THE EIKONAL

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Table of contents

Introduction

The representation of geometrical optics, as it is presented in the extensive literature on the subject, starts, in most cases, by developing the theory of lens systems. When the problems posed are treated satisfactorily, such a process is justified, in and of itself, since the lens instruments that serve to generate images on external objects, in fact, ultimately define the most important – and also the most difficult – topic in practical optics. Moreover, the restriction that one therefore imposes carries with it the drawback that, on the one hand, the true meaning of the assumptions and the theorems that one derives from them are occluded in places, and, on the other hand, the path to results of a general nature will be complicated. Giving a good explanation for this situation constitutes that chapter of geometrical optics that is usually referred to as the study of distinguished points of a lens system, and which I would, however, prefer to call the "first approximation" in the theory of optical images. Namely, if one discards the assumptions and line of reasoning of this chapter as inessential, as was first done by ABBE [1]), then what would remain is a train of thought that shall be presented in the following manner.

The starting point is the concept that geometrical optics always begins with, namely: Two spaces, ω and Ω, are mapped onto each other in a "ray-like" manner – i.e., any line σ in ω determines a line Σ in Ω and conversely. The two spaces then mean the first and last medium in any optical system, the σ are the rays that emanate from luminous points of the first medium, while the "conjugate" rays Σ define the corresponding advance of the light path in the latter medium. Rays that go through a point define a "homocentric" sheaf.

If one considers the sheaf of Σ that is conjugate to a homocentric σ-sheaf then, in general, it is "astigmatic" – i.e., non-homocentric – and the deviation from the homocentric union of rays is the "astigmatism" or "astigmacy" of the sheaf. Likewise, the σ-sheaf that is conversely associated with a homocentric Σ-sheaf is generally astigmatic. If the conjugate sheaves in ω and Ω are simultaneously homocentric as a result of special circumstances then they will define an "anastigmatic" pair of sheaves and the union of the two points is an "anastigmatic" point-pair. When nothing more specific is assumed about the ray-wise map of the two spaces ω, Ω then the following two cases relating to the appearance of anastigmatic points will be provisionally possible:

1. Anastigmatic points are completely absent.
2. They are present and isolated; let them be geometric locations where the lines, surfaces, and solids can be mapped.

If the latter case enters into consideration then anastigmatic bodies will be present, and the spatial domains that are taken from these bodies in ω and Ω will be mapped to each other, not only ray-wise, but also point-wise, in such a way that the union of points of the anastigmatic pairs of ray-sheaves will likewise determine the elements that are conjugate to each other under the point-wise map. If one, with ABBE, asks about the general properties of the point-wise map that is thus generated then this would imply – as one might confirm in CZAPSKI (loc. cit., page 24, et seq.) – that it is nothing but a

[1]) On this, confer the presentation in the book of S. CZAPSKI: "Theorie der optischen Instrumente nach ABBE," Breslau, E. Trewendt, 1893.

collineation between the two spaces considered. With this result, however, the problem that must be solved in the theory of optical images in the first approximation is already resolved essentially, since one now only needs to treat well-known properties of the collinear map when clad in optical garb in order to obtain all of the general theorems that one can infer in the first approximation. To them belong – e.g. – the theorems on focal points, principal points, cusps, and their corresponding planes, and furthermore, the theorems on the position of the object and image, as regards the magnification and brightness, on the action of the apertures and field of view, etc. In truth, these theorems are not by any means optical theorems, but belong to the general study of space or geometry. Likewise, their applicability to optics depends, not upon this or that particular property of the optical system that is under scrutiny, but upon two much more general things, namely, on the one hand, the concept that the first and last media can be mapped to each other ray-wise, and, on the other hand, the assumption that anastigmatic bodies exist, so astigmatism is absent, or at least, can be neglected in the first approximation. One will first set foot in the realm of real optics when one treats the problem of how maps of the type in question can be realized, whether rigorously or approximately.

The present line of inquiry that is being sketched, and which ABBE pursued, obviously renders in the first approximation of geometrical optics that which one strives to achieve in any scientific investigation, namely, the neat extraction of the logically necessary and sufficient assumptions; as a prize, this yields the fact that the totality of general theorems in this chapter of geometrical optics could be condensed, to a certain extent, into the one simple expression: "The object and the image are collinear."

In the book by CZAPSKI, the purely geometric theory of optical maps was truncated at the first approximation; the second approximation or the representation of astigmatic ray paths was already established for certain forms of optical systems. There is, however, no intrinsic basis for not extending the purely geometric examination to the second approximation. The following article shall show that such an advance on the ABBE train of thought is not only feasible, but also preferable. By means of it, it will be indeed possible to reduce the starting point for problems of a more general nature to the most mathematically simple form from the outset.

If one again starts with the concept that two spaces ω and Ω are mapped to each other ray-wise then, in the event that no further assumptions are added, one must first attend to the theorems that are true for such maps in general. When one dresses these theorems in optical clothing, they are likewise converted into many statements on the properties of the ray paths through any optical system, where naturally the value of these statements for practical purposes of geometrical optics can turn out to be quite different. Such a geometric theory will now also encompass maps that – at least, at present – are without interest in the study of optical instruments. The assumption that nothing further should be given besides the ray-wise map between ω and Ω is unnecessarily general, and allows one to single out a well-defined class in the totality of all possible ray-wise maps in the problems to be addressed here.

If one thinks of the first medium ω as being an arbitrary surface, and considers the normals to this surface as a sheaf of rays that, due to this way of generating them, we would like to briefly refer to as "surface normals" then two cases are possible, namely, the conjugate sheaf in the second medium Ω is or is not composed of surface normals, as well. The former case enters into view, as one knows, as long as the light path from the first to the last medium is consistent with the ordinary laws of refraction and reflection, and the statement of this property constitutes the content of the known theorem of MALUS. For that reason, we would like to refer to the requirement that every sheaf of surface normals of the first medium shall, in turn, generate a sheaf of surface normals in the last medium as the MALUS condition, and we can then arrange all ray-wise maps into two large groups, according to whether they do or do not fulfill the stated condition. Now, the cases in the theory of optical instruments in which MALUS's theorem is valid play only a subordinate role, and are left completely by the wayside. It then makes good sense for us to take the following two theorems as the starting point for a geometric theory of optical maps:

1. The first medium is mapped ray-wise to the last one.
2. The map satisfies the MALUS condition.

If one now asks about the general properties of the classes of maps that are present then the investigation of that question would yield the fact that any individual map is completely characterized by a definite mapping function with four variables, for which I will use the term "eikonal" in order to have a concise expression. Each map that satisfies MALUS's theorem is then associated with a definite eikonal, and conversely; all peculiarities of a given map find their counterpart in corresponding peculiarities of the eikonal. The eikonal is – and herein lies its main property – the generator of the equations for a contact transformation, through which any four determining data are coupled with each other, and which one might define to be the two conjugate lines σ and Σ. Thus, in the first approximation, the contact transformation enters into the picture in place of the collinearity relation.

It is perhaps not superfluous to clarify the role that the eikonal plays in the geometric theory of astigmatic ray paths by comparison with another part of applied mathematics. In the presentation of analytical mechanics, the problems in which HAMILTON's principle is valid are generally afforded a privileged role. This privileged status is well-founded, since the validity of HAMILTON's Ansatz allows one to treat all questions that are common to the envisioned class of problems in a unified manner. The concept of eikonal now plays an entirely similar role to the HAMILTON Ansatz in mechanics, but in the generally much narrower realm of geometrical optics; it delivers the simplest form of the Ansatz for the general treatment of general questions. The fact that the associated difficulties that any special problem gives rise to must be overcome by means of special tools that are developed for the individual cases is a separate matter. In regard to this, I would like to remark, in order to avoid any misunderstandings that it may take some time before its validity is confirmed by the examination of properties that all optical systems have in common, which will probably use purely numerical methods of calculation in real-world optics. Precisely on the grounds of the developments that will be given later on, I regard it as self-evident that the discovery of a much-needed purely analytical or algebraic replacement for the aforementioned numerical methods cannot be accomplished at all by means of the usual elementary tools that are at our disposal, although one also occasionally encounters tedious arguments in the literature that purport to have overcome these difficulties.

After the foregoing discussion, I now turn to the treatment of the problem that was suggested above. In order to avoid repetition, I will therefore begin with a series of assumptions, and then, in order to have everything in one place, briefly develop some well-known things to the extent that is necessary.

I.

Establishing the notations. Focal lines. Condition for a surface normal sheaf.

A given space ω will be referred to an arbitrarily chosen, right-angled system of axes (x, y, z); the y-axis and the z-axis shall be referred to as the *lateral axes* and the yz-plane, as the *base plane*. We write the equations for the points of a line σ – or, as we would also like to say, a *ray* σ – in the form:

$$\frac{x-0}{m} = \frac{y-h}{p} = \frac{z-k}{q}, \tag{1}$$

where the m, p, q mean the direction cosines and (0, h, k) is the location of the intersection of σ with the base plane. The four mutually independent quantities h, k, p, q are the necessary and sufficient determining data for σ, or, more briefly, the *ray coordinates*. The totality of σ defines a four-fold extended manifold. In order to abbreviate this somewhat cumbersome way of speaking, the symbol μ_n shall be used in order to refer to an n-fold extended manifold, without regard to its other qualities, so μ_0 is then the symbol for an infinite number of things. Thus, the σ collectively define an μ_4 .

If one imposes certain conditions on the σ then, depending upon the circumstances, a μ_0 , μ_1, μ_2, or a μ_3 will be selected from the μ_4 . The μ_1, μ_2, μ_3 can also be obtained when one thinks of the h, k, p, q as representing functions of 1, 2, 3 variable parameters, resp. A μ_1 of rays shall be referred to a *family* and a μ_2 as a *sheaf*. If the rays of a family or a sheaf go through a fixed point π then we call the structure *homocentric,* and refer to the common point π as the *vertex* of the family or the sheaf; if the structure is not homocentric then it shall be called *astigmatic.* If one thinks of an arbitrary sheaf of parallels τ as being drawn through the origin to the rays σ then it intersects a sphere of radius one centered at the origin at points whose coordinates are the m, p, q of unit rays. The spherical figure that is generated by the intersection points can serve as the representative of the sheaf of an aperture and shall be called the *aperture figure*.

If one selects an arbitrary ray σ_0 with the coordinates h, k, p, q from the sheaf then the rays of the sheaf that are infinitely close to σ_0 define an "elementary sheaf" with the "central ray" σ_0 ; the coordinates of the neighboring rays are given by:

$$h + dh, \qquad k + dk, \qquad p + dp, \qquad q + dq,$$

where the h, k, p, q are thought of as expressed in terms of functions of two varying parameters – say, α and β. The shortest distance between the central ray and the neighboring ones is, in general, of the same infinitely small order as the quantity:

$$\sqrt{d\alpha^2 + d\beta^2} \, ,$$

but it can be of higher order for special positions of the neighboring ray. If the latter is the case then one will say that the neighboring ray *intersects* the central ray and one will refer to the intersection point as the *focal point* of the elementary sheaf. If one defines:

$$dh = h_1\, d\alpha + h_2\, d\beta, \qquad dk = k_1\, d\alpha + k_2\, d\beta,$$

$$dm = m_1\, d\alpha + m_2\, d\beta, \qquad dp = p_1\, d\alpha + p_2\, d\beta, \qquad dq = q_1\, d\alpha + q_2\, d\beta,$$

and writes the ray equations (1) in the form:

$$x = \lambda\, m, \qquad y = h + \lambda\, p, \qquad z = k + \lambda\, q \tag{2}$$

then one will obtain the three conditions:

$$\left.\begin{aligned} 0 &= m\, d\lambda + \lambda m_1 d\alpha + \lambda m_2 d\beta, \\ 0 &= p\, d\lambda + (\lambda p_1 + h_1) d\alpha + (\lambda p_2 + h_2) d\beta, \\ 0 &= q\, d\lambda + (\lambda q_1 + k_1) d\alpha + (\lambda q_2 + k_2) d\beta, \end{aligned}\right\} \tag{3}$$

for the intersection of the central and neighboring rays, which, upon eliminating the $d\lambda$, $d\alpha$, $d\beta$, reduce to the one equation:

$$0 = \begin{vmatrix} m & \lambda m_1 & \lambda m_2 \\ p & \lambda p_1 + h_1 & \lambda p_2 + h_2 \\ q & \lambda q_1 + k_1 & \lambda q_1 + k_1 \end{vmatrix}. \tag{4}$$

If λ is determined from this quadratic equation then the coordinates of the focal point follow from (2) and the ratios of the $d\alpha$, $d\beta$ will follow from (3), which then determine the position of the intersecting neighboring rays.

If one searches for the focal point of all rays of the sheaf in question then its locus is a well-defined surface – the so-called *caustic* of the sheaf – that generally consists of two distinct sheets, due to the double-valuedness of λ. In order to obtain the equation of the caustic, one must eliminate the quantities λ, α, β from (2) and (4). If one preserves the parametric representation of the caustic that is given by (2) and (4) and sets:

$$dx = x_1\, d\alpha + x_2\, d\beta, \qquad dy = y_1\, d\alpha + y_2\, d\beta, \qquad dz = z_1\, d\alpha + z_2\, d\beta,$$

$$\lambda = \lambda_1\, d\alpha + \lambda_2\, d\beta$$

then the determinant:

$$D = \begin{vmatrix} m & x_1 & x_2 \\ p & y_1 & y_2 \\ q & z_1 & z_2 \end{vmatrix}$$

will go over to:

$$D = \begin{vmatrix} m & m\lambda_1 + \lambda m_1 & m\lambda_2 + \lambda m_2 \\ p & p\lambda_1 + \lambda p_1 + h_1 & p\lambda_2 + \lambda p_2 + h_2 \\ q & q\lambda_1 + \lambda q_1 + k_1 & q\lambda_1 + \lambda q_1 + k_1 \end{vmatrix},$$

from which it will follow, due to (4), that:

$$D = 0. \tag{5}$$

If one now considers (x, y, z) to be, on the one hand, a point of the caustic, and, on the other hand, the focal point of the elementary sheaf with the central ray (h, k, p, q) then equation (5) will say that the central ray and the normal to the caustic are perpendicular to each other, or that the central ray *contacts* the caustic. Thus, the sheaf in question represents the totality of all lines that contact the two sheets of the caustic. In special cases, one or both of the sheets can degenerate into caustic lines; furthermore, both sheets can reduce to a single point, which can happen with homocentric sheaves.

If one lays the x-axis of an elementary sheaf along the central ray, for the moment, and chooses the variable parameters to be the quantities p, q then, since only infinitely small values of the p, q can come under consideration when one neglects the quantities of higher order, one can define the equations:

$$m = 1, \qquad h = h_1\, p + h_2\, q, \qquad k = k_1\, p + k_2\, q,$$
$$y = (h_1 + h_2)\, p + h_1\, q, \qquad z = k_1\, p + (k_1 + k_2)\, q,$$
$$\Delta = (h_1 + h_2)\,(h_1 + h_2) - h_2\, k_1,$$
$$p\,\Delta = (k_2 + x)\, y - h_2\, z, \qquad q\,\Delta = -\,h_2\, z + (h_1 + x)\, z.$$

If one now establishes that in the limit of the elementary sheaf in question the aperture figure shall be an infinitely small circle that is described around the central ray with a radius ε then the extent of the p, q will be given by the condition:

$$p^2 + q^2 \leq \varepsilon^2.$$

If one further intersects the sheaf with a plane that is parallel to the base plane and lies at a distance x from the origin then the outline of the cross sectional figure is determined by the equation:

$$\varepsilon^2\,\Delta^2 = ((k_2 + x)\, y - h_2\, z)^2 + (k_1\, z - (h_1 + x)\, z)^2,$$

and is therefore an ellipse. For both abscissas that make Δ vanish, the ellipse will reduce to rectilinear line segments – viz., the so-called *focal lines*. The known construction of an elementary sheaf from its central ray and focal lines then follows from this immediately.

In the space ω, let $x = \varphi(x, y)$ be the equation of an arbitrary surface, so the normals to the surface will define a sheaf of rays; such sheaves shall be called *surface normals*. If one writes:

$$dx = d\varphi = \varphi_1\, dy + \varphi_2\, dz \tag{6}$$

then the ray coordinates of the normal that belongs to the point (x, y, z) will be determined by the equations:

$$\frac{m}{1} = \frac{p}{-\varphi_1} = \frac{q}{-\varphi_2}, \tag{7}$$

$$h = y - \frac{px}{m} = y + x\varphi_1, \qquad k = z - \frac{qx}{m} = z + x\varphi_2 . \tag{8}$$

If one defines $v = mx + py + qz$ then one will have:

$$dv = (m\,dx + p\,dy + q\,dz) + (x\,dm + y\,dp + z\,dq).$$

The first parenthesis on the right vanishes, on account of (6) and (7), and due to the fact that:

$$m^2 + p^2 + q^2 = 1, \qquad m\,dm + p\,dp + q\,dq = 0,$$

and with hindsight of (8), one obtains:

$$dv = h\,dp + k\,dq. \tag{9}$$

The expression $h\,dp + k\,dq$ is then a total differential for surface normal sheaves, and one has, if v, h, k are thought of as represented by functions of the p, q, that:

$$h = \frac{\partial v}{\partial p}, \qquad k = \frac{\partial v}{\partial q}, \qquad \frac{\partial h}{\partial q} = \frac{\partial k}{\partial p}. \tag{10}$$

This result may also be inverted. For a given sheaf, if the h, k are expressed as functions of the p, q then:

$$\frac{\partial h}{\partial q} = \frac{\partial k}{\partial p}, \tag{11}$$

so there exists a certain function $u(p, q)$ for which one has:

$$h = \frac{\partial u}{\partial p}, \qquad k = \frac{\partial u}{\partial q}, \qquad du = h\,dp + k\,dq. \tag{12}$$

Instead of the p, q, one can introduce new variables f, g by means of the equations:

$$m^2 (1 + f^2 + g^2) = 1, \qquad p = -m\,f, \qquad q = -m\,g, \tag{13}$$

define the function:

$$w(p, q) = -\frac{u(p,q)}{m} = -u(p, q)\sqrt{1 + f^2 + g^2}, \tag{14}$$

and then formulate the equations:

$$y = \frac{\partial w}{\partial f}, \qquad z = \frac{\partial w}{\partial g}, \qquad x = y\,f + z\,g - w. \tag{15}$$

We regard the system (15) as the parametric representation of a surface; (x, y, z) is a surface point and the f, g are the variable parameters. We now seek the relation between the h, k, p, q of the normal sheaf to this surface. To that end, one forms the expression:

$$v = x\,m + y\,p + z\,q,$$

in which the m, p, q are to be set equal to the direction cosines of the normal at (x, y, z). Now, it follows from (15) by differentiation that:

$$dx = f\,dy + g\,dz,$$

from which, one infers that:

$$\left.\begin{gathered} \frac{m}{1} = \frac{p}{-1} = \frac{q}{-g} \\ p = -mf, \quad q = -mg, \quad m^2(1 + f^2 + g^2) = 1. \end{gathered}\right\} \tag{16}$$

From this, one further obtains:

$$\begin{aligned} v &= m\,(x - y, f - zg) \\ &= -\,m\,w(f, g), \end{aligned}$$

or, when one compares (13) and (16) and observes (14):

$$v = u(p, q),$$

from which, one deduces the equations for the normal sheaf of the surface (15) in the form:

$$h = \frac{\partial u}{\partial p}, \qquad k = \frac{\partial u}{\partial q},$$

which is identical to that of the original pair of equations (12) for the chosen sheaf. Thus, if a sheaf satisfies the condition (11) then it is surface normal, and the system (15) will deliver the equation for the surface, as long as the function $u(p, q)$ is obtained by the quadrature:

$$u = \int (h\,dp + k\,dq).$$

With that, we can state the following theorem:

The necessary and sufficient condition for a sheaf of rays to be surface normal is given by the equation:

$$\frac{\partial h}{\partial q} = \frac{\partial k}{\partial p}, \tag{17}$$

which likewise states that $h\,dp + k\,dq$ is a total differential.

The preceding remark, which is quite simple, in principle, already includes the solution of the problem to be treated later on; from here, we proceed along a direct and long-established path to the properties of the eikonal.

Since parallel surfaces have the same normal sheaf, a surface normal sheaf is always associated with a family of parallel surfaces whose common normals generate the sheaf. As usual, we shall refer to each one of these surfaces as a *wave surface*. If one appeals to the known theorems on surface curvature then one will immediately obtain a series of properties of the surface normal sheaf. It does not then seem necessary to specify these theorems here; they are all connected with the remark that the caustic of such a sheaf is likewise the surface of the curvature center of its wave surface.

The reasoning up to now related to rays in a single space; we now go on to the consideration of two spaces at once. This might provoke two remarks of an extraneous nature. For the sake of clarity, I will often denote partial derivatives by simply adding an index; this will suffice whenever the schema for the sense of the indices is invoked in the form:

$$d\varphi(x, y, z, \dots) = \varphi_1\,dx + \varphi_2\,dy + \varphi_3\,dz + \dots$$

Furthermore, when it is appropriate, the following self-explanatory notation for the determinants that appear shall be employed:

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = (A\,B)_{12}\,, \qquad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = (A\,B\,C)_{123}, \text{ etc.,}$$

which is also useful for double indices; e.g.:

$$\begin{vmatrix} A_{\alpha\gamma} & B_{\beta\gamma} \\ A_{\alpha\delta} & B_{\beta\delta} \end{vmatrix} = (A_\alpha\,B_\beta)_{\gamma\delta}\,, \qquad \text{etc.}$$

II.

