i^k, \\ f_{il} f^{*kl} = f_{il}^* f^{*kl} = G \delta_i^k. \end{cases}$

One may easily verify the exactitude of these formulas in a Lorentz system, for example:

$$f_{1l} f^{2l} - f_{1l}^* f^{*2l} = f_{13} f^{23} + f_{14} f^{24} - f_{13}^* f^{*23} - f_{14}^* f^{*24}$$
$$= f_{13} f^{23} + f_{14} f^{24} - f^{24} f_{14} - f^{23} f_{34} = 0,$$
$$f_{1l} f^{*2l} = f_{13} f^{*23} + f_{14} f^{*24} = f_{13} f_{14} - f_{14} f_{31} = 0.$$

The most general Lagrangian will thus be:

$$\mathfrak{L} = L(A, B, C, G) \sqrt{-g},$$

In which L is an arbitrary function of the four elementary invariants.

At this point, we verify the exactitude of relation (3.12) by a simple algebraic reasoning [3].

Let $\mathfrak{J}$ be any of the four elementary invariants; the three quantities:

$$\frac{\partial \mathfrak{J}}{\partial g_{km}} g_{ml}, \qquad \frac{\partial \mathfrak{J}}{\partial f_{km}} f_{ml}, \qquad \frac{\partial \mathfrak{J}}{\partial \varphi_k} \varphi_l,$$

are homogenous functions of the g_{kl} of the same degree n_g as $\mathfrak{J}$ itself; the same is true with respect to f_{kl} and φ_k, whose degrees of homogeneity are n_f and n_φ, respectively. Moreover, one has:

(4.35) $\frac{\partial \mathfrak{J}}{\partial g_{km}} g_{ml} = n_g K_l^k, \qquad \frac{\partial \mathfrak{J}}{\partial f_{km}} f_{ml} = n_f K_l^k, \qquad \frac{\partial \mathfrak{J}}{\partial \varphi_k} \varphi_l = n_\varphi K_l^k,$

with *the same* K_l^k in the three cases.

Upon examining the structure of these invariants, one sees that they are all formed from products of the φ_k and f_{kl} multiplied by the corresponding product with upper indices. The number of upper indices is therefore equal to $\frac{1}{2} n_\varphi + n_f$, but, at the same time, it is obviously equal to the number of factors g^{kl} that are necessary to lower the indices. Since g^{kl} has degree –1 with respect to g_{kl} one will have:

(4.36) $- n_g = \frac{1}{2} n_\varphi + n_f.$

This being the case, we differentiate $\mathfrak{L}$; we obtain:

$$\mathfrak{T}^{kl} = \frac{\partial \mathfrak{L}}{\partial g_{kl}} + \frac{\partial \mathfrak{L}}{\partial g_{lk}} = \left(\frac{\partial \sqrt{-g}}{\partial g^{kl}} + \frac{\partial \sqrt{-g}}{\partial g^{lk}} \right) L + \sqrt{-g} \sum \left(\frac{\partial \mathfrak{J}}{\partial g_{kl}} + \frac{\partial \mathfrak{J}}{\partial g_{lk}} \right),$$

$$\mathfrak{T}^{kl} = \frac{\partial \mathfrak{L}}{\partial f_{kl}} - \frac{\partial \mathfrak{L}}{\partial f_{lk}} = \sqrt{-g} \sum \left(\frac{\partial \mathfrak{J}}{\partial f_{kl}} - \frac{\partial \mathfrak{J}}{\partial f_{lk}} \right),$$

$$\mathfrak{s}^k = \frac{\partial \mathfrak{L}}{\partial \varphi_k} = \sqrt{-g} \sum \frac{\partial \mathfrak{J}}{\partial \varphi_k},$$

in which the sum is taken over the four elementary invariants. By virtue of (4.35), (4.36), one deduces:

$$\mathfrak{s}^k \varphi_l + \mathfrak{p}^{km} f_{lm} + \mathfrak{T}^{km} g_{lm} = \sqrt{-g}\, g^{km} g_{ml} L + \sqrt{-g} \Sigma K_l^k (n_\varphi + 2n_f + 2n_g)$$
$$= L \delta_l^k \sqrt{-g} = \mathfrak{L} \delta_l^k,$$

in accord with (3.12).

5. Vectorial notation in ordinary space. - Naturally, Mie himself did not develop his theory by making use of the generalized invariance principle, which was introduced by Einstein some time later. He used only the vectorial notation of ordinary space, and postulated invariance only under Lorentz transformations. In my opinion, this restriction is not essential, and I believe that today there is still no reason to change the words that I published in an article in 1913 [1] in which I introduced the quadrivector and the six-component Minkowski vector, and in which I gave a simplified proof of Mie's conservation laws (which is, moreover, equivalent to the one that was used here, page ???). The words are the following:

"The universally recognized incompatibility between the differential equations of the electromagnetic field and the existence of points (electrons) at which the charge accumulates and persists without the benefit of external forces is closely related to the linear character of these equations; above all, it is therefore necessary to abandon the linear character of the fundamental equations. Mie has realized this program in the most general and most elegant manner that one may conceive of in the context of physics today, which has its origin in the analytical mechanics of Lagrange."

The reasons for which Mie's theory did not succeed in the context of classical physics reside in the scope of his object; we correct this point in the following section.

Obviously, the appearance of quantum theory has considerably modified this problem.

In what follows, we take the velocity of light equal to unity and adopt the Heaviside units for electromagnetic quantities.

With the ordinary vectorial notation, we have, on the one hand, the *spacetime vectors*:

$$(5.1) \quad \begin{cases} (x^1, x^2, x^3, x^4) = (x, y, z, t) & = (\mathbf{r}, t) & (coordinates, time), \\ (\varphi_1, \varphi_2, \varphi_3, \varphi_4) & = (\mathbf{a}, \varphi) & (vector\ and\ scalar\ potential), \\ (\mathfrak{s}^1, \mathfrak{s}^2, \mathfrak{s}^3, \mathfrak{s}^4) & = -(\rho\mathbf{v}, \rho) & (current\ and\ charge\ density), \end{cases}$$

and, on the other hands, the *six-component vector of the field*:

$$(5.2) \quad \begin{cases} (f_{23}, f_{31}, f_{12}) = & (\mathfrak{f}^{*14}, \mathfrak{f}^{*24}, \mathfrak{f}^{*34}) & = \mathbf{B}, \\ (f_{14}, f_{24}, f_{34}) = & (\mathfrak{f}^{*23}, \mathfrak{f}^{*31}, \mathfrak{f}^{*12}) & = \mathbf{E}, \end{cases}$$

$$(5.3) \quad \begin{cases} (\mathfrak{p}^{23}, \mathfrak{p}^{23}, \mathfrak{p}^{23}) = & (-\mathfrak{p}^*_{14}, -\mathfrak{p}^*_{24}, -\mathfrak{p}^*_{34}) & = \mathfrak{H}, \\ (\mathfrak{p}^{14}, \mathfrak{p}^{24}, \mathfrak{p}^{34}) = & (-\mathfrak{p}^*_{23}, -\mathfrak{p}^*_{31}, -\mathfrak{p}^*_{12}) & = -\mathfrak{D}. \end{cases}$$

With this notation, the field equations, (3.13) and (3.15), are:

$$(5.4) \quad \begin{cases} rot\mathbf{E} + \mathbf{B} = 0, & div\mathbf{B} = 0, \\ rot\mathfrak{H} - \mathfrak{D} = \rho\mathbf{v}, & div\mathfrak{D} = \rho. \end{cases}$$

The form of the Maxwell equations is preserved in any coordinate system [2]; this results from the fact that we have chosen to define (**B**, **E**) as a tensor and ($\mathfrak{H}$, $\mathfrak{D}$) as a *tensorial density*.

This difference disappears in a Lorentz system in which the *gik* are given by (3.7); one has:

$$(5.5) \quad \left. \begin{cases} (s_1, s_2, s_3) = (\rho\mathbf{v}, -\rho) \\ (p_{23}, p_{31}, p_{12}) = \mathfrak{H} = \mathbf{H}, \\ (\mathfrak{p}_{14}, \mathfrak{p}_{24}, \mathfrak{p}_{34}) = \mathfrak{D} = \mathbf{D}, \end{cases} \right\} \quad (Lorentz\ system),$$

which, with (5.2), forms the usual definition, which coincides with ours only in the case of special relativity.

The easiest manner of writing the tensor density of energy-momentum (3.12) consists of employing the matrix notation; one has:

$$(5.6) \quad (\mathfrak{T}^l_k) = \begin{vmatrix} \mathfrak{L} & 0 & 0 & 0 \\ 0 & \mathfrak{L} & 0 & 0 \\ 0 & 0 & \mathfrak{L} & 0 \\ 0 & 0 & 0 & \mathfrak{L} \end{vmatrix} + \begin{vmatrix} 0 & \mathfrak{H}_z & -\mathfrak{H}_y & -\mathfrak{D}_x \\ -\mathfrak{H}_z & 0 & -\mathfrak{H}_x & -\mathfrak{D}_y \\ \mathfrak{H}_y & \mathfrak{H}_x & 0 & -\mathfrak{D}_z \\ \mathfrak{D}_x & \mathfrak{D}_y & \mathfrak{D}_z & 0 \end{vmatrix}$$

$$\times \begin{vmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{vmatrix} + \begin{vmatrix} a_x \\ a_y \\ a_z \\ \varphi \end{vmatrix} (\rho\mathbf{v}_x, \rho\mathbf{v}_y, \rho\mathbf{v}_z, \rho),$$

or, explicitly:

(5.7)
$$\begin{cases}
\mathfrak{T}_1^1 = \mathfrak{L} - \mathfrak{H}_y B_y - \mathfrak{H}_z B_z + \mathfrak{D}_x E_x + \rho \mathfrak{v}_x a_x, \\
\mathfrak{T}_4^4 = \mathfrak{L} + \mathfrak{D}\mathbf{E} + \rho\varphi, \\
\mathfrak{T}_2^1 = \mathfrak{H}_x B_y + \mathfrak{D}_y E_x + \rho \mathfrak{v}_y a_x, \\
\mathfrak{T}_1^2 = \mathfrak{H}_y B_x + \mathfrak{D}_x E_y + \rho \mathfrak{v}_x a_y, \\
\mathfrak{T}_4^1 = \mathfrak{D}_z B_y - \mathfrak{D}_y B_z + \rho \mathfrak{v}_x \rho, \\
\mathfrak{T}_2^4 = \mathfrak{H}_z E_y - \mathfrak{H}_y E_z + \rho a_x.
\end{cases}$$

The relations are valid in any coordinate system. In this case, the symmetry of the tensor:

(5.8)
$$T^{kl} = g^{km} T_m^l = \frac{1}{\sqrt{-g}} g^{km} \mathfrak{T}_m^l$$

is not a simple condition; nevertheless, if one considers a Lorentz system with the g_{kl} that are given by (3.7), one will have:

(5.9) $T^{kl} = -T_l^k$ for $l = 1, 2, 3,$ and $T^{k4} = T_4^k$.

Now introduce the following vector notations:

(5.10)
$$\begin{cases}
(M) = \begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix} = Maxwell\ tensions, \\
U = energy\ density, \\
\mathbf{S} = Poynting\ vector.
\end{cases}$$

One may then write the following formulas, which have an immediate significance:

(5.11) $(T^{kl}) = \begin{vmatrix} (M) & -\mathbf{S} \\ -\mathbf{S} & U \end{vmatrix},$ $(T_k^l) = \begin{vmatrix} -(M) & -\mathbf{S} \\ \mathbf{S} & U \end{vmatrix},$

and then one has:

(5.12)
$$\begin{cases}
X_x = H_y B_y + H_z B_z - D_x E_x - \rho v_x a_x - L, \\
U = \mathbf{D} \cdot \mathbf{E} + \rho\varphi + L, \\
X_y = Y_x = -H_y B_x - D_x E_y - \rho v_x a_y = -H_x B_y - D_y E_x - \rho v_y a_x, \\
\mathbf{S} = (\mathbf{E} \times \mathbf{H}) + \rho \mathbf{a} = (\mathbf{D} \times \mathbf{B}) - \rho\varphi \mathbf{v}.
\end{cases}$$

In this case, the symmetry condition on T_{kl} takes the simple form:

(5.13)
$$\begin{cases} (\mathbf{H}\times\mathbf{B}) + (\mathbf{E}\times\mathbf{D}) = \rho(\mathbf{v}\times\mathbf{a}), \\ (\mathbf{E}\times\mathbf{H}) - (\mathbf{D}\times\mathbf{B}) = \rho(\mathbf{a} + \varphi\mathbf{v}). \end{cases}$$

The conservation laws (3.21) take the form:

(5.14)
$$\begin{cases} div\mathbf{X} + \dot{\mathbf{S}}_x = 0, \\ div\mathbf{S} + \dot{\mathbf{U}} = 0, \end{cases}$$

in which the rows of the matrix of the Maxwell tensor are considered as vectors $\mathbf{X} = (X_x, X_y, X_z)$, …

In a Lorentz system, the four elementary invariants are:

(5.15)
$$\begin{cases} A = \varphi^2 - \mathbf{a}^2, \\ B = [(\mathbf{a}\times\mathbf{B}) + \varphi\mathbf{E}]^2, \end{cases}$$

(5.16)
$$\begin{cases} F = \mathbf{B}^2 - \mathbf{E}^2, \\ G = \mathbf{B}\cdot\mathbf{E}. \end{cases}$$

The entire system of formulas is vacuous until one chooses the form of the function $L(A, B, F, G)$; once this function is fixed, one has:

(5.17)
$$\begin{cases} \rho = -\dfrac{\partial L}{\partial \varphi}, \quad \rho\mathfrak{v} = -\dfrac{\partial L}{\partial \mathbf{a}}, \\ \mathfrak{H} = \dfrac{\partial L}{\partial \mathbf{B}}, \quad \mathfrak{D} = -\dfrac{\partial L}{\partial \mathbf{E}}. \end{cases}$$

Mie tried to find a function L for which there exists a solution to the field equations that corresponds to a stable particle (electron) and has a spherically symmetric charge distribution. His attempt was not crowned with success; however, even if he had succeeded, it would still be unconvincing, except in the case for which this solution was imposed by a universal principle that reduces the extraordinary generality of the theory. However, such a principle is not known, as of yet.

Another characteristic of Mie's theory that raises three strong objections is the explicit appearance of the potentials ($\mathbf{a}$, φ) in the Lagrangian, and, as a consequence, in the field equations. It then follows that the absolute values of the potential must have a precise physical significance, which contradicts everything that is known about the electromagnetic field experimentally. Similarly, if we call into question the validity of an extrapolation of this knowledge to the atomic domain, then we may not escape the following difficulty: suppose that one has discovered a Lagrangian L such that one must find a solution of the field equations that corresponds to a particle. If L depends upon the potentials then this solution no longer exists when one displaces the particle in space to a

point where the potentials have a different value from the preceding one (different by a constant that constitutes potential difference between two positions at an infinite distance from the center of the particle). With these conditions, the theoretical explanation for the existence of particles, which defines the principal objective of Mie's theory, must be illusory.

In Maxwell's theory, since the physical laws have one degree higher they are insensitive to certain variations of the potential; they present what one calls *gauge invariance*: the physical laws do not change if one replaces φ_k by:

$$(5.18) \qquad \overline{\varphi}_k = \varphi_k + \frac{\partial \chi}{\partial x^k},$$

in which χ is an arbitrary function. This derives simply from the fact that the transformation (5.18) does not alter the expression (3.9) for the field components. When one derives Maxwell's equations from the corresponding variational principle, one confirms that the φ_k appear in an explicit fashion in L, but only in the form of a linear combination; L has the following form:

$$(5.19) \qquad L = L_0 + \varphi_i \mathfrak{s}^{0i} = L_0 - \rho^0(\mathbf{v}^0 \mathbf{a} - \varphi),$$

in which L_0 is a function of F and G that has the following form, in vacuo:

$$(5.20) \qquad L_0 = \tfrac{1}{2} F = \tfrac{1}{2}(B^2 - E^2).$$

The $\mathfrak{s}^{0i} = -(\rho^0 \mathbf{v}^0, \rho^0)$ are *given* functions of x, y, z, t that represent the density of charge and convection current, and satisfy the continuity equation:

$$(5.21) \qquad \frac{\partial \mathfrak{s}^{0i}}{\partial x^i} = 0.$$

The transformation (5.18) simply adds the term:

$$(5.22) \qquad \mathfrak{s}^{0i} \frac{\partial \chi}{\partial x^i} = \frac{\partial \mathfrak{s}^{0i} \chi}{\partial x^i} - \chi \frac{\partial \mathfrak{s}^{0i}}{\partial x^i} = \frac{\partial \mathfrak{s}^{0i} \chi}{\partial x^i},$$

to L, which constitutes a divergence, and does not make any contribution to the differential equations.

In reality, this case does not enter into the general theory that was discussed in the foregoing if we are given that L contains x, y, z, t explicitly, which appear in the given functions $\mathfrak{s}^{0i}$. It then follows that any conservation law of type (5.14) will no longer be valid; in the right-hand sides, instead of 0, the Lorentz force and power density will appear:

$$(5.23) \qquad \rho^0[\mathbf{H} + (\mathbf{v}^0 \times \mathbf{E})], \rho^0 \mathbf{v} \mathbf{E}.$$

Strictly speaking, the Maxwell theory is therefore not a particular case of the Mie theory.

If one applies the Maxwell theory to the problem of elementary particles, then − as described in the introduction − one is led to the concept of a mass of electromagnetic origin. The pioneering work of J. J. Thomson has been continued by a great number of researchers, and has attained its culmination in the electron theory of H. A. Lorentz [3]. In this theory, the electron is considered as a spherically symmetric charge distribution that is maintained in place by cohesive forces of unknown, but certainly not electromagnetic, origin. The dimensions of these particles must be finite, or else the proper energy would be infinite. This proper energy is, for the most part, of electromagnetic nature, but it also depends upon cohesive forces to a certain degree. As one knows nothing about these forces, the absolute value of the energy is itself unknown.

Meanwhile, it has been possible to account for the experimental fact that was discovered by Kaufmann [4] of the variation of mass with velocity. This problem has been treated by several authors, and, in particular, in a more complete fashion, by M. Abraham [5], under the hypothesis that the electric charge is fixed in a perfectly rigid material support. The hypothesis of an absolute rigidity, which was perfectly natural in the classical period of physics, is nevertheless becoming unacceptable with the introduction of the postulates of relative by Einstein. *Rigidity and relativity are contradictory notions.*

Somewhat later, Lorentz applied the Fitzgerald contraction to the electron and thus discovered his well-known formula that gives the mass as a function of velocity, a formula that is found to be in closer accord with the experimental results than that of Abraham. The Lorentz hypothesis may be expressed by saying that the electron is rigid in the coordinate system in which it is instantaneously at rest. 30 years ago, in one of my first articles, I showed that this relative rigidity exists for an entire class of motions that correspond to constantly accelerated motion in classical mechanics, and I then deduced [6] a generalization of the Lorentz formula for this non-uniform motion several years before Einstein began the systematic study of similar motions in his general theory of relativity. Langevin [7] has recently treated the same problem from the standpoint of general relativity, and has confirmed the formula that I gave.

These theories that concern the rigid or quasi-rigid electron are not satisfactory because they all suffer from the following difficulty: let E be energy, and let P, be the total electromagnetic momentum of the electron. Its electromagnetic mass may then be defined by either $\dfrac{E}{c^2}$ or $\dfrac{P}{c}$; now, one finds that these values do not coincide. The difference seems to result from the fact that the forces that shape the electron in the manner of a rigid body provide a certain contribution to the translational momentum without performing any internal work.

The only means of avoiding the difficulties that were provoked by the antagonism between relativity and rigidity seems to be the adoption of the *unitary field theory* of Mie, which represents an attempt to assimilate the cohesive forces with the electromagnetic forces into a field with a nonlinear connection. Nevertheless, we have just seen that this same theory has not succeeded in solving the problem, since Mie used the absolute values of the potentials in order to describe the discrepancies by starting with the linearity that gave the charge density.

6. Unitary electrodynamics with pointlike charges. – Nevertheless, the problem possesses a solution [1]. It is obtained by assuming that the potentials do not appear in the field laws, i.e., the $\mathfrak{L}$ depends only on f_{kl} and g_{kl}, and that:

$$(6.1) \qquad\qquad \mathfrak{s}^k = \frac{\partial \mathfrak{L}}{\partial \varphi_k} = 0.$$

Meanwhile, with these conditions it is indispensable to admit that there exist *point-like singularities that represent the particles.* One may then choose the function:

$$(6.2) \qquad\qquad \mathfrak{L}(g_{kl}, f_{kl}) = L(F, G)\sqrt{-g}\,,$$

in such a fashion that the proper electromagnetic energy of a point-like charge is finite; all of the contradictions that affect the old theories of the rigid electron disappear.

I have called the resulting theory the *unitary theory of the electromagnetic field,* in spite of the fact that the singularities correspond to distinct components of the field; in reality, Mie's ideal of a complete unitary theory was not attained for the good reason that it is impossible to attain. It is a question of appreciation, of knowing whether the preceding statements agree with the modified theory that I now present; nevertheless, for my own part, I prefer to retain the expression, since we have that in any case this theory offers the possibility of a purely electromagnetic explanation of mass. We shall discuss this question of the origin of mass later on.

Before we give an example of a function L that replaces the preceding conditions, we must consider the problem from another viewpoint that leads to the general formulation (6.2), of L, namely, the namely, the viewpoint of self-duality, as was defined in section 1 for an arbitrary variation.

The inspection of equations (5.4) for the electromagnetic field and the symmetry that one confirms between electric and magnetic vectors suggests the existence of an absolutely complete symmetry as a necessary condition in any unitary theory of fields and matter. This general condition is equivalent to that of self-duality; we express this analytically.

The first condition to replace is expressed precisely by (6.1). As far as the second one is concerned, we have seen that in the n-dimensional case the number of integrability conditions (3.15) is $\frac{1}{2}n(n-1)(n-2)$, which is equal to the number of Euler equations only if $n = 4$. The fact ([5]) that the self-duality of the electromagnetic field is possible only in our four-dimensional universe seems to be an extremely remarkable coincidence!

Meanwhile, one must not lose sight of the fact that self-duality has only a certain purely formal character; it is not fundamental, since one has not accounted for the singularities. Experience has shown us the existence of point-like electric charges, but we have not yet discovered isolated magnetic poles. Later on, we shall verify the manner by which one treats this dissymmetry.

([5]) As observed by my collaborator, P. Weiss.

In all of the reasoning that is concerned with the electromagnetic field equations, it is very convenient to use the notion of self-duality; we therefore dedicate the following statements to showing how one must construct the system of formulas that are dual to the ones that we have written up till now.

Introduce the function:

$$(6.3) \qquad \mathfrak{H} = \mathfrak{L} - \tfrac{1}{2}\,\mathfrak{p}^{ik} f_{ik} = \mathfrak{L} + \tfrac{1}{2}\,\mathbf{f}^{*lk}\, \overset{*}{p}_{lk} = \mathfrak{L} + \mathbf{E}\mathfrak{D} - \mathbf{B}\mathfrak{H},$$

Which one may call the *Hamiltonian of the system,* by analogy with the function that appears in point mechanics. By virtue of $(3.10b)$ and the fact that $\mathfrak{s}^k = 0$, we have:

$$(6.4) \qquad d\mathfrak{H} = \tfrac{1}{2}\,\mathfrak{T}^{ik} g_{lk} - \tfrac{1}{2} f_{ik}\, d\mathfrak{p}^{ik} = \tfrac{1}{2}\,\mathfrak{T}^{ik}\, dg_{lk} + \tfrac{1}{2} f^{*ik}\, d\overset{*}{p}_{ik}\,.$$

The *natural* independent variables for $\mathfrak{H}$ are therefore gik, $\overset{*}{p}_{lk}$; in accord with the general notion of duality, we admit that the $\overset{*}{p}_{lk}$ may be expressed by the *antipotentials* $\overset{*}{\psi}_k$:

$$(6.5) \qquad \overset{*}{p}_{lk} = \frac{\partial \overset{*}{\psi}_k}{\partial x^i} - \frac{\partial \overset{*}{\psi}_i}{\partial x^k}\,.$$

$\mathfrak{H}$ will have the form:

$$(6.6) \qquad \mathfrak{H} = H(P,\,Q)\sqrt{-g}\,,$$

in which P, Q are the elementary invariants of g_{ik}, $\overset{*}{p}_{lk}$ that correspond to F, G, as defined by (4.19) and (4.27), namely:

$$(6.7) \qquad P = \tfrac{1}{2}\,\overset{*}{p}_{ik}\,p^{*ik} = D^2 - H^2, \qquad Q = \tfrac{1}{4}\,\overset{*}{p}_{ik}\,p^{ik} = \mathbf{DH}.$$

A variational principle that uses the integral:

$$(6.7) \qquad I^* = \int_G \mathfrak{H}(g_{ik},\,\overset{*}{p}_{ik})\,dx\,,$$

leads to the equations:

$$(6.9) \qquad \frac{\partial \mathfrak{f}^{*kl}}{\partial x^k} = 0\,,$$

$$(6.10) \qquad \frac{\partial \overset{*}{p}_{ik}}{\partial x^l} + \frac{\partial \overset{*}{p}_{kl}}{\partial x^i} + \frac{\partial \overset{*}{p}_{li}}{\partial x^k} = 0\,.$$

The equations are indicated in the initial equations (3.13) and (3.15), which may be written in tensorial and vectorial notation, in the following fashion for $\mathfrak{s}^k = 0$:

$$(6.9a) \qquad \frac{\partial f_{ik}}{\partial x^l} + \frac{\partial f_{kl}}{\partial x^i} + \frac{\partial f_{li}}{\partial x^k} = 0 \quad \text{or} \qquad \text{rot } \mathbf{E} + \mathbf{B} = 0, \qquad \text{div } \mathbf{B} = 0,$$

$$(6.10a) \qquad\qquad \frac{\partial f_{li}}{\partial x^k} = 0 \qquad \text{or} \qquad \text{rot } \mathfrak{H} - \dot{\mathfrak{D}} = 0, \qquad \text{div } \mathfrak{D} = 0.$$

If we keep the g_{ik}'s constant then (3.10) and (6.4) become:

$$(6.4a) \qquad\qquad \begin{cases} d\mathfrak{L} = -\mathfrak{D}d\mathbf{E} + Hd\mathbf{B}, \\ d\mathfrak{H} = \mathbf{E}d\mathfrak{D} - \mathbf{B}d\mathfrak{H}. \end{cases}$$

The two dual representations differ from each other, not by their differential equations, but the choice of primary variables for the field:

$$(6.11) \qquad\qquad \begin{cases} \mathfrak{L}(\mathbf{E}, \mathbf{B}) \quad \cdots \quad \mathfrak{D} = -\dfrac{\partial \mathfrak{L}}{\partial \mathbf{E}}, \quad \mathfrak{H} = \dfrac{\partial \mathfrak{L}}{\partial \mathbf{B}}, \\[3mm] \mathfrak{H}(\mathfrak{D}, \mathfrak{H}) \quad \cdots \quad \mathbf{E} = \dfrac{\partial \mathfrak{H}}{\partial \mathfrak{D}}, \quad \mathbf{B} = -\dfrac{\partial \mathfrak{H}}{\partial \mathfrak{H}}. \end{cases}$$

The energy tensor is expressed in terms of L in the same fashion as with H:

$$(6.12) \qquad\qquad \mathfrak{T}^{ik} = \frac{\partial \mathfrak{L}}{\partial g_{ik}} + \frac{\partial \mathfrak{L}}{\partial g_{ki}} = \frac{\partial \mathfrak{H}}{\partial g_{ik}} + \frac{\partial \mathfrak{H}}{\partial g_{ki}}.$$

By virtue of (6.4), the diagonal elements of the mixed tensor $\mathfrak{T}_i^k$ that is given by (5.7), may be expressed in two perfectly equivalent ways:

$$(6.12a) \qquad\qquad \begin{cases} \mathfrak{T}_1^1 = \mathfrak{L} - \mathfrak{H}_y B_y - \mathfrak{H}_z B_z + \mathfrak{D}_x E_x = \mathfrak{H} - \mathfrak{D}_y E_y - \mathfrak{D}_z E_z + \mathfrak{H}_x B_x, \\ \mathfrak{T}_1^4 = \mathfrak{L} - \mathbf{DE} = \mathfrak{H} + \mathfrak{H}\mathbf{B}. \end{cases}$$

Since $U = \mathfrak{T}_4^4$ is the energy density [*see* (5.11)], the function, $\mathfrak{H}$, has a simple significance in the purely electric case. This circumstance often makes the use of $\mathfrak{H}$ more convenient than that of $\mathfrak{L}$.

In a Lorentz coordinate system, the energy tensor is expressed by (5.12), if one suppresses the terms in ρ. The symmetry relations, which are the expression of relativistic invariance, may be written:

$$(6.13) \qquad\qquad \begin{cases} (\mathbf{H} \times \mathbf{B}) = -(\mathbf{E} \times \mathbf{D}), \\ \mathbf{S} = (\mathbf{E} \times \mathbf{H}) = (\mathbf{D} \times \mathbf{B}). \end{cases}$$

This double expression for the Poynting vector is remarkable, and will be used later on to establish the equations of motion for a charged particle.

When one refuses to use the tensorial calculus at any price, which has, moreover, the advantage of making the covariance of the equations immediately obvious, one may likewise introduce other *fundamental functions* instead of $\mathfrak{L}$ and $\mathfrak{H}$. In particular, one of these representations is very useful for the application of the principles of quantum theory to electrodynamics. Indeed, the only means of attacking such a problem is to take the Hamiltonain of dynamics as the point of departure, i.e., energy; now, the function, H, that we have called *Hamiltonain* here is not, in general, identical with the energy density (this would be true only in the absence of the magnetic field, as have seen).

One confirms that, from (6.12a), the energy density is given in a Lorentz system by:

(6.13) $U = H + \mathbf{HB} = L + \mathbf{DE}$;

from this, by accounting for (6.4), one deduces:

(6.14a)
$$\begin{cases} dU = \mathbf{E}d\mathbf{D} + \mathbf{H}d\mathbf{B}, \\ \mathbf{E} = \dfrac{\partial U}{\partial \mathbf{D}}, \quad \mathbf{H} = \dfrac{\partial U}{\partial \mathbf{B}}, \end{cases}$$

in such a way that the function $U(\mathbf{D}, \mathbf{B})$ may be likewise employed as the foundation for the theory.

Finally, there exists another possibility, which is written in ordinary vectorial notation:

(6.15) $V = H - \mathbf{ED} = \mathbf{L} - \mathbf{BH}$,

(6.15a)
$$\begin{cases} dV = -\mathbf{D}d\mathbf{E} + \mathbf{B}d\mathbf{H}, \\ \mathbf{D} = -\dfrac{\partial V}{\partial \mathbf{E}}, \quad \mathbf{B} = \dfrac{\partial U}{\partial \mathbf{H}}; \end{cases}$$

however, the function $V(\mathbf{E}, \mathbf{H})$ has no immediate physical significance.

The interesting point in all of these representations of the theory is the fact that the field equations are certainly covariant, but the fundamental functions are not invariants. My collaborator, M. B. S. Madhava Rao, has studied this question, and has concluded with the following result [2]. Einstein and Mayer have shown that the group of Lorentz transformations may be separated into two more restricted groups; they are constructed from two types of quantities, the semi-vectors, which are covariant under the one of these restricted groups, but not under the entire Lorentz group. Madhava Rao has shown that relations (6.14a) and (6.15a) are covariant under these partial groups, and may therefore give us reasons to believe that the field equations are covariant under the complete group.

When one exhibits the discrepancies between the general theory of fields and that of Maxwell, it is convenient to introduce the tensor:

(6.16) $m^{kl} = f^{kl} - p^{kl}$ $[(m^{23}, m^{31}, m^{12}) = \mathbf{M}, (m^{14}, m^{24}, m^{34}) = \mathbf{P}]$,

and the vector:

(6.17) $\qquad \sigma^k = \dfrac{\partial m^{lk}}{\partial x^l}$ $\qquad\qquad [(\sigma^1, \sigma^2, \sigma^3, \sigma^4) = (-j, -\sigma)].$

In a Lorentz system, one has, with the ordinary vectorial notation:

(6.17a) $\qquad\qquad$ $\mathbf{M} = \mathbf{B} - \mathbf{H},$ $\qquad\qquad$ $\mathbf{P} = \mathbf{D} - \mathbf{E},$

which shows that $\mathbf{M}$ and $\mathbf{P}$ are the *densities of the electric and magnetic moment*; as a result, one has:

(6.18a) $\qquad\qquad$ $j = - \operatorname{rot} \mathbf{M} + \mathbf{P},$ $\qquad\qquad$ $\sigma = - \operatorname{div} \mathbf{P}.$

The field equations may be written in the Maxwell form:

(6.19)
$$
\begin{cases}
\dfrac{\partial f^{kl}}{\partial x^k} = \sigma^l & or \quad rot\mathbf{B} - \mathbf{E} = \mathbf{j}, \quad div\mathbf{E} = \sigma, \\[2mm]
\dfrac{\partial f^{*kl}}{\partial x^k} = 0 & or \quad rot\mathbf{E} - \mathbf{B} = 0, \quad div\mathbf{B} = 0.
\end{cases}
$$

Here, the right-hand sides are not *given* functions, but depend on the field components, according to the relations, (6.18a). One may call j and σ the current density and the *free* charge. $\mathbf{M}$ and $\mathbf{P}$ are the *vacuum polarizations*; this notion plays an important role in the attempts of Hesenberg to evaluate the differences from Maxwell's theory by starting with the relativistic wave mechanics of the electron (*see* sec. **13**, **14**).

It obviously possible to introduce an energy tensor that is a function of only the f^{kl}:

(6.20) $\qquad\qquad\qquad\qquad$ $S^l_k = \tfrac{1}{2} F \delta^l_k - f^{lm} f_{km},$

in which $F = \tfrac{1}{2} f_{kl} f^{kl}$ is the Lagrangian of Maxwell's theory; one obtains conservation laws by the usual procedure in the form:

(6.21) $\qquad\qquad\qquad\qquad$ $\dfrac{\partial S^l_k}{\partial x^l} = f_{kl} \sigma^l\, ;$

in the right-hand side the Lorentz force appears, along with the power of the free charge:

(6.22) $\qquad\qquad\qquad\qquad$ $\mathbf{E} + (j \times \mathbf{B})$ $\qquad$ and $\qquad$ $j\mathbf{E}.$

This shows that one must attempt to find the discrepancies associated with the classical values of the electromagnetic forces any time that one is concerned with fields that are essentially variable on a distance scale that is comparable to that of the diameter of the free charge distribution of a particle; this result will be confirmed later on.

In this theory, the *true* charges − i.e., the ones for which the integral $\int D_n d\sigma$, when taken over a closed surface, is finite − are always concentrated at a point.

A certain number of general results concerning the field and the energy of an isolated point-like particle at rest may be obtained without specifying the Lagrangian *L*.

In the electrostatic case, the field equations reduce to:

(6.23)   rot $\mathbf{E} = 0$,   div $\mathbf{D} = 0$.

The first one is satisfied identically for a spherically symmetric field of the form $\mathbf{E} = E_r \dfrac{\mathbf{r}}{r}$. The second becomes:

(6.23a) $\dfrac{d}{dr}(r^2 D_r) = 0,$

whose general solution is:

(6.24) $D_r = \dfrac{e}{4\pi}\dfrac{1}{r^2},$

in which the constant is chosen in such a fashion that for any closed surface that surrounds the origin one has:

(6.25) $\int D_r d\sigma = e.$

The field $\mathbf{D}$ is exactly the same as in Maxwell's theory.

Moreover, there exists a general relation between the value of the total energy *E*, and that of the electrostatic potential at the place where the point-like charge is found $\phi(0)$. To deduce this, we use the property that was mentioned before, that in the electrostatic case the energy density coincides with the Hamiltonain *H*, *I* such a way that:

(6.25) $E_0 = 4\pi \int_0^\infty H r^2 dr .$

We consider *H* as a function of $P = D^2$ ($Q = \mathbf{H}\mathbf{D}$ is annulled); we will have:

(6.27) $\dfrac{d}{dx}(r^3 H) = 3r^2 H + r^3 \dfrac{\partial H}{\partial D_r}\dfrac{dD_r}{dr};$

now, by virtue of (6.11) and (6.24):

$$\dfrac{\partial H}{\partial D_r} = E_r, \qquad\qquad \dfrac{dD_r}{dr} = -\dfrac{e}{2\pi}\dfrac{1}{r^3},$$

so:

(6.28) $H r^2 = \tfrac{1}{3}\dfrac{d}{dr}(r^3 H) + \dfrac{e}{6\pi} E_r .$

Since we assume that the integral (6.26) converges, H may not tend to infinity at the origin $r = 0$ more rapidly than $r^{-(2+\varepsilon)}$, $0 < \varepsilon < 1$; for $r = \infty$, we must make the hypothesis that H behaves like the energy density in Maxwell's theory, i.e., that it varies like r^{-4}. It then follows that $r^3 H$ is annulled for $r = 0$ and $r = \infty$. The electrostatic potential is given by:

(6.29) $$\varphi(r) = \int_r^\infty E_r \, dr,$$

and from (6.26) and (6.28), one obtains:

(6.30) $$E_0 = \tfrac{2}{3} e \varphi(0).$$

E_0 is the space integral of the 44 component of the energy tensor T_{kl}. It is easy to prove that the integrals of all the other components of T_{kl} are annulled for a spherically symmetric electrostatic field. Indeed, this is obvious for the integrals of the components T_{k4} that comprise the vector $\mathbf{S}$, which is annulled for $\mathbf{B} = 0$; it thus remains for us to prove this for the Maxwell tensor $M(X_x, \ldots, X_y, \ldots)$, which is given by (5.12):

(6.31) $$X_x = D_y E_y + D_z E_z - H, \qquad \ldots, \qquad X_y = D_x E_y, \qquad \ldots$$

H depends only upon $P = D^2$ and $\mathbf{E} = \dfrac{dH}{dP} 2\mathbf{D}$. The hypothesis of spherical symmetry signifies that $\mathbf{D} = f(r)\mathbf{r}$; therefore:

(6.32) $$\int X_y \, dv = -2 \int \frac{dH}{dP} f^2 xy \, dv = 0,$$

which is null since $\dfrac{dH}{dP} f^2$ depends only on r. Thus:

$$\int X_x \, dv = \int \left(\tfrac{2}{3} \mathbf{DE} - H \right) dv = 4\pi \int \left(\tfrac{4}{3} \frac{dH}{dP} D^2 - H \right) r^2 \, dr.$$

Since $\dfrac{dH}{dD_r} = 2 \dfrac{dH}{dP} D_r$, and since (6.23a) gives $\dfrac{dD_r}{dr} + \dfrac{2}{r} D_r = 0$, one may write (6.27) in the form:

$$3r^2 H - 4r^2 \frac{dH}{dP} D_r^2 = \frac{d}{dr}(r^3 H),$$

from which:

(6.33) $$\int X_x \, dv = -\frac{4\pi}{3} \int \frac{d}{dr}(r^3 H) \, dr = 0.$$

Thus, the integral of the energy tensor has only one non-null component, namely, the component E_0 with the indices 44. This theorem is known by the name of *Von Laue's*

theorem, since he first observed that *in order to consider the energy and momentum as a spacetime vector,* it is necessary that the conditions stated in the preceding theorem must be satisfied. In a Lorentz system that moves with the velocity v these components are equal to:

(6.34)
$$\mathbf{G} = \frac{E_0 \mathbf{v}}{\sqrt{1 - v^2}}, \qquad E = \frac{E_0}{\sqrt{1 - v^2}}.$$

7. **An example of a unitary theory of electromagnetic fields.** – The first known example of a function L that is compatible with the existence of point-like charges with finite proper energy was discovered by appealing to the following analogy:

The classical formula for kinetic energy $\frac{1}{2} mv^2$ fixes no upper limit on velocity. Now, the theory of relativity shows us that this is incorrect; in reality, the kinetic energy must be written: $m\left(\dfrac{1}{\sqrt{1 - v^2}} - 1 \right)$, an expression that corresponds to the Lagrangian $m(1 - \sqrt{1 - v^2})$, and which leads to an upper limit on velocity, which is taken to be unity here.

In electrodynamics, the classical Maxwell Lagrangian $L = \frac{1}{2}(B^2 - E^2)$ has a structure that is very analogous to that of kinetic energy $\frac{1}{2} mv^2$. This Lagrangian leads to an infinite proper energy for a point-like particle. We shall attempt to avoid this infinity by the same formal change as in relativistic mechanics by setting:

(7.1)
$$L = b^2 \left(\sqrt{1 + \frac{1}{b^2}(B^2 - E^2)} - 1 \right).$$

b is a universal constant − the *absolute field* – which may be taken as equal to unity in a number of problems. The constant, −1, was introduced so that the expression, (7.1), would reduce to that of Maxwell's theory $L = \frac{1}{2}(B^2 - E^2)$ for fields that are small (compared to b).

In electrostatic case, one has:

(7.2)
$$L = b^2 \left(\sqrt{1 - \frac{1}{b^2} E^2} - 1 \right),$$

(7.3)
$$D_r = -\frac{\partial L}{\partial E_r} = \frac{E_r}{\sqrt{1 - \dfrac{1}{b^2} E_r^2}},$$

(7.4)
$$E_r = \frac{D_r}{\sqrt{1 - \dfrac{1}{b^2} D_r^2}}.$$

By virtue of (6.24), one may write:

(7.5)
$$E_r = \frac{e}{4\pi} \frac{1}{\sqrt{r_0^4 + r^4}},$$

by setting:

(7.6)
$$r_0 = \sqrt{\frac{e}{4\pi b}}.$$

r_0 is a length that we call the *electron radius*. (7.5) shows that E_r is finite everywhere; for $r = 0$, its value is $b = \dfrac{e}{4\pi r_0^2}$, and for $\dfrac{r}{r_0} > 1$ it differs very little from the Coulomb field $\dfrac{e}{4\pi r^2}$. The potential is:

(7.7)
$$\varphi(r) = \int_r^\infty E_r \, dr = \frac{e}{4\pi r_0} f\left(\frac{r}{r_0}\right),$$

in which:

(7.8)
$$f(x) = \int_x^\infty \frac{dy}{\sqrt{1 + y^4}}.$$

By substituting $y = \tan\dfrac{\beta}{2}$, $x = \tan\dfrac{\xi}{2}$, one obtains:

(7.9)
$$f(x) = \tfrac{1}{2} \int_\xi^\pi \frac{d\beta}{\sqrt{1 - \tfrac{1}{2}\sin^2 \beta}} = f(0) - \tfrac{1}{2} F\left(\frac{1}{\sqrt{2}}, \xi\right),$$

in which $F(k, \xi)$ is Jacobi's elliptic integral of the first type for $k = \dfrac{1}{\sqrt{2}} = \sin\dfrac{\pi}{4}$:

(7.10)
$$F\left(\frac{1}{\sqrt{2}}, \xi\right) = \int_0^\xi \frac{d\beta}{\sqrt{1 - \tfrac{1}{2}\sin^2 \beta}}.$$

For $x = 0$, one has:

(7.11)
$$f(0) = F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) = 1.8541.$$

The potential (7.7) has its maximum at the origin and its value at this point is:

(7.12)
$$\varphi(0) = f(0)\frac{e}{4\pi r_0} = 1.8541 \frac{e}{4\pi r_0}.$$

The variation of the function, $f(x)$, is indicated in the figure 1; it has the same appearance as arc cot x. For example, one has:

$$\xi\left(\frac{1}{x}\right) = 2\arctan\frac{1}{x} = 2\left(\frac{\pi}{2} - \arctan x\right) = \pi - \xi(x) \, ;$$

on the other hand:

$$F\left(\frac{1}{\sqrt{2}}, \xi\right) + F\left(\frac{1}{\sqrt{2}}, \pi - \xi\right) = F\left(\frac{1}{\sqrt{2}}, \pi\right),$$

so:

(7.13) $$f(x) + f\left(\frac{1}{x}\right) = f(0).$$

As a consequence, it suffices to calculate the values of $f(x)$ between $x = 0$ and $x = 1$ (or between $\xi = 0$ and $\xi = \frac{\pi}{2}$).

To account for the differences between this theory and Maxwell's one may calulate the *free* charge density from formula (6.19), which must be:

(7.14) $$\sigma = \text{div } \mathbf{E} = \frac{1}{r^2}\frac{d}{dr}(r^2 E_r),$$

By substituting (7.5), one finds:

(7.15) $$\sigma = \frac{e}{4\pi r_0^2}\frac{2}{x(1+x^4)^{\frac{3}{2}}}, \qquad \left(x = \frac{r}{r_0}\right).$$

At a great distance, the density decreases like r^{-7}; at the center, it must be infinite like r^{-1}. The charge $4\pi\sigma r^2\,dr$ that is contained in the space between two spheres of radius r and $r + dr$ is null for $r = 0$, and, finally, the total charge may be written (by setting $x = \sqrt{\tan\varphi}$):

$$\int_0^\pi \sigma 4\pi r^2 dr = e\int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi = e \, ;$$

as one would expect, it is equal to e.

The total energy is calculated by means of (6.30); one obtains:

(7.16) $$E_0 = \frac{e}{4\pi r_0^2}\frac{2}{3}f(0) = \frac{e}{4\pi r_0^2}\,1.2361.$$

We may now confirm the hypothesis that $Hr^3 \to 0$ for $r \to 0$ and $r \to \infty$, do which we have made use of in the proof of (6.30). Indeed:

$$H = L + \mathbf{ED} = b^2\left(\sqrt{1 + \frac{D^2}{b}} - 1\right),$$

$$(7.17) \qquad r^3 H = b^2 \left[\sqrt{1 + \left(\frac{e}{4\pi b} \right)^2 \frac{1}{r^4}} - 1 \right] r^3 = \left(\frac{e}{4\pi} \right)^2 \frac{1}{r_0} (\sqrt{1 + x^4} - x^2)x,$$

$$\left(x = \frac{r}{r_0} \right),$$

and the function of x is precisely zero for $x = 0$ and $x = \infty$.

We have thus proved that the Lagrangian (7.1) leads to a finite energy for a point-like charge at rest. It is clear that we may equate this energy to the rest mass (multiplied by c^2, if we return to the usual units), without encountering the same obstacles as in the theory of the rigid electron, since out theory is invariant from the relativistic viewpoint.