Fundamental equations for a ray-wise map.

For the simultaneous consideration of two spaces ω and Ω, the quantities in ω and Ω that are associated with or "conjugate to" each other will be consistently denoted by the corresponding small and large symbols, respectively. Each of the two spaces will be referred to its own provisionally arbitrarily chosen system of axes (x, y, z) and (X, Y, Z), σ and Σ will be rectilinear rays whose coordinates (h, k, p, q) and (H, K, P, Q) will be defined as in the previous section, while m and M will mean the direction cosines along the x and X axes, resp. The spaces ω and Ω are the representatives of the first and last medium of any optical system, which, following the nomenclature, we will also call the *object space* and *image space*. The main characteristics of the customary geometric theory of optical maps now consist in the facts that the two spaces will be regarded, not as they are for most physical problems – namely, the totality of points in a μ_3 – but as a μ_4 of lines, and that one associates the lines σ and Σ that appear as spatial elements with each other pair-wise. We have referred to this arrangement above as the ray-wise map of ω onto Ω; any line σ in object space determines a line Σ in image space, and conversely. The H, K, P, Q are thus functions of the h, k, p, q, and the pair-wise map states that a system of equations of the form:

$$\left.\begin{array}{ll} H = A(h,k,p,q) & K = B(h,k,p,q) \\ P = C(h,k,p,q) & Q = D(h,k,p,q) \end{array}\right\} \tag{18}$$

exists. As long as nothing further is assumed about the map, the A, B, C, D can – with a restriction – be any functions of the h, k, p, q. The restriction is that any Σ must also determine a σ, so the system (18) must be soluble for the h, k, p, q. Therefore, the functional determinant of the A, B, C, D, when defined in terms of the h, k, p, q, which is the expression:

$$\Delta = \begin{vmatrix} \frac{\partial A}{\partial h} & \frac{\partial B}{\partial h} & \frac{\partial C}{\partial h} & \frac{\partial D}{\partial h} \\ \frac{\partial A}{\partial k} & \frac{\partial B}{\partial k} & \frac{\partial C}{\partial k} & \frac{\partial D}{\partial k} \\ \frac{\partial A}{\partial p} & \frac{\partial B}{\partial p} & \frac{\partial C}{\partial p} & \frac{\partial D}{\partial p} \\ \frac{\partial A}{\partial q} & \frac{\partial B}{\partial q} & \frac{\partial C}{\partial q} & \frac{\partial D}{\partial q} \end{vmatrix}, \tag{19}$$

cannot vanish identically. If Δ becomes zero or infinite for a special system of values for the h, k, p, q then that will correspond to a singular point of the map; an example of this is total reflection, *inter alia*.

The σ, Σ that are associated with each other by (18) are conjugate elements of the map; all of the other conjugate structures are composed from them. When it is preferable, we will also combine the two mutually conjugate rays into a single notion of the "light path;" the course of a light path is then given by its two components σ and Σ in object space and image space. As before, we refer to a μ_1 or μ_2 of light paths as a *family* or a *sheaf*, respectively.

In general, the properties of a family or a sheaf of rays that are important for the applications are lost under the map. Thus, the families or sheaves that are homocentric in ω are no longer homocentric, but astigmatic, in Ω, and the same thing is true for the transition from Ω to ω. If a family of light paths is homocentric in both spaces then we would like to speak of a *conical* family, because the σ and Σ will then define conical coverings (Kegelmäntel). The connecting points – or vertices – of the conjugate families of rays define a pair of conjugate conical points. In each of the two spaces ω and Ω, the conical points define a μ_0 , μ_1 , μ_2 , μ_3 , while the conical point-pairs can rise to a μ_4 , as the example of the refraction at a plane teaches us. For systems of prisms, one's efforts are usually directed towards generating lines at conical points.

If a sheaf of light paths is homocentric in both spaces then, as we mentioned earlier, we call it *anastigmatic*; the vertices of the two conjugate sheaves of rays will then define a pair of conjugate anastigmatic points. This type of point can appear as isolated or define a μ_1 , μ_2 , μ_3 . If the latter is the case then we shall speak of anastigmatic lines or surfaces or bodies. The anastigmatic relationship between lines, surfaces, or bodies likewise implies a point-wise map of these spatial forms onto each other. For systems of lenses, most of the attention is directed towards at least generating anastigmatic surface-pairs, since, as one may show, anastigmatic bodies are compatible with the properties of isotropic media only in a single trivial case. Moreover, by the conical relation, a point-wise map can come about between the conical points of the manifolds in question, so the loci of conical points in ω and Ω must be of the same dimension as the manifold of point-pairs, which does not always need to be the case, as the aforementioned example of a refracting plane teaches us.

If the infinitely-distant planes of the object space and image space define an anastigmatic surface-pair then we shall call the map *telescopic*. Obviously, all sheaves of parallel σ would then generate sheaves of parallel Σ in image space.

We will also apply the epithet "anastigmatic" to elementary sheaves, as long as they – necessarily by neglecting quantities of higher order – may be regarded as homocentric in ω and Ω simultaneously. For such sheaves, the focal lines on both sides come together at a point, as the behavior of the cross-sectional ellipses that were treated above shows.

The properties of a sheaf that are absent for an arbitrary map in general also include the relationship between ray and wave normals. In order for a surface normal σ-sheaf to generate such a Σ-sheaf in the image space, the partial derivatives of the *A*, *B*, *C*, *D* in (18) must fulfill no less than five equations of condition, as one may show.

III.

The MALUS condition in its first and second form.

When nothing further is given for a ray-wise map than the system of equations:

$$\left.\begin{aligned} H &= A(h,k,p,q), \quad K = B(h,k,p,q), \\ P &= C(h,k,p,q), \quad Q = D(h,k,p,q), \end{aligned}\right\} \tag{20}$$

which is indeed the analytical expression for any map, the investigation must be restricted in scope to certain properties of a ray structure in object space that are changed by the map, where the latter objective would constitute a proper classification. For the elementary sheaf whose features are exhausted by being given the focal lines, the question may answered without difficulty after one changes it, and I would like to at least present the result. One thinks of the sheaf-pair h and k as being represented by functions of p, q, and likewise H, K, by functions of the P, Q, and further sets:

$$dh = h_1\,dp + h_2\,dq, \qquad dk = k_1\,dp + k_2\,dq,$$
$$dH = H_1\,dP + H_2\,dQ, \qquad dK = K_1\,dP + K_2\,dQ,$$

$$h_1\,k_2 - h_2\,k_1 = (h\,k)_{12} = l, \qquad (H_1\,K_2 - H_2\,K_1) = (H\,K)_{12} = L,$$

$$dA = A_1\,dh + A_2\,dk + A_3\,dp + A_4\,dq, \qquad dB = B_1\,dh + B_2\,dk + B_3\,dp + B_4\,dq,$$
$$dC = C_1\,dh + C_2\,dk + C_3\,dp + C_4\,dq, \qquad dD = D_1\,dh + D_2\,dk + D_3\,dp + D_4\,dq,$$

and finally, if F, G mean any two of the four functions A, B, C, D, let:

$$[F\,G] = l(F\,G)_{12} + h_1(F\,G)_{14} + h_2(F\,G)_{31} + k_1(F\,G)_{24} + k_2(F\,G)_{23} + (F\,G)_{34}\,,$$

so one has:

$$\left.\begin{aligned} H_1[CD] &= [AD], \quad H_2[CD] = [CA], \\ K_1[CD] &= [BD], \quad K_2[CD] = [CB], \\ L[CD] &= [AB]. \end{aligned}\right\} \tag{21}$$

The quantities H_1, H_2, K_1, K_2, L are then piecewise linear functions of the h_1, h_2, k_1, k_2, l with a common denominator. Since the focal lines are determined by these two sequences of quantities, as we saw earlier, the system of equations (21) will answer the question of the change that the focal lines will experience under the map. I would not like to dwell further on the argument that follows from this, but turn myself to that class of maps that are, for the time being, of interest to geometrical optics alone. From now on, we consider only such maps that satisfy the aforementioned MALUS condition: The A, B, C, D shall thus be treated as the *surface normal sheaf in object space for which the map preserves surface normals*. This condition requires that the expression:

$$H\,dP + K\,dQ \tag{22}$$

must be a total differential, or, what amounts to the same thing, that it must be integrable, as long as this is the case with the expression:

$$h\,dp + k\,dq, \tag{23}$$

when the h and k have also been chosen to be functions of the p, q, as well. Since the integrability of (22) does not change when the H, K, P, Q are replaced with the A, B, C, D, instead of (22), we can also write:

$$A\,dC + B\,dD = a\,dp + b\,dq, \tag{24}$$

where:

$$\left.\begin{aligned} a &= A(C_1h_1 + C_2k_1 + C_3) + B(D_1h_1 + D_2k_1 + D_3), \\ b &= A(C_1h_2 + C_2k_2 + C_4) + B(D_1h_2 + D_2k_2 + D_4). \end{aligned}\right\} \tag{25}$$

The integrability condition for (24) is:

$$\frac{\partial b}{\partial p} - \frac{\partial a}{\partial q} = 0, \tag{26}$$

in which the h, k are thought of as functions of the p, q. The development of (26) gives, when suitably reduced:

$$\begin{aligned} 0 = l[(A\,C)_{12} + (B\,D)_{12}] &+ h_1[(A\,C)_{14} + (B\,D)_{14}] + h_2[(A\,C)_{31} + (B\,D)_{31}] \\ &+ k_1[(A\,C)_{24} + (B\,D)_{24}] + k_2[(A\,C)_{32} + (B\,D)_{32}] + (A\,C)_{34} + (B\,D)_{34}\,. \end{aligned}$$

The right-hand side will now vanish as long as the expression (23) is integrable. The vanishing must then result as long as one sets:

$$h = \frac{\partial v}{\partial p}, \qquad h_1 = \frac{\partial^2 v}{\partial p^2}, \qquad h_2 = \frac{\partial^2 v}{\partial p\,\partial q},$$
$$k = \frac{\partial v}{\partial q}, \qquad k_1 = \frac{\partial^2 v}{\partial p\,\partial q}, \qquad k_2 = \frac{\partial^2 v}{\partial q^2},$$

if $v(p, q)$ means an entirely arbitrary function. This is, however, possible only if the five conditions:

$$0 = (A\,C)_{12} + (B\,D)_{12}, \qquad 0 = (A\,C)_{14} + (B\,D)_{14},$$
$$0 = (A\,C)_{32} + (B\,D)_{32}, \qquad 0 = (A\,C)_{34} + (B\,D)_{34},$$
$$0 = (A\,C)_{31} + (B\,D)_{31} + (A\,C)_{24} + (B\,D)_{24}$$

are satisfied identically; i.e., for arbitrary, mutually independent values of the h, k, p, q. We write the conditions thus found, with the introduction of the auxiliary quantity:

$$E = (A\,C)_{13} + (B\,D)_{13} = (A\,C)_{24} + (B\,D)_{24}, \tag{27}$$

in the form:

$$\left.\begin{array}{ll}(AC)_{34}+(BD)_{34}=0 & (AC)_{12}+(BD)_{12}=0\\ (AC)_{42}+(BD)_{42}=-E & (AC)_{13}+(BD)_{13}=E\\ (AC)_{23}+(BD)_{23}=0 & (AC)_{14}+(BD)_{14}=0\end{array}\right\} \tag{28}$$

and call this system the first form of the Malus equations of condition; they include everything there is, as long as no further conditions are added that are allowed on the grounds of Malus's theorem.

With the system (28), we arrive at an extensively researched realm that is referred to as the study of *contact transformations*. For the further investigation, it is then only necessary for us to simply carry over the theorems of this study to our present study. Since contact transformations have still not reached a state of ironclad constitution in the textbooks, the results that are necessary here will be derived directly.

If one denotes eight arbitrarily-chosen quantities by:

$$F_1, F_2, F_3, F_4 \qquad \text{and} \qquad G_1, G_2, G_3, G_4,$$

and one linearly couples equations (28) with each other by means of six multipliers:

$$\begin{array}{ll}(F\,G)_{12}, & (FG)_{34},\\ (F\,G)_{13}, & (FG)_{42},\\ (F\,G)_{14}, & (FG)_{23}\end{array}$$

then one will obtain the combined equation:

$$(F\,G\,A\,C)_{1234} + (F\,G\,B\,D)_{1234} = -E[(F\,G)_{12} + (F\,G)_{24}],$$

from which, the original equations can once more emerge by specializing the F, G.

If one replaces the symbol-pair FG with the pairs:

$$AB, \quad AD, \quad CB, \quad CD, \quad AC, \quad BD,$$

successively then one will obtain the six new equations:

$$\begin{array}{ll}E[(A\,B)_{13} + (A\,B)_{24}] = 0, & E[(A\,D)_{13} + (A\,D)_{24}] = 0,\\ E[(C\,B)_{13} + (C\,B)_{24}] = 0, & E[(C\,D)_{13} + (C\,D)_{24}] = 0,\end{array}$$

$$\begin{aligned}E[(A\,C)_{13} + (A\,C)_{24}] &= -(A\,B\,C\,D)_{1234},\\ E[(B\,D)_{13} + (B\,D)_{24}] &= -(B\,D\,A\,C)_{1234}.\end{aligned}$$

The last two equations give, when summed:

$$2(A\,B\,C\,D)_{1234} = E[(A\,C)_{13} + (B\,D)_{13} + (A\,C)_{24} + (B\,D)_{24}],$$

from which, on account of (27), it follows that the functional determinant of the map is:

$$\Delta = (A\,B\,C\,D)_{1234} = E^2. \tag{29}$$

Since Δ may not vanish identically, one can now also write the equations thus found as:

$$\left.\begin{array}{ll} (AB)_{13} + (AB)_{24} = 0 & (BC)_{13} + (BC)_{24} = 0 \\ (AC)_{13} + (AC)_{24} = 0 & (BD)_{13} + (BD)_{24} = 0 \\ (AD)_{13} + (AD)_{24} = 0 & (CD)_{13} + (CD)_{24} = 0. \end{array}\right\} \tag{30}$$

Starting from this system, we can once more go backwards and arrive at the conditions (28). To that end, if u and v mean any two functions of the h, k, p, q then we next introduce the symbol (u, v) by the equation:

$$(u, v) = \left(\frac{\partial u}{\partial h}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial h}\right) + \left(\frac{\partial u}{\partial k}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial k}\right),$$

or, when it is written with indices:

$$(u, v) = (u\,v)_{13} + (u\,v)_{24}\,. \tag{31}$$

With that, system (30) can be put into the form:

$$\left.\begin{array}{c} (A, B) = (A, D) = (C, B) = (C, D) = 0, \\ (A, C) = (B, D) = E. \end{array}\right\} \tag{32}$$

One now thinks of the four mapping functions A, B, C, D as being subject to either the condition (30) or the condition (32), with the additional demand that the functional determinant:

$$\Delta = (A\,B\,C\,D)_{1234}$$

does not vanish, and E is a possibly indeterminate function of the h, k, p, q. If F and G mean any two functions of the variables h, k, p, q, and one defines, with the partial derivatives F_1, F_2, …, the determinant product:

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{vmatrix} \times \begin{vmatrix} F_3 & F_4 & -F_1 & -F_2 \\ G_3 & G_4 & -G_1 & -G_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

then this is, on the one hand, equal to $\Delta(F\,G)_{34}$, and, on the other hand, when one multiplies it out, it is equal to the determinant:

$$\begin{vmatrix} (AF) & (AG) & A_3 & A_4 \\ (BF) & (BG) & B_3 & B_4 \\ (CF) & (CG) & C_3 & C_4 \\ (DF) & (DG) & D_3 & D_4 \end{vmatrix}.$$

If one replaces the symbol-pair FG with the sequence of six pairs:

$$AB, \quad AC, \quad AD, \quad BC, \quad BD, \quad CD$$

then, with the abbreviation $\delta = \Delta - E^2$, one will obtain the six equations:

$$0 = \delta(A\ B)_{34} = \delta(A\ C)_{34} = \delta(A\ D)_{34} = \delta(B\ C)_{34} = \delta(B\ D)_{34} = \delta(C\ D)_{34} ,$$

from which, it follows that:

$$\delta = 0, \qquad E^2 = \Delta, \tag{33}$$

because the simultaneous vanishing of the six determinants $(AB)_{34}$, …, and also the vanishing of Δ, would be deduced from it. If one now further couples the six conditions (30) with each other linearly by means of the six multipliers:

$$\begin{matrix} (C\ D)_{\alpha\beta} , & (A\ D)_{\alpha\beta} , \\ (D\ B)_{\alpha\beta} , & (C\ A)_{\alpha\beta} , \\ (B\ C)_{\alpha\beta} , & (A\ B)_{\alpha\beta} , \end{matrix}$$

where the α, β mean any two of the indices 1, 2, 3, 4, then one will obtain the combined equation:

$$(A\ B\ C\ D)_{13\alpha\beta} + (A\ B\ C\ D)_{24\alpha\beta} = E[(D\ B)_{\alpha\beta} + (C\ A)_{\alpha\beta}],$$

which yields the earlier conditions (28), with no further assumptions, as long as one replaces the index pair $\alpha\beta$ with the pairs 12, 13, 14, 23, 24, 34, in succession. With that, the conditions (30) and (28) are equivalent to each other.

We refer to the system:

$$\left.\begin{aligned} 0 = (A, B) = (A, D) = (C, B) = (C, D) \\ E = (A, C) = (B, D) \end{aligned}\right\} \tag{34}$$

as the second form of the Malus conditions.

For the three arbitrary functions u, v, w of h, k, p, q, the following well-known identity is true:

$$(u, (v, w)) + (v, (w, u)) + (w, (u, v)) = 0,$$

as is verified by direct computation. If one replaces the u, v, w in this with A, B, C then, from (34), it will become:

$$(A, 0) + (B, (C, A)) + (C, 0) = 0,$$

from which, $(E, B) = 0$. If one treats the three other combinations ABD, ACD, BCD in the same way then one will obtain, in all, four conditions:

$$0 = (E, A) = (E, B) = (E, C) = (E, D). \tag{35}$$

If one thinks of the function u as not being expressed in terms of the h, k, p, q directly, but in terms of any sort of coupling φ, ψ, … of those variables, then the relation:

$$(u, v) = \frac{\partial u}{\partial \varphi}(\varphi, v) + \frac{\partial u}{\partial \psi}(\psi, v) + \ldots$$

will follow from the definition of the symbol (u, v). If one thinks of the variables in $u = f(h, k, p, q)$ as being expressed in terms of the H, K, P, Q by means of the mapping equations then one will obtain an expression $g(H, K, P, Q)$; if one introduces A, B, C, D into this in place of the H, K, P, Q then one will arrive at the identity transformation:

$$u = f(h, k, p, q) = g(A, B, C, D).$$

Upon considering such a transformation, one has:

$$(u, A) = \frac{\partial u}{\partial A}(A, A) + \frac{\partial u}{\partial B}(B, A) + \frac{\partial u}{\partial C}(C, A) + \frac{\partial u}{\partial D}(D, A).$$

One obtains similar equations when one chooses B, C, or D to be the second element in the bracket symbol. With hindsight of (34), it follows from this that:

$$\left.\begin{array}{ll} (u, A) = -E\dfrac{\partial u}{\partial C} & (u, B) = -E\dfrac{\partial u}{\partial D} \\ (u, C) = E\dfrac{\partial u}{\partial A} & (u, D) = E\dfrac{\partial u}{\partial B}. \end{array}\right\} \tag{36}$$

With this, equations (35) give:

$$0 = \frac{\partial E}{\partial A} = \frac{\partial E}{\partial B} = \frac{\partial E}{\partial C} = \frac{\partial E}{\partial D}; \tag{37}$$

i.e., E is either independent of the h, k, p, q or it is constant.

IV.

The spatial indices and their relation to the refraction quotients.