Meanwhile, we may use this property to determine the absolute field or the equivalent length, r_0, by identifying our point-like charge with the electron. Upon passing to electrostatic units, we must leave the factor 4π aside in (7.17); by setting $E_0 = mc^2$, one then obtains:

$$(7.18) \qquad r_0 = 1.2361 \frac{e^2}{m_0 c^2} = 3.47 \ \% \ 10^{-13} \text{ cm.,}$$

and

$$(7.19) \qquad b = \frac{e}{r_0^2} = 3.96 \times 10^{15} \text{ E.S.U.}$$

The order of magnitude for the *electron radius* r_0 is correct. The value of the field b is enormous; all of the experimentally realizable fields are negligible compared to it. Therefore, one must not expect to measure an arbitrary effect that is directly related to the existence of a *finite* maximum field [1]. The differences between this theory and Maxwell's theory are noticeable only for distances of order 10^{-13} cm., i.e., for nuclear dimensions. This result has great importance; it shows that the true electrodynamical laws that govern the elementary processes of nuclear transformations may be very complicated, or else one would be forced to confirm the differences with the Maxwell theory when one is confined to examining extra-nuclear phenomena.

Furthermore, one may not solve the field equations for the case of several point-like charges at definite positions in a rigorous fashion. However, Pryce [2] has proved that *the field is uniquely determined by the charges.* We write the relation between **D** and **E** [(7.3), (7.4)] in the following form, in which we have set $b^{-2} = \alpha$:

$$(7.20) \qquad \mathbf{D} = \frac{\mathbf{E}}{\sqrt{1 - \alpha E^2}}, \qquad\qquad \mathbf{E} = \frac{\mathbf{D}}{\sqrt{1 + \alpha D^2}}.$$

One may remark that for $\alpha = -1$ one obtains the equation for minimal surfaces in four-dimensional space; indeed, the integral that must be varied:

$$(7.21) \qquad \int \sqrt{1 + E^2}\, dx dy dz = \int \sqrt{1 + (grad\,\varphi)^2}\, dx dy dz,$$

represents the area of the surface, $\varphi = \varphi(x,\ y,\ z)$. The theorem of uniqueness to which we alluded above is likewise true for the geometric problem.

Let φ_1, φ_2 be two solutions to (6.23) that correspond to the vectors $\mathbf{E}_1$, $\mathbf{D}_1$, and $\mathbf{E}_2$, $\mathbf{D}_2$; it is convenient for all that follows to define a vector and a scalar by the formulas:

(7.22) $\mathbf{G} = (\varphi_2 - \varphi_1)(\mathbf{D}_1 - \mathbf{D}_2),$

(7.23) $u = (\mathbf{E}_1 - \mathbf{E}_2)(\mathbf{D}_1 - \mathbf{D}_2).$

The proof of the uniqueness theorem then reduces to that of the following lemma:

LEMMA. – *One has $u \geq 0$ at any point. The equality is valid only for:*

$$\mathbf{E}_1 = \mathbf{E}_2.$$

u may be written in one of the following two forms:

$$u = (\mathbf{E}_1 - \mathbf{E}_2)\left(\frac{\mathbf{E}_1}{\sqrt{1 - \alpha E_1^2}} - \frac{\mathbf{E}_2}{\sqrt{1 - \alpha E_2^2}} \right),$$

$$u = (\mathbf{D}_1 - \mathbf{D}_2)\left(\frac{\mathbf{D}_1}{\sqrt{1 + \alpha D_1^2}} - \frac{\mathbf{D}_2}{\sqrt{1 + \alpha D_2^2}} \right).$$

If one can prove the lemma by using the first of these two expressions with a choice of α then one may likewise prove it for $-\alpha$; indeed, it will suffice to exchange both the $\mathbf{D}$'s and $\mathbf{E}$'s in them and use the second expression. Therefore, it is not necessary to treat the case of the minimal surface $\alpha = -1$ separately.

Take $\alpha = 1$; one will have:

$$2u = (\mathbf{E}_1 - \mathbf{E}_2)\left[(\mathbf{E}_1 + \mathbf{E}_2)\left(\frac{1}{\sqrt{1 - E_1^2}} - \frac{1}{\sqrt{1 - E_2^2}} \right) + (\mathbf{E}_1 - \mathbf{E}_2)\left(\frac{1}{\sqrt{1 - E_1^2}} + \frac{1}{\sqrt{1 - E_2^2}} \right) \right]$$

$$= (\mathbf{E}_1^2 - \mathbf{E}_2^2)\left(\frac{1}{\sqrt{1 - E_1^2}} - \frac{1}{\sqrt{1 - E_2^2}} \right) + (\mathbf{E}_1 - \mathbf{E}_2)^2 \left(\frac{1}{\sqrt{1 - E_1^2}} + \frac{1}{\sqrt{1 - E_2^2}} \right).$$

If $1 > E_1^2 \geq E_2^2$ then one will have $\dfrac{1}{\sqrt{1 - E_1^2}} \geq \dfrac{1}{\sqrt{1 - E_2^2}}$, and conversely. Therefore, the

first term is never negative, and one may annul it only if $E_1^2 = E_2^2$; the second one is

essentially non-negative, and becomes zero only when $\mathbf{E}_1 = \mathbf{E}_2$. The lemma is therefore proved ([6]).

From (7.22), one obtains:

$$\text{div } \mathbf{G} = \text{div}[(\varphi_2 - \varphi_1)(\mathbf{D}_1 - \mathbf{D}_2)]$$
$$= (\mathbf{D}_1 - \mathbf{D}_2)\,\text{grad}(\varphi_2 - \varphi_1) + (\varphi_2 - \varphi_1)\,\text{div}(\mathbf{D}_1 - \mathbf{D}_2).$$

Since $\mathbf{D}_1$ and $\mathbf{D}_2$ are solutions of div $\mathbf{D} = 0$, and $\mathbf{E}_1$, $\mathbf{E}_2$ are given by:

$$\mathbf{E}_1 = -\text{grad }\varphi_1, \qquad\qquad \mathbf{E}_2 = -\text{grad }\varphi_2,$$

one has:

(7.24) $$\text{div } \mathbf{G} = (\mathbf{D}_1 - \mathbf{D}_2)(\mathbf{E}_1 - \mathbf{E}_2) = u.$$

Now let R be a region of three-dimensional space that is bounded by a surface Σ on which the value of φ is given; one thus has $\varphi_2 = \varphi_1$, and as a consequence, $\mathbf{G} = 0$. By virtue of Gauss's theorem, when applied to (7.24), one will have:

(7.25) $$\int_R u\,dxdydz = \int_\Sigma G_n d\sigma = 0,$$

thus $u = 0$ in R since u is non-negative. In this case, the lemma demands that $\mathbf{E}_1 = \mathbf{E}_2$ in R; since $\varphi_2 = \varphi_1$ on Σ, one will likewise have $\varphi_2 = \varphi_1$ in R.

In the same fashion, one may prove that $\mathbf{E}_1 = \mathbf{E}_2$ when one is given the value of the normal component of $\mathbf{D}$ on Σ, since one likewise has $G_n = 0$ in this case; nonetheless, φ_1 and φ_2 are not equal, but differ by a constant.

Nevertheless, the most important problem for physical applications is not concerned with the questions of boundary conditions, but wit the treatment of point-like charges.

One may prove the following results:

Assume that the positions and values of a set of charges are given. If the vector D is of order r^{-2} in a neighborhood of one charge and at infinity then equations (6.23) *have one and only one solution in all of space.*

To show this take R to be the space between a sphere of large radius ρ that includes all of the charges in its interior and a series of small spheres of radius ε that are centered on each of the charges and do not intersect each other. By virtue of Gauss's theorem, which is applied by always considering the exterior normal, one may write:

([6]) One may easily see that a sufficient condition for this lemma in an arbitrary unitary theory is:

$$\frac{\partial L}{\partial F} > 0, \qquad\qquad 2\frac{\partial^2 L}{\partial F^2}F + \frac{\partial L}{\partial F} > 0.$$

This condition expresses the ideas that $\mathbf{D}$ and $\mathbf{E}$ have the same direction, and that their values increase or decrease together.

$$\int_R u \, dxdydz = \int_\rho G_n d\sigma - \sum \int_\varepsilon G_n d\sigma \,.$$

Since the magnitude of the charge is $e = \int D_n d\sigma$, one has:

$$\int_\varepsilon (D_1 - D_2)_n \, d\sigma = 0 \,.$$

In the neighborhood of a definite charge $\varphi = \varphi^0 + O(\varepsilon^{-1})$; therefore:

$$\int_\varepsilon G_n d\sigma = \int_\varepsilon [\varphi_1^0 - \varphi_2^0 + O(\varepsilon^{-1})](D_1 - D_2)_n \, d\sigma$$
$$= (\varphi_1^0 - \varphi_2^0) \int (D_1 - D_2)_n \, d\sigma + \int O(\varepsilon^{-1}) d\sigma \,,$$

and since $d\sigma = \varepsilon^2 d\omega$ in which $d\omega$ is the element of the unit sphere, one will have:

$$\int_\varepsilon G_n d\sigma = O(\varepsilon) \,.$$

By hypothesis, D is $O(\varepsilon^{-2})$ at infinity; from (7.20), one concludes that $\mathbf{E} = O(\varepsilon^{-2})$, hence $\varphi = O(\varepsilon^{-1})$. The contribution of the large sphere is therefore:

$$\int_\rho G_n d\sigma = \int_\rho O(\rho^{-3}) d\sigma = O(\rho^{-1}) \,;$$

hence:

$$\int_R u \, dxdydz = O(\varepsilon) + O(\rho^{-1}) \,.$$

We make ε go to zero, $\varepsilon \to 0$, and ρ to infinity, $\rho \to \infty$; the right-hand side goes to zero. Since the integral is non-negative and non-decreasing, it must, consequently, be zero. The lemma then demands, as before, that one have $\mathbf{E}_1 = \mathbf{E}_2$ at every point of R.

Following my suggestions, Pryce has succeeded in completely solving the *electrostatic problem in two dimensions* [3]. The fundamental idea that he used is precisely that of using the equivalence of this problem with that of minimal surfaces to the best advantage. There exists a method of solution for the latter problem, which was described by Weierstrass, and which consists of representing *x, y,* and φ by analytic functions of a complex parameter. One may transpose this method without adding anything and apply it to the electrostatic problem in two dimensions. We confine ourselves to indicating the solution.

Let $f(\xi)$ be an arbitrary function of the complex parameter ξ.

One has:

(7.26)
$$\begin{cases} x = \Re \int \left(\dfrac{1}{\varepsilon^2} + 1 \right) f(\xi) d\xi, \quad y = \Re \int i \left(\dfrac{1}{\varepsilon^2} - 1 \right) f(\xi) d\xi, \\[2ex] \qquad\qquad \varphi + i\psi = 2 \int f(\xi) \dfrac{d\xi}{\xi}, \end{cases}$$

in which $\Re A$ signifies the real part of the complex number A.

The force that is exerted on a closed domain of the *xy*-plane that is bounded by a curve is expressed by means of the line integrals:

(7.27)
$$\begin{cases} X = \int [X_{11} \cos(v,x) + X_{12} \cos(v,x)]ds, \\ Y = \int [X_{21} \cos(v,x) + X_{22} \cos(v,x)]ds, \end{cases}$$

in which v is the exterior normal to the element ds. Into this formula we introduce the expressions, (5.12), for the Maxwell tensions (for the two-dimensional case); the expression for the components of the force as a function of the parameter ξ will be:

(7.28)
$$\begin{cases} X = \Re\left(-\frac{i}{\pi}\int_C f(\xi)d\xi\right), \\ Y = \Re\left(-\frac{1}{\pi}\int_C f(\xi)d\xi\right). \end{cases}$$

The integrals must be taken over the closed curve C that bounds the domain considered. When this domain does not contain any charge, X and Y are null, by virtue of Cauchy's theorem. When this domain does contain a charge, X and Y have finite values that represent the forces that are exerted on it (and which are due to the other charges). In the following paragraph, we discuss the problem of forces in detail and examine, in particular, the extent to which formula (7.27) follows from fundamental hypotheses.

The choice of the function $f(\xi)$ is subject to a number of restrictions that follow from an exact analysis of the physical significance of the magnitudes that are expressed by means of this function. One finds that $|\xi|$ [1, i.e., that the representative point ξ must be found in the interior of or on the circle $\xi = 1$, and that $f(\xi)$ must be real for any $|\xi| = 1$; hence, $f(\xi)$ must not have (regular) zeroes in $|\xi| = 1$; it may have zeroes on the circle, $|\xi| = 1$, but they must be simple. $f(\xi)$ may be multi-valued; on one of the corresponding Riemann surfaces $f(\xi)$ must be infinite at the point $\xi = 0$, while remaining finite for all of the other sheets. Finally, the developments must be well defined.

When one chooses a function $f(\xi)$ that satisfies these conditions, one may show, in general, that the field $\mathbf{D}$ is given by $\mathbf{D} = D\dfrac{\mathbf{r}}{r}$ in the neighborhood of a charge, with:

$$D = \frac{2|F(\theta)|}{r}, \qquad F(\theta) = f(e^{i\theta}),$$

in such a way that the charge is represented by:

(7.29)
$$e = -\frac{1}{\pi}\int_0^{2\pi} F(\theta)d\theta.$$

In this case, the forces are given by the following formulas:

$$(7.30) \qquad X = -\frac{1}{\pi} \int_0^{2\pi} F(\theta) \cos \theta d\theta, \qquad Y = -\frac{1}{\pi} \int_0^{2\pi} F(\theta) \sin \theta d\theta.$$

The simplest choice is $f(\xi) = \text{constant}$, namely:

$$(7.31) \qquad f(x) = -\frac{1}{2} a \qquad (a = \text{real}).$$

In this case, the integrals of (7.26) are easily calculated, and give:

$$(7.31a) \qquad \varphi = a \sin^{-1}\left(\frac{r}{a}\right), \qquad \psi = a\theta \qquad (e = a),$$

in which r, θ are polar coordinates. This is the solution, in two dimensions, of the problem of one point-like charge [and corresponds to the one that was previously given for three dimensions, namely (7.7)].

In order to represent the case of *two point-like charges,* one must take $f(x)$ to be a function with two manifestations whose type depends on whether one is dealing with charges of the same or opposite signs, namely:

for two charges of the same signs:

$$(7.32) \quad f(\xi) = k \left\{ \left[a + a' - \xi - \frac{1}{\xi} \right] \right.$$
$$\left. \times \left[\left(2a - \xi - \frac{1}{\xi} \right)\left(2a' - \xi - \frac{1}{\xi} \right) \right]^{-\frac{1}{2}} + 1 \right\} \qquad \text{with} \quad (a, a' > 1),$$

and *for two charges with opposite signs*:

$$(7.33) \quad f(\xi) = k \left\{ \left[2a \cos\alpha - \xi - \frac{1}{\xi} \right] \right.$$
$$\left. \times \left[4(a^2 - \sin^2\alpha) - 4a\left(\xi + \frac{1}{\xi} \right)\cos\alpha + \left(\xi + \frac{1}{\xi} \right) \right]^{-\frac{1}{2}} + 1 \right\} \qquad (a > 0).$$

The expressions for the potential and force are too complicated to be reproduced here. Meanwhile, consider the particular case of *two charges with equal but opposite signs*; in this case, $f(\xi)$ has the following simple form:

(7.34)
$$f(x) = \frac{k'}{\sqrt{2a - \xi - \dfrac{1}{\xi}}}.$$

The field is analogous to that of a classical dipole, except in a neighborhood of the charges. It gives the relatively simple expressions for the charge, force, and distance:

(7.34a)
$$\begin{cases} e = \dfrac{k'}{\pi} \displaystyle\int_0^{2\pi} \dfrac{d\theta}{\sqrt{2(a-\cos\theta)}}, \\[2ex] X = \dfrac{k'}{\pi} \displaystyle\int_0^{2\pi} \dfrac{\cos\theta\, d\theta}{\sqrt{2(a-\cos\theta)}}, \\[2ex] r = 2k' \displaystyle\int_0^{\pi} \dfrac{a+1-(a-1)\cos\theta}{\sqrt{a+3-(a-1)\cos\theta}}\, d\theta. \end{cases}$$

One sees that $\dfrac{X}{e}$ and $\dfrac{r}{e}$ depend only upon the constant a; by eliminating it, one obtains an expression for $\dfrac{X}{e}$ as a function of $\dfrac{e}{r}$ that reduces to $X = \dfrac{e^2}{r}$ at a great distance, as one would expect for a two-dimensional problem with a logarithmic potential. At small distances, the forces remain finite and tend to $X = e$.

This latter result may be generalized for the case of the problem of two different charges; in this case, X tends to the absolute value of the smallest charge.

Pryce has shown the existence of an interesting solution that corresponds to a singular line of finite length, and which is given by:

(7.35) $$f(\xi) = -\frac{a\sqrt{\xi}}{2(1+\xi)}$$ (a real).

At a great distance, it has the character of a dipole solution. Finally, Pryce has shown that the function:

(7.36) $$f(\xi) = -k\left(\frac{2\xi}{1-2a\xi+\xi^2}\right)^2$$ ($a > 0$),

corresponds to a point-like charge in a constant external field E. He found:

(7.36a) $$E = \frac{1}{r}, \qquad e = \frac{2ka}{\sqrt{a^2-1}}, \quad X = \frac{2k}{\sqrt{a^2-1}} = eE;$$

the force is equal to the product of the charge and the field, as in Maxwell's theory. We shall verify later on (sec. 10) that this result may be deduced from much more general hypotheses.

Unfortunately, it seems impossible to transpose the method in the case of two dimensions to that of three dimensions. Aside from the problem of a point-like charge at rest, there is only one other problem that one may treat in three-dimensional space, at least by approximation methods: it is *the problem of the stationary field of a constant current that is filamentary and circular.* I have suggested the study of the case of an *annular singularity* in the hope that it may provide us with a model that is applicable to the proton.

The energy of this singularity may have no particular value since we are given that it depends on the radius of the circle and the intensity of the current. Nevertheless, this ambiguity is reduced by applying the quantum postulate, which demands that the kinetic moment be a multiple of $h / 2\pi$. I hope that the solution to this problem will give us the possibility of accounting for the high value of the mass of the proton. M. B. S. Madhava Rao has taken pains to carry out the complicated calculations that this solution demands [4]. One may perform them completely, due to the fact that the nonlinearity of the equations affects the solution only the immediate neighborhood of the ring, the interior of a torus, to whose exterior one may apply Maxwell's equations. Now, on the one hand, one knows the solution to Maxwell's equations for an annular surface quite well, and, on the other, that the solution in the interior of a torus may be found by a simple approximation method; it is therefore perfectly possible to obtain an approximate solution that satisfies the equations at every point and from it, one can calculate the total energy E, the kinetic moment M, and the magnetic moment m. The final result is the following:

$$(7.37) \qquad E = \frac{4e^2}{a}\log\left(\frac{16a}{r_1}\right), \quad M = 4e^2 v \log\left(\frac{16a}{r_1}\right), \qquad m = \frac{aev}{2},$$

in which e is the total charge, which is uniformly distributed on the ring of radius a, v is the velocity of rotation, and

$$r_1 = 16\pi\left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2}},$$

in which r_0 is the radius of the electron that was defined by (7.6), (7.18).

The primary consequence that one may infer is the following one. One has:

$$(7.38) \qquad \frac{m}{M} = \frac{e}{2E} = \frac{e}{2\mu},$$

in which $E = m$ is the rest mass; this relation is identical to the one that exists between the magnetic moment and angular momentum of an electron orbital in classical theory. Therefore, such an annular singularity certainly does not constitute a model for a particle with "spin," for which this ratio is equal to twice the preceding value.

By taking v close to c (the velocity of light) and equal to $r_0 / 17$, one obtains a mass around 1840 times that of the electron (a point-like charge), and, at the same time, an angular momentum equal to h; however, this state of the ring does not constitute a stable equilibrium position.

This result shows that the nonlinear theory of electromagnetic fields does not provide a solution to the problem of two fundamental elementary masses [5].

8. Other examples of unitary theories. – We now examine the question of knowing whether one justify the choice of the function, (7.1), by reasons that are more fundamental than the ones that we gave, or if there exist other functions that are just as well adapted, or better, to the objective we are pursuing.

Infeld has proposed an argument [1] that leads to a function that slightly different from (7.1) and coincides with it in the electrostatic case; the calculation of the field and the energy of a point-like charge thus remains the same as in the present situation.

Infeld's idea is closely related to the concept that Einstein constantly followed that the gravitational field and the electromagnetic field are, in reality, two aspects of one and the same "unified" field u_{kl}.

We assume that this field is arbitrary from the standpoint of its symmetry characteristics; in any case, it may be additively separated into a symmetric part and an antisymmetric part:

$$u_{kl} = g_{kl} + f_{kl}, \quad g_{kl} = g_{lk}, \qquad f_{kl} = -f_{lk}.$$

The first part describes the metric, or gravitational, field, and the second part, the electromagnetic field. The Lagrangian will be the simplest function that makes the integral I invariant under any arbitrary transformation; this well-known function is the square root of the determinant of u_{kl} (taken with the – sign):

$$\sqrt{-\,|\,u\,|} = \sqrt{-\,|\,g+f\,|} = \sqrt{-\,|\,g\,|}\sqrt{|\,1+fg^{-1}\,|}\,.$$

Now, from sec. 4, we have already studied the matrix, $a(1) = 1 + fg^{-1}$; we have seen that, from (4.18):

$$|\,a(1)\,| = |\,1+fg^{-1}\,| = 1 + F - G^2.$$

If we wish that L should coincide with $\frac{1}{2} F$ for weak fields, then it will suffice to subtract 1 from the preceding expression; one thus obtains:

(8.1) $$L = \sqrt{1 + F - G^2} - 1.$$

This expression differs from (7.1) by the term $- G^2 = - (\mathbf{E} \cdot \mathbf{B})^2$, which is annulled for an electrostatic field. Meanwhile, since *any* function of F, G is likewise permissible as a Lagrangian L, we may imagine the term $- G^2$ being replaced by $-\lambda G^2$, in which λ is an arbitrary constant; for $\lambda = 0$, one obtains (7.1) (with $b = 1$). In the sequel, we will always employ the Lagrangian (8.1); by suppressing the terms in G^2, we obtain the results that correspond to (7.1).

The argument that led to (8.1) is not very convincing since, by treating g_{ik} and f_{ik} on the same plane, one must obtain the electromagnetic equations as well as the gravitational equations by means of the same Lagrangian. Another objection to the process that consists of simply adding the two types of fields to obtain u_{kl} is the enormous difference between their values, as expressed in the usual units. Nevertheless, we shall study the function, (8.1), because of its remarkable properties.

A first property is the following: the Hamiltonain that corresponds to $\sqrt{1+F-G^2} - 1$ is exactly that same function, with the F and G replaced with the dual invariants P and Q (6.7). One easily finds that one has the relations:

$$(8.2) \qquad \frac{1+F^2-G^2}{1+G^2} = \frac{1+Q^2}{1+P^2-Q^2} \quad \text{and} \quad G=Q$$

between F, G and P, Q, in which:

$$(8.3) \qquad H = \sqrt{1+P-Q^2} - 1.$$

Likewise, the other fundamental functions U and V, (6.14) and (6.15), resp., have simple values:

$$(8.4) \qquad \begin{cases} U = \sqrt{1+D^2+B^2+S^2} - 1, \\ V = \sqrt{1+E^2+H^2+S^2} - 1, \end{cases}$$

in which S^2 is the square of the Poynting vector [*see* (6.13)]:

$$(8.4a) \qquad S^2 = (\mathbf{D} \times \mathbf{B})^2 = (\mathbf{E} \times \mathbf{H})^2.$$

Another interesting form for the theory was discovered by Schrödinger [2]. It employed the complex combinations:

$$(8.5) \qquad \mathfrak{F} = \mathbf{B} - i\mathbf{D}, \quad \mathfrak{G} = \mathbf{E} + i\mathbf{H},$$

which form a true six-component vector; one confirms that the appropriate Lagrangian is the following:

$$(8.6) \qquad \mathfrak{L}' = \frac{\mathfrak{F}^2 - \mathfrak{G}^2}{\mathfrak{F}\mathfrak{G}};$$

its partial derivatives with respect to $\mathbf{F}$ and $\mathbf{G}$ are precisely equal to the complex conjugates of $\mathfrak{F}$ and $\mathfrak{G}$:

$$(8.7) \qquad \mathfrak{F}^* = \frac{\partial \mathfrak{L}'}{\partial \mathfrak{G}'}, \quad \mathfrak{G}^* = \frac{\partial \mathfrak{L}'}{\partial \mathfrak{F}'}.$$

Schrödinger has proved these relations by a detailed study of Lorentz transformations. He has shown that there exists a "normal" Lorentz system in which all of the four vectors, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{F}^*$, $\mathfrak{G}^*$, are parallel at a given point; when one develops the first theory in this particular system, one may easily prove that it is identical to the one that is obtained by means of the preceding complex representation.

The most remarkable property of $\mathfrak{L}'$ is its character as a rational function: it is simply the quotient of two invariants. Nevertheless, this advantage is purely formal; the square root appears again when one wishes to perform a concrete calculation. Schr`dinger has likewise drawn attention to the fact that this complex representation permits two types of fields; the one, for which the two vectors, **F** and **G** are parallel in the normal Lorentz system, and the other, for which the same vectors are anti-parallel. There is no valid argument that permits us to exclude the one or the other of these cases. Now, in the real form of the theory, this difference corresponds to the possibility of taking the + sign or the – sign under the radical; it therefore seems that one must admit the possibility of this double sign in the original theory [3].

Madhava Rao [4] has undertaken a detailed comparison of the formulas of Schrödinger with the formulas of our theory.

P. Weiss [5] has proposed another complex form for the theory that is essentially new. He introduces the space vectors $\mathbf{B} \pm i\mathbf{E}$ and $\mathbf{D} \pm i\mathbf{H}$, which *do not* collectively define a six-component vector. He proves that the group of orthogonal transformations of *complex* three-dimensional vectors is isomorphic to the group of transformations of vectors with six *real* components (i.e., the antisymmetric second rank tensors in four-dimensional space) that is induced by the Lorentz group. This theorem permits us to present the entire theory in a perfectly symmetrical form.

Weiss then criticized the Lagrangian (8.1), and proposed to replace it by the real part of the following complex function of the invariants F and G:

$$(8.8) \qquad \sqrt{1+2(F+iG)} \; - 1;$$

nevertheless, his arguments do not seem convincing to me.

The only reasonable proposal for the adoption of a general principle from which the choice of Lagrangian must follow uniquely was made by Infeld [6]; it demands that the *field components, $f_{kl}(\mathbf{B}, \mathbf{E})$, present no singularities.*

Consider a point-like charge with one of the Lagrangians (7.1) and (8.1), which coincide in the electrostatic case, moreover. In these two cases, the point at which the charge is found is a singular point not only for **D**, which varies like e / r^2, but also for **E**, since all of the components of **E** jump from –1 to +1 (in natural units for which $b = 1$), when one traverses the singularity along a line. This singularity is inoffensive since it does not alter the character of the energy that its magnitude is finite. Nevertheless, it is interesting to search for a Lagrangian for which **E** and its first derivatives are everywhere continuous, including the central point of a spherically symmetric field.

Infeld has discovered a similar function by a systematic study that I would like to sketch out in what follows. He starts with the hypothesis that the complete symmetry of the theory, which is expressed by its character as a self-dual theory, must allow us to use a variational principle that simultaneously contains the two groups of self-dual tensors.

He introduces an action function T that depends on $f_{kl}(\mathbf{E}, \mathbf{B})$ and $p^{*}_{kl}(\mathbf{D}, \mathbf{H})$, naturally by the intermediary of the invariants F and P ($G = Q$ is excluded). The f_{kl} must be derived from potentials and the p_{kl}, from "anti-potentials," in such a fashion that they satisfy two groups of integrability conditions:

(8.9)
$$\frac{\partial f^{*kl}}{\partial x^{l}} = 0, \qquad\qquad \frac{\partial p^{kl}}{\partial x^{l}} = 0 .$$

In turn, one restricts the generality of the function $T(F, P)$ by imposing the condition that f_{kl} and p^{kl} must be "canonically conjugate," i.e., that one has:

(8.10)
$$\begin{cases} p^{kl} = \dfrac{\partial T}{\partial f^{kl}} = \dfrac{\partial T}{\partial F} 2 f^{kl}, \\[2ex] f^{*kl} = \dfrac{\partial T}{\partial p^{*}_{kl}} = \dfrac{\partial T}{\partial P} 2 p^{*kl}. \end{cases}$$

These relations must permit us to calculate the p^{*}_{kl} as functions of the f_{kl}. Meanwhile, their number is equal to $2 \times 6 = 12$, and, as a consequence, in order for them to be compatible it is necessary that T satisfy certain conditions that are expressed by:

(8.11)
$$4 \frac{\partial T}{\partial F} \frac{\partial T}{\partial P} = 1, \qquad\qquad \frac{\partial T}{\partial F} F + \frac{\partial T}{\partial P} P = 0 .$$

The latter condition shows that T is a homogenous function of degree zero in F and P – in other words, that it depends uniquely upon:

(8.11a)
$$\varepsilon = \sqrt{-\frac{F}{P}} .$$

A simple discussion of the characteristics of $T(\varepsilon)$ leads to the following result: In order for E to be continuous at a point where $D \approx \dfrac{1}{r^{2}}$, it must be of the form:

(8.12)
$$T = - \log \varepsilon + 1 + \varepsilon + \dots,$$

in which the terms that are indicated by dots contain higher powers of ε.

When one neglects these terms, one obtains the simplest $T(\varepsilon)$; the corresponding Hamiltonian is:

(8.13)
$$H = \tfrac{1}{2} \log(1 + P).$$

The Lagrangian has a more complicated form, but the formula that relates p_{kl} and f_{kl} is simple, and is expressed by means of P or F:

(8.14) $$f_{kl} = \frac{p_{kl}}{1+P}, \qquad p_{kl} = \frac{-1 \pm \sqrt{1+4F}}{2F} f_{kl}.$$

The sign of the radical must be chosen in such a fashion that the continuity condition for the field components is satisfied. The critical points are defined by $1 + 4F = 0$; at these critical points - which form a surface, in general – F attains its minimum $F = -\frac{1}{4}$, and the radical disappears. In order to preserve continuity, the sign of the radical must be changed when one traverses a surface $F = -\frac{1}{4}$. Since $p_{kl} \to f_{kl}$ at a great distance from the charges, we see that it must take the "plus" sign in these regions; therefore, there exists a surface $-\frac{1}{4} = F$ that surrounds a charge, in whose interior it must take the "minus" sign.

This reasoning is confirmed by the result of actual calculation for the problem of an isolated point-like charge. Instead of (8.13), we write:

(8.13a) $$H = \frac{b^2}{2} \log\left(1 + \frac{P}{b^2}\right);$$

with $P = D^2$, $D_r = \dfrac{e}{4\pi} \dfrac{1}{r^2}$, one obtains:

(8.15) $$E_r = \frac{e}{4\pi r_0^2} \frac{x^2}{1+x^4} \qquad \left(x = \frac{r}{r_0}; r_0 = \sqrt{\frac{e}{4\pi b}} \right).$$

The surface, in whose interior the sign of the radical in (8.14) changes, is a sphere whose radius is defined by $(E_r/b)^2 = \left(\dfrac{x^2}{1+x^4}\right)^2 = \frac{1}{4}$, namely, $x = 1$. The free charge density is:

(8.16) $$\sigma = \operatorname{div} \mathbf{E} = \frac{e}{4\pi r_0^2} \frac{4x}{(1+x^4)^2};$$

it is annulled for $r = 0$, and, for large r it decreases like r^{-7}, exactly as in the case of sec. 7. One may easily verify that the integral of σ over all space is equal to e.

For the potential, one finds:

(8.17) $$\varphi(r) = \frac{e}{4\pi r_0} \int_x^\infty \frac{y^2}{1+y^4} dy .$$

Its value for $r = 0$ is:

(8.18) $$\varphi(0) = \frac{e}{4\pi r_0} \frac{x\sqrt{2}}{4} = \frac{e}{4\pi r_0} 1.11,$$

and, by virtue of (6.30), the energy is:

$$(8.19) \qquad E_0 = \frac{e^2}{4\pi r_0} \frac{\sqrt{2}\pi}{6} = \frac{e^2}{4\pi r_0} 0.741.$$

The numerical coefficient is noticeably smaller than in (7.16).

As one easily sees, there exists no solution to a point-like magnetic charge. This theory therefore accounts for the empirical confirmation that electricity and magnetism are not interchangeable. This fact seems to constitute the most serious argument in favor of the continuity hypothesis on **E**; indeed, since the singularity of **D** may not be avoided, the existence of a discontinuity in **E** has no great importance, although it remains inoffensive, i.e., it leaves energy finite.

9. The conditions for dynamic equilibrium [1]. – Now consider the problem of the motion of a point-like charge under the action of an external electromagnetic field.

Before we do that, we must explain what we mean by the latter expression – "external electromagnetic field" – which necessitates certain clarifications in our nonlinear theory, in which fields may no longer be superposed. It is obvious that we intent "external field" to mean the field at an infinite distance from a point-like charge, which must be considered as having been given. Naturally, this is possible only if we are concerned with doing an approximate calculation; indeed, if the particle moves with a variable velocity, it will emit waves that will be functions of the motion and which one may not consider as having been given. Therefore, we are principally concerned with "quasi-stationary" motions, here, for which one may neglect the reaction that is produced by the emission of waves.

We make no particular hypothesis concerning the Lagrangian; we only admit that the field, its energy, and momentum all have finite values.

One may endlessly debate the question of whether the establishment of new equations of motion does or does not demand new hypotheses. In our first article, Infeld and myself treated this problem by appealing to a variational principle; we have formulated this principle in such a manner that the integral must be stationary for not only variations of the field in space, but also for variations of the world-lines of the charge.

Leaving aside the question of whether the mathematical method that have used is correct or not, we continue to think that out viewpoint is justified, in principle, and that if one wants to make it completely acceptable, only a slight alteration is sufficient. Feenberg [2] has raised the following objection: having proved that one may determine a field for any arbitrary motion of the charge, he concluded from this that the actual motion must be fixed among all of the other possible ones by means of supplementary hypotheses. Pryce made remarks that were analogous to the preceding ones. I think that this objection falls short of the mark. In order to show this, I would like to consider the very simple example of a problem in the calculus of variations that has one trait in common with ours, namely, the following problem: *to find the shortest distance from a point P to a line L.* The common trait consists of the fact that one may not give the position of the point *Q* of closest approach to *P* on the line; in our problem it corresponds to the unknown motion of the charge.

By varying the integral that gives the distance, one obtains:

1. A differential equation that expresses that the extremal must be a straight line;
2. A boundary condition that expresses that the extremal must be normal to *L*.

The viewpoint that was adopted by Feenberg leads us to neglect the boundary conditions that follow from the variational problem. To give the motion of the charge is the same as arbitrarily giving the angle by which the extremal must cut the line *L*. For a given angle – for example, 45° – there exists a solution to the problem, namely, the curve *PR* (*fig.* 2), which is rectilinear (therefore, it satisfies the differential equation), and which cuts the line *L* at 45°. Nevertheless, it does not realize an extremum of the initial integral; indeed, the length of any curve that is analogous to *C* (*fig.* 2) (a straight line from *P* to a neighborhood of *L*, which it must intersect at an angle of 45°) is obviously smaller than *PR*. The lower limit of the length of all these curves is the length of the perpendicular *PQ*; this does not belong to the category of the preceding curves since it does not cut *L* at 45°. From this analysis, it results that the integral has no minimum when one imposes this arbitrary condition. Such a minimum exists only for the "natural" boundary conditions, which express that the angle with *L* must be 90°.

In a similar fashion, the solution to the field equations for an arbitrary motion of the charge does not necessarily provide an extremum for integral *I* (or, more correctly, "does not necessarily annul the first variation," assuming that the integral is not positive definite). We must find what the natural boundary conditions are, and see whether they correspond to a motion of the charge. If this is not the case, then we must conveniently modify the integral we start with and use the new boundary conditions.

In section 3, we calculated the variation on the boundary for an arbitrary four-dimensional surface; we found, see (3.23), (3.24), and (3.24*a*), that:

(9.1)
$$\delta I = \int_{S} (\mathfrak{X}_{k}\,\delta x^{k} + \mathfrak{P}^{k}\,\delta \varphi_{k})\,du \,,$$

with

(9.2)
$$\mathfrak{X}_{k} = \mathfrak{T}_{k}^{i} N_{i}\,, \qquad\qquad \mathfrak{P}^{k} = \mathfrak{p}^{ik} N_{i}\,.$$

The surface *S* in our problem is a tube of infinitely small section that surrounds the world-line of a singularity (a singularity of **D**, since **E** may be continuous, *cf.* sec. 8). We introduce parameters on *S* in the form of time t_0 along the world-line, and two other parameters (polar coordinates) that define the position of the surface element in two dimensions $d\sigma$ of an infinitesimal sphere *s* in three-dimensional space t_0 = const. Therefore, on the surface of sphere, one has:

$$\frac{\partial x^1}{\partial t_0} = \frac{\partial x^2}{\partial t_0} = \frac{\partial x^3}{\partial t_0} = 0\,, \qquad\qquad \frac{\partial x^4}{\partial t_0} = 1\,,$$

and the definition (1.12) of N_i gives:

(9.3)
$$(N_1, N_2, N_3)\,du = \mathbf{n}\,d\sigma\,dt_0, \qquad\qquad N_4 = 0,$$

in which **n** is the unit vector that is directed along the normal to the sphere *s*, and $d\sigma$ is its surface element.

We now introduce the vectors:

(9.4) $\mathbf{X}(X_x, X_y, X_z)$; $\mathbf{Y}(Y_x, Y_y, Y_z)$; $\mathbf{Z}(Z_x, Z_y, Z_z)$,

and the integrals are taken over the sphere *s* (and indicated by an *O*):

(9.5) $\int_O \mathbf{Xn} d\sigma = F_x$, $\int_O \mathbf{Yn} d\sigma = F_y$, $\int_O \mathbf{Zn} d\sigma = F_z$,

(9.6) $\int_O (\mathfrak{H} \times \mathbf{n}) d\sigma = \boldsymbol{\gamma}$, $\int_O \mathfrak{D} \mathbf{n} d\sigma = \varepsilon$, $\int_O \mathbf{Sn} d\sigma = f$.

We may write (9.1) in ordinary vectorial notation in the form:

(9.7) $\delta I = -\int (\mathbf{F} \delta\mathbf{r} + f \delta t + \boldsymbol{\gamma} \delta\mathbf{a} + \varepsilon \delta\varphi) dt_0$,

in which **F** is the vector with components (9.5) and **r** is the radius vector with components *x, y, z*. $\delta\mathbf{r}$ and δt are the variations of the world-line and $\delta\mathbf{a}$, $\delta\varphi$ are the variations of the potentials on the world-line.

The natural boundary conditions are:

(9.8) $\mathbf{F} = 0$, $f = 0$,
and
(9.9) $\boldsymbol{\gamma} = 0$, $\varepsilon = 0$.

From the definition (9.6), one sees that (9.9) signifies *the complete absence of charge and current.*

Therefore, a field that is determined by a point-like charge does not correspond to a null first variation of *I*. In fact, the integral *I* does not permit the existence of any point-like charge; for this to be the case, one must correct the statement of the variational principle by adding another integral to the primitive integral, one which is taken over the world-line of the singularity. Since the proper time of the world-line is $d\tau = \sqrt{1 - v^2} \, dt_0$, one will have:

(9.10) $I = \int \mathfrak{L} dx + e \int u^k \varphi_k d\tau$.

In this expression, the constant *e* represents the charge of the singularity, and u^k is the world-vector of velocity, which is related to the ordinary vector by the relations:

(9.11) $(u^1, u^2, u^3) = \dfrac{\mathbf{v}}{\sqrt{1 - v^2}}$, $u^4 = \dfrac{1}{\sqrt{1 - v^2}}$.

The line integral may be likewise written:

(9.12)
$$\int u^k \varphi_k \, d\tau = \int (\mathbf{va} + \varphi) dt_0 \,.$$

By varying **a** and φ in this new integral, one confirms that the natural boundary conditions (9.9) must be replaced by the following:

(9.13)
$$\gamma \cdot \int_O (\mathfrak{H} \times \mathbf{n}) d\sigma = e\mathbf{v}, \qquad \varepsilon \cdot \int_O \mathfrak{D}\mathbf{n}\, d\sigma = e,$$

whereas the first conditions (9.8), as well as the field equations, do not change.

We must therefore consider (9.10) as the exact expression of the integral that must be varied; (9.13) expresses that the charge generates the field. (9.8) represents the conditions for dynamic equilibrium, which one may deduce from the equations of motion ([7]).

Obviously, we are making a new hypothesis when we admit that (9.8) remains the same as before and does not need to be modified. This hypothesis amounts to admitting that *there does not exists a mass that is concentrated at a point,* but that any mass is of electromagnetic origin, and, as such, is distributed over the field that surrounds the charge.

The discovery of the neutron (and the existence of the neutrino, if it is confirmed) has cast a shadow of doubt on the truth of this assumption. Meanwhile, the essential problem for us is not the question of the electromagnetic origin of mass, but that of the elimination of the infinite masses that Maxwell's theory implies. We estimate that this problem is completely solved by the considerations that were discussed in the preceding sections and by the deduction of the laws of motion that we shall give. In the latter, all mass has an electromagnetic origin; if a supplementary one exists, it will suffice to simply add it to the latter. Personally, I believe that the concept of the mass representing the proper energy of the field of a singularity will be preserved, just as one must abandon the usual electromagnetic theory in favor of a more satisfying one (for example, in favor of the theory that is based on the hypothesis of the neutrino, which was suggested by de Broglie [4] and developed by Jordan and Kronig [5]).

Having arrived at this point, we must discuss the influence of gravitation on electromagnetic phenomena in more detail.

B. Hoffman [6] has solved the system of simultaneous equations (3.5), (3.13), and (3.15) of gravitation and electromagnetism in the case of a spherically symmetric field and with a function L of the form (7.1) or (8.1). The calculations are too long for us to reproduce them here; we shall content ourselves by giving the results, in which we shall simplify certain important points, moreover. Hoffmann included the cosmological term in the equations for gravitation and assumed the existence of magnetic poles that coincide

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([7]) Pryce has proved that by choosing a dual variational principle (6.8), and replacing the integrability conditions (6.10) by:
$$\frac{\partial p^{ik}}{\partial x^k} = j^i,$$

in which j^1 designates the charge-current vector density, one obtains not only the same field equations (6.9), but also the conditions (9.13) of dynamic equilibrium [3].

with point-like electric charges; we ignore the terms in the equations that correspond to these hypotheses.

One may show that there exists a system of coordinates in which the world-line element has the following form:

$$(9.14) \qquad ds^2 = A\, dt^2 - A^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

and in which the electromagnetic field of a point-like charge is exactly the same as the one that was given in section 7. This amounts to saying that by a convenient choice of coordinate system gravitation has absolutely no influence on the electromagnetic field of a particle at rest. This result is not surprising if one recalls that the field equations have the form of the generalized Maxwell equations [*cf.* sec. 5, (5.4)] *in any coordinate system,* provided that one considers (**E**, **B**) as a tensor and ($\mathfrak{D}$, $\mathfrak{H}$) as a tensorial density.

The magnitude A – viz., Einstein's gravitational potential – is given by:

$$(9.15) \qquad A(r) = 1 - \frac{2}{r}[m_0 + m(r)],$$

in which m_0 is an integration constant, and:

$$(9.16) \qquad m(r) = 4\pi \int_0^r U r^2 dr,$$

represents the fraction of the electromagnetic energy that is contained in the interior of a sphere of radius r; for the Lagrangian, (7.1), it is equal to:

$$(9.17) \qquad 4\pi \int_0^r U r^2 dr = 4\pi \left(\frac{e}{4\pi}\right)^2 \frac{1}{r_0} \int_0^{\frac{r}{r_0}} (\sqrt{1+x^2} - x^2) dx,$$

and this expression is proportional to r for small r; therefore, its contribution $\dfrac{m(r)}{r}$ to the potential A remains finite for $r \to 0$.

For a null charge $m(r) = 0$, one has the well-known ds of Schwarzschild, with $A = 1 - \dfrac{2m_0}{r}$, which must be infinite for $r = 0$. m_0 represents the gravitational mass of the uncharged sphere, which signifies that the geodesics of space that are defined by the preceding ds coincide approximately with the trajectories of a body that takes the form of a point of mass m_0 that moves according to Newton's law.

We now constrain the motion of the test particle to region such that the distance r from the center of our particle is large compared to r_0; in this case, formula (9.15) tells us that the charge of the particle contributes to the gravitational mass of a quantity $m(\infty) = m$, which is equal to the total electromagnetic energy that we have called E_0 up till now. The effective mass decreases exactly as in Newton's theory when the test

particle approaches the center of the particle: the energy density that is distributed over a spherical shell contributes nothing to the gravitational action at the points in its interior.

The postulate that the coefficients g_{kl} of the elementary world-interval all remain finite, like the electromagnetic field f_{kl} leads to the condition $m_0 = 0$. The mass is then of strictly electromagnetic nature, but, at the same time, it constitutes the gravitational mass, in the sense that was analyzed above.

The center of the particle is not a regular point, even if $m_0 = 0$; indeed, the limit:

$$(9.18) \qquad \lim_{r \to 0} \frac{m(r)}{r} = 4\pi \left(\frac{e}{4\pi} \right)^2 \frac{1}{r_0^2} ,$$

is finite. The length of the circumference of radius 1 that is centered on the particle is thus not equal to 2π, but $\dfrac{2\pi}{A}$; the geometry at the center of the particle is therefore non-euclidean.

This inconvenience may be avoided by the use of the Hamiltonian (8.13) of Infeld, in which the field is everywhere continuous. Infeld and Hoffmann [7] have calculated the field, which is simultaneously electromagnetic and gravitational, to which it corresponds. The result is the same as before: there exists a coordinate system in which the expression for any spherically symmetric electrostatic field is the same, whether in the presence of gravitation or not; the world-interval has the form (9.14), with (9.15) and (9.16). If the gravitational forces must be finite then $m_0 = 0$, then they must likewise be continuous; the value of $m(r)$ is:

$$(9.19) \qquad m(r) = 4\pi \left(\frac{e}{4\pi} \right)^2 \frac{1}{r_0} \frac{1}{2} \int_0^{\left(\frac{r}{r_0} \right)} x^2 \log\left(1 + \frac{1}{x^4} \right) dx ,$$

and tends to zero for $r \to 0$. Therefore, A $\to$ 1 for $r \to 0$, and the geometry in the neighborhood of the center is Euclidean.

We may summarize the results above by saying that they confirm our opinion that gravitation adds nothing to the structure and cohesion of elementary particles. Einstein's gravitational equations may be combined with the electromagnetic equations without altering them. This result is very satisfying if one reflects that Einstein's theory has a rather formal character since it simply adds the curvature term to the electromagnetic Lagrangian and does not explain the gravitational constant. This latter problem will find its solution only much later on. The importance of Einstein's general invariance postulate in the electromagnetic theory resides not in the link that it establishes with gravitation; rather, it appears in the consequences that were discussed in section 5.