By the introduction of the MALUS condition, we are only required to insure that every surface normal σ-sheaf in object space becomes a similar sheaf in image space; one will not, however, arrive at the fact that this property can be inverted with no further assumptions, so each surface normal Σ-sheaf is conjugate to such a sheaf in object space. It will be shown that this invertibility is a necessary consequence of the original assumption. If one employs the H, K, P, Q in place of the h, k, p, q as the independent variables in the symbol (u, v) then this will be indicated by a prime thus:

$$(u, v)' = \left(\frac{\partial u}{\partial H}\frac{\partial v}{\partial P} - \frac{\partial u}{\partial P}\frac{\partial v}{\partial H}\right) + \left(\frac{\partial u}{\partial K}\frac{\partial v}{\partial Q} - \frac{\partial u}{\partial Q}\frac{\partial v}{\partial K}\right). \tag{37}$$

One then obtains:

$$\begin{aligned}(u, v) &= \frac{\partial u}{\partial H}(H, v) + \frac{\partial u}{\partial K}(K, v) + \frac{\partial u}{\partial P}(P, v) + \frac{\partial u}{\partial Q}(Q, v) \\ &= \frac{\partial u}{\partial H}(A, v) + \frac{\partial u}{\partial K}(B, v) + \frac{\partial u}{\partial P}(C, v) + \frac{\partial u}{\partial Q}(D, v),\end{aligned}$$

and from this, with (36):

$$(u, v) = E\left(\frac{\partial u}{\partial H}\frac{\partial v}{\partial C} - \frac{\partial u}{\partial P}\frac{\partial v}{\partial A}\right) + E\left(\frac{\partial u}{\partial K}\frac{\partial v}{\partial D} - \frac{\partial u}{\partial Q}\frac{\partial v}{\partial B}\right).$$

When one writes H, K, P, Q for the A, B, C, D that they are analogous to, this equation gives:

$$(u, v) = E(u, v)', \tag{38}$$

or, when one sets $e = 1 : E$:

$$(u, v)' = e(u, v). \tag{39}$$

If one substitutes all pair-wise couplings of the h, k,p, q for u and v, and computes (u, v) using the original defining equation then one will get:

$$\left.\begin{array}{c}(h, k)' = (h, q)' = (p, k)' = (p, q)' = 0 \\ (h, p)' = (k, q)' = e.\end{array}\right\} \tag{40}$$

If one considers the original mapping equations:

$$H = A, \qquad K = B, \qquad P = C, \qquad Q = D$$

as having been solved for the h, k, p, q and written in the form:

$$\begin{array}{ll} h = a(H, K, P, Q), & k = b(H, K, P, Q), \\ p = c(H, K, P, Q), & q = d(H, K, P, Q) \end{array}$$

then the ray-wise map from Ω to ω, or the original inverse map, will be expressed by it, and the system (40) will go to:

$$\left.\begin{array}{c}(a,b)' = (a,d)' = (c,b)' = (c,d)' = 0 \\ (a,c)' = (b,d)' = e.\end{array}\right\} \tag{41}$$

This is, however, the MALUS condition for the inverse map, and indeed in the second form. We can then say: If the map from ω to Ω satisfies MALUS's theorem then this will also be true immediately for the inverse map, or the map from Ω to ω; the constant E goes into the reciprocal value under the inverse map.

The quantity E then takes the form of the square root of the functional determinant Δ and, like the functions A, B, C, D, can depend upon the choice of coordinate axes. In order to investigate this, we think of there being a second system of axes (X' Y' Z') and denote the space that it refers to by Ω'; correspondingly, all quantities that refer to Ω' shall take on a prime. If one introduces the symbol ($\omega\Omega$) for the map from ω to Ω, for the moment, then we will next have three maps ($\omega\Omega$), ($\omega\Omega'$), ($\Omega\,\Omega'$), and the three inverses ($\Omega\,\omega$), ($\Omega'\omega$), ($\Omega'\,\Omega$) that are associated with them. Since MALUS's theorem shall be valid for the first map ($\omega\Omega$), it must then be true for the other five, with no further assumptions. If one denotes the corresponding E-constants with $E(\omega\Omega)$, $E(\omega\Omega')$, … then one has:

$$1 = E(\omega\Omega)\, E(\Omega\omega) = E(\omega\Omega')\, E(\Omega'\omega) = E(\Omega\,\Omega')\, E(\Omega'\Omega). \tag{42}$$

If one poses the mapping equations for ($\omega\Omega'$) in the form:

$$\left.\begin{array}{ll}H' = A'(h,k,p,q), & K' = B'(h,k,p,q), \\ P' = C'(h,k,p,q), & Q' = D'(h,k,p,q),\end{array}\right\} \tag{43}$$

and recalls the meaning of the symbol $(u, v)'$ then one will have:

$$E(\omega\Omega) = (H, P), \qquad E(\omega\Omega') = (H', P'), \qquad E(\Omega\,\Omega') = (H', P')',$$

from which, with (38), it will follow that:

$$E(\omega\Omega') = E(\omega\Omega') \cdot E(\Omega\,\Omega'). \tag{44}$$

In order to ascertain the effect of a change in the coordinate axes (X Y Z), one must calculate the symbol $(H', P')'$.

If we now subject the coordinate axes to a parallel displacement, under which the new origin will be shifted to:

$$X = a, \quad Y = b, \quad Z = c,$$

then one will have:

$$X' = X - a, \qquad Y' = Y - b, \qquad Z' = Z - c.$$

We pose the equations of one and the same ray, as referred to Ω and Ω', namely:

$$Y = H + \frac{PX}{M}, \qquad Z = K + \frac{QX}{M},$$
$$Y' = H' + \frac{P'X'}{M'}, \qquad Z' = K' + \frac{Q'X'}{M'},$$

so one has:

$$\left.\begin{aligned} M' = M \quad P' = P \quad Q' = Q \\ H' = H + \frac{Pa}{M} - b \quad K' = K + \frac{Qa}{M} - c. \end{aligned}\right\} \tag{45}$$

If one calculates $(H', P')'$ then it becomes:

$$(H', P')' = 1,$$

such that a displacement of the coordinate axes in Ω will not change the value of E.

If we now consider a rotation around the origin then we will have to set:

$$\begin{aligned} X' &= \alpha X + \beta Y + \gamma Z, & M' &= \alpha M + \beta P + \gamma Q, \\ Y' &= \alpha' X + \beta' Y + \gamma' Z, & P' &= \alpha' M + \beta' P + \gamma' Q, \\ Z' &= \alpha'' X + \beta'' Y + \gamma'' Z, & Q' &= \alpha'' M + \beta'' P + \gamma'' Q, \end{aligned}$$

where the α, β, γ, … mean the direction cosines of the new axes (X', Y', Z') with respect to the old ones (X, Y, Z). If one writes the equations for a ray, when referred to the old axes, in the form:

$$X = \rho M, \qquad Y = \rho P + H, \quad Z = \rho Q + K$$

then one will have:

$$\begin{aligned} X' &= \rho M' + \beta H + \gamma K = \rho' M', \\ Y' &= \rho P' + \beta' H + \gamma' K = \rho' P' + H', \\ Z' &= \rho Q' + \beta'' H + \gamma'' K = \rho' Q' + K', \end{aligned}$$

from which, it will follow that:

$$\left.\begin{aligned} H' &= \beta' H + \gamma' K - \frac{P'}{M'}(\beta H + \gamma K), \\ K' &= \beta'' H + \gamma'' K - \frac{Q'}{M'}(\beta H + \gamma K). \end{aligned}\right\} \tag{46}$$

Substitution in $(H', P')' = 1$ gives, after an appropriate reduction, in turn:

$$(H', P')' = 1;$$

thus, E also remains unchanged under a rotation. If one then follows through the same argument for the inverse maps and the coordinate change in ω then, with consideration of (42), one will come to the conclusion that the constant E is independent of the choice of coordinate axes, so it only relates to the other properties of the map $(\omega\Omega)$.

If the arbitrary maps $(\omega\Omega)$, $(\Omega\,\Omega')$ are given for the three spaces ω, Ω, and Ω' then the composed map $(\omega\Omega')$ and the three inverse maps will be determined by them. If two maps that are not inverse to each other satisfy the MALUS conditions then this will also be true for the remaining maps. For the constant E, (38) yields:

$$E(\omega\Omega') = E(\omega\Omega) \cdot E(\Omega\,\Omega'). \tag{47}$$

This relation may be extended immediately. One thinks of a series of n spaces ω_1, ω_2, …, ω_n as given, with each one being mapped to the next one, while fulfilling the MALUS condition, so the composed maps $(\omega_\alpha\;\omega_\beta)$, which also satisfy the MALUS condition, are determined by way of the maps $(\omega_1\;\omega_2)$, $(\omega_2\;\omega_3)$, …, $(\omega_{n-1}\;\omega_n)$, and one obtains by repeated application of (47):

$$E(\omega_1\;\omega_n) = E(\omega_1\;\omega_2) \cdot E(\omega_2\;\omega_3) \ldots E(\omega_1\;\omega_n). \tag{48}$$

One now thinks of each space ω_n as being associated with a certain constant "index" Γ_α, and chooses the numerical value of this spatial index such that:

$$E(\omega_1\;\omega_2) = \frac{\Gamma_1}{\Gamma_2}, \qquad E(\omega_2\;\omega_3) = \frac{\Gamma_2}{\Gamma_3}, \qquad \ldots, E(\omega_{n-1}\;\omega_n) = \frac{\Gamma_{n-1}}{\Gamma_n},$$

in which obviously one of the Γ's – e.g., Γ_1 – can be chosen arbitrarily. It then follows from (48) that:

$$E(\omega_1\;\omega_n) = \frac{\Gamma_1}{\Gamma_n},$$

or, more generally, that:

$$E(\omega_\alpha\;\omega_\beta) = \frac{\Gamma_\alpha}{\Gamma_\beta}. \tag{49}$$

In the case of optics, the indices Γ are connected in a simple way with the refraction exponents of the individual media or spaces. An optical system is, for our considerations, regarded as a series of spaces ω_1, ω_2 , … that are bounded by definite refracting or reflecting surfaces; the individual refractions or reflections then generate the ray-wise map of each individual space to the one that immediately follows it. Since a reflection can be regarded as a refraction with the refraction ratio – 1, it suffices to treat the case of a single refraction; the relations (47) to (49) then immediately given the values of the Γ for a series of refractions. If one therefore now thinks of the object space ω and the image space Ω as adjacent to each other along a surface Φ, and that furthermore n and N are the indices of refraction of the two spaces then one will have to calculate the quantity:

$$E = (H, P),$$

which is independent of the ray coordinates, as well as the position of the coordinate axes. If one lets the axes (x, y, z) and (X, Y, Z) coincide for both spaces, lays the x-axis along the normal to a point π of Φ, the base plane in the tangent plane to π and the lateral axes in the directions of the principal curvatures then one can express the equation of the surface in the form:

$$x = \varphi(x, y) = \frac{1}{2}(\alpha y^2 + \beta z^2) + \gamma y^2 + \ldots,$$

from which, we then define:

$$dx = \varphi_1\, dy + \varphi_2\, dz, \qquad \varphi_1 = a\, y + \ldots, \;\; \varphi_2 = b\, z + \ldots$$

The incident ray σ, the refracted ray Σ, and the incident perpendicular pass through a given point (x, y, z) of the surface. From the law of refraction, the expressions:

$$NM - nm, \qquad NP - np, \qquad NQ - nq$$

are proportional to the direction cosines of the incident perpendicular, and these will be, in turn, proportional to:

$$1, -\varphi_1, -\varphi_2,$$

so one may set:

$$\frac{NM - nm}{1} = \frac{NP - np}{-\varphi_1} = \frac{NQ - nq}{-\varphi_2},$$

or:

$$NM = nm + \lambda, \qquad NP = np - \lambda\,\varphi_1, \qquad NQ = nq - \lambda\,\varphi_2 .$$

Furthermore, according to whether the point (x, y, z) is attributed to one ray or the other, one will have:

$$y = h + \frac{xp}{m} = H + \frac{xP}{M}$$

or:

$$z = k + \frac{xq}{m} = K + \frac{xQ}{M},$$

resp.

The differentiation of these equations gives:

$$N\,dM = n\,dm + d\lambda,$$
$$N\,dP = n\,dp - \varphi_1\,d\lambda - \lambda\,d\varphi_1,$$
$$N\,dQ = n\,dq - \varphi_1\,d\lambda - \lambda\,d\varphi_2,$$

$$dy = dh + \frac{p}{m}dx + x\,d\left(\frac{p}{m}\right) = dH + \frac{P}{M}dx + x\,d\left(\frac{P}{M}\right),$$

$$dz = dk + \frac{q}{m}dx + x\,d\left(\frac{q}{m}\right) = dK + \frac{Q}{M}dx + x\,d\left(\frac{Q}{M}\right),$$

from which, one obtains the equations:

$$dx = \varphi_1\, dy + \varphi_2\, dz, \qquad d\varphi_1 = \alpha\, dy + \ldots, \qquad d\varphi_2 = \beta\, dz + \ldots$$

Since E is independent of the h, k, p, q, we can use any special ray as the basis for the calculation of E. We choose the ray in the x-axis to be that ray and then set:

$$p = q = P = Q = 0, \qquad m = M = 1,$$
$$h = k = H = K = 0, \qquad x = y = 0, \qquad \varphi_1 = \varphi_2 = 0,$$

and then obtain from this:

$$N = n + \lambda,$$

$$N\, dP = n\, dp - \lambda\, \alpha\, dy, \qquad N\, dQ = n\, dq - \lambda\, \beta\, dz,$$

$$dx = 0, \qquad dy = dh = dH, \qquad dz = dk = dK,$$

or:

$$dH = dh, \qquad dK = dk,$$

$$dP = -\frac{N-n}{N}\alpha\, dh + \frac{n}{N}dp\,, \quad dQ = -\frac{N-n}{N}\beta\, dk + \frac{n}{N}dq\,.$$

From this, it then follows that:

$$\frac{\partial H}{\partial h} = 1, \qquad \frac{\partial P}{\partial h} = \alpha\,\frac{n-N}{N}, \qquad \frac{\partial H}{\partial k} = 0, \qquad \frac{\partial P}{\partial k} = 0,$$

$$\frac{\partial H}{\partial p} = 0, \qquad \frac{\partial P}{\partial p} = \frac{n}{N}, \qquad \frac{\partial H}{\partial q} = 0, \qquad \frac{\partial P}{\partial q} = 0,$$

and, when one then calculates E:

$$E = (H, P) = \frac{n}{N}\,. \tag{50}$$

Therefore, if no reflections appear in an optical system with the series of media ω_1, ω_2, …, and the corresponding exponents of refraction are n_1, n_2, …, when compared to empty space, then one will have the following relation for any two indices Γ:

$$\Gamma_\alpha : \Gamma_\beta = n_\alpha : n_\beta,$$

and one can set the Γ equal to the corresponding n with no further assumptions. For the case of a reflection, one must write:

$$E = -1, \qquad n = -N,$$

in (50), and in a given optical system one may accordingly set:

$$\Gamma_\alpha = \pm\, n_\alpha \,,$$

in general, where the sign is chosen to be + or – according to whether the ray path up to the medium ω_α includes an even or odd number of reflections, respectively.

In summation, we may now state the following theorem:

If the two spaces ω *and* Ω *are mapped to each other ray-wise by the system of equations:*

$$\left.\begin{aligned} H &= A(h,k,p,q), \quad K = B(h,k,p,q), \\ P &= C(h,k,p,q), \quad Q = D(h,k,p,q) \end{aligned}\right\} \tag{51a}$$

then the necessary and sufficient condition for the fulfillment of the Malus condition is given by the system of equations:

$$\left.\begin{aligned} (H,K) = (H,Q) = (P,K) = (P,Q) = 0, \\ (H,P) = (K,Q) = E = \frac{n}{N}, \end{aligned}\right\} \tag{51b}$$

in which the indices n, N of the two spaces mean certain constants that will be essential for the map and independent of the choice of coordinate axes, and which go over to the indices of refraction of the two spaces, which are taken to be positive or negative.

If one would like to keep the mapping equations in the form (51a) for the further investigations then one will always have to consider the conditions (51b) along with them, which would make the reasoning extremely tedious. This grievance may be circumvented if one gives the mapping equations a form from the outset in which the MALUS condition is fulfilled by the form itself. The means for achieving this is the introduction of a certain generating function – viz., the eikonal – to the presentation of which, I now proceed.

V.

Definition of the eikonal. Critical determinant.

If one forms the differential expression:

$$dS = n(p\,dh + q\,dk) + N(H\,dP + K\,dQ) \tag{52}$$

from the variables of a map then if one considers the mapping equations (51a), one can write it in the form:

$$dS' = n(p\,dh + q\,dk) + N(A\,dC + B\,dD), \tag{53}$$

from which, one can derive a third form:

$$dS'' = \alpha\,dh + \beta\,dk + \gamma\,dp + \delta\,dq \tag{54}$$

by further development, in which:

$$\left.\begin{aligned} \alpha &= np + N(AC_1 + BD_1), \quad & \gamma &= N(AC_3 + BD_3), \\ \beta &= nq + N(AC_2 + BD_2), \quad & \delta &= N(AC_4 + BD_4). \end{aligned}\right\} \tag{55}$$

With that, one computes the expressions:

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial k} - \frac{\partial\beta}{\partial h} &= N(CA)_{12} + N(DB)_{12}, & \frac{\partial\beta}{\partial p} - \frac{\partial\gamma}{\partial k} &= N(CA)_{23} + N(DB)_{23}, \\ \frac{\partial\alpha}{\partial p} - \frac{\partial\gamma}{\partial h} &= N(CA)_{13} + N(DB)_{13} + n, & \frac{\partial\beta}{\partial q} - \frac{\partial\delta}{\partial k} &= N(CA)_{24} + N(DB)_{24} + n, \\ \frac{\partial\alpha}{\partial q} - \frac{\partial\delta}{\partial h} &= N(CA)_{14} + N(DB)_{14}, & \frac{\partial\gamma}{\partial q} - \frac{\partial\delta}{\partial p} &= N(CA)_{34} + N(DB)_{34}. \end{aligned}\right\} \tag{56}$$

Due to (28) and (51b), the right-hand sides of these equations vanish, so it follows that the left-hand sides are also zero; i.e., the expression dS' is a total differential, or, more briefly, dS' is integrable. This result may be inverted. One thinks the expression dS' as being formed from four possibly arbitrary functions A, B, C, D, and the two likewise possibly arbitrary constants n, N, and when dS'' is described in this way the left-hand sides of (56) will vanish if dS'' is integrable. However, the vanishing of the right-hand sides that follows from this will once more lead to the MALUS condition. The latter is then equivalent to the integrability of dS''.

The integration of dS'' delivers a certain function $F(h, k, p, q)$ for S''. We would now like to assume, for the moment, that when the last two of the mapping equations (51a):

$$P = C(h, k, p, q), \qquad Q = D(h, k, p, q) \tag{57}$$

are solved for p, q, (51a) can then be brought into the form:

$$\left.\begin{array}{ll} np = \varphi(h,k,P,Q), & nq = \psi(h,k,P,Q), \\ NH = \Phi(h,k,P,Q), & NK = \Psi(h,k,P,Q). \end{array}\right\} \tag{58}$$

If one expresses the p, q in terms of the h, k¸ P, Q then dS'' will be converted into dS, in which the integrability has not changed, while $F(h, k, p, q)$ will be converted into an expression $E(h, k, P, Q)$, and one will have:

$$dE(h, k, P, Q) = dS = n(p\,dh + q\,dk) + N(H\,dP + K\,dQ),$$

from which, it will follow that:

$$np = \frac{\partial E}{\partial h}, \qquad nq = \frac{\partial E}{\partial k}, \qquad NH = \frac{\partial E}{\partial P}, \qquad NK = \frac{\partial E}{\partial Q}. \tag{59}$$

This system must now be identical with (58), since otherwise one would obtain from the combination of (58) and (59) – i.e., as a result of the mapping equations (51a) and the conditions (51b) – at the very least, an equation of the form:

$$0 = f(h, k, P, Q),$$

which is again the assumption that was made to begin with. With that, we have the theorem: If the mapping equations can be written in the form (58) then the right-hand sides will be equal to the partial derivatives of a certain function $F(h, k, P, Q)$.

Conversely, one now thinks of a function $E(h, k, P, Q)$ as being given arbitrarily and forms the system of equations (59) from it. If E is chosen in such a way that the equations (59) can be solved for the H, K, P, Q or the h, k, p, q then a certain ray-wise map will be defined by (59). For it, the expression:

$$dS = n(p\,dh + q\,dk) + N(H\,DP + K\,dQ)$$

will be a total differential, namely, it will equal dE, so it will further follow that the dS'' that follows from dS will be integrable, so the MALUS condition will also be fulfilled. It then suffices that the mapping equations can be written in the form (59), in order to insure that the map fulfills the MALUS conditions. In this, the function E plays the role of a generating function for the mapping equations; from now on, we will refer to such a function as the *eikonal* of the map in question.

In the foregoing argument, it was assumed that equations (57) could be solved for the p, q. In order to recognize the possibilities that thus emerge more completely, let it first be remarked that the expression $(CD)_{34}$ is the functional determinant of the C, D with respect to the p, q. If this expression, which we will call the *critical determinant* of the eikonal $E(h, k, P, Q)$, does not vanish identically then the equations (57) are soluble for the p, q and one can put the map into the form (58), from which, the existence of the eikonal $E(h, k, P, Q)$ is likewise established. Conversely, $(CD)_{34}$ cannot vanish identically when the eikonal exists. If $(CD)_{34}$ did vanish identically then at least one relation of the form:

$$0 = f(h, k, P, Q)$$

would follow, which is incompatible with the existence of the system of equations (59). We can then say: The eikonal $E(h, k, P, Q)$ does or does not exist according to whether the critical determinant $(CD)_{34}$ does not or does vanish identically, respectively.

The determinant $(CD)_{34}$ can now vanish under certain circumstances, which can be verified most simply by an example, such:

$$H = A \equiv -p, \quad K = B \equiv -q, \quad P = C \equiv h, \qquad Q = D \equiv k, \qquad n = N.$$

The eikonal $E(h, k, P, Q)$ does not exist then, since the mapping equations cannot be solved for the H, K, p, q. In this case, however, there always exists another form for the eikonal that enters the picture in place of the missing eikonal. Namely, in the MALUS condition equations, one can perform certain exchanges in the variables h, k, p, q and H, K, P, Q, under which these equations do not change. As a consequence of this, the integrability of the differential expressions dS, dS', dS'' also remains when one performs the same exchanges in them. If one denotes a substitution that takes the quantities x_1, x_2, x_3, … into y_1, y_2, y_3, … in the usual notation by a symbol:

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ y_1 & y_2 & y_3 & \cdots \end{pmatrix}$$

then the substitutions:

$$\begin{pmatrix} h & p \\ -p & h \end{pmatrix}, \quad \begin{pmatrix} k & q \\ -q & k \end{pmatrix}, \quad \begin{pmatrix} H & P \\ -P & H \end{pmatrix}, \quad \begin{pmatrix} K & Q \\ -Q & K \end{pmatrix}, \tag{60}$$

whether they are taken individually or in combination, have the aforementioned character. They are, as one can confirm most simply by direct trial and error, likewise the only ones that change dS without altering the MALUS equations. The application of the substitutions (60) to dS then produces 16 different forms, which, if we make the abbreviation:

$$\frac{p, q, H, K}{h, k, P, Q} = n(p\,dh + q\,dk) + N(H\,dP + K\,dQ), \tag{61}$$

we can summarize in the following table:

$$\left.\begin{array}{llll}
[1]=\dfrac{p,q,H,K}{h,k,P,Q} & [2]=\dfrac{h,q,H,K}{-p,k,P,Q} & [3]=\dfrac{p,k,H,K}{h,-q,P,Q} & [4]=\dfrac{h,k,H,K}{-p,-q,P,Q} \\
[5]=\dfrac{p,q,-P,K}{h,k,H,Q} & [6]=\dfrac{h,q,-P,K}{-p,k,H,Q} & [7]=\dfrac{p,k,-P,K}{h,-q,P,K} & [8]=\dfrac{h,k,-P,K}{-p,-q,H,Q} \\
[9]=\dfrac{p,q,H,-Q}{h,k,P,K} & [10]=\dfrac{h,q,H,-Q}{-p,k,P,K} & [11]=\dfrac{p,k,H,-Q}{h,-q,P,K} & [12]=\dfrac{h,k,H,-Q}{-p,-q,P,K} \\
[13]=\dfrac{p,q,-P,-Q}{h,k,H,K} & [14]=\dfrac{h,q,-P,-Q}{-p,k,H,K} & [15]=\dfrac{p,k,-P,-Q}{h,-q,H,K} & [16]=\dfrac{h,k,-P,-Q}{-p,-q,H,K}.
\end{array}\right\} \tag{62}$$

Under an application of the substitutions (60), these sixteen formulas go into each other, and thus define a closed group. In order to the find the corresponding critical determinants, one must perform the same substitutions in $(CD)_{34}$ that take the eikonal form [1] into the fifteen other forms. This gives the following table for the critical determinants, which corresponds to the summary (62) term-by-term:

$$\left.\begin{array}{llll}
(CD)_{34} & (CD)_{14} & (CD)_{23} & (CD)_{12} \\
(AD)_{34} & (AD)_{14} & (AD)_{23} & (AD)_{12} \\
(CB)_{34} & (CB)_{14} & (CB)_{23} & (CB)_{12} \\
(AB)_{34} & (AB)_{14} & (AB)_{23} & (AB)_{12}.
\end{array}\right\} \tag{63}$$

Whenever an eikonal form is absent from (62) because it is not possible, a zero appear in the corresponding location in (63), and conversely. We would now like to show that no more than three zeroes can appear in a row or column of (63). For instance, if one has:

$$(CD)_{34} = (CD)_{14} = (CD)_{23} = (CD)_{12} = 0$$

then, due to the identity:

$$(CD)_{12}\,(CD)_{34} + (CD)_{13}\,(CD)_{42} + (CD)_{14}\,(CD)_{23} \equiv 0,$$

one would also have:

$$(CD)_{12}\,(CD)_{34} = 0,$$

which, when coupled with the MALUS condition equation:

$$0 = (CD) = (CD)_{13} + (CD)_{24}\,,$$

would immediately give:

$$(CD)_{12} = (CD)_{34}\,,$$

i.e., the six determinants that one forms from the C_α and D_α would all be zero, from which, the vanishing of:

$$\Delta = (ABCD)_{1234} = 0$$

would follow. Since this is inadmissible, and since one may, moreover, immediately carry over the line of reasoning employed to the remaining rows and columns of (63) by an application of (60), this would yield the fact that at least one term in every row and column of (63) must be different from zero. Correspondingly, at least one possible eikonal form appears in each row and column of (62), such that the number of the eikonals that actually exist for a given map amounts to at least four. One can see the fact that the remaining maps can produce only four eikonals most simply by a concrete example, such as, for instance:

$$H = A \equiv h, \qquad K = B \equiv k, \qquad P = C \equiv p, \qquad Q = D \equiv q.$$

If one examines the cases with only four eikonals more closely then, as I would like to discuss shortly, the maps for the eikonal [1] will take the form:

$$E(h, k, P, Q) = P(\alpha h + \beta) + Q(\gamma k + \delta) + \varepsilon h + \zeta k + \eta,$$

where the a, b, … mean constants; the eikonals are summarized for the other possible cases correspondingly.

Since the vanishing of a critical determinant imposes a special condition on the A, B, C, D by way of MALUS's theorem, one can say that *in general* – i.e., under an arbitrarily selected map – all sixteen eikonals will be present. The sixteen forms do not all generally possess the same value in a given case for the application. The lateral axes of the coordinates appear in the eikonals [1], [4], [13], [16] in the same way, whether in the line segments h, k or H, K, or the direction quantities p, q or P, Q. These forms will then have an advantage when the map has no essential peculiarities in the various directions around the x-axis. Examples of this are: the ordinary system of lines, the water droplets in rainbows, the Earth's atmosphere, etc. For the remaining twelve eikonal forms, a noteworthy exception comes about in regard to the occurrence of the variables between the lateral axes in at least one of the two spaces ω, Ω. These forms can be advantageous when the map itself also exhibits corresponding exceptions along the lateral axes. Examples of this are systems of prisms and cylindrical lenses. For the investigations of a general nature, one usually makes do with the four forms [1], [4], [13], and [16], since ultimately one can already manage with [1]. In order to overlook when one of these four forms is absent, one must inspect the geometric meaning of the vanishing of the determinants:

$$(CD)_{34}\ , \ (CD)_{12}\ , \ (AB_{34}\ , \ (AB)_{34}\ .$$

I begin with the case $(AB)_{34} = 0$, which belongs to [13]. This condition says that at least one equation of the form:

$$0 = f(h, k, H, K) \tag{64}$$

exists between the h, k, H, K. If h and k are constant then (64) is the equation of a curve in the base plane of Ω. A homocentric σ-sheaf whose vertex lies in the base plane of ω thus generates a Σ-sheaf whose rays go through a definite curve in the YZ-plane that is independent of the h, k. One of the two sheets of the caustic of the Σ-sheaf then degenerates, and indeed, into the curve in question. As a consequence, the curvature lines in the associated family of waves consist of circles. On the other hand, if one makes H and K constants then one will arrive at similar theorems, except that the roles of ω and Ω have been switched. The two base planes are thus surfaces of conical points, and there exists a μ_3 of conical point-pairs.

If $(CD)_{34} = 0$ then there exists at least one equation of the form:

$$0 = f(h, k, P, Q).$$

The homocentric sheaf "h, k constant" generates a caustic in Ω, one of whose sheets degenerates into an infinitely distant curve. As a consequence, the associated waves are developable surfaces. Furthermore, a caustic belongs to a parallel Σ-sheaf in ω, one of whose sheets degenerates into a curve in the yz-plane. The case where $(AB)_{12}$ vanishes leads to the same results as the foregoing, except that the roles of ω and Ω have been switched.

Since in the three cases treated the property that the degenerate curves must lie in the base plane can be removed immediately by a change of coordinate axes, one see that for any map one can bring about the existence of the eikonal forms [1], [13], [16] by a suitable choice of axes. Things are different for [4] when $(CD)_{12} = 0$. In this case, there exists at least one equation of the form:

$$0 = f(p, q, P, Q).$$

Parallel sheaves in ω generate a caustic in Ω with an infinitely distant degenerate curve, and the same is true for parallel Σ-sheaves in the space ω.

For further investigations, it is preferable to bring the sixteen different forms into a unified schema, which will now be presented.

VI.

Relations between the eikonals of a map. The expression Θ. Composition of eikonals.

The individual eikonal will come about by integrating the differential expressions that are summarized in (62):

$$dS = \frac{p, q, H, K}{h, k, P, Q} = n(p\,dh + q\,dk) + N(H\,dP + K\,dQ),$$

……………………………………..

The quantities beneath the line include the independent variables of the eikonal whose differentials appear in dS; the quantities above the line are the coefficients of these differentials, up to the factors $\pm n$, $\pm N$. In all sixteen cases, one obtains the first, second, third, and fourth independent variables when one selects one element from the series of four pairs of quantities:

$$(hk), \quad (pq), \quad (HK), \quad (PQ). \tag{65}$$

If one calls the sequence of chosen elements t, u, T, U then the place of the corresponding eikonal in the table (62) is determined completely by the symbol:

$$E(t, u, T, U).$$

Thus, the four variables of the sequences will likewise be associated with the four lateral axes in the order y, z, Y, Z. If one further denotes the sequence of variables that remain in (65) after the choice of t, u, T, U by v, w, V, W then one will have:

$$dE(t, u, T, U) = \pm\, n\, v\, dt \pm n\, w\, du \pm N\, V\, dT \pm N\, W\, dU,$$

in which the rule for the sign has yet to be given. Now, some of the eight variables of the map are linear quantities or line segments like h, k, H, K and some of them are direction quantities like p, q, P, Q, and one sees that the sign + or – appears in dS according to whether the differentials dt, du, dT, dU are taken from the sequence dh, dk, dP, dQ or the sequence dp, dq, dH, dK, resp. Thus, the sign $\varepsilon(x)$ takes the value + 1 or – 1 according to whether x is a line segment or direction quantity, resp., so one generally has:

$$dE(t, u, T, U) = n\, \varepsilon(t)\, v\, dt + n\, \varepsilon(t)\, w\, du - N\, \varepsilon(t)\, V\, dT - N\, \varepsilon(U)\, W\, dU, \tag{66}$$

from which, the mapping equations follow immediately in the form:

$$\left.\begin{aligned} n\,\varepsilon(t)v &= \frac{\partial E}{\partial t}, & n\,\varepsilon(u)w &= \frac{\partial E}{\partial u}, \\ -N\,\varepsilon(T)V &= \frac{\partial E}{\partial T}, & -N\,\varepsilon(U)W &= \frac{\partial E}{\partial U}. \end{aligned}\right\} \tag{67}$$

The symbol E is thus the sign for an operation whose form obviously depends upon the map in question, as well as on the choice of independent variables; the result of the operation is then a function whose form likewise depends upon the two stated things.

Equations (67) say that a contact transformation exists between any four quantities:

$$nh, nk, p, q \qquad \text{and} \qquad NH, NK, P, Q.$$

In this theorem, one finds the actual origin of all the properties to be sought that are common to the maps considered here. As long as the map remains undetermined, the expression E can possess any arbitrary form, with the restriction that the equations (67) must represent an actual map, so the system (67) must be soluble for the t, u, v, w as well as the T, U, V, W. The necessary and sufficient condition for this consists in saying that the determinant:

$$\frac{\partial^2 E}{\partial t\,\partial T}\frac{\partial^2 E}{\partial u\,\partial U}-\frac{\partial^2 E}{\partial t\,\partial U}\frac{\partial^2 E}{\partial u\,\partial T} \tag{68}$$

must vanish identically.

If $E(t, u, T, U)$ and $E(t_1, u_1, T_1, U_1)$ are two different eikonals for the same map then the initial term in the difference:

$$dE(t, u, T, U) - d\,E(t_1, u_1, T_1, U_1)$$

will contribute an amount to both dE that is equal to:

$$n\,\varepsilon(t)\,v\,dt - n\,\varepsilon(t_1)\,v_1\,dt_1\,,$$

which we write in the form:

$$n\frac{\varepsilon(t)+\varepsilon(t_1)}{2}(v\,dt - v_1 dt_1) + n\frac{\varepsilon(t)-\varepsilon(t_1)}{2}(v\,dt + v_1 dt_1)\,.$$

The product:

$$[\varepsilon(t) + \varepsilon(t_1)] \cdot [v\,dt - v_1\,dt_1]$$

will always be equal to zero since only the following two cases are possible:

$$\begin{array}{lll} \text{I:} & t = v_1, & v = t_1, \\ \text{II:} & t = t_1, & v = v_1; \end{array}$$

for I, the first factor vanishes, while for II, the second one does. If one further writes:

$$\begin{aligned}[]
[\varepsilon(t) - \varepsilon(t_1)] \cdot [v\,dt + v_1\,dt_1] &= [\varepsilon(t) - \varepsilon(t_1)] \cdot\; d(tt_1) \\
&\quad + [\varepsilon(t) - \varepsilon(t_1)] \cdot [(v - t_1)\,dt + (v_1 - t)\,dt_1]
\end{aligned}$$

then the second summand on the right will vanish in both cases I and II, such that its contribution to the difference in question of the dE in all cases will be the expression:

$$n\frac{\varepsilon(t)-\varepsilon(t_1)}{2}d(tt_1).$$

From this, when one understands that inconsequential constants are always ignored in E, one deduces that:

$$E(t, u, T, U) - E(t_1, u_1, T_1, U_1) =$$

$$n\frac{\varepsilon(t)-\varepsilon(t_1)}{2}tt_1 + n\frac{\varepsilon(u)-\varepsilon(u_1)}{2}uu_1 - N\frac{\varepsilon(T)-\varepsilon(T_1)}{2}TT_1 - N\frac{\varepsilon(U)-\varepsilon(U_1)}{2}UU_1. \tag{69}$$

Naturally, the relation between two eikonals for the same map is given by this, under the assumption that both of them exist. Moreover, equation (69) can make sense even when this assumption is not fulfilled. One needs only to observe that the eikonals were originally generated as functions of h, k, p, q by integrating an expression of the form:

$$dE = \alpha\, dh + \beta\, dk + \gamma dp + \delta\, dq.$$

If one now thinks of functions thus obtained as replacing the E, and correspondingly expresses the T, U, T_1, U_1 as functions of the h, k, p, q then (69) will become an identity.

If one denotes the derivatives of the eikonal $E(t, u, T, U)$ by indices according to the schema:

$$dE = E_1\, dt + E_2\, du + E_3\, dT + E_4\, dU,$$
$$dE_\alpha = E_{\alpha 1}\, dt + E_{\alpha 2}\, du + E_{\alpha 3}\, dT + E_{\alpha 4}\, dU,$$

then it will follow from the mapping equations:

$$\left.\begin{aligned} n\varepsilon(t)v &= E_1 & n\varepsilon(t)w &= E_2 \\ -N\varepsilon(T)V &= E_3 & -N\varepsilon(U)W &= E_4 \end{aligned}\right\} \tag{70}$$

by differentiation that:

$$\left.\begin{aligned} n\varepsilon(t)dv &= E_{11}dt + E_{12}du + E_{13}dT + E_{14}dU \\ n\varepsilon(t)dw &= E_{21}dt + E_{22}du + E_{23}dT + E_{24}dU \\ -N\varepsilon(T)dV &= E_{31}dt + E_{32}du + E_{33}dT + E_{34}dU \\ -N\varepsilon(U)dW &= E_{41}dt + E_{42}du + E_{43}dT + E_{44}dU. \end{aligned}\right\} \tag{71}$$

Solving for dT, dU, dV, dW, with the use of the aforementioned determinant relation, gives:

$$\left.\begin{aligned} (E_1E_4)_{12}\,dT &= -(E_1E_4)_{12}\,dt - (E_2E_4)du + n\varepsilon(t)E_{24}dv - n\varepsilon(u)E_{14}dw, \\ (E_3E_4)_{12}\,dU &= \quad (E_1E_3)_{12}\,dt + (E_2E_3)du + n\varepsilon(t)E_{23}dv + n\varepsilon(u)E_{13}dw, \end{aligned}\right\} \tag{72}$$

$$\left.\begin{aligned}-N\varepsilon(T)(E_3E_4)_{12}\,dV &= (E_1E_2E_3)_{134}\,dt+(E_1E_2E_4)_{234}\,du\\ &\quad+n\varepsilon(T)(E_3E_2)_{34}\,dv+n\varepsilon(u)(E_1E_3)_{34}\,dw,\\ -N\varepsilon(U)(E_3E_4)_{12}\,dW &= (E_1E_2E_4)_{134}\,dt+(E_1E_2E_4)_{234}\,du\\ &\quad+n\varepsilon(t)(E_4E_2)_{34}\,dv+n\varepsilon(u)(E_1E_4)_{34}\,dw.\end{aligned}\right\}\tag{73}$$

If one denotes the derivatives of T, U, V, W with respect to t, u, v, w by indices according to the schema:

$$dT = T_1\,dt + T_2\,du + T_3\,dv + T_4\,dw$$

then one will obtain the T_α, U_α, V_α, W_α – i.e., the partial derivatives of the previously employed mapping functions A, B, C, D with respect to the h, k, p, q – by dividing the dt, du, dv, dw out of equations (72) and (73). Conversely, in order to express the $E_{\alpha\beta}$ in terms of the T_α, U_α, V_α, W_α, we would like to denote the left-hand sides in (70) by (1), (2), (3), (4), so we write:

$$(1) = n\,\varepsilon(t)\,v,\qquad (2) = n\,\varepsilon(u)\,w,\qquad (3) = -N\,\varepsilon(T)\,V,\qquad (4) = -N\,\varepsilon(U)\,W,$$

which makes:

$$d(\alpha) = E_{\alpha1}\,dt + E_{\alpha2}\,du + E_{\alpha3}\,dT + E_{\alpha4}\,dU,$$

from which, after dividing out the dt, …, the system of four equations follows:

$$\left.\begin{aligned}\frac{\partial(\alpha)}{\partial t} &= E_{\alpha1} + E_{\alpha3}T_1 + E_{\alpha4}U_1,\\ \frac{\partial(\alpha)}{\partial u} &= E_{\alpha2} + E_{\alpha3}T_2 + E_{\alpha4}U_2,\\ \frac{\partial(\alpha)}{\partial v} &= E_{\alpha3}T_3 + E_{\alpha4}U_3,\\ \frac{\partial(\alpha)}{\partial w} &= E_{\alpha3}T_4 + E_{\alpha4}U_4.\end{aligned}\right\}\tag{74}$$

Its solution yields:

$$\left.\begin{aligned}(TU)_{34}E_{\alpha1} &= (TU)_{34}\frac{\partial(\alpha)}{\partial t}+(TU)_{41}\frac{\partial(\alpha)}{\partial v}+(TU)_{13}\frac{\partial(\alpha)}{\partial w},\\ (TU)_{34}E_{\alpha2} &= (TU)_{34}\frac{\partial(\alpha)}{\partial u}+(TU)_{42}\frac{\partial(\alpha)}{\partial v}+(TU)_{23}\frac{\partial(\alpha)}{\partial w},\\ (TU)_{34}E_{\alpha3} &= U_4\frac{\partial(\alpha)}{\partial t}-U_3\frac{\partial(\alpha)}{\partial w},\\ (TU)_{34}E_{\alpha4} &= -T_4\frac{\partial(\alpha)}{\partial v}+T_3\frac{\partial(\alpha)}{\partial w}.\end{aligned}\right\}\tag{75}$$

The $E_{\alpha\beta}$ with unequal indices will then be determined in two ways; setting the expressions thus obtained equal to each other yields nothing but the MALUS conditions.