1. The derivatives of the Lagrangian with respect to the g_{kl} are the components of the energy tensor, which is consequently symmetric;

2. The derivatives of the potentials appear only in the antisymmetric combinations f_{ik} .

10. Equations of motion [1]. –The natural boundary conditions (9.8) give us the possibility of integrating the conservation laws for energy and momentum (5.14) over an arbitrary three-dimensional domain without accounting for the singularities. By integrating over all space, one obtains:

(10.1)
$$\begin{cases} \int_\infty \mathbf{Xn}d\sigma + \dfrac{dG_x}{dt} = 0, \\ \qquad \cdots \\ \int_\infty \mathbf{Sn}d\sigma + \dfrac{dE}{dt} = 0, \end{cases}$$

in which the symbol ∞ under the $\int$ sign indicates a closed surface at infinity, and:

(10.2)
$$\mathbf{G} = \int \mathbf{S}dv, \qquad E = \int U dv,$$

are the *momentum* and *total energy,* resp.

One may establish the equations in the case for which the external field does not vary appreciably over a distance that is large compared to the radius of the electron.

It suffices to treat the case in which the infinitude of the *total* field reduces to a constant field that represents the *external* field in a neighborhood of the electron.

In all of what follows, we use a Lorentz system; therefore, we no longer distinguish between vectors and vector densities. It is convenient to take $\mathbf{D}$ and $\mathbf{B}$ as independent variables.

Let $\mathbf{D}^{(e)}$, $\mathbf{B}^{(e)}$ represent the constant "external" field, and let $\mathbf{D}^{(i)}$, $\mathbf{B}^{(i)}$ represent the "interior" or proper field of the electron. One has:

(10.3)
$$\lim_\infty \mathbf{D}^{(i)} = 0, \qquad \lim_\infty \mathbf{B}^{(i)} = 0,$$

and the total field is:
(10.4)
$$\mathbf{D} = \mathbf{D}^{(i)} + \mathbf{D}^{(e)}, \qquad \mathbf{B} = \mathbf{B}^{(i)} + \mathbf{B}^{(e)}.$$

Define the field, $\mathbf{E}^{(e)}$, $\mathbf{H}^{(e)}$, by:

(10.5)
$$\mathbf{E}^{(e)} = \left(\frac{\partial U}{\partial \mathbf{D}}\right)_e, \qquad \mathbf{H}^{(e)} = \left(\frac{\partial U}{\partial \mathbf{B}}\right)_e,$$

in which the index e specifies that the variables in the functions considered are $\mathbf{D}^{(e)}$ and $\mathbf{B}^{(e)}$.

One may then separate this total field $\mathbf{E}$, $\mathbf{H}$ into an exterior field $\mathbf{E}^{(e)}$, $\mathbf{H}^{(e)}$, and an interior field $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$:

(10.6)
$$\mathbf{E} = \mathbf{E}^{(e)} + \mathbf{E}^{(i)}, \qquad \mathbf{H} = \mathbf{H}^{(e)} + \mathbf{H}^{(i)};$$

the $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$ depend not only upon the $\mathbf{D}^{(e)}$, $\mathbf{B}^{(e)}$, but one has, in any case:

$$\lim_{\infty} \mathbf{E}^{(i)} = 0, \qquad\qquad \lim_{\infty} \mathbf{H}^{(i)} = 0.$$

The interior field differs from the external one only by a constant field; it therefore satisfies the fundamental axioms. Moreover, it satisfies conditions (9.8) and (9.13) at the point where one finds the charge since all of the surface integrals are annulled for a constant field.

One finds that the Poynting vector is:

(10.7) $$\mathbf{S} = \mathbf{S}^{(e)} + (\mathbf{D}^{(e)} \times \mathbf{B}^{(e)}) + (\mathbf{D}^{(i)} \times \mathbf{B}^{(e)}) + \mathbf{S}^{(i)},$$

in which $\mathbf{S}^{(i)} = (\mathbf{D}^{(i)} \times \mathbf{B}^{(i)})$ is the internal Poynting vector and $\mathbf{S}^{(e)}$ is a constant that may be ignored; indeed, it provides only a constant term in the expression for the momentum and this does not change (10.1). We therefore have:

(10.8) $$\mathbf{G} = \mathbf{G}^{(i)} + \left(\mathbf{D}^{(e)} \times \int \mathbf{B}^{(i)} dv\right) - \left(\mathbf{B}^{(e)} \times \int \mathbf{D}^{(i)} dv\right).$$

The derivative of energy with respect to time is:

$$\frac{dE}{dt} = \frac{d}{dt} \int U dv = \int \left(\frac{\partial U}{\partial \mathbf{B}} \mathbf{B} + \frac{\partial U}{\partial \mathbf{D}} \mathbf{D}\right) dv$$
$$= \int (\mathbf{HB} + \mathbf{ED}) dv,$$

by virtue (6.14). By substituting (10.4) and (10.6), one obtains:

(10.9) $$\frac{dE}{dt} = \int (\mathbf{H}^{(e)}\mathbf{B}^{(i)} + \mathbf{E}^{(e)}\mathbf{D}^{(i)} + \mathbf{H}^{(i)}\mathbf{B}^{(i)} + \mathbf{E}^{(i)}\mathbf{D}^{(i)}) dv.$$

Now define the internal energy by:

(10.10) $$\mathbf{E}^{(i)} = \int dt \int (\mathbf{H}^{(i)}\mathbf{B}^{(i)} + \mathbf{E}^{(i)}\mathbf{D}^{(i)}) dv,$$

which depends on the motion of the charge and the external field. If the latter is weak, it tends to the equilibrium value:

(10.10a) $$\mathbf{E}^{(i)} = \int U(\mathbf{B}^{(i)}, \mathbf{D}^{(i)}) dv.$$

One may then write (10.9) and the derivative of (10.8) with respect to time in the following form:

$$(10.11) \quad \begin{cases} \dfrac{dE}{dt} = \dfrac{dE^{(i)}}{dt} + \mathbf{H}^{(e)} \int \mathbf{B}^{(i)} dv + \mathbf{E}^{(e)} \int \mathbf{D}^{(i)} dv, \\[2ex] \dfrac{d\mathbf{G}}{dt} = \dfrac{d\mathbf{G}^{(i)}}{dt} + \left(\mathbf{D}^{(e)} \times \int \mathbf{B}^{(i)} dv \right) - \left(\mathbf{B}^{(e)} \times \int \mathbf{D}^{(i)} dv \right). \end{cases}$$

Now calculate the surface integrals of (10.1). The quantities inside the $\int$ sign may be developed in $\mathbf{D}^{(i)}$, $\mathbf{B}^{(i)}$, or $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$, since the latter are very small compared to the external field over a surface of any sort that is situated sufficiently far away.

The components of the vector $\mathbf{X}$ may be written [*cf.* (5.12)]:

$$(10.12) \quad \begin{cases} X_x = H_y B_y + H_z B_z + D_y E_y + D_z E_z - U, \\ X_y = -H_x B_y - E_x D_y \\ X_z = -H_x B_z - E_x D_z. \end{cases}$$

We substitute (10.4) and (10.6) in these formulas, and neglect the terms of second order; by taking (6.14*a*) into account, one easily obtains:

$$(10.13) \quad (\mathbf{Xn}) = (\mathbf{X}^{(e)}\mathbf{n}) + \mathbf{H}_x^{(e)} (\mathbf{B}^{(i)}\mathbf{n}) - \mathbf{E}_x^{(e)} (\mathbf{D}^{(i)}\mathbf{n})$$
$$+ [\mathbf{B}^{(e)} \times (\mathbf{n} \times \mathbf{H}^{(i)})]_x + [\mathbf{D}^{(e)} \times (\mathbf{n} \times \mathbf{E}^{(i)})]_x.$$

For $\mathbf{S}$, by using the expression $(\mathbf{E} \times \mathbf{H})$ one obtains:

$$(10.14) \quad (\mathbf{Sn}) = (\mathbf{S}^{(e)}\mathbf{n}) + \mathbf{E}^{(e)}(\mathbf{H}^{(i)} \times \mathbf{n}) - \mathbf{H}^{(e)}(\mathbf{E}^{(i)} \times \mathbf{n}).$$

Now consider the equations of the internal field:

$$\begin{aligned} \text{rot } \mathbf{H}^{(i)} - \mathbf{D}^{(i)} = 0, \qquad & \text{div } \mathbf{D}^{(i)} = 0, \\ \text{rot } \mathbf{E}^{(i)} + \mathbf{B}^{(i)} = 0, \qquad & \text{div } \mathbf{B}^{(i)} = 0, \end{aligned}$$

and integrate them over all space, to the exclusion of a small sphere that has the charge at its center. The values of the corresponding surface integrals are given by (9.13); one obtains:

$$(10.15) \quad \begin{cases} \int_\infty (\mathbf{n} \times \mathbf{H}^{(i)}) d\sigma = e\mathbf{v} + \int_\infty \mathbf{D}^{(i)} dv, \quad \int_\infty \mathbf{n} \mathbf{D}^{(i)} d\sigma = e, \\[2ex] \int_\infty (\mathbf{n} \times \mathbf{E}^{(i)}) d\sigma = -\int_\infty \mathbf{B}^{(i)} dv, \qquad \int_\infty \mathbf{n} \mathbf{B}^{(i)} d\sigma = 0. \end{cases}$$

Now integrate (10.13) over a surface that is situated at infinity; the constant terms contribute nothing, and we find, by accounting for (10.15), that:

$$(10.16) \quad \begin{cases} \int_\infty (\mathbf{Xn})d\sigma = -e[\mathbf{E}_x^{(e)} + (\mathbf{v}\times\mathbf{B}^{(e)})_x] \\ \qquad + \left(\mathbf{B}^{(e)}\times\int\mathbf{D}^{(i)}dv\right)_x - \left(\mathbf{D}^{(e)}\times\int\mathbf{B}^{(i)}dv\right)_x \\ \int_\infty (\mathbf{Sn})d\sigma = -ev\mathbf{E}^{(e)} \\ \qquad -\mathbf{E}^{(e)}\int\mathbf{D}^{(i)}dv - \mathbf{H}^{(e)}\int\mathbf{B}^{(i)}dv \end{cases}$$

By substituting (10.11) and (10.16) in (10.1), one sees that all of the terms that contain volume integrals are pairwise equal and opposite, and consequently disappear. The final result is:

$$(10.17) \quad \begin{cases} \dfrac{d\mathbf{G}^{(i)}}{dt} = e\left[\mathbf{E}^{(e)} + \left(\mathbf{v}\times\mathbf{B}^{(e)}\right)\right], \\ \dfrac{d\mathbf{E}^{(i)}}{dt} = ev\mathbf{E}^{(e)}. \end{cases}$$

These equations are the *well-known equations of motion of the Lorentz theory of the electron* if one replaces $\mathbf{E}^{(i)}$, $\mathbf{G}^{(i)}$ with the corresponding values for stationary motion that are given by (6.34). The system (10.17) remains nonetheless valid even if the motion may not be considered as quasi-stationary; in this case, one must correct the values of $\mathbf{G}^{(i)}$ and $\mathbf{E}^{(i)}$ in order to account for the emitted waves. Feenberg has calculated the higher-order approximations and has shown that the correction terms of the first order correspond to the well-known classical radiation reaction exactly.

I have tried to generalize these formulas in another sense, by assuming that the particle possesses not only a point-like charge, but also a magnetic (or electric) dipole [2]. The two corresponding moments are related to the field by formulas that are analogous to the conditions (9.13), in which the radius vector appears inside the integrals taken over infinitely small spheres considered, for example:

$$\int_O \mathbf{r}(\mathbf{nD})d\sigma = e\mathbf{r}_0 + \mathbf{p},$$

in which $\mathbf{r}_0(t)$ is the radius vector of the singularity, and $\mathbf{p}$ is the electric moment. This method provides us with the supplementary equations of motion that relate the angular momentum $\mathbf{M}$ and the center of electric energy $\mathbf{P}$ of the field to the dipole moments $\mathbf{m}$, $\mathbf{p}$, whether magnetic or electric:

$$(10.18) \quad \begin{cases} \dfrac{d\mathbf{M}^{(i)}}{dt} = \left(\mathbf{B}^{(e)}\times\mathbf{m}\right) - \left(\mathbf{E}^{(e)}\times\mathbf{p}\right), \\ \dfrac{d\mathbf{P}^{(i)}}{dt} = \left(\mathbf{E}^{(e)}\times\mathbf{m}\right) + \left(\mathbf{B}^{(e)}\times\mathbf{p}\right). \end{cases}$$

Kramers [3] has postulated the existence of the same equations by taking a purely descriptive viewpoint; he assumes, moreover, that the antisymmetric tensor $(\mathbf{M}^{(i)}, \mathbf{P}^{(i)})$ is proportional to the tensor $(\mathbf{m}, \mathbf{p})$, and he has proved that this hypothesis leads to the

following result: *if the particle at rest has a magnetic moment, m$_0$, and a null electric moment then the ratio* $\dfrac{m_0}{M_0^{(i)}}$ *of the magnetic moment to the angular momentum will be equal to* $\dfrac{e}{c\mu_0}$ *, in which m$_0$ is the rest mass*; this value is twice the value of the analogous ratio for the orbital motion of the electron.   To him, this result seemed to provide a sufficient basis upon which to develop a classical theory of the spinning electron.

It is not possible for me to discuss this last point here. Equations (10.18) are probably correct, but the manner by which I have obtained them from the nonlinear field theory is not defensible. The reason is simple: *it is impossible to choose and to employ arbitrary types of singularities.* In section 9, we saw that each singularity must satisfy certain "natural" boundary conditions, and not merely conditions that one may impose arbitrarily.

For any field theory of the type considered, these conditions have the form (9.13), i.e., they are uniquely compatible with the existence of a point-like charge. One may not assume the existence of a supplementary point-like moment (electric, magnetic, or both) without radically modifying the structure of the theory, which consists of field equations. The experimental proof of spin indicates that similar modifications are becoming necessary. Up to the present, I have made no attempts in this direction because success seems doubtful to me as long as one remains in the context of the classical theory without introducing quantum considerations.

Here, we shall stop this discussion of the classical treatment of nonlinear electrodynamics; we briefly recall the results:

It is possible to generalize Maxwell's equations in such a manner as to render the proper energy of a point-like charge finite. Nevertheless, there exists an infinitude of possible generalizations that replace this condition, each of which leads to another "structure" for the electron and other numerical factors. Both fundamental laws of the classical theory of the electron, anmely, Maxwell's equations for the field that is "external" to the electron, and the expression for the Lorentz force that is produced by an external field are valid in any case [4]. Therefore, nonlinear electrodynamics does not alter the essential characteristics of the old theories in any manner, while eliminating their "black hole" - the infinite proper energy – and permitting the introduction of the hypothesis of a purely electromagnetic mass. Nevertheless, it gives no indication of the effective structure of the electron; the "black hole" is therefore transformed into a "question mark."

The problem that is now posed is that of knowing whether one may respond to this by using quantum theory.

CHAPTER II

QUANTUM THEORY

11. Quantum dynamics of continuous media. – The "quantization" of electrodynamics is as old as quantum theory itself; indeed, the idea of a quantum appeared for the first time in Planck's theory of radiation, i.e., in the statistical theory of electromagnetic waves.

Between 1900 and 1906, during the first developmental period of quantum theory, one could not precisely establish whether the quantum corresponded to a property of the exchange of energy between matter and radiation or a characteristic of electromagnetic radiation itself. The decisive step towards the second hypothesis was made by Einstein, who gave an explanation for photoelectric phenomena that was based on the concept of *light quanta,* or *photons,* and ultimately provided arguments in favor of the existence of the latter that were derived from the analysis of the fluctuations of the radiated energy. The next step was made by Debye, who deduced Planck's formula from purely electromagnetic arguments, while appealing to neither electrons nor atoms; his procedure basically consists of a "quantization" of the method by which Rayleigh and Jeans found their law of radiation. One develops the radiation field in a Fourier series and considers each of the terms as an oscillator that obeys Planck's law for quanta. This method constitutes the point of departure for all of the improvements that were ultimately made to the quantum theory of radiation; we shall confine ourselves here to mentioning just one of them.

Debye's field quantization provided the value for the mean energy (Planck's formula), but did not account for the fluctuations of the energy. Einstein showed that one may calculate the fluctuations for an arbitrary system by means of a general formula of statistical mechanics due to Gibbs, provided that one knows the expression for the total energy of the system as a function of temperature. The application of this method to Planck's formula gives a result that is in flagrant contradiction to the one that is obtained directly by means of the classical theory of light; a new term appears, which Einstein interpreted as a proof of the existence of quanta of luminous corpuscles.

This disaccord was eliminated by the discovery of quantum mechanics. In our first work on the mechanics of matrices, Heisenberg, Jordan, and myself [1] treated this problem and found a formula that agrees with the statistical formula of Einstein. Later on, Heisenberg [2] criticized the derivation of this formula because of an infinite term that one encounters along the way, and showed how what must be done to avoid it: it suffices to assume that the cavity in which one calculates the energy fluctuations has no perfectly well defined boundary, but that it possesses a degree of uncertainty as to its position in space. With this correction, which is intimately related to general conditions of uncertainty in quantum mechanics, one obtains a complete accord between the results of statistical thermodynamics and those of the quantum theory of waves.

A great step forward was made by Dirac, who systematically applied the rules of quanta to the system of "radiation + material particles" and obtained Einstein's well known formulas that give the probability of emission and absorption of an atom in a radiation field. These formulas are the quantum equivalent of the classical laws of

emission and absorption that Planck used to begin with; they directly obey that formula without encountering the obstacles that one did in the original theory, which are based on a mélange of classical and quantum conditions that contradict each other, strictly speaking.

In fact, Dirac's theory gives us much more, namely, a complete theory of the emission, of absorption, dispersion, and diffusion of light by electrons or atoms (the Rayleigh, Raman, and Compton effects), and finally, of the deeper nature of the spectral lines (Weisskopf, Wigner, Fermi). Meanwhile, this admirable theory [3], which accounted for a very large number of phenomena, encountered serious obstacles. Dirac's method consisted of considering the binding forces between matter and radiation to be small perturbations of the uncoupled system of atom and field. The first application provided reasonable results that were confirmed by experiments, but the second one and the approximations of higher order lead to divergent integrals. They are related to the number of different frequencies of a radiating field that are present in a given frequency interval, and are analogous to the infinite proper energy of this field. Nevertheless, the analogy can go no further: whereas the energy at absolute zero appears as a constant and may be simply omitted, the higher-order infinites seem inevitable.

The most general attempt to establish a quantum theory of wave fields that was in accord with the principle of relativistic invariance was made by Heisenberg and Pauli [4]. It consisted of a systematic generalization of the classical theory that we discussed in section 1, and contains the theory of Dirac, as well as all the previous ones, as special cases. It likewise provided an appropriate method for the quantization of nonlinear electrodynamics; this quantization was carried out by Infeld and myself in the case of a pure radiation field [5], and by Pryce in the case of fields that contain point-like charges [6].

The problem that interests us here is solely that of the electromagnetic mass, and for that problem all of the work that has been done gives only slightly satisfying results. The nonlinear equations, which were accepted to be useful in making the classical proper energy of an electron finite, have not succeeded in achieving the same objective in quantum theory. In any case, it has been impossible to prove whether this objective has been achieved or not, since the equations are too complicated to be solved even in the simplest case.

For this reason, I will not reproduce that theory here, which is quite extensive, moreover, but I will confine myself to brief indications concerning its principal characteristics.

The application of the principles of quantum theory to a system depends, in the first place, on the possibility of expressing its laws in the schema of Hamiltonian dynamics, for which we know the rules that permit us to pass from a classical law to the corresponding quantum law.

The variational principle that we took as our point of departure in section 1 is therefore no longer useful.

In point mechanics one may replace the Lagrangian by another fundamental function, which is called the Hamiltonian and which represents the energy of the material point considered; meanwhile, in the dynamics of continuous media there exists no simply and unambiguous relation of the type that allowed us to proceed in a manner that is analogous to the preceding one. For example, in electrodynamics the function H that we have called

the "Hamiltonian" does not represent either the field energy or the energy density at all. If we would like to apply the usual quantization rules then we must use the total energy as the fundamental function, which is the space integral of the 44 component of a second rank tensor. This signifies that we are constrained to abandon the symmetry in the formulas with respect to x^1, x^2, x^3, x^4, as well as the use of tensor calculus. With these conditions, relativistic invariance is no longer immediate, and it must proved separately ([8]). In the sequel, we therefore treat the coordinate $x^4 = t$ separately, and intend the expression "a point" to mean a point ordinary three-dimensional space:

$$(x^1, x^2, x^3) = (x, y, z).$$

Consider the continuous medium envisioned to be the limit of a discontinuous system. For this, divide three-dimensional space completely into identical cells $\Delta x = \Delta x^1 \Delta x^2 \Delta x^3$, which we enumerate by three integers l_1, l_2, l_3. Replace any continuous function z^α by a discontinuous function that has the constant value z_l^α in the cell (l_1, l_2, l_3). The space derivatives $z_k^\alpha = \dfrac{\partial z^\alpha}{\partial x^k}$ will be replaced by difference quotients:

$$\frac{\Delta z^\alpha}{\Delta x^k} = \frac{z_{l+1}^\alpha - z_l^\alpha}{\Delta x^k},$$

in which $l + 1$ represents the neighboring cell in the direction x^k ($k = 1, 2, 3$).

Denote the derivative with respect to time z_4^α by $\dot{z}^\alpha$. The integral (1.4) is replaced by the sum:

$$(11.1) \qquad \overline{\mathfrak{L}} = \sum_l \mathfrak{L}\left(z_l^\alpha, \frac{z_{l+1}^\alpha - z_l^\alpha}{\Delta x}, \dot{z}_l^\alpha \right) \Delta x = \sum_l \mathfrak{L}_l \Delta x,$$

which is taken over these cells, and the function $\overline{\mathfrak{L}}$ is considered to be the Lagrangian of a mechanical system that has the z_l^α for its variables. The moments are:

$$(11.2) \qquad p_{\alpha l} = \frac{\partial \overline{\mathfrak{L}}}{\partial \dot{z}_l^\alpha} = \frac{\partial \mathfrak{L}_l}{\partial \dot{z}_l^\alpha} \Delta x = P_{\alpha l} \Delta x,$$

in which

$$(11.3) \qquad P_{\alpha l} = \frac{\partial \mathfrak{L}}{\partial \dot{z}_l^\alpha}, \qquad P_\alpha = \frac{\partial \mathfrak{L}}{\partial \dot{z}^\alpha}.$$

z_l^α appears not only in the l term of the sum (11.1), but also in the neighboring $l + 1$ terms; for this reason, one has:

([8]) An invariant process that avoids these difficulties and makes relativistic invariance obvious was given by Weiss [7].

$$\frac{1}{\Delta x}\frac{\partial \overline{\mathfrak{L}}}{\partial z_l^{\alpha}} = \frac{\partial \mathfrak{L}_l}{\partial z_l^{\alpha}} - \left[\left(\frac{\partial \mathfrak{L}_l}{\partial z_l^{\alpha}}\right)_l - \left(\frac{\partial \mathfrak{L}_l}{\partial z_l^{\alpha}}\right)_{l+1}\right]\frac{1}{\Delta x^1} - \cdots;$$

in the limit, $\Delta x \to 0$:

(11.4)
$$\lim \frac{1}{\Delta x}\frac{\partial \overline{\mathfrak{L}}}{\partial z_l^{\alpha}} \to \frac{\partial \mathfrak{L}_l}{\partial z_l^{\alpha}} - \sum_{k=1,2,3}\frac{\partial}{\partial x^k}\left(\frac{\partial \mathfrak{L}_l}{\partial z_k^{\alpha}}\right) = \frac{\delta \overline{\mathfrak{L}}}{\delta z^{\alpha}}.$$

This formula shows that the process of differentiating the discontinuous function envisioned gives us, in the limit, "the functional derivative," which disappears in the Euler equation of the continuous system (2.1).