If the values of the four ray coordinates t, u, T, U are given for a given eikonal $E(t, u, T, U)$ then the values of the four remaining coordinates v, w, V, W are also determined from the four mapping equations (70), and therefore the two conjugate rays σ, Σ and the light path (σ, Σ). Accordingly, we would like to regard the combination (t, u, T, U) as the light path coordinates and briefly speak of the light path (t, u, T, U). Now, should a light path go through the points $p(x, y, z)$ and $\Pi(X, Y, Z)$ in the object and image spaces, resp., then one would obtain the condition for that situation if one added the four extra equations:

$$y = h + \frac{xp}{m}, \quad z = k + \frac{xq}{m}, \quad Y = H + \frac{XP}{M}, \quad Z = K + \frac{XQ}{M} \tag{76}$$

to the four mapping equations. Solving (70) and (76) for the eight ray coordinates would then determine the light path that goes through π and Π. If one now expresses the v, w, V, W on the left-hand side of (70) in terms of the x, y, z, X, Y, Z and t, u, T, U by means of (76) then carrying out the calculation of the sixteen different cases will yield that the left-hand sides are equal to the partial derivatives of certain expressions. This situation allows us to give the conditions that were given by (70) and (76) a simple form. It may suffice to suppress the intermediate computations and give only the result. One first defines:

$$\left.\begin{aligned} l &= \frac{x}{m} = \sqrt{x^2 + (y-h)^2 + (z-k)^2}, \\ l' &= kp + \frac{x}{m} = py + \sqrt{1-p^2}\cdot\sqrt{x^2+(z-k)^2}, \\ l'' &= kq + \frac{x}{m} = qz + \sqrt{1-q^2}\cdot\sqrt{x^2+(y-k)^2}, \\ l''' &= xm + yp + zq, \end{aligned}\right\} \tag{77}$$

where these expressions are regarded as functions of h and k, p and k, h and q, p and q, respectively, and furthermore:

$$\left.\begin{aligned} pdh + qdk &= -dl, \quad & hdp - qdk &= dl', \\ pdh - kdq &= -dl'', \quad & hdp + kdq &= dl'''. \end{aligned}\right\} \tag{78}$$

One imagines the corresponding expressions for the quantities L, L', L'', L''' in image space Ω as being added to equations (77) and (78). One then further defines the following table of expressions $F(t, u, T, U)$, which correspond to the sixteen eikonals in the table (62) term-by-term:

$$\left.\begin{array}{llll}
[1]: & F(h,k,P,Q) = -nl + NL''', & [2]: & F(p,k,P,Q) = -nl' + NL''', \\
[3]: & F(h,q,P,Q) = -nl'' + NL''', & [4]: & F(p,q,P,Q) = -nl''' + NL'', \\
[5]: & F(h,k,H,Q) = -nl + NL'', & [6]: & F(p,k,H,Q) = -nl' + NL'', \\
[7]: & F(h,q,H,Q) = -nl'' + NL'', & [8]: & F(p,q,H,Q) = -nl''' + NL'', \\
[9]: & F(h,k,P,K) = -nl + NL', & [10]: & F(p,k,P,K) = -nl' + NL', \\
[11]: & F(h,q,P,K) = -nl'' + NL', & [12]: & F(p,q,P,K) = -nl''' + NL', \\
[13]: & F(h,k,H,K) = -nl + NL, & [14]: & F(p,k,H,K) = -nl' + NL, \\
[15]: & F(h,q,H,K) = -nl'' + NL, & [16]: & F(p,q,H,K) = -nl''' + NL.
\end{array}\right\} \tag{79}$$

If one now defines the following expression, which includes all sixteen cases:

$$\Theta(t, u, T, U) = E(t, u, T, U) - F(t, u, T, U) \tag{80}$$

then the system of eight equations (70) and (76) can be replaced with the four conditions:

$$0 = \frac{\partial\Theta}{\partial t} = \frac{\partial\Theta}{\partial u} = \frac{\partial\Theta}{\partial T} = \frac{\partial\Theta}{\partial U}, \tag{81}$$

which say that the light path (t, u, T, U) should go through the two points $\pi(x, y, z)$ and $\Pi(X, Y, Z)$.

In order to have everything in one place for later applications, we derive a rule for the eikonal of a composition of maps. Let three spaces ω_1, ω_2, ω_3 with indices n_1, n_2, n_3, resp., be given, along with the coordinates:

$$h_1, k_1, p_1, q_1, \qquad h_2, k_2, p_2, q_2, \qquad h_3, k_3, p_3, q_3$$

of the three rays σ_1, σ_2, σ_3, resp., of a light path. For the eikonal of the three maps $(\omega_1\,\omega_2)$, $(\omega_2\,\omega_3)$, $(\omega_3\,\omega_1)$, one chooses three variables t_1, t_2, t_3 from the variable pairs $(h_1\ p_1)$, $(h_2\ p_2)$, $(h_3\ p_3)$, resp., and likewise the variables u_1, u_2, u_3 from the pairs $(h_1\ q_1)$, $(h_2\ q_2)$, $(h_3\ q_3)$, resp., while the quantities that remain in the pairs will be correspondingly denoted, as before, by v_1, v_2, v_3 and w_1, w_2, w_3, resp. If one now thinks, on the basis of the choice that was made, of the three eikonals of the three maps as being:

$$\Phi'' = E(t_1, u_1, t_2, u_2), \quad \Phi = E(t_2, u_2, t_3, u_3), \quad \Phi' = E(t_3, u_3, t_1, u_1)$$

then it will follow immediately from the defining equation (66) that:

$$d\Phi + d\Phi' + d\Phi'' = 0,$$

from which, we will further obtain, since the additive constants in the Φ do not enter in:

$$\Phi + \Phi' + \Phi'' = 0. \tag{82}$$

If one further sets down the three times four mapping equations, using (67), in which obviously the v, w happen to be expressed in two different ways, then, by eliminating the v, w one will obtain the relations:

$$\left.\begin{aligned} 0 &= \frac{\partial}{\partial t_1}(\Phi' + \Phi'') = \frac{\partial}{\partial t_2}(\Phi'' + \Phi) = \frac{\partial}{\partial t_3}(\Phi + \Phi''), \\ 0 &= \frac{\partial}{\partial u_1}(\Phi' + \Phi'') = \frac{\partial}{\partial u_2}(\Phi'' + \Phi) = \frac{\partial}{\partial u_3}(\Phi + \Phi''). \end{aligned}\right\} \tag{83}$$

Since only two conditions can exist between the six variables t, u, of the six equations (83), four of of them are a consequence of the remaining two.

For applications, it is convenient to give the foregoing formulas a somewhat different form by ignoring the symmetry. If one thinks of the maps $(\omega_1\ \omega_2)$ and $(\omega_2\ \omega_3)$ as being given, and the composed map $(\omega_1\ \omega_3)$ as being constructed from them then one must derive, from the given eikonals:

$$\Psi = E(t_1, u_1, t_2, u_2), \qquad \Psi' = E(t_2, u_2, t_3, u_3),$$

the eikonal:

$$\Psi'' = E(t_1, u_1, t_3, u_3)$$

for the composed map. Since:

$$\Psi = \Phi'', \qquad \Psi' = \Phi, \qquad \Psi'' = -\Phi',$$

one can, from (82) and (83), set down the equations:

$$\Psi'' = \Psi + \Psi', \qquad 0 = \frac{\partial}{\partial t_2}(\Psi + \Psi') = \frac{\partial}{\partial u_2}(\Psi + \Psi'),$$

$$\frac{\partial \Psi''}{\partial t_1} = \frac{\partial \Psi}{\partial t_1}, \quad \frac{\partial \Psi''}{\partial u_1} = \frac{\partial \Psi}{\partial u_1}, \quad \frac{\partial \Psi''}{\partial t_3} = \frac{\partial \Psi'}{\partial t_3}, \quad \frac{\partial \Psi''}{\partial u_3} = \frac{\partial \Psi'}{\partial u_3}.$$

This yields the following rule: *From the eikonals of the two given maps* $(\omega_1\ \omega_2)$ *and* $(\omega_2\ \omega_3)$, *if one forms the expression:*

$$S = E(t_1, u_1, t_2, u_2) + E(t_2, u_2, t_3, u_3) \tag{85}$$

and eliminates the variables t_2 *and* u_2 *in them with the help of the conditions:*

$$0 = \frac{\partial S}{\partial t_2}, \qquad 0 = \frac{\partial S}{\partial u_2} \tag{86}$$

then S will go to the eikonal $E(t_1, u_1, t_3, u_3)$ *of the composed map* $(\omega_1\ \omega_3)$; *furthermore, one will have:*

$$\left.\begin{aligned}
\frac{\partial}{\partial t_1}E(t_1,u_1,t_3,u_3) &= \frac{\partial}{\partial t_1}E(t_1,u_1,t_2,u_2), & \frac{\partial}{\partial u_1}E(t_1,u_1,t_3,u_3) &= \frac{\partial}{\partial u_1}E(t_1,u_1,t_2,u_2),\\
\frac{\partial}{\partial t_3}E(t_1,u_1,t_3,u_3) &= \frac{\partial}{\partial t_3}E(t_2,u_2,t_3,u_3), & \frac{\partial}{\partial u_3}E(t_1,u_1,t_3,u_3) &= \frac{\partial}{\partial u_3}E(t_2,u_2,t_3,u_3).
\end{aligned}\right\} \quad (87)$$

For this rule, it is obviously essential that the second variable pair in the eikonal to $(\omega_1\ \omega_2)$ be identical to the first variable pair in the eikonal to $(\omega_2\ \omega_3)$. If this condition were not fulfilled then, upon taking (69) into account, a similar rule could be posed, which is, however, substantially more complicated.

The prescription that is included in (85) and (86) may be generalized with no further assumptions. If one is given the maps:

$$(\omega_1\ \omega_2),\ (\omega_2\ \omega_3),\ \ldots,\ (\omega_{r-1}\ \omega_r)$$

for the r spaces $\omega_1, \omega_2, \ldots, \omega_r$, with the eikonals:

$$E(t_1, u_1, t_2, u_2),\ E(t_2, u_2, t_3, u_3),\ \ldots,\ E(t_{r-1}, u_{r-1}, t_r, u_r), \text{ resp.}$$

then one will obtain the eikonal $E(t_1, u_1, t_r, u_r)$ for the composed map $(\omega_1\ \omega_r)$ when one eliminates the variables $t_2, u_2, \ldots, t_{r-1}, u_{r-1}$ in the expression:

$$S = E(t_1, u_1, t_2, u_2) + \ldots + E(t_{r-1}, u_{r-1}, t_r, u_r),$$

with the help of the equations:

$$0 = \frac{\partial S}{\partial t_2} = \frac{\partial S}{\partial u_2} = \ldots = \frac{\partial S}{\partial t_{r-1}} = \frac{\partial S}{\partial u_{r-1}}\,.$$

VII.

Anastigmatic bodies and surfaces. Parametric representation. Cosine and sine theorem.

With the foregoing developments, I will break away from the preparations of a general nature and go on to the applications of the theorems that were found. For the first situation, I might treat the question of anastigmatism, which, together with the resolution of chromatic aberration (Farbenfehler), constitutes the real difficulty in practical optics.

If anastigmatic bodies appear in the ray-wise map of ω to Ω, so the sheaf that is homocentric in ω is, in turn, homocentric in image space, then the point-wise map of the two spaces onto each other that it generates will likewise be collinear, which was already suggested in the Introduction for the proof of the CZAPSKI representation. Thus, the two cases of the affine – or *telescopic* – and the actual collinear maps can be separated from each other. For the affine map, the relation between the conjugate points $\pi(x, y, z)$ and $\Pi(X, Y, Z)$ may, by a suitable choice of coordinates axes, be brought into the form:

$$X = a\,x, \qquad Y = b\,y, \qquad Z = c\,z, \tag{88}$$

where the a, b, c mean the essential constants for the map. By comparison, for the actual collinear map one arrives, by a suitable choice of axes, at the equations:

$$X = \frac{a}{x}, \qquad Y = \frac{by}{x}, \qquad Z = \frac{cz}{x}, \tag{89}$$

where the a, b, c, in turn, mean constants. Next, the mapping equations between the ray coordinates h, k, p, q and H, K, P, Q are presented, which connect the x, y, z and the X, Y, Z by the equations:

$$\left.\begin{aligned} y &= h + \frac{xp}{m}, & z &= k + \frac{xq}{m}, \\ Y &= H + \frac{XP}{M}, & Z &= K + \frac{XQ}{M}. \end{aligned}\right\} \tag{90}$$

In the case of the telescopic map, one obtains from (88) and (90), by eliminating the y, z, Y, Z:

$$b\left(h + \frac{xp}{m}\right) = H + \frac{axP}{M}, \qquad c\left(k + \frac{xq}{m}\right) = K + \frac{axQ}{M},$$

from which, since x may be chosen arbitrarily along a light path, one has:

$$H = bh, \qquad K = ck, \qquad \frac{P}{M} = \frac{bp}{am}, \qquad \frac{Q}{M} = \frac{cq}{am}.$$

With the abbreviation:

$$\mu^2 = (am)^2 + (bp)^2 + (cq)^2 = a^2 + (b^2 - a^2)\, p^2 + (c^2 - a^2)\, q^2,$$

one then obtains the mapping equations in the explicit form in terms of *H*, *K*, *P*, *Q*:

$$H = b\,h, \qquad K = c\,k, \qquad P = \frac{bp}{\mu}, \qquad Q = \frac{cq}{\mu}. \tag{91}$$

If one calculates the values of the symbols (*H*, *K*), … in the MALUS conditions according to (51b) then this will yield:

$$(H, K) = 0, \qquad (H, P) = E = \frac{b^2}{\mu} - b^2 p^2 \frac{b^2 - a^2}{\mu^3},$$

$$(H, P) = -\,bcpq\frac{b^2 - a^2}{\mu^3}, \qquad (K, P) = -bcpq\frac{c^2 - a^2}{\mu^3},$$

$$(K, Q) = E = \frac{c^2}{\mu} - b^2 q^2 \frac{c^2 - a^2}{\mu^3}, \qquad (P, Q) = 0.$$

Now, if an actual map is present then the *a*, *b*, *c* must be different from zero; the vanishing of the expressions (*H*, *Q*) and (*K*, *P*) that is required by MALUS's theorem would then lead to the conditions:

$$b^2 = a^2, \qquad c^2 = a^2, \qquad E = \mu = \pm\, a. \tag{92}$$

If one treats equations (89) and (90), which belong to the actual collinear map in a similar way, then upon carrying out the intermediate computations the map will be represented by the equations:

$$\nu^2 = a^2 + (bh)^2 + (ck)^2,$$

$$H = \frac{bp}{m}, \qquad K = \frac{cq}{m}, \qquad P = \frac{bh}{\nu}, \qquad Q = \frac{ck}{\nu}.$$

The application of (51b) gives the two conditions:

$$0 = (H, Q) = b^3chk\frac{1-q^2}{(m\nu)^3} - bchk\frac{a^2 + b^2h^2}{(m\nu)^3},$$

$$0 = (K\, P) = b^3chk\frac{1-p^2}{(m\nu)^3} - bchk\frac{a^2 + c^2k^2}{(m\nu)^3},$$

which must be fulfilled identically; i.e., for arbitrary values of the *h*, *k*, *p*, *q*. This is, however, not possible, since the *b*, *c* cannot be equal to zero; the MALUS conditions then lead to a contradiction. *Therefore, Malus's theorem allows for anastigmatic bodies only in the case where the point-wise map possesses the form:*

$$X = \pm \mu x, \qquad Y = \pm \mu y, \qquad Z = \pm \mu z,$$

which is then a geometrical similarity. This case is realized, e.g., by reflection in a plane. For the sake of simplicity, it is not necessary to go further, but only to treat the form that is most important for practical optics, namely, for the purposes of dealing with microscopes, cameras, and telescopes, as well as eyeglasses, which do not cause one to treat geometrically similar maps of bodies. The foregoing result – which is well-known, moreover – may also be derived in a geometric way when one shows that for collinear maps MALUS's theorem couples the assumption of anastigmatic bodies with the property of angle preserving, which is the case only for geometrical similarities.

From what we just said, in the description of actual maps, except for the limiting cases, optics must restrict oneself to the anastigmatism that comes about only between specific surfaces. In order to see the extent to which this is mathematically possible, we consider the following case: Let any surfaces φ and Φ be given arbitrarily in ω and Ω, which we would like to briefly refer to as the object space and image space. These two surfaces are associated with each other in some way; i.e., let each point $\pi(x, y, z)$ in φ be conjugate to a point $\Pi(X, Y, Z)$ in Φ, and conversely. Analytically, this will be expressed by saying that the coordinates x, y, z and X, Y, Z are expressed by certain functions of two varying parameters α and β. Having established that, one then forms the expression:

$$\Gamma = - n(m\,x + p\,y + q\,z) + N(M\,X + P\,Y + Q\,Z) + \psi(\alpha, \beta), \tag{93}$$

where ψ means an arbitrarily chosen function of the parameters α, β, and the rectilinear coordinates are thought of as functions of the two parameters. We now establish a relationship between the directions of the two rays σ, Σ by the pair of equations:

$$\left.\begin{aligned} 0 &= \frac{\partial\Gamma}{\partial\alpha} = -n\left(m\frac{\partial x}{\partial\alpha} + p\frac{\partial y}{\partial\alpha} + q\frac{\partial z}{\partial\alpha}\right) + N\left(M\frac{\partial X}{\partial\alpha} + P\frac{\partial Y}{\partial\alpha} + Q\frac{\partial Z}{\partial\alpha}\right) + \frac{\partial\psi}{\partial\alpha}, \\ 0 &= \frac{\partial\Gamma}{\partial\beta} = -n\left(m\frac{\partial x}{\partial\beta} + p\frac{\partial y}{\partial\beta} + q\frac{\partial z}{\partial\beta}\right) + N\left(M\frac{\partial X}{\partial\beta} + P\frac{\partial Y}{\partial\beta} + Q\frac{\partial Z}{\partial\beta}\right) + \frac{\partial\psi}{\partial\beta}. \end{aligned}\right\} \tag{94}$$

If the ray σ is given then the point $\pi(x, y, z)$ and the parameter pair α, β will be determined in the object surface φ, as well as the direction m, p, q; furthermore, the point $\Pi(X, Y, Z)$ in Φ will be given by this, and the direction M, P, Q will be given by (94), and therefore also the ray Σ. Equations (94) thus define a ray-wise map, which generally does not have to satisfy MALUS's theorem. If one lets the ray σ vary in such a way that $\pi(x, y, z)$ remains fixed, while the direction m, p, q varies, then $\Pi(X, Y, Z)$ will also remain fixed, while the direction M, P, Q of Σ will change; φ and Φ then define a pair of anastigmatic surfaces for the map in question. The intersection points of the conjugate rays with the base planes of ω and Ω are given by:

$$h = y - \frac{xp}{m}, \qquad k = z - \frac{xq}{m}, \qquad H = Y - \frac{XP}{M}, \quad K = Z - \frac{XQ}{M}. \tag{95}$$

If one now eliminates the parameters a, b from Γ by means of equations (94) then the Γ will be converted into a certain function Δ of the four variables p, q, P, Q, and, from (94), the total differential of Δ will assume the form:

$$d\Delta = -\, n\left(y - \frac{xp}{m}\right)dp - n\left(z - \frac{xq}{m}\right)dq + N\left(Y - \frac{XP}{M}\right)dP - N\left(Z - \frac{XQ}{M}\right)dQ\,,$$

which, from (95), will lead to the equations:

$$-\,nh = \frac{\partial\Delta}{\partial p}, \qquad -\,nk = \frac{\partial\Delta}{\partial q}, \qquad NH = \frac{\partial\Delta}{\partial P}, \qquad NK = \frac{\partial\Delta}{\partial Q}. \tag{96}$$

The map considered thus possesses an eikonal Δ of the form [4] or $E(p, q, P, Q)$, from which the validity of MALUS's theorem again follows. Thus, anastigmatic maps of surfaces are not only mathematically possible, but one can also construct infinitely many such maps, even though the two surfaces and the type of point-wise relationship between them is prescribed arbitrarily by means of the arbitrary function $\psi(\alpha, \beta)$.

The equations (94) include an important relation between conjugate rays of an anastigmatic pair of sheaves. Since the parameter-pair α, β determines the conjugate points π, Π in the object space and image space completely, we can also denote these points by $\pi(\alpha, \beta)$ and $\Pi(\alpha, \beta)$. If one lets the parameters vary, not independently of each other, but with a condition existing between them, then π and Π will trace out certain curves in the two surfaces that are conjugate point-by-point. In particular, we consider the two well-defined curve families in φ, Φ that are given by the equation:

$$\psi(\alpha, \beta) = \text{constant}, \tag{97}$$

and seek the tangent t in the family of curves that go through $\pi(\alpha, \beta)$ that contacts the family at this point. It will be written:

$$dx = x_1\,d\alpha + x_2\,d\beta, \qquad dy = y_1\,d\alpha + y_2\,d\beta, \qquad dz = z_1\,d\alpha + z_2\,d\beta,$$
$$d\varphi = \psi_1\,d\alpha + \psi_2\,d\beta,$$

and if one further lets $\cos(st)$ be the cosine of the angle between two directions s and t then one has, when dt means the arc length element of the curve in π:

$$dt\cos(tx) = dx = x_1\,d\alpha + x_2\,d\beta,$$
$$dt\cos(ty) = dy = y_1\,d\alpha + y_2\,d\beta,$$
$$dt\cos(tz) = dz = z_1\,d\alpha + z_2\,d\beta,$$

where the $d\alpha$ and $d\beta$ have to satisfy the condition:

$$0 = \psi_1\,d\alpha + \psi_2\,d\beta.$$

One thus obtains, if g means a proportionality factor:

$$g\cos(tx) = x_1\,\psi_2 - x_2\,\psi_1, \qquad g\cos(ty) = y_1\,\psi_2 - y_2\,\psi_1, \qquad g\cos(tz) = z_1\,\psi_2 - z_2\,\psi_1,$$

$$g^2 = \psi_2^2(x_1^2 + y_1^2 + z_1^2) - 2\psi_1\psi_2(x_1x_2 + y_1y_2 + z_1z_2) + \psi_1^2(x_2^2 + y_2^2 + z_2^2)\,.$$

In the same way, we define the relations for the tangent T to the curve through $\Pi(\alpha\beta)$, namely:

$$G\cos(XY) = X_1\,\psi_2 - X_2\,\psi_1, \quad G\cos(TY) = Y_1\,\psi_2 - Y_2\,\psi_1, \quad G\cos(TZ) = Z_1\,\psi_2 - Z_2\,\psi_1,$$

$$G^2 = \psi_2^2(X_1^2 + Y_1^2 + Z_1^2) - 2\psi_1\psi_2(X_1X_2 + Y_1Y_2 + Z_1Z_2) + \psi_1^2(X_2^2 + Y_2^2 + Z_2^2)\,.$$

From the equations (94), it follows, when one combines it linearly with the factors ψ_2 and $-\psi_1$, that:

$$\begin{aligned} &n[m(x_1\psi_2 - x_2\psi_1) + p(y_1\psi_2 - y_2\psi_1) + q(z_1\psi_2 - z_2\psi_1)] \\ &= N[M(X_1\psi_2 - X_2\psi_1) + P(Y_1\psi_2 - Y_2\psi_1) + Q(Z_1\psi_2 - Z_2\psi_1)], \end{aligned}$$

or:

$$ng\cos(t\sigma) = NG\cos(T\,\Sigma). \tag{98}$$

The quantities that appear in this equation relate to the directions of the conjugate rays σ and Σ in the anastigmatic sheaf pair with the vertices $\pi(\alpha\beta)$ and $\Pi(\alpha\beta)$, and to the location of the elements of the two surfaces φ and Φ that belong to π and Π, resp., and finally, to the arc length elements of the families of curves that are determined by (97) and lie in $d\varphi$ and $d\Phi$. Since the quantities g, G are regarded as constant inside the two surface elements $d\varphi$, $d\Phi$, resp., one obtains the theorem: Inside of the sheaf, through which the two surface elements $d\varphi$, $d\Phi$ are mapped anastigmatically onto each other, the quotient of the two cosines $\cos(t\sigma)$ and $\cos(T\Sigma)$ is constant for conjugate rays. I would like to briefly refer to this theorem as the *cosine theorem*. The celebrated sine theorem is included in it as a special limiting case. In order to show this, we put equations (94) into a somewhat different form. If one thinks of the base planes of ω, Ω as lying in the contact planes of $d\varphi$, $d\Phi$, and further, the x-axis and X-axis are the normals to these surface elements, then one can, if the coordinates y, z are chosen for the parameters α, β, and the lateral axes are chosen suitably, set:

$$\begin{array}{lll} x_1 = 0, & y_1 = 1, & z_1 = 0, \\ x_2 = 0, & y_2 = 0, & z_2 = 1, \\ X_1 = 0, & Y_1 = a, & Z_1 = 0, \\ X_2 = 0, & Y_2 = 0, & Z_2 = b, \end{array}$$

where a and b mean the lateral expansions for the maps of $d\varphi$ and $d\Phi$, resp. If one further sets:

$$\begin{array}{lll} m = \cos r, & p = \sin r\cos s, & q = \sin r\sin s, \\ M = \cos R, & P = \sin R\cos S, & Q = \sin R\sin S, \end{array}$$

where the r, R can be regarded as the elevations and the s, S, as the azimuths of σ, Σ, resp., then one will obtain from (94):

$$- N\, a \sin R \cos S + n \sin r \cos s = \psi_1,$$
$$- N\, b \sin R \sin S + n \sin r \sin s = \psi_2\,,$$

$$\frac{N \sin R}{n \sin r} = \frac{\psi_2 \cos s - \psi_2 \sin s}{a\psi_2 \cos S - b\psi_2 \sin S}; \tag{99}$$

the sine quotient of the two elevations thus depends upon the azimuths, in general. Now, if the parameters ψ_1 and ψ_2 vanish simultaneously for the value pair in question then equations (98) and (99) will lose their meaning, since they assume an indeterminate form. In this case, one can, however, write:

$$\frac{N \sin R}{n \sin r} = \frac{\cos s}{a \cos S} = \frac{\sin s}{b \sin S}. \tag{100}$$

As long as the a and b are different, the sine quotient thus depends, in turn, on the azimuth; by comparison, if $a = b$ then one will have:

$$s = S, \qquad \frac{N \sin R}{n \sin r} = \frac{1}{a}. \tag{101}$$

This is the sine theorem, as it was discovered by – e.g. – CZAPSKI (loc. cit., page 102).

In the cosine theorem and its corollary, the sine theorem, one treats, as the foregoing shows, a theorem not of optics, but of line geometry; it is valid for the ray-wise maps that arise from a contact transformation, as long as anastigmatic surface elements appear.

VIII.

Simple cases of anastigmatic surfaces. Tangent theorem.

As long as nothing more specific is assumed about the two surfaces φ, Φ and the arbitrary function ψ, equations (93) and (94), which are employed for the construction of the eikonal, cannot be further consolidated. By comparison, one can represent the eikonal explicitly as long as the two anastigmatic surfaces φ, Φ consist of planes. Depending upon whether these two planes, which we would like to distinguish from each other as the *object plane* and the *image plane*, lie at finite points or infinite ones, one can identify the following four cases:

	Object plane	Image plane
I.	at infinity	at infinity
II.	at infinity	finite
III.	finite	at infinity
IV.	finite	finite.

(102)

If one likewise takes the planes to be the base planes, assuming that they lie at finite points, then this will yield the following table:

$$\left.\begin{array}{llllll} \text{Case} & \text{I}: & \text{for} & p, q \text{ constant} & \text{that makes} & P, Q \text{ constant} \\ " & \text{II}: & " & p, q \text{ constant} & " & H, K \text{ constant} \\ " & \text{III}: & " & h, k \text{ constant} & " & P, Q \text{ constant} \\ " & \text{IV}: & " & h, k \text{ constant} & " & H, \text{K constant.} \end{array}\right\} \tag{103}$$

Should the mapping equations:

$$H = A(h, k, p, q), \qquad K = B(h, k, p, q), \qquad P = C(h, k, p, q), \qquad Q = D(h, k, p, q)$$

have the property that for constant p, q the value pair P, Q happens to be constant then the variables h, k must be absent from C and D; i.e., with the previously-employed notation, the partial derivatives must satisfy:

$$C_1 = C_2 = D_1 = D_2 = 0.$$

The repetition of this argument leads to the table:

$$\left.\begin{array}{lll} \text{Case} & \text{I.} & C_1 = C_2 = D_1 = D_2 = 0, \\ " & \text{II.} & A_1 = A_2 = B_1 = B_2 = 0, \\ " & \text{III.} & C_3 = C_4 = D_3 = D_4 = 0, \\ " & \text{IV.} & A_3 = A_4 = B_3 = B_4 = 0. \end{array}\right\} \tag{104}$$

Since one can obtain cases II, III, IV from I by the previously-employed substitutions:

$$\left.\begin{array}{c}\begin{pmatrix} H & K & P & Q \\ -P & -Q & H & K \end{pmatrix} \begin{pmatrix} h & k & p & q \\ -p & -q & h & k \end{pmatrix} \\[2em] \begin{pmatrix} h & k & p & q & H & K & P & Q \\ -p & -q & h & k & -P & -Q & H & K \end{pmatrix}\end{array}\right\} \tag{105}$$

it would then suffice for the further calculations to pursue the details only in case I.

In the table of critical determinants (63), if the terms:

$$(CD)_{14}, \qquad (CD)_{23}, \qquad (CD)_{12}, \qquad (AD)_{12}, \qquad (CB)_{12}$$

vanish for case I, due to the first row of (104), then it will follow that the terms $(CD)_{34}$, $(AB)_{12}$, are certainly not identically zero, and the eikonals $E(h, k, P, Q)$ and $E(p, q, H, K)$ certainly do not exist. If one employs, as is sufficient, only the form [16] – or $E(p, q, H, K)$ – then the right-hand sides of the mapping equations:

$$-NP = \frac{\partial E}{\partial H}, \qquad -NQ = \frac{\partial E}{\partial K}$$

will be free of H and K, so E will be a linear function of H and K of the form:

$$E = H\,\alpha(p, q) + K\,\beta(p, q) + \gamma(p, q),$$

where α, β, γ mean any functions of the p, q, with the restriction that the determinant presented in (68) does not vanish, so the two functions α and β of the p, q will be independent of each other. Upon making the substitutions (105), this yields the following eikonal forms and mapping equations:

$$\text{I.}\quad \left\{\begin{array}{l} [16]:\quad E(p,q,H,K) = H\alpha(p,q) + K\beta(p,q) + \gamma(p,q), \\ \quad -NP = \alpha(p,q), \quad -nh = H\dfrac{\partial\alpha}{\partial p} + K\dfrac{\partial\beta}{\partial p} + \dfrac{\partial\gamma}{\partial p}, \\ \quad -NQ = \beta(p,q), \quad -nh = H\dfrac{\partial\alpha}{\partial q} + K\dfrac{\partial\beta}{\partial q} + \dfrac{\partial\gamma}{\partial q}, \end{array}\right\} \tag{106}$$

$$\text{II.}\quad \left\{\begin{array}{l} [4]:\quad E(p,q,P,Q) = P\alpha(p,q) + Q\beta(p,q) + \gamma(p,q), \\ \quad NH = \alpha(p,q), \quad -nh = P\dfrac{\partial\alpha}{\partial p} + Q\dfrac{\partial\beta}{\partial p} + \dfrac{\partial\gamma}{\partial p}, \\ \quad NK = \beta(p,q), \quad -nk = P\dfrac{\partial\alpha}{\partial q} + Q\dfrac{\partial\beta}{\partial q} + \dfrac{\partial\gamma}{\partial q}, \end{array}\right\} \tag{107}$$

$$\text{III.}\quad \left\{\begin{array}{l} [13]: \quad E(h,k,H,K) = H\alpha(h,k) + K\beta(h,k) + \gamma(h,k), \\ -NP = \alpha(h,k), \quad np = H\dfrac{\partial\alpha}{\partial h} + K\dfrac{\partial\beta}{\partial h} + \dfrac{\partial\gamma}{\partial h}, \\ -NQ = \beta(h,k), \quad nq = H\dfrac{\partial\alpha}{\partial k} + K\dfrac{\partial\beta}{\partial k} + \dfrac{\partial\gamma}{\partial k}, \end{array}\right\} \tag{108}$$

$$\text{IV.}\quad \left\{\begin{array}{l} [1]: \quad E(h,k,P,Q) = P\alpha(h,k) + Q\beta(h,k) + \gamma(h,k), \\ NH = \alpha(h,k), \quad np = P\dfrac{\partial\alpha}{\partial h} + Q\dfrac{\partial\beta}{\partial h} + \dfrac{\partial\gamma}{\partial h}, \\ NK = \beta(h,k), \quad nq = P\dfrac{\partial\alpha}{\partial k} + Q\dfrac{\partial\beta}{\partial k} + \dfrac{\partial\gamma}{\partial k}. \end{array}\right\} \tag{109}$$

Case I is realized by any prism theorem; the anastigmatic relationship between the two infinitely-distant planes will be employed by any spectroscope in order to obtain the sharpest possible slit image by illuminating the collimator and eyepiece.

Case II corresponds to the ideal one for the purposes of a telescope or a camera that is photographing infinitely-distant objects. Case III is the inversion of II, and represents the ideal collimator. One thinks of case II as being realized by a centered system of lenses, so when the image in the focal plane of the image space is correctly drawn – i.e., perspective to the infinitely-distant object – for a suitable choice of coordinates, the following relation must exist:

$$\alpha(p, q) = NH = \frac{Nap}{m}, \qquad \beta(p, q) = NK = \frac{Naq}{m},$$

where a means the focal length of the image space. One will then have:

$$\begin{aligned} -nh &= NaP\frac{1-q^2}{m^3} + NaQ\frac{pq}{m^3} + \quad + \frac{\partial\gamma}{\partial p}, \\ -nk &= NaP\frac{pq}{m^3} \qquad + NaQ\frac{1-p^3}{m^3} + \frac{\partial\gamma}{\partial q}. \end{aligned}$$

The right-hand sides, and therefore the h, k, can be independent of the p, q only when:

$$P = Q = 0, \qquad \frac{\partial\gamma}{\partial p} = \text{constant}, \qquad \frac{\partial\gamma}{\partial q} = \text{constant}.$$

Therefore, a strictly anastigmatic focal point, but not an anastigmatic focal plane, can appear in the object space of such a system. Now, since such a focal plane must be present when the system of lenses is symmetric, this gives us that for a symmetric construction the correctness of the picture and the anastigmatism contradict each other. One of them can be attained only at the cost of the other. Were the anastigmatism in both focal planes achieved for such an objective then one would be dealing with a union of cases II and II, and the h, k would have to depend upon the P, Q linearly, and likewise, the H, K would depend linearly upon the p, q. Since the system is centered, this would lead, by a suitable choice of axes, to mapping equations of the form:

$$NH = ap, \qquad NK = aq,$$
$$-nh = aP, \qquad -nk = aQ,$$

and to an eikonal:

$$E(p, q, P, Q) = a(pP + qQ). \tag{110}$$

Thus, in order obtain the picture in the focal plane of the image space, one must then project the infinitely-distant object centrally to a sphere of radius $a : N$, and then from the sphere image, parallel to the figure axis of the objective, in order to form the orthographic projection onto the focal plane ([1]).

Case IV corresponds to the ideal microscope objective. Should the map of the two planes make them geometrically similar to each other, then one must have, by a suitable choice of coordinates:

$$\left.\begin{aligned} H &= ah, \quad np = NaP + \frac{\partial \gamma}{\partial h}, \\ K &= ak, \quad nq = NaQ + \frac{\partial \gamma}{\partial k}, \end{aligned}\right\} \tag{111}$$

where a means the linear expansion. If one now thinks of any fixed point with the abscissas x_0 and X_0 on the x-axis and X-axis as having been selected in ω and Ω then one can consider the lines from x_0 to the points (h, k) and from X_0 to the conjugate image points (H, K) to be corresponding projection rays. Now, should these lines also always be components of the same light path, then the equations:

$$\frac{x_0 - 0}{m} = \frac{0 - h}{p} = \frac{0 - k}{q}, \qquad \frac{X_0 - 0}{M} = \frac{0 - H}{P} = \frac{0 - K}{Q}$$

must be satisfied, so with the abbreviations:

$$s^2 = x_0^2 + h^2 + k^2, \qquad S^2 = X_0^2 + H^2 + K^2 = X_0^2 + a^2 (h^2 + k^2),$$

one would have:

$$m = \frac{x_0}{s}, \qquad p = -\frac{h}{s}, \qquad q = -\frac{k}{s},$$
$$M = \frac{X_0}{S}, \qquad P = -\frac{H}{S}, \qquad Q = -\frac{K}{S}.$$

When this is substituted in equations (111), that will give:

([1]) The fact that there are good camera objectives with symmetric arrangements is obviously no contradiction to the remarks above, since here we are always dealing with the purely mathematical anastigmatism, which is linked to no limitations on the aperture of the sheaf.

$$\frac{\partial\gamma}{\partial h} = \frac{Na^2h}{S} - \frac{nh}{s}, \qquad \frac{\partial\gamma}{\partial k} = \frac{Na^2k}{S} - \frac{nk}{s},$$

from which, it follows by integration that:

$$\gamma(h, k) = NS - ns = N\sqrt{X_0^2 + a^2(h^2 + k^2)} - n\sqrt{x_0^2 + h^2 + k^2}\ . \tag{112}$$

Conversely: When γ possesses the form (112) there will exist an "orthoscopic" pair of points x_0 and X_0 on the x-axis, and one will have:

$$\frac{p}{m} : \frac{P}{M} = \frac{q}{m} : \frac{Q}{M} = \frac{X_0}{x_0 a}.$$

The so-called *tangent theorem*, in its strict form, is included in this (cf., CZAPSKI, pp. 111). Its validity is linked with the fact that γ possesses the form (112).

IX.

Anastigmatic elementary sheaf. Reduced form of the condition.

The parametric representation of an eikonal with a prescribed anastigmatic surface that was developed in equations (93) to (96) was chosen in such a way that it led to the eikonal form $E(p, q, P, Q)$. This representation can be carried over to any of the remaining forms with no further assumptions. Let the mutually associated points $\pi(x, y, z)$ and $\Pi(x, y, z)$ of the prescribed anastigmatic surfaces be once more defined in such a way that the rectangular coordinates would be represented as functions of two variable parameters α, β, and furthermore let $F(t, u, T, U)$ be one of the expressions that were summarized in Table (79), which includes the coordinates of π, Π, in addition to the light coordinates t, u, T, U. One next defines the expression:

$$\Gamma = F(t, u, T, U) + \psi(a, b),$$

where ψ means an arbitrary function of the α, β. If one now eliminates the two parameters from Γ with the help of the equations:

$$0 = \frac{\partial\Gamma}{\partial\alpha}, \qquad 0 = \frac{\partial\Gamma}{\partial\beta}$$

then Γ will go over to an eikonal $F(t, u, T, U)$ for which the surface points π, Π are conjugate anastigmatic points. The proof will proceed, step-by-step, in the same way as the previous case.

In the present manner of representation, seven arbitrarily-chosen functions of the two parameters α, β next enter the picture, namely, the six rectangular coordinates and the function ψ. If one chooses two of the coordinates – e.g., y and z – then five arbitrary functions will still be remain. This situation shows that a considerable latitude is already present from the requirement of anastigmatic surfaces; it is self-explanatory that the latitude in the requirement that only anastigmatic curves or isolated anastigmatic points should appear is even greater. I shall not pursue the latter two cases here, which are much less important for optics than the anastigmatic surfaces, but treat the question of the existence of the stated surfaces from another angle.

Instead of constructing the eikonal of the prescribed anastigmatic surface, one can also pose the question of which conditions the eikonal must satisfy if anastigmatism is to be present. One can pursue this in different ways.

If the eikonal $E(t, u, T, U)$ is represented explicitly as a function of its variables then, from (80), the expression:

$$\Theta(t, u, T, U) = E(t, u, T, U) - F(t, u, T, U)$$

that we defined by means of the system of equations (81), or:

$$0 = \frac{\partial\Theta}{\partial t} = \frac{\partial\Theta}{\partial u} = \frac{\partial\Theta}{\partial T} = \frac{\partial\Theta}{\partial U}, \tag{113}$$

will give the conditions for the light ray (t, u, T, U) to go through the points $\pi(x, y, z)$, $\Pi(X, Y, Z)$ that lie in ω, Ω. If the point-pair π, Π is given arbitrarily then one will obtain the light ray that goes through it when one solves equations (113) for the light ray coordinates. In general, the number of solutions that appear in this way is finite; i.e., equations (113) are independent of each other. Optically speaking, this comes down to saying: An eye with sufficiently small pupils that one finds in an image space generally sees illuminated points of the object space as isolated illuminated points, and indeed, in the directions that are given by the light rays. For definite positions of the points π, Π, however, the case in which the number of solutions is infinitely large can also come about, so for the desired light ray one will obtain a manifold μ_1 or μ_2 ; the π, Π will then define pairs of conical or astigmatic points. The search for such distinguished points is then equivalent to the discussion of solutions of (113), when one considers the cases where, of the four equations, one or two of them will be consequences of the remaining ones. For the actual implementation of this discussion, it is naturally required that the eikonal be actually given, since one does not arrive at the simplest formulation of the problem in the other case.