The Lagrange equations of motion of (11.1), namely:

(11.5)
$$\frac{d}{dt}p_{\alpha l} - \frac{\partial \overline{\mathfrak{L}}}{\partial z_l^{\alpha}} = 0,$$

are likewise equivalent to the Euler equations, (2.1); indeed, one may write them in the form above when one takes (11.3) and (11.4) into account:

(11.6)
$$\dot{P}_{\alpha} + \sum_{k=1,2,3}\frac{\partial}{\partial x^k}\left(\frac{\partial \mathfrak{L}}{\partial z_k^{\alpha}}\right) - \frac{\partial \mathfrak{L}}{\partial z^{\alpha}} = \sum_{k=1,\cdots,4}\frac{\partial}{\partial x^k}\mathfrak{L}_{z_k^{\alpha}} - \mathfrak{L}_{z^{\alpha}} = 0.$$

We may now pass to the Hamiltonian form of dynamics and define the total energy, or Hamiltonian, by:

(11.7)
$$\overline{H} = \sum_l p_{\alpha l}\dot{z}_l^{\alpha} - \overline{\mathfrak{L}} = \sum_l (P_{\alpha l}\dot{z}_l^{\alpha} - \mathfrak{L}_l)\Delta x \to \int (P_{\alpha}\dot{z}_{\alpha} - \mathfrak{L})dx.$$

The variation of $\overline{H}$ is:

$$\delta \overline{H} = \sum_l (p_{\alpha l}\delta \dot{z}_l^{\alpha} + \dot{z}_l^{\alpha}\delta p_{\alpha l} - \delta \mathfrak{L}_l)\Delta x,$$

in which:

$$\delta \mathfrak{L}_l = \frac{\partial L}{\partial z_l^{\alpha}}\delta z_l^{\alpha} + \frac{\partial L}{\partial \dot{z}_l^{\alpha}}\delta \dot{z}_l^{\alpha}.$$

Therefore, by virtue of (11.2) and (11.3):

$$\delta \overline{H} = \sum_l \left(-\frac{\partial L}{\partial z_l^{\alpha}}\delta z_l^{\alpha} + \dot{z}_l^{\alpha}\delta P_{\alpha}\right)\Delta x,$$

or:

(11.8)
$$\frac{\delta \overline{H}}{\delta z^{\alpha}} = -\frac{\delta \overline{\mathfrak{L}}}{\delta z^{\alpha}}, \qquad \frac{\delta \overline{H}}{\delta P^{\alpha}} = \dot{z}^{\alpha}.$$

By combining these equations with (11.6) one obtains the canonical equations:

$$(11.9) \qquad \dot{z}^\alpha = \frac{\delta \bar{H}}{\delta P^\alpha}, \qquad\qquad \dot{P}_\alpha = -\frac{\delta \bar{H}}{\delta z^\alpha}.$$

The previous argument proves that one may consider a continuous medium as the limiting case of a discontinuous system in all respects. With these conditions, the quantization method is obviously the following:

From the mechanical viewpoint, we have the commutation law:

$$(11.10) \qquad p_{\alpha l} z^{\beta l'} - z^{\beta l'} p_{\alpha l} = \frac{h}{2\pi i} \delta_l^{l'} \delta_\alpha^\beta,$$

in which l, l' are the indices of two different cells. In relation (11.10) we may pass directly to the limit $\Delta x \to 0$, since the $p_{\alpha l}$ are, by definition, proportional to the volume Δx of the cell [*see* (11.2)]. To do this, we first multiply (11.10) by a discontinuous function $f_{l'}$, which is constant in each cell and approaches a continuous function $f(x^1, x^2, x^3)$ in the limit, in such a manner that:

$$\sum_{l'} f_{l'} \Delta x \to \int f dx .$$

By summing l' over a spatial volume V' one obtains:

$$\sum_{l' \, in V'} f_{l'} \Delta x \left[\frac{p_{\alpha l}}{\Delta x} z^{\beta l'} - z^{\beta l'} \frac{p_{\alpha l}}{\Delta x} \right] = \frac{h}{2\pi i} \delta_{\alpha\beta} \begin{cases} f_l & \text{if } l \text{ is in } V', \\ 0 & \text{if } l \text{ is not in } V'. \end{cases}$$

In the limit, $\Delta x \to 0$, one will have:

$$(11.11) \qquad \int_{V'} f(x') dx' \{ P_\alpha(x) z^\beta(x')$$

$$-z^\beta(x') P_\alpha(x) \} = \frac{h}{2\pi i} \delta_{\alpha\beta} \begin{cases} f(x) & \text{if } l \text{ is in } V', \\ 0 & \text{if } l \text{ is not in } V'. \end{cases}$$

in which x represents a point (x^1, x^2, x^3), and x' another.

It is convenient to describe formulas of this type in an abbreviated form by using the symbolic Dirac delta function δ, which is defined by:

$$(11.12) \qquad \int_a^b f(x) \delta(x) dx = \begin{cases} f(0) & \text{if } \quad x = 0 \text{ is in } (a,b), \\ 0 & \text{if } \quad x = 0 \text{ is not in } (a,b). \end{cases}$$

It follows that we may assume that $\delta(-x) = \delta(x)$.

We use the same symbol $\delta(x)$ to denote a three-dimensional function that is actually equal to product $\delta(x^1)\,\delta(x^2)\,\delta(x^3)$. By means of the bracket notation:

(11.13) $$FG - GF = [F,\,G],$$

one may write the commutation laws in the following form [8]:

(11.14) $$[P_\alpha(x), z^\beta(x')] = \frac{h}{2\pi i}\,\delta_\alpha^\beta\,\delta(x - x').$$

These laws must be completed by the following ones, which are obvious:

(11.15) $$[z^\alpha(x), z^\beta(x')] = 0,\quad [P_\alpha(x), P_\beta(x')] = 0.$$

We remark that the time variable $x^4 = t$ always has the same value in all of the functions that appear in (11.14) and (11.5).

By applying the following well known formulas of quantum mechanics to the system of cells envisioned, and passing to the limit, one obtains the equations of motion:

(11.16) $$\dot{z}_\alpha = \frac{2\pi i}{h}[\bar{H}, z_\alpha],\qquad \dot{P}_\alpha = \frac{2\pi i}{h}[\bar{H}, P^\alpha].$$

One may directly show that these equations are formally identical to the Euler equations, (2.1), of the classical theory. From this it then results that there exists a first integral of energy $\bar{H} = $ const., and corresponding first integrals that express the conservation of momentum (which ignore here, because we have considered them no further in the general classical theory of section 1). Finally, one may prove that if the commutation conditions (11.14) and (11.15) are valid at a given instant t_0 they will be likewise valid at any other later instant t.

12. Quantum electrodynamics. – We apply the preceding general theory to the particular case of the electromagnetic field; the unknown functions z^α will be the potentials φ_k, and we must take $\mathfrak{L}$ to be a function of the field components $f_{kl} = \varphi_{l,k} - \varphi_{k,l}$.

A serious difficulty is presented when one wishes to form the moments:

(12.1) $$P^k = \frac{\partial \mathfrak{L}}{\partial \varphi_{4,k}} = \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}_k}.$$

One finds:

(12.2) $$P^k = \frac{\partial \mathfrak{L}}{\partial \varphi_{4,k}} = \frac{\partial \mathfrak{L}}{\partial f_{4,k}} - \frac{\partial \mathfrak{L}}{\partial f_{k,4}} = P^{4,k} = -\mathbf{D} \qquad (k = 1, 2, 3),$$

but also:

(12.3)
$$P^4 = \frac{\partial \mathfrak{L}}{\partial \varphi_{4,4}} = 0.$$

This relation, which expresses the fact that the moment of the scalar potential is null, corresponds to a remarkable degeneracy of the electrodynamical Lagrangian, and leads to considerable complications. They result from the fact that (in classical theory) the three space components P_1, P_2, P_3 must satisfy the relation:

(12.4)
$$\sum_{k=1,2,3} \frac{\partial P^k}{\partial x^k} = -div\mathbf{D} = 0 \,,$$

and that, as a consequence, one may not directly apply the commutation conditions (11.14) and (11.15). Indeed, one will have:

(12.4)
$$\begin{cases} [\varphi_k(x), \varphi_l(x')] = 0, \\ [P^k(x), P^l(x')] = 0, \\ [P^k(x), \varphi_l(x')] = \frac{h}{2\pi i} \delta_l^k \delta(x - x'). \end{cases}$$

Meanwhile, the latter gives:

$$\left[\sum_{k=1,2,3} \frac{\partial P^k}{\partial x^k}, \varphi_l(x') \right] = \frac{h}{2\pi i} \frac{\partial}{\partial x^l} \delta(x - x') \,.$$

The left-hand side is null, by virtue of (12.4), and the right-hand side is not. Therefore, condition (12.4) – i.e., one of the Maxwell equations – is in contradiction with the commutation laws.

One may imagine several methods by which one may avoid this difficulty. The most reasonable one seems to be the one that advocates the use of formulas (12.5) alone as a heuristic means of discovering the true commutation conditions of the field components [sec. 11; 4], which are not in contradiction with (12.4), as we shall prove in a moment.

By differentiating (12.5) with respect to *x, y, z,* one obtains:

(12.6)
$$\begin{cases} [B_x(x), B_y(x')] = 0, & \cdots \\ [D_x(x), D_y(x')] = 0, & \cdots \\ [D_x(x), B_x(x')] = 0, & \cdots \\ [D_y(x), B_z(x')] = -[D_z(x), B_y(x')] = \frac{h}{2\pi i} \frac{\partial \delta}{\partial x}, & \cdots \end{cases}$$

By forming the bracket of div **D**, one obtains:

$$[B_x(x), div\mathbf{D}] = \frac{h}{2\pi i}\left(\frac{\partial^2 \delta}{\partial y \partial z} - \frac{\partial^2 \delta}{\partial z \partial y}\right) = 0 \ ;$$

the contradiction has disappeared [1].

The variables that appear in the commutation conditions, (12.6), are precisely the components of the vectors $\mathbf{D}$, $\mathbf{B}$. This incites us to consider them as fundamental variables instead of the potentials φ_k. The opportunity to make this choice manifests itself in the fact that the "natural" variables of the total energy $E = \int U dv$ (which is the fundamental quantity of quantum theory) are precisely the vectors $\mathbf{B}$ and $\mathbf{D}$; the other vectors $\mathbf{E}$ and $\mathbf{H}$ may be deduced by differentiation:

$$(12.7) \qquad\qquad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}, \qquad\qquad \mathbf{H} = \frac{\partial U}{\partial \mathbf{B}} \ .$$

By systematically taking this viewpoint, Infeld and myself have developed a coherent theory of nonlinear quantum electrodynamics. True, in the beginning we had hoped to likewise obtain the laws of motion for the charges − i.e., the singularities − in this fashion; this hope vanished, like Mie's hopes of constructing a strictly classical unitary theory of the electromagnetic field. Later on, we shall examine the main point of this particular problem in detail. For the moment, I will first sketch quantum electrodynamics in the absence of point-like charges, and then add several observations about the manner that seems to me to be the best for studying the general case.

It is clear that one must renounce the complete symmetry between space and time coordinates; it is nevertheless possible to write the equations of motion in the form of a symmetric set by adding similar equations to (11.16) that relate the space derivatives and the total momentum.

Consider $\mathbf{D}$ and $\mathbf{B}$ as operators that operate on a certain function (more precisely, on a functional), which we shall not write explicitly. Having done this, we adopt the Heisenberg viewpoint, and not that of Schrödinger. In the Heisenberg picture, the operators depend on the space and time variables, and the law of variation in our case is the following one.

Let $F(\mathbf{D}, \mathbf{B})$ be an arbitrary function of the field, and let:

$$(12.8) \qquad\qquad E = \int U dv \ , \qquad\qquad \mathbf{G} = \int \mathbf{S} dv \ ,$$

be the energy and momentum of this field; the variation of F in space and time is given by:

$$(12.9) \qquad\qquad \frac{\partial F}{\partial t} = \frac{2\pi i}{h}[E, F], \qquad \frac{\partial F}{\partial x} = -\frac{2\pi i}{h}[G_x, F], \quad \dots$$

These *laws of motion*, when completed by the commutation conditions (12.6) are the fundamental equations of the theory.

The first question that one poses is that of whether the system is invariant under Lorentz transformations.

This will not be the case if one chooses the function (**D**, **B**) arbitrarily. We have previously seen (sec. 6, pp. 192) that the invariance condition is the symmetry of the energy tensor, which may be expressed by the following two vector equations:

(12.10) $$\begin{cases} \mathbf{S} = \mathbf{D} \times \mathbf{B} = \mathbf{E} \times \mathbf{H}, \\ \quad\; \mathbf{D} \times \mathbf{E} = \mathbf{H} \times \mathbf{B}. \end{cases}$$

By replacing **E** and **H** with their expressions (12.7) in these relations, one obtain six partial differential equations for $U(\mathbf{D}, \mathbf{B})$ that constitute necessary and sufficient conditions for the invariance of the system considered, Any function U that is derived from an invariant Lagrangian $L(\mathbf{E}, \mathbf{B})$ naturally satisfies these conditions (12.10).

The second question to examine is that of the invariance of the commutation laws, (12.6). This invariance was proved by Heisenberg and Pauli in the case of Maxwell's theory in their fundamental work, which was previously cited. Infeld and myself gave a general proof that was not, however, completely satisfactory, but it has been since improved by Infeld [2] and, independently, by Pryce [3]. The method that was employed consists of applying an infinitesimal Lorentz transformation to the anti-symmetric tensors (**E**, **B**) and (**D**, **H**), as well the variables x, y, z, t and x', y', z', t' that appear in the commutation laws, and proving that the latter do not change. The actual calculations are too complicated to be reproduced here.

Often, it is more convenient to replace the commutation laws with formulas that contain arbitrary functions in place of the δ function. To accomplish this, Infeld and myself proceeded in the following fashion:

Let F and G be arbitrary functions of **D** and **B**; form the space integrals $\int F dv$ and $\int G dv$, which are taken over an arbitrary domain V. One has:

(12.11) $$\frac{4\pi i}{h}\left[\int F dv, \int G dv\right] = \int \left\{\frac{\partial F}{\partial \mathbf{B}}\left(\nabla \times \frac{\partial G}{\partial \mathbf{D}}\right) - \frac{\partial F}{\partial \mathbf{D}}\left(\nabla \times \frac{\partial G}{\partial \mathbf{B}}\right)\right\} dv$$

$$= -\int \left\{\frac{\partial G}{\partial \mathbf{B}}\left(\nabla \times \frac{\partial F}{\partial \mathbf{D}}\right) - \frac{\partial G}{\partial \mathbf{D}}\left(\nabla \times \frac{\partial F}{\partial \mathbf{B}}\right)\right\} dv,$$

in which $=$ is the vectorial operator $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$. By virtue of $[a, b] = -[b, a]$, the two right-hand sides of (12.11) must be identical, which restricts the arbitrariness of the admissible functions. They must be such, that the integral:

$$\int \left\{\left(\frac{\partial F}{\partial \mathbf{B}} \times \frac{\partial G}{\partial \mathbf{D}}\right) - \left(\frac{\partial F}{\partial \mathbf{D}} \times \frac{\partial G}{\partial \mathbf{B}}\right)\right\} \mathbf{n} \, d\sigma,$$

is annulled when taken over the boundary of the domain V.

One obtains the form (12.6) of the commutation laws by taking:

$$F = B_x, \ldots, \qquad G = \delta D_x, \qquad \ldots$$

If one then takes:

$$F = B_x, \ldots, \qquad \text{or } D_x, \ldots; \qquad G = U \text{ or } S_x, S_x, S_x,$$

and if one accounts for (12.7), one obtains, by starting with the equations of motion (12.9):

(12.12)
$$\begin{cases} \mathbf{B} = -(\nabla \times \mathbf{E}), & \mathbf{D} = (\nabla \times \mathbf{H}), \\ div\mathbf{B} = 0, & div\mathbf{D} = 0, \end{cases}$$

for the entire set of field equations.

By taking:

$$F = dU, \qquad G = U,$$

one obtains:

$$\dot{U} = -\,\text{div}(\mathbf{E} \times \mathbf{H}) = -\,\text{div } \mathbf{S},$$

i.e., the law of conservation of energy in differential form; in the same fashion, one may prove the conservation of momentum.

The total *angular momentum* of the field is defined by:

(12.13)
$$\mathbf{M} = \int (\mathbf{r} \times \mathbf{S})dv,$$

and the *center of energy* **q** by:

(12.14)
$$\tfrac{1}{2}(E\mathbf{q} + \mathbf{q}E) = \int \mathbf{r}U\,dv.$$

The corresponding commutation laws are the following:

(12.15)
$$\begin{cases} \mathbf{M} \times \mathbf{M} = \dfrac{h}{2\pi i}\mathbf{M}, \\ \dot{\mathbf{q}} = [E, \mathbf{q}] = \mathbf{G}E^{-1}, \\ \ddot{\mathbf{q}} = [E, \dot{\mathbf{q}}] = 0, \end{cases}$$

and the other relations between **q**, **G**, **M** are known, exactly as if they acted on material points.

A well-known reason, when applied to the first of equations (12.15) gives us that the proper values of the total kinetic moment are either 0, ±1, ±2, ±3, ..., or $\pm\dfrac{1}{2}, \pm\dfrac{3}{2}, \pm\dfrac{5}{2}, \ldots$ (the unit is $\dfrac{h}{2\pi i}$).

We have believed from the beginning – Infeld and myself – that it is possible to explain spin by considering it to be a quantum state of the electromagnetic field with a half-integer number; to this end, we defined an "internal kinetic moment," i.e., a moment with respect to the center of energy **q** that presents certain properties that are analogous to those of spin. Nevertheless, this identification is erroneous, as Pauli has remarked. Pryce rigorously proved that that the electromagnetic angular momentum has only integer proper values even if one admits the existence of singularities (without spin) [4].

This result clearly shows that it is impossible to explain the existence of particles by means of nonlinear field theory without their explicit introduction, and without attributing them to convenient spin variables.

The problem has been treated by Heisenberg and Pauli for the particular case of a Maxwell field (for which $U = \frac{1}{2}(D^2 + B^2)$), and by Pryce for the case of the general field theory [with $U(\mathbf{D}, \mathbf{B})$ arbitrary and subject to only the conditions of relativistic invariance] [3].

In this latter theory, the position of a singularity is represented by a radius vector $\boldsymbol{\xi}^{(n)}$ with the components $\xi_x^{(n)}, \xi_y^{(n)} \xi_z^{(n)}$; moreover, one attributes each of them to an "intrinsic moment" $\boldsymbol{\pi}^{(n)}$, which is *not* related to the velocity by the simple well-known relation that contain sthe mass, since here one does not introduce mass for the particles. Pryce systematically described the commutation laws between these quantities and the field components. We shall not dwell on these exceedingly complicated considerations here; we content ourselves by indicating the final results, which are relatively simple.

Exactly as in the theory of uncharged fields, there are three groups of fundamental laws:

1. The expressions for the energy and momentum:

(12.16)
$$\begin{cases} E = \int U(\mathbf{D}, \mathbf{B}) dv - \sum_n \boldsymbol{\alpha}^{(n)} \boldsymbol{\pi}^{(n)}, \\ \mathbf{G} = \int (\mathbf{D} \times \mathbf{B}) dv + \sum_n \boldsymbol{\pi}^{(n)}; \end{cases}$$

in these formulas, $\boldsymbol{\alpha}^{(n)}(\alpha_x^{(n)}, \alpha_y^{(n)}, \alpha_z^{(n)})$ are a group of Dirac operators (3 matrices with 4 rows and 4 columns) that satisfy:

(12.17)
$$\alpha_x^{(n)} \alpha_y^{(m)} + \alpha_y^{(m)} \alpha_x^{(n)} = 2\delta_{xy}\delta_{mn}.$$

2. The equations of motion for an arbitrary quantity F:

(12.18)
$$\frac{\partial F}{\partial t} = -\frac{2\pi i}{h}[E, F], \qquad \frac{\partial F}{\partial x} = -\frac{2\pi i}{h}[G_x, F], \quad \dots$$

3. The commutation laws:

$$(12.19) \quad \begin{cases} [D_y(x), B_z(x')] = \dfrac{h}{2\pi i}\dfrac{\partial \delta}{\partial x}, & \cdots, \\[2mm] [D_x(x), \pi_x^{(n)}] = -e\delta(x - \xi^{(n)}), & \cdots, \\[2mm] [\xi_x^{(n)}, \pi_x^{(n)}] = \dfrac{h}{2\pi i}, & \cdots, \\[2mm] [\pi_y^{(n)}, \pi_z^{(n)}] = \dfrac{h}{2\pi i}\, e\displaystyle\int \delta(x - \xi^{(n)}) B_x(x)\,dv, & \cdots, \end{cases}$$

in addition, all of the quantities that were not mentioned commute with them. One may show that the set of these equations is invariant from the relativistic standpoint.

The energy operator of (12.16) has the form of the energy operator for a Dirac electron, in which the mass term was replaced by the energy of the electromagnetic field.

One may now show that these laws generally contain all of the ones that must be included, namely, Maxwell's equations with a point-like charge and corresponding current, as expressed in terms of δ functions:

$$(12.20) \quad \begin{cases} \mathbf{D} = -rot\mathbf{H} = e\displaystyle\sum_n \delta(x - \xi^{(n)})\alpha^{(n)}, & div\mathbf{D} = \displaystyle\sum_n \delta(x - \xi^{(n)}), \\[2mm] \mathbf{B} + rot\mathbf{E} = 0, & div\mathbf{B} = 0; \end{cases}$$

in turn, the well-known fact from Dirac's theory that the velocity of the particle n is $\boldsymbol{\alpha}^{(n)}$:

$$(12.21) \qquad\qquad \boldsymbol{\xi}^{(n)} = \boldsymbol{\alpha}^{(n)},$$

and finally the Lorentz equations of motion in the following form:

$$\boldsymbol{\pi}^{(n)} = e\int \delta(x - \xi^{(n)})[\mathbf{E} + (\boldsymbol{\xi}^{(n)} \times \mathbf{B})]dv.$$

One may then show that the components of the momentum $\mathbf{G}$ commute with E, and thus constitute first integrals of motion.

The total angular momentum becomes:

$$\mathbf{M} = \int (\mathbf{r} \times \mathbf{S})dv + \sum_n (\boldsymbol{\xi}^{(n)} \times \boldsymbol{\pi}^{(n)}) + \frac{1}{2}\frac{h}{2\pi}\sum_n \boldsymbol{\sigma}^{(n)},$$

in which $\sigma_x^{(n)} = \alpha_y^{(n)} \cdot \alpha_z^{(n)}$, ... are the components of spin; from this, it follows that $\mathbf{M}$ is likewise a first integral. The term that does not contain spin has integer proper values; consequently, the total angular momentum behaves exactly as it must for a system of particles that each possesses spin.

The main problem with this theory is that of the existence of finite proper values for the energy of an isolated particle, and their calculation. Nevertheless, no solution to this problem has been obtained up till now.

We do not know whether this theory, which is formally the counterpart of the classical non-linear field theory, does or does not contain the quantum solution to the problem of electromagnetic mass. Meanwhile, in any case, whether one solves this problem or not, the theory that we have presented here may not be considered as satisfactory, for the following reasons.

When one takes U to have the Maxwell expression $\frac{1}{2}(D^2 + B^2)$, one falls back to the quantum electrodynamics of Heisenberg and Pauli, which contains Dirac's theory of radiation as a particular case. All of the infinite terms that make this theory unacceptable appear again, and it does not seem that one may remover them by accounting for the terms that are nonlinear in **D** and **B**.

The discovery of uncharged particles casts doubt on the purely electromagnetic origin of mass. The hypothesis of the existence of point-like dipoles (or multipoles) is in contradiction with the "natural" boundary conditions, and, consequently, may not be accepted, just as in the classical field theory. The problem of a correct generalization of the theory that makes the existence of these dipoles (or multipoles) appear normally has not been correctly studied.

The analysis of nuclear processes has revealed the existence forces of a new type in regions of the order of magnitude one "electron radius." We now know that forces of a character that is unknown up till now appear between two neutrons or between a neutron and a proton that are situated at a small distance from each other. Fermi's theory of β-disintegration introduced another, completely new, type of interaction that simultaneously introduces four particles, a neutron, a proton, an electron, and a neutrino. We may not reasonably hope to obtain a correct value for the electromagnetic mass if the fundamental laws of nature are of a very different type from the electromagnetic type in precisely the region where all of the energy of the particle is concentrated.

Pryce's equations contain two constants: the elementary charge and Planck's constant h. Now, we know that these two constants are not independent; they are related by $\frac{2\pi e^2}{ch} = \frac{1}{137}$; this combination of constants e, h, c is a dimensionless number: viz., Sommerfeld's "fine structure constant." It seems very improbable that one may develop a theory of particles that applies to only the electron, while neglecting the existence of heavier particles whose mass is 1840 times higher. A satisfactory theory of elementary particles must explain the values of the two dimensionless numbers 137 and 1840.

We are very far from such a theory. Some attempts in this sense have points in common with nonlinear field theory; we shall examine them in the last section of this work.

13. Dirac's theory of the electron and nonlinear electrodynamics. – The quantum theory of the electron permits us to attack the problem of nonlinear electrodynamics in an entirely new manner. The importance of the attempts of Dirac, Heisenberg, and their collaborators resides in the fact that they were the first to theoretically justify the relation that exists between the elementary charge and Planck's constant, in other words, the value of the fine structure constant. In the sequel, we shall rapidly sketch this theory.

The relativistic wave equation of an electron is derived from the classical equations of motion, which, as we have seen, are a consequence of a nonlinear electrodynamical law

that is defined by an arbitrary Lagrangian.   These equations may be written in the following canonical form:

(13.1) $$\mathbf{r} = \frac{\partial W}{\partial \mathbf{p}}, \qquad \mathbf{p} = -\frac{\partial W}{\partial \mathbf{r}},$$

with the Hamiltonian:

(13.2) $$W = \{(\mathbf{p} - e\mathbf{a})^2 + m_0^2\}^{\frac{1}{2}} + e\varphi,$$

in which $\mathbf{a}$, φ are the vector and scalar potentials.   By performing the derivations in (13.1), one obtains:

$$\dot{x} = \frac{p_x - ea_x}{\sqrt{(\mathbf{p} - e\mathbf{a})^2 + m_0^2}},$$

$$\dots$$

$$\dot{p}_x = \frac{e}{\sqrt{(\mathbf{p} - e\mathbf{a})^2 + m_0^2}}$$

$$\times \left\{ (p_x - ea_x)\frac{\partial a_x}{\partial x} + (p_y - ea_y)\frac{\partial a_y}{\partial y} + (p_z - ea_z)\frac{\partial a_z}{\partial z} \right\} - e\frac{\partial \varphi}{\partial x},$$

$$\dots$$

From the first group of equations, one deduces that:

(13.3) $$(\mathbf{p} - e\mathbf{a})^2 + m_0^2 = \frac{m_0^2}{1 - v^2},$$

therefore:

(13.4) $$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2}} + e\mathbf{a};$$

with these values, the second group becomes:

(13.5) $$\frac{d}{dt}\left(\frac{m_0 \dot{x}}{\sqrt{1 - v^2}} + ea_x \right) = e\left\{ \dot{x}\frac{\partial a_x}{\partial x} + \dot{y}\frac{\partial a_y}{\partial x} + \frac{\partial a_z}{\partial x} - \frac{\partial \varphi}{\partial x} \right\}, \dots$$

By using the relations between potentials and fields, namely,

$$\mathbf{E} = -\nabla \varphi - \mathbf{a}, \qquad \mathbf{B} = \text{rot } \mathbf{a},$$

these equations are equivalent to:

$$(13.6) \qquad \frac{d}{dt}\left(\frac{m_0 \mathbf{v}}{\sqrt{1-v^2}}\right) = e\{\mathbf{E} - (\mathbf{v}\times\mathbf{B})\},$$

in accordance with (10.17), (6.34).

Write (13.2) in the symmetric form:

$$(13.7) \qquad (W - e\varphi)^2 - (\mathbf{p} - e\mathbf{a})^2 = m_0^2.$$

One may employ this relation to make the wave theory of matter that was proposed by Schrödinger agree with the principle of relativity, by considering W and $\mathbf{p}$ to be operators of the form:

$$(13.8) \qquad W = \frac{-\hbar}{i}\frac{\partial}{\partial t}, \qquad p_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, \qquad \dots \qquad \left(\hbar = \frac{h}{2\pi}\right),$$

that operate on a wave function $\psi(x, y, z, t)$.

One calls the wave equation that is thus obtained the equation of Klein and Gordon [1]. This equation does not satisfy all of the conditions that are required by the statistical interpretation of quantum mechanics [2]. One confirms that the probability density for the presence of a particle is not positive definite; the equation contains second derivatives with respect to time, which is in contradiction with the fundamental statistical postulate that requires the function, ψ, to be completely determined by its initial values; finally, this equation does not account for the spin of the electron. These objections led Dirac to replace it with his celebrated relativistic equation [3]:

$$(13.9) \qquad \{W - e\varphi - \boldsymbol{\alpha}(\mathbf{p} - e\mathbf{a}) - \beta m_0\}\psi = 0,$$

in which $\boldsymbol{\alpha}(\alpha_x, \alpha_y, \alpha_z)$, b are four matrices with four rows and four columns that satisfy the commutation laws:

$$(13.9a) \qquad \begin{cases} \alpha_x \alpha_y + \alpha_y \alpha_x = 0, \\ \quad\dots \\ \alpha_x \beta + \beta \alpha_x = 0, \\ \quad\dots \end{cases}$$

This equation is linear in W (i.e., of first order in d/dt), and permits us to bring the mechanical and magnetic properties of the spin into perfect accord with experiments.

The two equations (13.7) and (13.9) present the grave defect of leading to negative proper values for energy. For example, in the case of the absence of a field ($\mathbf{a} = 0$, $\varphi = 0$) one will have a solution by taking ψ proportional to:

$$e^{\frac{i}{\hbar}(Wt - \mathbf{p}\mathbf{r})},$$

where W and $\mathbf{p}$ are constants that are related by:

$$W^2 - \mathbf{p}^2 = m_0^2 .$$

It is clear that for a given $\mathbf{p}$, W may have two values that are equal and of opposite sign $W = \pm\sqrt{\mathbf{p}^2 + m_0^2}$.

Now, this is precisely the defect that insures the triumph of this theory. Dirac proposed to assume that all of the negative energy states are occupied by electrons that exist everywhere, but are not observed, in the hopes that they would not contribute to formation of observable electromagnetic fields. An external action may nevertheless raise an electron from a negative energy state to another state of positive energy, and thus make it observable; at the same time, the "hole" that is thus created in the infinite set of negative energy electrons becomes likewise observable, and experiments must reveal it to us in the form of a positively charged particle. Dirac first believed that these "holes" must represent protons [4], despite the difficulty with this identification that was presented by the fact that the mass of the proton is 1840 times greater than the mass of the electron. Meanwhile, in this same epoch, Anderson discovered the existence of positive electrons in cosmic rays [5]; Dirac immediately recognized that these experiments constituted a confirmation of his theory.

The fundamental idea is the following: in the universe (or, at least, in the part that is known), there exists a surplus of positive energy electrons that constitute the external envelope of the atoms. It may happen that as a result of a collision between electrons or photons one of the negative energy electrons is aroused from its sleep and given a positive energy: one then observes the appearance of a *pair* that is composed of an electron and a "positron." A positron has a mean lifetime of short duration; indeed, it may combine with any electron that it encounters along the way and emit energy ($2m_0c^2$, plus the kinetic energy of the two particles). The result of this combination is light – more precisely, *two* photons – at least if one wants the principle of conservation of momentum to be satisfied. In the language of the theory of "holes," this signifies that the electron jumps into a place in the vacuum that is unoccupied by a negative energy electron, and may no longer be observed.

This theory permits us to foresee, for example, the minimum value of a photon's energy if it is to give rise to a pair, or, on the contrary, to make the annihilation of an electron and a positron possible; this then permits us to calculate the probability of materialization and dematerialization under the catalytic action of an external (nuclear) field, etc.; the results are in amazing accord with experiments [8].

The difficulties that this theory encounters stem from the fact that the unobservable electrons are infinite in number, being found in the negative energy states, so they themselves provide an infinite contribution to the values of the energy density, the current, and the charge, and on first glance this contribution is devoid of any physical significance. Dirac himself [9], as well as Heisenberg [10], have proposed formal rules for the elimination of these infinite terms. Recently, Weisskopf [11] has simplified these rules considerably, and has shown that by starting with essentially physical postulates it is possible to isolate, and in a unique and well-defined manner, the only finite partial sums that are susceptible to a physical interpretation among the divergent sums that formally represent the charge, current, energy, etc. In the sequel, we shall follow Weisskopf.

We now arrive at the point where this theory joins up with that of nonlinear electrodynamics.

From the standpoint of the observability of phenomena, the production of pairs, or the inverse process, has meaning in vacuo since the infinitude of electrons that are found in the negative energy states are unobservable. It is true that the appearance of these processes depends on the presence of an external electromagnetic field; meanwhile, the fields do not actually take part in the exchanges of energy in their presence, and play only the role of catalysts, as we have already pointed out.

The theory therefore demands that the phenomena of absorption or emission of light in the presence of a constant external field must have meaning in vacuo. Now, this demand is in contradiction with the law of superposition, which is derivable from the linear Maxwell theory; indeed, according to that theory, a light wave may traverse any portion of space that is devoid of matter without obstruction, independently of the presence or absence of an arbitrary electromagnetic field in this space.

We must therefore expect to find results that are different from the ones that the Maxwell theory provides; the general characteristic of these results may be glimpsed by establishing an analogy with a similar situation that is well known in the theory of atoms. Consider a light wave that traverses a substance whose atoms are all found in the fundamental state. If the spectrum of the light wave contains a frequency that corresponds to the difference in energy between an excited state and the fundamental state of the atom then this frequency will be absorbed and the atom excited. If the light does not contain frequencies that are sufficiently elevated to excite it to the first excited state of the atom then there will be no absorption; meanwhile, this does not signify in any sense that the substance in question exerts no actual influence on light wave that traverses it. The atom is polarized by the light at its proper frequency; there is diffusion and, in turn, the interference of diffused waves, dispersion, an effect which may be likewise described by attributing a dielectric constant that is greater than one to the substance traversed.

Now, in the relativistic theory of the electron, the vacuum behaves in a fashion that is analogous to that of the preceding hypothetical substance; indeed, in this theory, the "vacuum" is not absolutely devoid of matter, since it is, on the contrary, replaced by an infinite number of electrons in stationary states, namely, each of them are in the lowest possible state that permits the Pauli exclusion principle. A light wave of sufficient energy ($> 2m_0c^2$) may be absorbed by the passage of an electron into an excited state of positive energy. Nonetheless, if the energy is not sufficient for the production of a similar jump then the light wave will diffuse, in exactly the same manner as the one that were analyzed above. This process may likewise be described as a virtual double jump, there and back, between the fundamental state and the excited state. The general electronic configuration of the vacuum does not change, and there is no production of pair; nevertheless, the vacuum acquires an additional energy of polarization that is a function of the field of the luminous wave.

Heisenberg and his collaborators, Euler and Kockel [12], have developed this theory under the hypothesis of a very slight variation of the external field in space and time (field of the luminous wave); to be precise, under the hypothesis that any component F satisfies the following conditions:

$$\frac{\hbar}{mc} \mid grad\ F \mid \ll \mid F \mid, \qquad\qquad \frac{\hbar}{mc} \mid \frac{\partial F}{\partial t} \mid \ll \mid F \mid,$$

in which c is the velocity of light ([9]).

We assume the same hypotheses, and, moreover, we assume that the density of radiation is small enough that there are no potential differences that are higher than $2m_0c^2$; with these conditions, we may reasonably assume that there is no production of pairs.

Another simplifying hypothesis is the one that consists of neglecting the interaction of the electrons in vacuo. In reality, it is impossible to separate this "internal" field from the external electromagnetic field of the light wave; strictly speaking, one must therefore consider the method that we use to be analogous to the method of the self-consistent field that was used by Hartree [13] to calculate the electronic orbits in the atom; one knows that in this manner of proceeding every electron is treated separately under the action of the field, which is the sum of the external field and the mean field that is produced by the other electrons. A more exact analysis of the question must make use of the method of second quantization of the electronic waves; this will nevertheless be too complicated, and will make another type of divergent process appear that is completely foreign to the problem that we consider.

We shall perform the calculations for the case of the Dirac electron. Nevertheless, it is remarkable to confirm that the same theory may be developed for an electron that satisfies the Klein-Gordon equation, the difference being that the statistics that the electrons must obey will be that of Dirac-Fermi in the first case and Bose-Einstein in the second. Pauli and Weisskopf [14] have shown that a scalar theory of the electron without spin, which is based on the Klein-Gordon equation, is perfectly possible, and Weisskopf applied that to the problem of the polarization of the vacuum; in fact, this latter effect depends only upon the existence of negative energy states, and has nothing to do with spin. Nevertheless, since the electron actually possesses spin, the preceding scalar theory has the character of a very abstract theoretical development that is much too far from reality.

14. Theory of holes and polarization of the vacuum. – We assume that the total energy density of the vacuum is composed of two parts:

(14.1) $$U = U^0 + U';$$

the first one U^0 represents the Maxwell density (in electrostatic units):

(14.2) $$U^0 = \tfrac{1}{2}(E^2 + B^2),$$

and the second one U' is the contribution of the "vacuum electrons:"

([9]) In what follows, we shall no longer set $c = 1$, in order to show precisely the manner by which the various constants depend on c.

(14.3) $U' = \sum_i U_i'.$

The contribution a single electron is given by:

(14.4) $U_i' = (\psi_i, w\psi_i),$

in which w is the internal energy of the electron and ψ_i is the proper function that describes the state in which it is found; the scalar product of the two functions ψ and φ simply signifies that one forms the sum over the spin index:

(14.5) $(\psi, \varphi) = \sum_k \psi^{*k}\varphi^k.$

If the electron has no spin, we will have:

$$W = \frac{m_0 c^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} = W - e\varphi,$$

in which W is the total energy, or Hamiltonian, which is given by (13.2).

For an electron without spin, we must use (13.9) instead of (13.2). We introduce the velocity of light c and notate the charge, which is negative, by $-e$ ($e > 0$), instead of e; we will have:

(14.6) $w = W + e\varphi = \alpha(c\mathbf{p} + e\mathbf{a}) + \beta m_0 c^2.$

ψ_i will be a proper function of the operator W, so:

(14.7) $W\psi_i = W_i\psi_i;$

W_i is the corresponding proper value. The sum (14.3) is composed of a part that gives the total energy W, a part that we designate by:

(14.8) $\tilde{U} = \sum_i W_i(\psi_i, \psi_i),$

and the second part has its origin in the term $e\varphi$, and is equal to:

(14.9) $\sum_i (\psi_i, e\varphi\psi_i).$

This last part may be reduced to another one that depends on $\tilde{U}$. Indeed, if we integrate $(\psi_i, e\varphi\,\psi_i)$ over all space then we obtain the diagonal element $e\varphi_{ii}$. On the other

hand, if we replace φ with $\lambda\varphi$, and write $W = W_0 - \lambda e\varphi$, then we will have $\lambda e\varphi_{ii} = -\lambda\dfrac{\partial W_{ii}}{\partial\lambda}$.

We now solve the wave equation in the case of a constant electric field **E**; we may then equate the value E of the latter with the parameter λ. Consequently, the integral of the expression (14.9) over all space is $-E\dfrac{dW_{ii}}{dE}$, and this expression must likewise be valid for the densities whenever the external field is constant. We therefore have:

$$(14.10) \qquad\qquad U' = \tilde{U} - E\frac{d\tilde{U}}{dE}.$$

The calculation of U' is therefore effected in the following manner: we first determine the negative proper values W_i and the corresponding proper functions ψ_i for an electron. Then, we form the density $\tilde{U}$ of (14.8) by taking the sum over all the negative states, and finally, we perform the indicated operation in (14.10).

The calculation may be simplified even more if one takes the case of a particular external field; since the final result must enjoy relativistic invariance, we may easily pass from the this particular case to the general case of an arbitrary field by applying a Lorentz transformation.

Heisenberg and Euler have chosen the particular case of a constant magnetic field superposed with a constant electric field that is parallel to the latter. This hypothesis gives rise to certain difficulties; indeed, in a constant field that is as small one desires there exist considerable potential difference between points that are situated at great distance from each other. On the other hand, the well-known "Klein paradox" [1] requires that any potential difference that is greater than $2m_0c^2$ may give rise to a pair, i.e., may provoke the jump from an electron in a negative state to one in a positive state. Now, we would like to completely exclude the case of pair production. To avoid this difficulty, Weisskopf took an electric field that is parallel to a magnetic field and slightly periodic in space, with the potential:

$$(14.11) \qquad\qquad \varphi = \varphi_0 e^{\frac{ig}{h}x} + \varphi_0^* e^{-\frac{ig}{h}x}.$$

Then, he proceeds in the following manner: He first solves the Dirac equation under the hypothesis that only the magnetic field $\mathbf{B} = (B_x, 0, 0)$ exists. This problem may be treated rigorously. It is characterized by the possibility of separating the partial differential equation into three other terms that contain only one of the three independent variables x, y, z; two of them have a continuous spectrum, and the third one has a discontinuous spectrum of the type of the harmonic oscillator. This corresponds to the fact that in classical theory, the motion of an electron in a magnetic field gives rise to a helix. The proper functions are given by:

$$(14.12) \qquad \psi_i = \frac{a(p_x)}{2\pi h} e^{\frac{i}{\hbar}[(xp_x + zp_z + \hbar W_n(p_x))]} \left(\frac{b}{2\hbar^2}\right)^{\frac{1}{2}} H_n(\eta),$$

in which η is a linear function of y and b is proportional to the magnetic field:

$$(14.13) \qquad \eta = \left(y + \frac{2\hbar}{b} p_z\right)\sqrt{\frac{b}{2\hbar^2}} \qquad \text{and} \qquad b = \frac{2e\hbar}{c} B.$$

$H_n(\eta)$ is the nth proper function of the harmonic oscillator, normalized to unity. $a(p_x)$ is a "spinor" with two components. The negative energy is given by:

$$(14.14) \qquad W_n(p_x) = -c\sqrt{p_x^2 + m_0^2 c^2 + b\left(n + \frac{1-\sigma}{2}\right)},$$

in which σ is another spinor that has the proper values, +1 and −1, that correspond to the two possible positions of the electron spin.

One substitutes (14.12) and (14.14) into (14.8). The sum must be taken over $n = 0, 1,$ ..., $\sigma = +1, -1$, and over the continuous domain of p_x, p_y, ranging from $-\infty$ to $+\infty$. The relation (14.13) shows that when one integrates over p_z the variable y disappears; the result is (writing p instead of p_z):

$$(14.15) \qquad \tilde{U}^0 = \left(\frac{b}{8\pi^2 \hbar^3}\right) \sum_{\sigma=-1}^{+1} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} W_n(p)\,dp.$$

Set:

$$x = b\left(n + \frac{1-\sigma}{2}\right).$$

x is null for $n = 0$, $\sigma = 1$, but takes the *same* value $b, 2b, 3b, \ldots$, for *two* combinations of n and σ; for example, $x = 2b$ for $n = 2$, $s = 1$, as well as for $n = 1$, $\sigma = -1$. We thus have:

$$(14.16) \qquad \sum_{\sigma=-1}^{+1} \sum_{n=0}^{\infty} W_n = F(0) + 2\sum_{n=1}^{\infty} F(nb),$$

in which:

$$(14.17) \qquad F(x) = -c\sqrt{p^2 + m_0^2 c^2 + x}.$$

It is clear that the sum (14.16) is divergent; nevertheless, one may transform it in such a manner that the physical significance of the divergent terms is obvious; if we apply the MacLaurin-Euler summation formula, this gives:

$$(14.18) \qquad F(0) + 2\sum_{n=1}^{\infty} F(nb) = \frac{1}{b}\left\{\int_0^{\infty} F(x)dx + \sum_{k=1}^{\infty}(-1)^k \frac{B_k}{(2k)!}b^{2k}F^{(2k-1}-(0)\right\},$$

since all of the derivatives $F^{(k)}(x)$ of $F(x)$, namely:

$$(14.19) \qquad F^{(1)}(x) = -\frac{c}{2}(p^2 + m_0^2 c^2 + x)^{-\frac{1}{2}}, \qquad F^{(2)}(x) = \frac{c}{4}(p^2 + m_0^2 c^2 + x)^{-\frac{3}{2}},$$

is annulled when $x \to \infty$. B_k is the Bernoulli number of rank k.

By substituting (14.18) into (14.15), one obtains the energy density $\tilde{U}$ of the vacuum electrons in a magnetic field **B** expressed by means of a power series in a quantity b that is proportional to B.

We isolate the only part of $\tilde{U}$ that can have a physical significance; Weisskopf characterized it by the following conditions:

1. *The energy density of the electrons must be annulled, along with the field.*

This signifies that the zeroth-order term in the development (14.15), has no physical significance; now, this is precisely the term that contains the divergent integral of $F(x)$.

2. *The vacuum polarizability must be annulled, just as it is in the case of the absence of a field.*

By "vacuum polarizability" we mean the factor α that measures the degree of disagreement with Maxwell's theory in the equation:

$$\mathbf{H} = \frac{\partial U}{\partial \mathbf{B}} = \mathbf{B} + \frac{\partial U'}{\partial \mathbf{B}} = \mathbf{B}(1 + \alpha).$$

α is a function of **B**, and our postulate consists of saying that there is no constant term in the development of this function in powers of $|\mathbf{B}|$. In this case, $\dfrac{\partial U'}{\partial \mathbf{B}} = \alpha\mathbf{B}$ contains no term of first order in $\tilde{U}$, nor a term of second order in $|\mathbf{B}|$.