Another path that generally comes under consideration only for the eikonals with anastigmatic pairs of surfaces is the following one: A parametric representation with five arbitrary functions for this class of eikonals will be obtained by means of the system of equations that was given recently:

$$\Gamma = F(t, u, T, U) + \psi(a, b), \qquad 0 = \frac{\partial\Gamma}{\partial\alpha} = \frac{\partial\Gamma}{\partial\beta}.$$

If one now takes the partial derivatives of the eikonal:

$$\Gamma = E(t, u, T, U)$$

with respect to t, u, T, U up to an order such that the arbitrary functions can be eliminated, and then actually carries out the elimination then one will obtain the partial differential equations whose common solutions define the eikonal that we imagine. Instead of this direct path, I prefer to follow a detour, which, however, possesses the advantage that the meaning of the individual relations emerges more directly. When one temporarily restricts oneself to it, one thus arrives at only the investigation of the behavior of the elementary sheaves. For this, it suffices to employ only the eikonal form $E(p, q, P, Q)$, which leads to the simplest formulas; the restriction that is introduced may be subsequently lifted by a simple argument. We thus set, from (80):

$$\Theta(p, q, P, Q) = E(p, q, P, Q) + n(xm + yp + zq) - N(X\,M + YP + ZQ).$$

For the partial derivatives with respect to the p, q, P, Q, we employ the following notation of the indices according to the schema:

$$df(p, q, P, Q) = f_1\,dp + f_2\,dq + f_3\,dP + f_4\,dQ,$$

in which one has:

$$\left.\begin{aligned} &m_1 = -\frac{p}{m} \qquad m_2 = -\frac{q}{m} \qquad m_3 = m_4 = 0, \\ &M_1 = M_2 = 0, \quad M_3 = -\frac{P}{M} \quad M_4 = -\frac{Q}{M}, \end{aligned}\right\} \tag{115}$$

$$\left.\begin{aligned} &m_{11} = -\frac{q^2-1}{m^3} \quad m_{12} = -\frac{pq}{m^3} \quad m_{22} = \frac{p^2-1}{m^3}, \\ &M_{33} = \frac{Q^2-1}{M^3}, \quad M_{34} = -\frac{PQ}{M^3} \quad M_{44} = -\frac{P^2-1}{M^3}. \end{aligned}\right\} \tag{116}$$

We introduce the abbreviation:

$$(\Theta_1\ \Theta_2\ \Theta_3\ \Theta_4)_{1234} = \vartheta \tag{117}$$

for the determinant that is defined by the $\Theta_{\alpha\beta}$, and set the sub-determinants of third order equal to:

$$\frac{\partial\vartheta}{\partial\Theta_{\alpha\beta}} = \vartheta_{\alpha\beta}\,. \tag{118}$$

The conditions that for a light ray L, with the light path coordinates p, q, P, Q, both rays $\sigma(p, q, P, Q)$ and $\Sigma(p, q, P, Q)$ go through the points $\pi(x, y, z)$ and $\Pi(X, Y, Z)$, resp., are given by the four equations:

$$\left.\begin{array}{ll} 0 = \Theta_1 \equiv E_1 + n(xm_1 + y), & 0 = \Theta_2 \equiv E_2 + n(xm_2 + z), \\ 0 = \Theta_3 \equiv E_3 - N(XM_1 + Y), & 0 = \Theta_4 \equiv E_4 - N(XM_1 + Z). \end{array}\right\} \tag{119}$$

If these four equations were fulfilled by the pieces L, π, Π under consideration then an infinitely close light ray with the coordinates:

$$p + dp, \qquad q + dq, \qquad P + dP, \qquad Q + dQ$$

would not generally go through the two points π, Π, and furthermore, in order for this to be the case, the displacements dp, dq, dP, dQ would have to satisfy the four conditions:

$$0 = d\Theta_\alpha = \Theta_{\alpha 1}\, dp + \Theta_{\alpha 2}\, dq + \Theta_{\alpha 3}\, dP + \Theta_{\alpha 4}\, dQ \qquad (\alpha = 1, 2, 3, 4),$$

which, when written out, would take the form:

$$\left.\begin{array}{l} 0 = (E_{11} + nxm_{11})dp + (E_{12} + nxm_{12})dq + E_{13}dP + E_{14}dQ, \\ 0 = (E_{21} + nxm_{21})dp + (E_{22} + nxm_{22})dq + E_{23}dP + E_{24}dQ, \\ 0 = E_{31}dp + E_{32}dq + (E_{33} - NXM_{33})dP + (E_{44} - NXM_{14})dQ, \\ 0 = E_{31}dp + E_{32}dq + (E_{33} - NXM_{33})dP + (E_{44} - NXM_{14})dQ. \end{array}\right\} \tag{120}$$

If the determinant of this linear system, namely:

$$\vartheta = (\Theta_1\ \Theta_2\ \Theta_3\ \Theta_4)_{1234} = (E_1 + nxm_1,\ E_2 + nxm_2,\ E_3 - NXM_3,\ E_4 - NXM_4)_{1234}\,, \tag{121}$$

were non-zero then it would follow from (120) that:

$$0 = dp = dq = dP = dQ\,,$$

which would say that none of the light rays that neighbor L go through π, Π. Should L be intersected by a neighboring light ray in object space and image space then ϑ would have to vanish. In this case, one of the four equations (120) would be superfluous as a consequence of the other three, and the three remaining equations would determine the ratios dp, dq, dP, dQ, and thereby the light ray or its neighbor with the stated property. If one develops ϑ then one will obtain an expression that is quadratic in x and X, and thus, an expression of the form:

$$0 = \vartheta = X^2(\alpha x^2 + \beta x + \gamma) + X(\alpha' x^2 + \beta' x + \gamma') + \alpha'' x^2 + \beta'' x + \gamma, \tag{122}$$

where the coefficients depend upon $E_{\lambda\mu}$, $m_{\lambda\mu}$, $M_{\lambda\mu}$. From this, π (or Π) are chosen arbitrarily, so Π (or π) is then determined, and, in fact, in a two-to-one manner, in general. If one imagines picking out all of the light rays that neighbor L that define an elementary homocentric sheaf with vertex π then the locus of the focal lines on the conjugate Σ-sheaf is nothing but the two points Π that are determined by (120), and the corresponding statement is true when, conversely, Π is given and π is determined from it by (122). If one regards x and X in the doubly-quadratic equation (122) as rectangular point coordinates in a plane then the equation will define a certain curve of fourth degree. The double points of these curves determine those locations on the light ray L where one finds an X that doubly covers x and an x that doubly covers X, and thus, where the elementary sheaf in question is homocentric or anastigmatic on both sides. The condition for the anastigmatism of an elementary sheaf with the central light ray L is then given by the three conditions:

$$\vartheta = 0, \qquad \frac{\partial \vartheta}{\partial x} = 0, \qquad \frac{\partial \vartheta}{\partial X} = 0, \tag{123}$$

where the last two, when developed, assume the form:

$$0 = m_{11}\vartheta_{11} + 2\, m_{12}\vartheta_{12} + m_{22}\vartheta_{22}, \qquad 0 = M_{33}\vartheta_{33} + 2\, M_{34}\vartheta_{34} + M_{44}\vartheta_{44}\,. \tag{124}$$

The further examination of (124) would lead to the theorem that all of the $\vartheta_{\alpha\beta}$ vanish. One can, however, arrive at the same result more concisely by the following argument: If an anastigmatic elementary sheaf should exist along L with the vertices π, Π then infinitely many systems of values must exist for the ratios of the dp, dq, dP, dQ that satisfy the equations (120); therefore, one of these equations must then imply the other two, which immediately leads to the equations:

$$0 = \vartheta_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, 3, 4), \tag{124}$$

which include self-evident conditions, due to the symmetry of the determinant ϑ. Of these equations, we would first like to employ these four:

$$\vartheta_{13} = \vartheta_{23} = \vartheta_{14} = \vartheta_{24} = 0, \tag{125}$$

and for the moment, a special position of the coordinate axes. If one defines the x-axis and the X-axis to be parallel to the rays σ and Σ, resp., of the light ray in question L then one will have:

$$\begin{aligned} &p = q = P = Q = 0, && m = M = 1, \\ &m_{11} = m_{22} = M_{33} = M_{44} = -1, && m_{12} = M_{34} = 0. \end{aligned}$$

If, as is always possible, one further defines the lateral axes in such a way that:

$$E_{12} = E_{34} = 0$$

then the four conditions (125), when developed, will assume the form:

$$\begin{aligned} (E_{22} - nx)\,(E_{44} + NX)\,E_{13} &= E_{24}\,(E_1\,E_2)_{34}\,, \\ (E_{22} - nx)\,(E_{33} + NX)\,E_{14} &= -\,E_{23}\,(E_1\,E_2)_{34}\,, \\ (E_{11} - nx)\,(E_{44} + NX)\,E_{23} &= -\,E_{14}\,(E_1\,E_2)_{34}\,, \\ (E_{11} - nx)\,(E_{33} + NX)\,E_{24} &= E_{13}\,(E_1\,E_2)_{34}\,, \end{aligned}$$

from which one can, with the abbreviations:

$$\rho = E_{13}\,E_{14} + E_{23}\,E_{24}\,, \qquad \rho' = E_{13}\,E_{23} + E_{14}\,E_{24}\,,$$

derive the series of relations:

$$\frac{E_{44}+NX}{E_{33}+NX} = -\frac{E_{14}E_{24}}{E_{13}E_{23}}, \qquad \frac{E_{22}-nx}{E_{11}-nx} = -\frac{E_{23}E_{24}}{E_{13}E_{23}},$$

$$\left.\begin{aligned} \rho nx &= E_{11}E_{23}E_{24} + E_{22}E_{13}E_{14}, & -\rho' NX &= E_{33}E_{14}E_{24} + E_{44}E_{13}E_{23},\\ \rho(E_{11}-nx) &= (E_{11}-E_{22})E_{13}E_{14}, & \rho'(E_{33}+NX) &= (E_{33}-E_{44})E_{13}E_{23},\\ \rho(E_{22}-nx) &= -(E_{11}-E_{22})E_{23}E_{24}, & \rho'(E_{44}+NX) &= -(E_{33}-E_{44})E_{14}E_{24}. \end{aligned}\right\} \tag{126}$$

If one substitutes the expressions that were found for x, X in all of the $\vartheta_{\alpha\beta}$ and ϑ then, with the abbreviation:

$$\Phi = \frac{E_{11}-E_{22}}{\rho}\,\frac{E_{33}-E_{44}}{\rho'}\;E_{13}\,E_{14}\,E_{23}\,E_{24} - (E_1\,E_2)_{34}\,, \tag{127}$$

one will get:

$$\left.\begin{aligned} \rho'\vartheta_{11} &= E_{23}E_{24}(E_{33}-E_{44})\Phi, & \rho'\vartheta_{13} &= -E_{24}\Phi,\\ \rho'\vartheta_{22} &= -E_{13}E_{14}(E_{33}-E_{44})\Phi, & \rho'\vartheta_{23} &= E_{14}\Phi,\\ \rho'\vartheta_{33} &= E_{14}E_{24}(E_{11}-E_{22})\Phi, & \rho'\vartheta_{14} &= E_{23}\Phi,\\ \rho'\vartheta_{44} &= -E_{13}E_{23}(E_{11}-E_{22})\Phi, & \rho'\vartheta_{24} &= -E_{13}\Phi, \end{aligned}\right\} \tag{128}$$

$$\vartheta_{12} = \vartheta_{34} = 0, \qquad \vartheta = \Phi^2.$$

From this, the necessary and sufficient condition for the existence of an anastigmatic elementary sheaf along the light ray L is given by the vanishing of the expression Φ. If this condition is fulfilled then one will find the two united points $\pi(x, y, z)$, $\Pi(X, Y, Z)$ from the equations:

$$\begin{aligned} \rho\, nx &= E_{11}\,E_{23}\,E_{24} + E_{22}\,E_{13}\,E_{14}\,, & -\rho'\,NX &= E_{33}\,E_{14}\,E_{24} + E_{44}\,E_{13}\,E_{23}\,,\\ ny &= -E_1 - nxm_1\,, & NY &= E_3 - NXM_3\,,\\ nz &= -E_2 - nxm_2\,, & NZ &= E_4 - NXM_4\,. \end{aligned}$$

The form of the result thus found is coupled with the special choice of coordinate axes; this restriction shall now be lifted, and we shall likewise take the opportunity to discuss certain limiting cases.

X.

General form of the conditions for anastigmatic elementary sheaves

One thinks of the coordinate axes as generally being arbitrary, and correspondingly, the element of the determinant ϑ is fixed. The condition that all sub-determinants $\vartheta_{\alpha\beta}$ should vanish can also be presented in the following form: It will be fixed with the four variables x_1, x_2, x_3, x_4, and the $\Theta_{\alpha\beta}$ as coefficients of the quadratic form:

$$
\begin{aligned}
R &= \sum_{\alpha,\beta} \Theta_{\alpha\beta} x_\alpha x_\beta \\
&= \sum_{\alpha,\beta} E_{\alpha\beta} x_\alpha x_\beta + ny(m_{11}x_1^2 + 2m_{12}x_1x_2 + m_{22}x_2^2) - NX(M_{33}x_3^3 + 2M_{34}x_3x_4 + M_{44}x_4^2), \\
&\qquad\qquad (\alpha, \beta = 1, 2, 3, 4).
\end{aligned}
$$

The vanishing of the $\vartheta_{\alpha\beta}$ then says that the form R can be represented as the sum of just two squares. This property remains unchanged when one performs a linear transformation on the form or introduces new variables z_1, z_2, z_3, z_4 in place of the x_1, x_2, x_3, x_4 by means of the substitution:

$$
\left.
\begin{aligned}
x_1 &= \alpha_1 z_1 + \beta_1 z_2, \quad & x_2 &= \alpha_2 z_1 + \beta_2 z_2, \\
x_3 &= \alpha_3 z_3 + \beta_3 z_4, \quad & x_4 &= \alpha_4 z_3 + \beta_4 z_4.
\end{aligned}
\right\} \tag{129}
$$

On just this basis then the determinant that is defined by the coefficients of the transformed form will then vanish, as well as its sub-determinants of third order. Furthermore, this behavior of the transformed form implies the same property for the original form.

The transformation (129) should now be chosen in such a way that:

$$
\begin{aligned}
m_{11}x_1^2 + 2m_{12}x_1x_2 + m_{22}x_2^2 &= -2\, z_1\, z_2\,, \\
M_{33}x_3^2 + 2M_{34}x_3x_4 + M_{44}x_4^2 &= -2\, z_3\, z_4\,.
\end{aligned}
$$

This comes about when one sets:

$$
\left.
\begin{aligned}
\alpha_1\sqrt{2} &= (m + ipq)\sqrt{\frac{m}{m^2+p^2}}, \qquad & \beta_1\sqrt{2} &= (m - ipq)\sqrt{\frac{m}{m^2+p^2}}, \\
\alpha_2\sqrt{2} &= -i\sqrt{m(m^2+p^2)}, & \beta_2\sqrt{2} &= i\sqrt{m(m^2+p^2)}, \\
\alpha_1\sqrt{2} &= (M + iPQ)\sqrt{\frac{M}{M^2+P^2}}, & \beta_3\sqrt{2} &= (M - iPQ)\sqrt{\frac{M}{M^2+P^2}}, \\
\alpha_4\sqrt{2} &= -i\sqrt{M(M^2+P^2)}, & \beta_4\sqrt{2} &= i\sqrt{M(M^2+P^2)}.
\end{aligned}
\right\} \tag{130}
$$

The part of R that depends upon $E_{\alpha\beta}$ assumes the form:

$$\sum_{\alpha,\beta} E_{\alpha\beta} x_\alpha x_\beta = \sum_{\alpha,\beta} G_{\alpha\beta} z_\alpha z_\beta ,$$

where:

$$\left.\begin{aligned}
G_{11} &= E_{11}\alpha_1\alpha_1 + E_{12}(\alpha_1\alpha_2 + \alpha_2\alpha_1) + E_{22}\alpha_2\alpha_1, \\
G_{12} &= E_{11}\alpha_1\beta_1 + E_{12}(\alpha_1\beta_2 + \alpha_2\beta_1) + E_{22}\alpha_2\beta_2, \\
G_{22} &= E_{11}\beta_1\beta_1 + E_{12}(\beta_1\beta_2 + \beta_2\beta_1) + E_{22}\beta_2\beta_1,
\end{aligned}\right\} \tag{131}$$

$$\left.\begin{aligned}
G_{33} &= E_{33}\alpha_3\alpha_3 + E_{34}(\alpha_3\alpha_4 + \alpha_4\alpha_3) + E_{44}\alpha_4\alpha_4, \\
G_{34} &= E_{33}\alpha_3\beta_3 + E_{34}(\alpha_3\beta_4 + \alpha_4\beta_3) + E_{44}\alpha_4\beta_4, \\
G_{44} &= E_{33}\beta_3\beta_3 + E_{34}(\beta_3\beta_4 + \beta_4\beta_3) + E_{44}\beta_4\beta_4,
\end{aligned}\right\} \tag{132}$$

$$\left.\begin{aligned}
G_{13} &= E_{13}\alpha_1\alpha_3 + E_{23}\alpha_2\alpha_3 + E_{14}\alpha_1\alpha_4 + E_{24}\alpha_2\alpha_4, \\
G_{23} &= E_{13}\beta_1\alpha_3 + E_{23}\beta_2\alpha_3 + E_{14}\beta_1\alpha_4 + E_{24}\beta_2\alpha_4, \\
G_{14} &= E_{13}\alpha_1\beta_3 + E_{23}\alpha_2\beta_3 + E_{14}\alpha_1\beta_4 + E_{24}\alpha_2\beta_4, \\
G_{24} &= E_{13}\beta_1\beta_3 + E_{23}\beta_2\beta_3 + E_{14}\beta_1\beta_4 + E_{24}\beta_2\beta_4.
\end{aligned}\right\} \tag{133}$$

If one subsequently calls the determinant of the transformed form ϑ then since:

$$R = \sum_{\alpha,\beta} G_{\alpha\beta} z_\alpha z_\beta - 2n\, z_1\, z_2 + 2N\, z_3\, z_4 ,$$

one will have:

$$J = \begin{vmatrix} G_{11} & G_{21} - nx & G_{31} & G_{41} \\ G_{12} - nx & G_{22} & G_{32} & G_{42} \\ G_{13} & G_{23} & G_{33} & G_{43} + NX \\ G_{14} & G_{24} & G_{34} + NX & G_{44} \end{vmatrix}, \tag{134}$$

where ϑ is, in turn, symmetric, since $G_{\alpha\beta} = G_{\beta\alpha}$. Since, from previous statements, we know from the outset that the vanishing of the ten new sub-determinants $\vartheta_{\alpha\beta}$ leads only to a condition between the $E_{\alpha\beta}$ or the $G_{\alpha\beta}$ that is free of x, X, we can make an arbitrary choice of the $\vartheta_{\alpha\beta}$ for further investigation. We next employ the two conditions:

$$\vartheta_{11} = 0, \qquad \vartheta_{22} = 0,$$

which lead to the two equations:

$$\left.\begin{aligned}
0 &= G_{22}(G_{34} + NX)^2 - 2G_{23}G_{24}(G_{34} + NX) + G_{33}G_{24}^2 + G_{44}G_{23}^2 - G_{22}G_{33}G_{44}, \\
0 &= G_{11}(G_{34} + NX)^2 - 2G_{13}G_{14}(G_{34} + NX) + G_{33}G_{14}^2 + G_{44}G_{13}^2 - G_{11}G_{33}G_{44}.
\end{aligned}\right\} \tag{135}$$

In the same way, one obtains from ϑ_{33} and ϑ_{44} :

$$\left.\begin{aligned} 0 &= G_{44}(G_{12}-nx)^2 - 2G_{41}G_{42}(G_{12}-nx) + G_{11}G_{42}^2 + G_{22}G_{41}^2 - G_{44}G_{11}G_{22}, \\ 0 &= G_{33}(G_{12}-nx)^2 - 2G_{31}G_{32}(G_{12}-nx) + G_{11}G_{32}^2 + G_{22}G_{31}^2 - G_{33}G_{11}G_{22}. \end{aligned}\right\} \tag{136}$$

From (135), one next derives the proportion:

$$\frac{(G_{34}+NX)^2}{u} = \frac{-2(G_{34}+NX)^2}{v} = \frac{1}{w},$$

where:

$$u = \begin{vmatrix} G_{23}G_{24} & G_{33}G_{24}^2 + G_{44}G_{23}^2 - G_{22}G_{33}G_{44} \\ G_{13}G_{14} & G_{33}G_{14}^2 + G_{44}G_{13}^2 - G_{11}G_{33}G_{44} \end{vmatrix}, \tag{137}$$

$$v = \begin{vmatrix} G_{33}G_{24}^2 + G_{44}G_{23}^2 & G_{22} \\ G_{33}G_{14}^2 + G_{44}G_{13}^2 & G_{11} \end{vmatrix}, \qquad w = \begin{vmatrix} G_{22} & G_{23}G_{24} \\ G_{11} & G_{13}G_{14} \end{vmatrix}. \tag{138}$$

From this, one obtains *X*, and by switching the indices, also *x*, in the form:

$$\left.\begin{aligned} 2(G_{34}+NX) &= \frac{G_{11}G_{33}G_{24}^2 + G_{11}G_{44}G_{23}^2 - G_{22}G_{33}G_{14}^2 - G_{22}G_{44}G_{13}^2}{G_{11}G_{23}G_{24} - G_{22}G_{13}G_{14}}, \\ 2(G_{12}-nx) &= \frac{G_{11}G_{33}G_{24}^2 - G_{11}G_{44}G_{23}^2 + G_{22}G_{33}G_{14}^2 - G_{22}G_{44}G_{13}^2}{G_{33}G_{14}G_{24} - G_{44}G_{13}G_{23}}. \end{aligned}\right\} \tag{139}$$

The proportion above further leads to the condition:

$$0 = \Psi = vv - uw,$$

which is free of *x*, *X*. In order to represent this more clearly, we introduce the abbreviations:

$$\left.\begin{aligned} \Gamma_{13} &= (G_{11}G_{33} - G_{13}G_{13})G_{24}, \quad \Gamma_{23} = (G_{22}G_{33} - G_{23}G_{23})G_{14}, \\ \Gamma_{14} &= (G_{11}G_{44} - G_{14}G_{14})G_{23}, \quad \Gamma_{24} = (G_{22}G_{44} - G_{24}G_{24})G_{13}, \\ \Gamma &= G_{13}G_{14}G_{23}G_{24}, \end{aligned}\right\} \tag{140}$$

and thus obtain the sequence from:

$$\begin{aligned} v &= \Gamma_{13}\,G_{24} + \Gamma_{14}\,G_{23} - \Gamma_{23}\,G_{14} - \Gamma_{24}\,G_{13} = (\Gamma_3\,G_4)_{12} + (\Gamma_4\,G_3)_{12}, \\ uw &= (G_{13}\,G_{23} - \Gamma_{23}\,G_{13})\,(\Gamma_{14}\,G_{24} - \Gamma_{24}\,G_{14}) + \Gamma(G_3\,G_4)_{12}\,(G_3\,G_4)_{12} \\ &= (\Gamma_3\,G_4)_{12}\,(\Gamma_3\,G_4)_{12} + \Gamma(G_3\,G_4)_{12}\,(G_3\,G_4)_{12}, \end{aligned}$$

$$\Psi = [(\Gamma_3G_4)_{12} + (\Gamma_4G_3)_{12}] - 4(\Gamma_3G_3)_{12}\,(\Gamma_4G_4)_{12} - 4\Gamma(G_3G_4)_{12}\,(G_3G_4)_{12}, \tag{141}$$

$$\Psi = [(\Gamma_3G_4)_{12} - (\Gamma_4G_3)_{12}] - 4(\Gamma_3\Gamma_4)_{12}\,(G_3G_4)_{12} - 4\Gamma(G_3G_4)_{12}\,(G_3G_4)_{12}. \tag{142}$$