We must therefore eliminate the terms $k = 1$ in (14.18) that give rise to terms in b^2 in $\tilde{U}$. Indeed, the integral of these terms over p diverges, as one may see by analyzing the behavior of $F^{(1)}(x)$ in (14.19). The terms of higher order give rise to convergent integrals, as one sees in (14.18); we are thus led to the following finite expression for $\tilde{U}$:

$$(14.19) \qquad \tilde{U}^0 = \frac{1}{4\pi^2\hbar^3}\sum_{k=2}^{\infty}(-1)^k \frac{B_k}{(2k)!}b^{2k}\int_{-\infty}^{+\infty} F^{(2k-1)}dp$$

$$= -\frac{c}{4\pi^2\hbar^3}\sum_{k=2}^{\infty}(-1)^k \frac{B_k}{(2k)!}b^{2k}\frac{1\cdot 3\cdots(4k-5)}{2^{2k-1}}\int_{-\infty}^{+\infty}\frac{dp}{(p^2 + m_0^2 c^2)^{\frac{4k-3}{2}}}.$$

This power series may be expressed with the aid of hyperbolic cotangents by the following formula:

(14.20)
$$\tilde{U}^0 = -\frac{1}{8\pi^2} m_0^2 c^2 \left(\frac{m_0 c}{\hbar}\right)^3 \int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta} \left[\mathfrak{B}\,\eta \coth(\eta\mathfrak{B}) - 1 - \frac{\eta^2}{3}\mathfrak{B}^2\right];$$

$\mathfrak{B}$ is the magnetic field compared to the critical field b_1:

(14.21)
$$B = \frac{B}{b_1};$$

the critical field is given by:

(14.22)
$$b_1 = \frac{m_0^2 c^3}{e\hbar} = b_0 \frac{e^2}{\hbar c} = \frac{b_0}{137},$$

in which b_0 is the Coulomb field of the charge e at a distance that is equal to the electron radius:

(14.23)
$$b_0 = \frac{e}{r_0^2}, \qquad m_0 c^2 = \frac{e}{r_0}.$$

Now calculate the contribution of the electric field that is given by (14.11). Weisskopf proceeded in two stages. First, he assumed that the field is sufficiently small that one may apply the ordinary perturbational method, which thus gives the first terms in the development into a power series in E; then, he determined the general law of the desired law by considering its relativistic invariance.

We content ourselves by discussing the essential characteristics of this method of calculation without reproducing the details. The terms in the second approximation $\overline{U^{(2)}}$ to $\tilde{U}$, which are due to the electric potential V, are proportional to:

$$\frac{g^2}{\hbar^2} V_0^2 = \frac{1}{2}\overline{E}^2,$$

in which $g/\hbar$ is the frequency that appears in (14.11), and in which the bar signifies that one takes the mean over all space. (14.10) then shows that:

(14.24)
$$U' = \tilde{U}^{(0)} + \overline{U^{(2)}} - E\frac{\overline{dU^{(2)}}}{dE} = \tilde{U}^0 - \overline{U^{(2)}}.$$

The calculation gives:

(14.25)
$$\overline{U^{(2)}} = \frac{1}{8\pi^2} m_0 c^2 \left(\frac{m_0 c}{\hbar}\right)^3 \frac{1}{3}\mathfrak{E}^2 \int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta}[\mathfrak{D}\coth(\eta\mathfrak{D}) - 1],$$

in which:

(14.26)
$$\mathfrak{E} = \frac{E}{b_1}.$$

Now write the first terms of the development of in terms of $\mathfrak{D}$ and $\mathfrak{E}$ that is deduced from (14.20) and (14.25), namely:

(14.27)
$$\begin{cases} \tilde{U}^0 = -\frac{1}{360\pi^2} m_0 c^2 \left(\frac{m_0 c}{\hbar}\right)^3 \left(\mathfrak{D}^4 + \frac{4}{7}\mathfrak{D}^6 + \cdots\right), \\ \overline{U^{(2)}} = -\frac{1}{360\pi^2} m_0 c^2 \left(\frac{m_0 c}{\hbar}\right)^3 \left(5\mathfrak{E}^2\mathfrak{D}^2 - 2\mathfrak{E}^2\mathfrak{D}^4 + \cdots\right), \end{cases}$$

These formulas are valid for fields **E**, **B** that are parallel to each other; one may generalize to arbitrary fields by means of the following argument.

As we know, from section 4, that the energy density U may not be an arbitrary function of **E**, **B** (or of **D**, **H**), the postulate of relativity imposes certain conditions that are not, moreover, very simple. On the other hand, we know that one may choose the Lagrangian L arbitrarily as a function of the elementary invariants $E^2 - B^2$ and **EB**; when one is given L, one obtains U by means of the equation:

(14.28)
$$U = \mathbf{ED} - L = \mathbf{E}\frac{\partial L}{\partial \mathbf{E}} - L.$$

Let:
(14.29)
$$L = L^0 + L^1,$$

be the development of L, in which:

(14.30)
$$\begin{cases} L^0 = \frac{1}{8}(E^2 - B^2), \\ L^1 = \alpha(E^2 - B^2)^2 + \beta(\mathbf{EB})^2 + \xi(E^2 - B^2)^3 + \zeta(E^2 - B^2)(\mathbf{EB})^2 + \cdots \end{cases}$$

By starting with (14.28), one obtains:

(14.31)
$$\begin{aligned} U^1 = {}&\alpha(E^2 - B^2)(3E^2 + B^2) + \beta(\mathbf{EB})^2 \\ &+ \xi(E^2 - B^2)^2(5E^2 + B^2) + \zeta(\mathbf{EB})^2(3E^2 - B^2) + \ldots \end{aligned}$$

It suffices to compare (14.27) and (14.31) to obtain the values of $\alpha, \beta, \xi, \zeta, \ldots$:

$$(14.32) \quad \begin{cases} \alpha = \dfrac{1}{360\pi^2} m_0 c^2 \left(\dfrac{m_0 c}{\hbar}\right)^3 \dfrac{1}{b_1^4} = \dfrac{1}{360\pi^2} \dfrac{e^4 \hbar}{m_0^4 c^7} = \dfrac{1}{360\pi^2} \dfrac{\hbar c}{e^2} \dfrac{1}{b_1^4} \\[2mm] \beta = 7\alpha, \\[2mm] \xi = \dfrac{1}{360\pi^2} m_0 c^2 \left(\dfrac{m_0 c}{\hbar}\right)^3 \dfrac{4}{7} \dfrac{1}{b_1^6} = \dfrac{1}{360\pi^2} \dfrac{e^6 \hbar^3}{m_0^8 c^{13}} = \dfrac{1}{360\pi^2} \left(\dfrac{\hbar c}{e^2}\right)^3 \dfrac{1}{b_0^4} \\[2mm] \zeta = \dfrac{13}{2} \xi. \end{cases}$$

Some analogous invariance considerations may likewise be employed to determine the terms of U' that are of higher order in E. The fact that L depends only on $E^2 - B^2$ and (**E·B**), signifies, for example, that there exists a numerical relationship between the coefficients E^k and B^k.

If one designates the real and imaginary parts of A by RA and IA, respectively, the final result may be written:

$$(14.33) \quad L' = \frac{b_1^2}{8\pi^2} \frac{e^2}{\hbar c} \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3}$$

$$\times \left[i\eta(\mathfrak{E}\mathfrak{D}) \frac{\cos \eta R\left\{\sqrt{\mathfrak{E}^2 - \mathfrak{D}^2 + 2i(\mathfrak{E}\mathfrak{D})}\right\}}{\cos \eta I\left\{\sqrt{\mathfrak{E}^2 - \mathfrak{D}^2 + 2i(\mathfrak{E}\mathfrak{D})}\right\}} + 1 + \frac{\eta^2}{3}(\mathfrak{E}^2 - \mathfrak{D}^2) \right].$$

This expression was discovered by Heisenberg and Euler [2] with the aid of a method that was much more complicated than the preceding one, which may be described as a theory of the "diffusion of light by light."

We therefore see that the theory of "holes" leads us to a well-defined Lagrangian for the electromagnetic field. We are now confronted with the task of comparing the various Lagrangians that have introduced in the preceding section with the aim of making it possible for mass to be assimilated by the electromagnetic energy of a point-like charge.

This latter concept is obviously I contradiction with the theory that we just developed and in which the mass of the electron is a fundamental constant of the wave equation that constituted our point of departure. We may therefore hope to find a complete accord between the results of the two theories, even if they are dissimilar in their basic hypotheses; nevertheless, there is reason to be satisfied if they may be made to coincide in the domain in which both of them do not differ too much from a common viewpoint, namely, in the domain of Maxwell's equations.

We will thus have to compare the development (14.30) with that of a Lagrangian of a type that was studied in the preceding sections. As we have seen, there is a considerable ambiguity in the choice of that Hamiltonian, but we may reasonably hope that the coefficients of the developments of all the functions do not differ much from each other. With these conditions, take the function that we have studying all along.

(14.34) $$L = \frac{b^2}{4\pi} \left[1 - \sqrt{1 + \frac{B^2 - E^2}{b^2} - \frac{(\mathbf{BE})}{b^4}} \right] = L^0 + L^1 + \dots,$$

which we write with the – sign in electrostatic units, in order to be in accord with thee formulas of Weisskopf. The developments give:

(14.35)
$$\begin{cases} L^0 = \dfrac{1}{8\pi}(E^2 - B^2), \\[2mm] L^1 = \dfrac{1}{32\pi b^2}\left[(E^2 - B^2)^2 + 4(\mathbf{EB})^2 \right]. \end{cases}$$

The comparison with (14.30) gives:

(14.36) $$\alpha = \frac{1}{32\pi b^2}, \quad \beta = 4\alpha.$$

One already confirms that there is a divergence between the two theories: In the "hole" theory, one has $\frac{\beta}{\alpha} = 7$ (formula 14.32), and in this one one has $\frac{\beta}{\alpha} = 4$. Nevertheless, if we press on and we postulate the identity of the α coefficients in both theories then we may write the relation:

(14.37) $$\frac{\hbar c}{e^2} = \frac{45\pi}{4} \frac{b_0^2}{b^2}.$$

On the other hand, b is related to the "electronic radius" by $b = \frac{e}{r_0^2}$ and r_0 to the proper mass by $m_0 c^2 = \frac{e^2}{r_0} 1.2361$, in such a way that:

(14.38) $$b = \frac{e}{\left(\dfrac{e^2}{m_0 c^2} 1.2361 \right)^2} = \frac{b_0}{(1.2361)^2}.$$

By substituting in (14.37), one finds that:

$$\frac{\hbar c}{e^2} = \frac{45\pi (1.2361)^4}{4} = 82.$$

This result, which is due to Euler and Kockel [3], is of considerable interest. It shows that by postulating the identity of the departures from Maxwell's equations with terms in

developments that are obtained, on the one hand, by means of the quantum theory of the vacuum, and, on the other, the classical theory of the electromagnetic mass, one is led to a relation between charge and the Planck constant; in other words, one obtains a numerical value for the fine structure constant. This value is very weak, and the experimental value is $\frac{137}{82} = 1.67$ times greater [4]; nevertheless, the order of magnitude is correct and the process of calculation shows how a dimensionless number of that order may be presented in the form of a combination of other factors, such as the factors 45, etc., in (14.38), for example.

This reasoning constitutes the first known indication of a theoretical explanation for the "mysterious" number 137, if we leave aside the speculations of Eddington to the effect that this number will be an integer given by the formula $\frac{n^2(n^2+1)}{2}+1$, for $n = 4$.

We now establish a comparison between the two theories for higher values of the field.

We remark that, in the first place, the "characteristic field" b_1 is much smaller than the absolute field of the unitary theory; indeed, from (14.22) and (14.38), one has:

$$(14.40) \qquad b_1 = \frac{b_0}{137} = \frac{(1.2361)^2}{137}b = \frac{b}{90}.$$

Then, we observe that for large values of **E** the higher-order terms in the development of the expression (14.33) become:

$$(14.41) \qquad L' \to \frac{e^2}{24\pi^2\hbar c}E^2 \log\frac{E}{b_1}$$

(and a similar expression for large values of **B**). The ration between this additional term and the corresponding term $\frac{1}{4\pi}E^2$ in Maxwell's theory is:

$$(14.42) \qquad \frac{L^1}{L^0} \to \frac{1}{3\pi}\frac{e^2}{\hbar c}\log\frac{E}{b_1} = \frac{1}{1290}\log\frac{E}{b_1}.$$

This result proves that the nonlinear terms that the theory of holes introduces constitute only weak corrections to the Maxwell theory, even for very strong fields. A Lagrangian of the type (14.33) is therefore of no utility for the problem of electromagnetic mass. This negative result must not be surprising; indeed, it suffices for us to recall that the origin of that Lagrangian is the wave equation of the electron, and that this equation already contains the mass as the coefficient of an essential term.

15. Final remarks. – The preceding monograph examined only one facet of the general problem of adapting physical theories to the results of modern experimental discoveries. Nonlinear electrodynamics may be considered as a *superstructure* for the classical Maxwell theory; meanwhile, there exists an *infrastructure* whose role is to reduce Maxwell's laws to a simpler set of equations; I spoke of the theory of light that is based on the hypothesis of the neutrino, such as was imagined by de Broglie and completed Jordan and Kronig. At this point in time, we may only guess which of these concepts will be victorious in the end, and we may not know whether they are finally absorbed by another concept that embraces both of them. One thing seems certain: the fact that it fails to give us a general principle that regulates phenomena at the nuclear level. Theoretician need to follow experimental research attentively and to analyze their results with care, in the hopes of finding a clue concerning the nature of this unknown principle. The object of this monograph was precisely to give a similar analysis of the actual state of electrodynamics.

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INTRODUCTION

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Sec. 1. - 1. General discussions of the calculus of variations and functional calculus

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M. FRICHET, *Ann. Di Math.,* 3rd series, t. XI, 1905, pp. 187.
G. PRANGE, *Die Hamilton-Jacobische Theorie fhr Doppelintegrale* (*Dissertation,* Göttingen, 1915).
M. G. JUVET, *Sur une équation aux dérivées fonctionelles partielles et sur une généralisation du théorème de Jacobi* (Thesis, Paris, 1926).

Sec. 2. - 1. F. KLEIN, *Göttinger Nachr.,* 1918, pp. 171 and 394.

Sec. 3. - 1. G. MIE, *Ann. D. Phys.,* 4th series, Bd. 37, 1912, pp. 511; 39, 1912, pp. 1; 40, 1913, pp. 1.

2. Various treatments of tensorial calculus:

H. Weyl, *Raum, Zeit, Materie,* 5th ed., Berlin, 1923.
W. Pauli, *Relativitätstheorie* (*Encyclop. D. Math. Wiss.,* V, t. 19, pp. 539).
A. EDDINGTON, *The Mathematical Theory of Relativity,* Cambridge, 1922.
J. A. SCHOUTEN, *Der Ricci-Calchl,* Berlin, 1924.
J. LEVI-CIVITA, *The Absolute Differential Calculus,* London, 1935.

3. The bibliography of this subject is too vast to be described here.
4. See, for example, the article of PAULI in the *Encyclopédie allemande,* that was cited in No. 2, pp. 269.
5. The reasoning at the end of sec. 3 is not rigorous because of the appearance of the arbitrary variational functions, δx^k, as coefficients of the $\mathfrak{U}_k^l$ or $\mathfrak{T}_k^l$ in the expression, (3.23). In the case of unitary electrodynamics (see sec. 6), Weiss has recently developed a calculation that rigorously leads to the $\mathfrak{T}_k^l$ instead of the $\mathfrak{U}_k^l$, and which will be published shortly. His argument is based on "gauge invariance" [see sec. 5 (5.18)], which is a characteristic property of unitary electrodynamics. In the general case of Mie's electrodynamics, there is no "gauge invariance;" hence, such a calculation is impossible. This fact may be regarded as one more reason against Mie's theory, along with the other ones that were discussed at the end of sec. 5.

Sec. 4. - 1. The proof that corresponds to the two relations, (4.14), in the case of the gravitational field was sketched by Klein in the article that was cited in sec. 2, no. 1. One may likewise find the second condition, (4.15), in the article of PAULI, and in several other publications.
2. See, for example, sec. 48 of A. EDDINGTON (op. cit. in sec. 3, 2).

3. P. WEISS, in his previously-cited thesis, has proved conditions (3.12) by another algebraic method in the case where L depends only on F and G. The proof of the text does not seem to have been indicated in any known publication.

Sec. 5. - 1. M. BORN, *Göttinger Nachr.,* 1914, pp. 1.
2. This fact, which was clearly explained in the work of WEYL (sec. 3, no. 2) and in the article of PAULI, was the object of a detailed examination by VAN DANTZIG (*Proc. Cambr. Phil. Soc.,* 1934, pp. 421).
3. The most complete discussion of this theory is found in the articles of LORENTZ in the *Encyclop. d. Math. Wiss.,* V, t. 3.
4. W. KAUFMANN, *Gött. Nachr.,* 1901, pp. 143; 1902, pp. 291; 1903, pp. 90.
5. The research of ABRAHAM has been collected in his well-known work: *Theorie der Elektrizität,* 3rd Ed., 1914, Berlin; one will find the corresponding bibliography in volume II.
6. M. BORN, *Ann. D. Phy.,* 4th series, t. 30, 1909, pp. 1.
7. P. LANGEVIN, *Jubilée de Marcel Brillouin,* pp. 426.

Sec. 6. - 1. M. BORN, *Nature,* t. 132, 1933, pp. 282. *Proc. Roy. Soc. A,* t. 143, 1934, pp. 410.
M. BORN and L. INFELD, *Nature,* t. 132, 1933, pp. 970, 1004; *Proc. Roy. Soc. A,* t. 144, 1934, pp. 425.
J. FRENKEL, *Proc. Roy. Soc. A,* t. 146, 1934, pp. 930.
2. B. S. MADHAVA RAO, *Proc. Indian Ac. Sciences,* vol. 4, 1936, pp 436.

Sec. 7. - 1. S. SCHUBIN an A. SMIRNOV, *C. R. Acad. Sciences de l'U. R. S. S.,* vol. 1, 10th series, 1935, pp. 69, have treated some particular cases.
2. M. H. L. PRYCE, *Proc. Cambr. Phil. Soc.,* vol. 31, 1935, pp. 625.
3. M. H. L. PRYCE, *Proc. Cambr. Phil. Soc.,* vol. 31, 1935, pp. 50.
4. B. S. MADHAVA RAO, *Proc. Indian Ac. Sciences,* vol. 4, 1936, pp 355.
5. Since 1907, J. Stark has suggested the hypothesis of an electron with axial symmetry, and has tried to experimentally confirm this. Nevertheless, his efforts have not been crowned with success, because he ignores the explanation that was discovered by Pauli, Uhlenbeck, and Goudsmit, of the multiplicity of spectral lines due to spin. Recently, Stark has renewed his attempts (*Phys. Zeitschr.,* t. 38, 1937, pp. 269), by trying to establish a relationship between the axial structure of the electron and the superconductivity of metals, but without first looking for a mathematical form to his thoughts. Moreover, if he tries to do this by using the classical theory, to which he is a strict partisan, then he will encounter certain difficulties of the same order or even more considerable than those that doomed the classical theory of the electron. The calculations of Madhava Rao constitute, in some way, the realization of the program that was traced out by Stark; he clearly showed that any classical theory of the annular electron is incapable of representing the experimental facts, and shows the reason for that failure.

Sec. 8. - 1. The suggestion of Infeld was developed in the second article of M. BORN and L. INFELD, cited in sec. 6, 1.

2. E SCHRÖDINGER, *Proc. Roy. SOC., A,* t. 150, 1935, pp. 465.
3. The double sign in the radical that represents the energy density in an electromagnetic field is analogous to the double sign in the energy of an electron in the relativistic treatment. In the latter case, Dirac has shown how one may eliminate it by the introduction of spin matrices. He is tempted to try to apply the same procedure to the energy of an electromagnetic field; I have succeeded in showing that for a given point of space, it is possible to eliminate the square root without using any other matrices than the ones that were used by Dirac (M. BORN, *Proc. Cambr. Phil. Soc.,* t. 32, 1936, pp. 102). The general theory then demands that the independent spin matrices be assigned to each of the points of space, and it has not been possible for me to develop this idea sufficiently enough that it should bear fruit.
4. B. S. MADHAVA RAO, *Proc. Indian Ac. Sciences,* vol. 4, 1936, pp 575.
5. P. WEISS, *Proc. Cambr. Phil.. Soc.,* vol. 33, 1937, pp. 79.

6. L. INFELD, *Proc. Cambr. Phil. Soc.,* vol. 32, 1936, pp. 127; vol. 33, 1937, pp. 70; L. INFELD and B. HOFFMANN, *Phys, Rev.,* vol. 51, 1937, pp. 765.

Sec. 9. - 1. See the articles of Born, Infeld, Frenkel, that were cited in sec, 6, 1.
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    5. P. JORDAN, *Zs. F. Phys.,* t. 93, 1935, pp. 464; t. 98, 1936, pp. 759; t. 99, 1936, pp. 109; t. 100, 1936, pp. 569.
    6. B. HOFFMANN, *Quart. Journ. of Math.,* vol. 6, 1935., pp. 149; *Phys. Rev.,* vol. 47, 1935, pp. 877.
    7. L. INFELD and B. HOFFMANN, cited in sec. 8, 6.

Sec. 10.- 1. The method used in the text is indicated in the article of M. Born, *Proc. Indian Acad. Sciences,* vol. 3, 1936, part. I, pp. 8; part. II, pp. 85.
    2. This attempt is found in the articles cited in sec. 1.
    3. H. A. KRAMERS, *Physica,* t. 1, 1934, pp. 825; ZEEMAN, *Verhandlingen,* 1935, pp. 403.
    4. G. RACAH, has proved that the Lorentz force is the only one that may be derived from a variational principle, provided that it does not depend on derivatives of the coordinates of order higher than one (*Rend. Dei Lincei,* vol. 25, 1937, pp. 223).

Sec. 11. - 1. M. BORN, W. HEISENBERG, and P. JORDAN, *Zeitschr. F. Phys.,* t. 35, 1925, pp. 557.
    2. W. HEISENBERG, *Ber. D. Sächs. Ac. d. Wiss.,* vol. 83, 1931, pp. 5.
    3. One will find clear and complete discussions of this theory in:
        E. FERMI, *Rev. of Modern Physics,* vol. 4, 1932, pp. 87;
        W. HEITLER, *The Quantum Theory of Radiation,* Oxford, 1936;
        L. NORDHEIM, *Théorie des chocs et du rayonnement* (*Ann. De l'Institut Poincaré,* 1936).
    4. W. HEISENBERG and W. PAULI, *Zeitschr. F. Phys.,* t. 56, 1929, pp. 1; t. 59, 1930, pp. 168.
    5. M. BORN and L. INFELD, *Proc. Roy. Soc.,* (A), vol. 147, 1934; pp. 522; vol. 150, 1935, pp. 141.
    6. M. H. L. PRYCE, *Proc. Roy. Soc.,*(A), vol. 155, 1936, pp. 597; vol. 159, 1935, pp. 355.
    7. In his thesis and in the *Proc. Roy. Soc.,* (A), t. 156, 1936, pp. 192, P. Weiss has developed a direct method of deducing the commutation laws by introducing a new definition of the "canonically conjugate" variables that is valid for systems with more than one independent variable. This method has the advantage of being absolutely invariant from the relativistic standpoint, and exhibits the important role that played by the total energy in the quantization method of Heisenberg and Pauli.
    8. P. WEISS has given (in his article cited in sec. 7) a generalization of the commutation laws that is valid when, instead of a t = const. section of spacetime, one is given an arbitrary three-dimensional section. Likewise, see V. FOCK, *Phys. Zeitschr. d. Sowjetunion,* vol. 6, 1934, pp. 425.

Sec. 12. - 1. The method of P. WEISS (cited in sec. 11, 7, 8) shows the profound reason for the degeneracy that was mentioned in the text, and the disappearance of the corresponding moment of the scalar potential. P. Weiss has likewise established new commutation laws that relate the potentials and the total charge, but has made no application of them.
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In the first article cited in sec. 13, 12.

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