The form (142) for the expression Ψ reveals the fact that Ψ does not change when one performs the substitution of indices:

$$\begin{pmatrix} 1, & 2, & 3, & 4 \\ 3, & 4, & 1, & 2 \end{pmatrix}.$$

For that reason, the use of equations (136) instead of (135) yields precisely the same expression. Therefore, the desired condition for the appearance of an anastigmatic elementary sheaf can be presented in the form:

$$\Psi = 0, \tag{143}$$

which is valid for arbitrary coordinate axes; it then includes the second-order derivatives of the eikonal $E(p, q, P, Q)$, in addition to the p, q, P, Q. The point-pair π, Π of the two vertices of the elementary ray sheaves is obtained from (139), in combination with the equations:

$$y = h + \frac{xp}{m}, \quad z = k + \frac{xq}{m}, \quad Y = H + \frac{XP}{M}, \quad Z = K + \frac{XQ}{M}.$$

In the foregoing calculations, it was tacitly assumed that the pair of equations (135) and (136) have only one root in common with each other, such that along the light ray L in question only one anastigmatic elementary sheaf appears. This case is, in fact, regarded as the general one, since, as will presently be shown, the appearance of two or more anastigmatic sheaves generates even more conditions beyond the ones that are contained in (143). Should two anastigmatic sheaves exist along L then, due to the geometric meaning of equations (135) and (136), both pairs of equations would have two common roots, so the equations of each pair would have to agree in their coefficients, up to a factor. This would lead to the relations:

$$\left.\begin{aligned} \frac{G_{23}G_{24}}{G_{22}} = \frac{G_{13}G_{14}}{G_{11}}, \quad \frac{G_{33}G_{24}^2 + G_{44}G_{23}^2}{G_{22}} = \frac{G_{33}G_{14}^2 + G_{44}G_{13}^2}{G_{11}}, \\ \frac{G_{41}G_{42}}{G_{44}} = \frac{G_{31}G_{32}}{G_{33}}, \quad \frac{G_{11}G_{42}^2 + G_{22}G_{41}^2}{G_{44}} = \frac{G_{11}G_{32}^2 + G_{22}G_{31}^2}{G_{33}}. \end{aligned}\right\} \tag{144}$$

The expressions in (137) and (138) that are denoted by u, v, w vanish in this case, and likewise the expression Ψ, while the formulas presented in (139) for the x, X assume the form 0 : 0. If one next sets:

$$\left.\begin{aligned} G_{11} = \lambda G_{13}G_{14}, \quad G_{33} = \mu G_{31}G_{32}, \\ G_{22} = \lambda G_{23}G_{24}, \quad G_{44} = \mu G_{41}G_{42} \end{aligned}\right\} \tag{145}$$

then the four conditions (144) will be fulfilled identically such that now, in place of the condition $\Psi = 0$, the two equations that arise from (145) by eliminating λ and μ will appear. If one introduces the abbreviations:

$$\Gamma = G_{13}\,G_{14}\,G_{23}\,G_{24}\,, \qquad\qquad \Delta = G_{13}\,G_{24} + G_{14}\,G_{23} \tag{146}$$

then the quadratic equations for the x, X will assume the form:

$$0 = \lambda(G_{34} + NX)^2 - 2(G_{34} + NX) + \mu(\Delta - \lambda\mu\Gamma),$$
$$0 = \mu(G_{12} - nx)^2 - 2(G_{12} - nx) + \lambda(\Delta - \lambda\mu\Gamma),$$

whose solution is contained in the formulas:

$$\left.\begin{array}{c} \nu^2 = (1-\lambda\mu G_{13}G_{24})(1-\lambda\mu G_{14}G_{23}), \\ \lambda(G_{12} + NX) = 1 \pm \nu, \quad \mu(G_{12} - nx) = 1 \pm \nu. \end{array}\right\} \tag{147}$$

In order to correctly associate the pairs of roots x', x'' and X', X'' for x, X, one can subject one of the unused sub-determinants to – say – the condition:

$$\vartheta_{14} = 0,$$

which, when developed, and with hindsight of (145), (146), (147), will lead to the equation:

$$(G_{12} - nx)(G_{34} + NX) = G_{13}\,G_{24}\,[\mu(G_{12} - nx) + \lambda(G_{34} + NX) - 2] + \frac{1-\nu^2}{\lambda\mu},$$

which will then imply that one must set:

$$\left.\begin{array}{ll} \lambda(G_{34} + NX') = 1+\nu, & \lambda(G_{34} + NX'') = 1-\nu, \\ \mu(G_{12} - nx') = 1-\nu, & \mu(G_{12} - nx'') = 1+\nu, \end{array}\right\} \tag{148}$$

where x' and X' belong to conjugate vertices, and likewise x'' and X''.

Since equations (135) and (136) are of only second degree, if more than two point-pairs of the desired type are present along a light ray then one must have infinitely many of them. This requires that the four equations (135) and (136) must be fulfilled for arbitrary x, X, so one must have:

$$\left.\begin{array}{c} G_{11} = G_{22} = G_{33} = G_{44} = 0, \\ G_{13}G_{14} = G_{23}G_{24} = G_{13}G_{23} = G_{14}G_{24} = 0. \end{array}\right\} \tag{149}$$

Now, from (133), one has:

$$(G_1\,G_2)_{34} = \begin{vmatrix} E_{13}\alpha_1 + E_{23}\alpha_2 & E_{14}\alpha_1 + E_{24}\alpha_2 \\ E_{13}\beta_1 + E_{23}\beta_2 & E_{14}\alpha_1 + E_{24}\beta_2 \end{vmatrix} \cdot \begin{vmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{vmatrix}$$

$$= (E_1\,E_2)_{34} \cdot (\alpha\beta)_{12} \cdot (\alpha\beta)_{34} = (E_1\,E_2)_{34} \cdot (i\,m^2) \cdot (i\,M^2).$$

Since the determinant $(E_1\ E_2)_{34}$ cannot vanish, except for the excluded points of the map (cf. (68)), this will also be true for $(G_1\ G_2)_{34}$. If one now sets – for example – G_{13} equal to zero in (149) then G_{14} and G_{23} must be non-zero, from which, the vanishing of G_{24} will follow, moreover. However, one has, from (149):

$$\text{either} \qquad G_{13} = G_{24} = 0 \qquad \text{or} \qquad G_{14} = G_{23} = 0. \tag{150}$$

The expressions for the x, X in (139) will then be plainly indeterminate; in their place will appear the relation between the conjugate x, X that the still-unused sub-determinants $\vartheta_{\alpha\beta}$ yield. If one next chooses the first of the two cases (150) then the determinant ϑ will have the form:

$$\vartheta = \begin{vmatrix} 0 & G_{21} - nx & 0 & G_{41} \\ G_{12} - nx & 0 & G_{32} & 0 \\ 0 & G_{23} & 0 & G_{43} + NX \\ G_{14} & 0 & G_{34} + NX & 0 \end{vmatrix}.$$

The sub-determinants ϑ_{11}, ϑ_{22}, ϑ_{33}, ϑ_{44} vanish identically, as one could foresee, and likewise, the ϑ_{13} and ϑ_{24} vanish identically, while the other ones lead to the equation:

$$0 = (G_{12} - nx)(G_{34} + NX) - G_{14}\, G_{23}\,, \tag{151a}$$

and for the second case (150), the following equation enters in its place:

$$0 = (G_{12} - nx)(G_{34} + NX) - G_{13}\, G_{24}\,. \tag{151b}$$

A simple example of the aforementioned special case is given by refraction through a sphere. Under this map, there will be two pairs of anastigmatic elementary sheaves along any light ray L that does not go through the center of the sphere; the vertices of the other pairs are found in the well-known aplanatic spheres. If the light path L goes through the center then there are infinitely many anastigmatic elementary sheaves along L.

The investigation up to now employed the eikonal $E(p, q, P, Q)$, so it is valid only for the maps that possessed this eikonal. This restriction shall now be dropped. Now, we shall treat the conditions that pertain to the latter development that we carried out in order for a light path L to be intersected by infinitely neighboring ones in object space, as well as in image space. If one poses the mapping equations in their original form:

$$\begin{aligned} 0 &= \alpha \equiv A(h, k, p, q) - H, & 0 &= \beta \equiv B(h, k, p, q) - K, \\ 0 &= \gamma \equiv C(h, k, p, q) - P, & 0 &= \delta \equiv D(h, k, p, q) - Q, \end{aligned}$$

and replaces the h, k, H, K with the expressions:

$$h = y - \frac{xp}{m} = y + xm_1\,, \qquad\qquad k = z - \frac{xq}{m} = z + xm_2\,,$$

$$H = Y - \frac{XP}{M} = Y + XM_3\,, \qquad\qquad K = Z - \frac{XQ}{M} = Z + XM_4$$

then one will obtain the conditions for the light path L or (p, q, P, Q) to go through the two points $\pi(x, y, z)$ and $\Pi(X, Y, Z)$. If one writes the derivatives of the $\alpha, \beta, \gamma, \delta$ with respect to the indices 1, 2, 3, 4 according to the schema:

$$d\alpha = \alpha_1\, dp + \alpha_2\, dq + \alpha_3\, dP + \alpha_4\, dQ$$

then the desired conditions for the anastigmatic elementary sheaf will be given by the notion that not only the determinant:

$$\eta = (\alpha\beta\gamma\delta)_{1234}\,,$$

but also the sub-determinants:

$$\frac{\partial\eta}{\partial\alpha_1}, \quad \frac{\partial\eta}{\partial\beta_1}, \quad \frac{\partial\eta}{\partial\gamma_1}, \quad \frac{\partial\eta}{\partial\delta_1} \qquad (\lambda = 1, 2, 3, 4)$$

must vanish. With this Ansatz, the previous calculations can be repeated step-by-step. Any equation in the new calculation corresponds to a definite equation in the previous ones, and conversely. The transition between two associated equations is then obtained directly when one employs the relations between A, B, C, D, and the eikonals in the relations developed in (70) to (75). One next obtains a condition $\Psi = 0$, in which not only the ray coordinates, but also the first-order derivatives appear, but the rectangular coordinates of the two vertices of the two sheaves π, Π are missing. From this, the explicit expressions for the loci of the π, Π emerge, expressed in terms of the quantities that enter into Ψ.

The foregoing manner of representation is independent of the assumption that a definite eikonal form – e.g., $E(p, q, P, Q)$ – also actually exists for the map in question. The Ansatz above is even useful for the ray-wise maps that do not satisfy the MALUS theorem, and thus cannot possess any eikonal. Furthermore, one can, in turn, introduce each of the sixteen eikonals that exist for the map in question by means of equations (70) to (75) using the representation that starts with the A, B, C, D. The restriction that arises from the use of the form $E(p, q, P, Q)$ thus reduces to the one that one must, if need be, transform the previously-developed expressions to another eikonal form before using them in the transition.

XI.

Classification of eikonals.

If the expression Ψ were constructed for a given eikonal $E(t, u, T, U)$, under the guidance of the foregoing section, then the vanishing of Ψ would be the necessary and sufficient condition for an anastigmatic elementary sheaf to exist along the light path (t, u, T, U). If π, Π are the vertices of this sheaf in object space and image space then one will think of π as the vertex of a homocentric σ-sheaf that therefore corresponds to a well-defined Σ-sheaf in image space, together with the corresponding caustic and wave family that is associated with it. Now, the vanishing of Ψ also says that the individual waves of the family referred to will be cut by the light ray in question at the umbilical points. The corresponding statement will be true when one seeks the conjugate σ-sheaf to a homocentric Σ-sheaf with the vertex Π.

The Ψ that belongs to a particular type of eikonal $E(t, u, T, U)$ admits the totality of eikonals of this type, and correspondingly, the associated maps divide into three large groups, according to whether:

1) Ψ reduces to a non-zero constant a.
2) Ψ is identically equal to zero.
3) Ψ is an arbitrary function of the light coordinates t, u, T, U.

In the first case, the condition $\Psi = 0$ will not be fulfilled by any light rays; there exists no anastigmatic elementary sheaf and, *a fortiori*, there is also no anastigmatic point-pair. The eikonals of this group are defined to be the solutions of the second-order partial differential equation $\Psi = a$.

In the second group, Ψ vanishes for any light ray, so an anastigmatic elementary sheaf exists along any light ray. This sheaf and the vertices that belong to it define a manifold μ_4 . Since space includes only a μ_3 of points, any π must be the vertex of infinitely many elementary sheaves. Thus, one must distinguish two cases. If the π define a μ_3 or a solid then any point of this solid is the vertex of a μ_1 of elementary sheaves. The σ-sheaf with such a π will be its vertex generates waves in image space that possess a μ_1 of umbilical points or an umbilical point curve in image space. On the contrary, if the π define a μ_2 or a surface then any π is the vertex of a μ_2 of elementary sheaves. The homocentric sheaf with such a π as its vertex generates waves in the image space with a μ_2 of umbilical points; i.e., these waves are spheres, and the surface of points π that one imagines is one component of an anastigmatic pair of surfaces. These case in which the π reduces to a μ_1 or a μ_0 cannot come about for the present group since if that were true then the manifold of the anastigmatic elementary sheaf would be at most a μ_3 . On the contrary, the locus of the points π can very well include isolated curves or points, in addition to the stated bodies or surfaces.

In order to decide which of the two cases we discussed enter into the picture, one must remember that from the pervious examination, along with the expression Ψ, also the loci of the π, Π would be included in the parametric representation:

$$x = \alpha(t, u, T, U), \qquad y = \beta(t, u, T, U), \qquad z = \gamma(t, u, T, U), \tag{152}$$

$$X = \mathrm{A}(t, u, T, U), \qquad Y = \mathrm{B}(t, u, T, U), \qquad Z = \Gamma(t, u, T, U). \tag{153}$$

Now, should the vertices π of the elementary sheaf define a surface – which, from the above, would have the same property as Π – then the four variables t, u, T, U would appear in the functions α, β, γ in only two contexts, so the four functional determinants of the α, β, γ that are defined by any three of the variables t, u, T, U would vanish identically. Of the four equations that arise, two of them are a consequence of the remaining ones; thus, when anastigmatic surfaces are present, two more conditions will be added to the condition that $\Psi \equiv 0$. Since the functions α, β, γ in (152) include the derivatives of the eikonal up to second order, the additional conditions will represent two third-order differential equations, which, together with the second-order equation $\Psi = 0$, will define the eikonals that have anastigmatic surfaces.

In the third of the aforementioned groups, Ψ can vanish only for certain light rays that define a μ_3 . The vertices π can define a μ_3 or μ_2 or μ_1, while the anastigmatic points define at most a μ_1. The appearance of anastigmatic surfaces is then excluded from this group.

The anastigmatic bodies are still missing from the foregoing discussion. As one easily sees, they belong to the second group as a limiting case and appear when the expression for x is plainly undetermined in (152).

The conditions for the appearance of anastigmatic surfaces have been essentially developed. Up till now, the expression Ψ was not entirely concise, while the parametric representation for this class of eikonals that was given before in (114) simply failed to suffice. However, one generally comes to lucid equations for the cases of anastigmatic plane pairs that are most important in optics. The explicit representation that is given in (106) to (109) shows that for the forms [1], [4], [13], [16], the eikonal $E(t, u, T, U)$ must satisfy the conditions:

$$0 = \frac{\partial^2 E}{\partial T^2} = \frac{\partial^2 E}{\partial T\,\partial U} = \frac{\partial^2 E}{\partial U^2}, \tag{154}$$

as long as the coordinate axes were chosen suitably. Moreover, when one goes over to the theory of optical instruments, the difficulties that one must then overcome belong to a circle of problems that is essentially different from the ones that were treated up to now. Since this situation, as far as I know, requires a fairly complex, specialized examination, I will restrict myself to only a brief sketch of it. Questions such as the following ones then arise:

Are ray-wise maps realizable by means of optical processes – e.g., refraction and reflection?

What are the essential properties of eikonals that arise from one, two, three, etc., refractions?

What are the properties of eikonals when one is treating refraction at centered spherical surfaces?

Does the number of constant parameters that enter into the eikonals possess a limiting value or does it go beyond all limits with the increasing number of refractions?

The resolution of these and similar questions is necessary when one wishes to essentially leave behind the *general* theory of optical systems for the more contemporary view of things. As long as practical optics is composed of a summary of a great number of deviations from the single path of numerical tests that is practicable at this point in time, the progress that has actually been achieved will essentially be the result of an art that is acquired by years of instruction and experience, whose details one must always pick up anew, but which cannot be handed down in the form of an edifice of finished and generally valid theorems, as with other theoretical domains that have been worked out completely. Undoubtedly, the power of contemporary microscope and camera objectives rests upon the unconscious pause to consider certain general laws whose rigorous formulation would provide the insight into the true basis for the results that have been obtained. An example of this is the proof that was carried out by ABBE that the sine law in optics was already obeyed, and thus unconsciously, before his own discovery of it ([1]).

With these remarks, we may now treat some applications that illustrate how one uses eikonals in special cases.

([1]) See the remarks of CZAPSKI above on page 103.

XII.

Eikonal of a refracting surface.

The first example that shall be examined is refraction at a surface. Instead of presenting the equations for this directly, we will first explore the conditions for the two components σ and Σ of a light path to intersect each other under a ray-wise map. If one lets the system of axes (X, Y, Z) in the image space Ω coincide with the system of axes (x, y, z) in object space ω then the equations:

$$\left.\begin{aligned} y &= h + \frac{xp}{m} = H + \frac{xP}{M}, \\ z &= k + \frac{xq}{m} = K + \frac{xQ}{M} \end{aligned}\right\} \tag{155}$$

must exist between the ray intersection (x, y, z) and the ray coordinates h, k, p, q and H, K, P, Q, from which it follows that:

$$\begin{gathered} H - h = x\left(\frac{p}{m} - \frac{P}{M}\right), \qquad K - k = x\left(\frac{q}{m} - \frac{Q}{M}\right), \\ x = \frac{H-h}{\dfrac{p}{m} - \dfrac{P}{M}} = \frac{K-k}{\dfrac{q}{m} - \dfrac{Q}{M}}. \end{gathered} \tag{156}$$

If one employs the eikonal $E(p, q, P, Q)$ and defines the mapping equations:

$$-nh = \frac{\partial E}{\partial p}, \qquad -nk = \frac{\partial E}{\partial q}, \qquad NH = \frac{\partial E}{\partial P}, \qquad NK = \frac{\partial E}{\partial Q}$$

then one will obtain the following linear differential equation for E from (156):

$$\left(n\frac{\partial E}{\partial P} + N\frac{\partial E}{\partial p}\right)\left(\frac{q}{m} - \frac{Q}{M}\right) = \left(n\frac{\partial E}{\partial Q} + N\frac{\partial E}{\partial q}\right)\left(\frac{p}{m} - \frac{P}{M}\right).$$

If one introduces the new variables:

$$e = NM - nm, \qquad f = NP - np, \qquad g = NQ - nq$$

then E can be represented as a function of e, f, g and one of the original variables – e.g., p – in the form:

$$E(p, q, P, Q) = \varphi(e, f, g, p).$$

The differential equation for φ will then assume the form:

$$N\frac{\partial\varphi}{\partial p}\left(\frac{q}{m}-\frac{Q}{M}\right)=0,$$

i.e., p cannot be included explicitly, but only the three quantities e, f, g. Conversely, if the eikonal possesses the form $\varphi(e, f, g)$ then the conjugate rays will intersect each other. By means of the equation:

$$E(p, q, P, Q) = \varphi(e, f, g),$$

one then defines the mapping equations:

$$\left.\begin{aligned}
h &= -\frac{1}{n}\frac{\partial E}{\partial p} = -\frac{p}{m}\frac{\partial\varphi}{\partial e}+\frac{\partial\varphi}{\partial f}, \quad & k &= -\frac{1}{n}\frac{\partial E}{\partial q} = -\frac{q}{m}\frac{\partial\varphi}{\partial e}+\frac{\partial\varphi}{\partial g},\\
H &= \frac{1}{N}\frac{\partial E}{\partial P} = -\frac{P}{M}\frac{\partial\varphi}{\partial e}+\frac{\partial\varphi}{\partial f}, \quad & K &= \frac{1}{N}\frac{\partial E}{\partial Q} = -\frac{Q}{M}\frac{\partial\varphi}{\partial e}+\frac{\partial\varphi}{\partial g},
\end{aligned}\right\} \tag{157}$$

and if one introduces the h, k, H, K into (156) and (155) then this will give two expressions for the x, y, z, whose values agree, namely:

$$x = \frac{\partial\varphi}{\partial e}, \qquad y = \frac{\partial\varphi}{\partial f}, \qquad z = \frac{\partial\varphi}{\partial g}. \tag{158}$$

The foregoing equations (158) define the locus of the points $\Pi(x, y, z)$ at which the two conjugate rays cut each light path. Since the totality of light paths define a μ_1, but the possible rays through a point defines a μ_2 , the locus of points Π will then be a μ_3 or a μ_2 .

If, as we shall first assume, the equations (158) are independent of each other in the e, f, g then the e, f, g can be expressed in terms of the x, y, z, so one will have that Π is a μ_3 . In this case, the expression:

$$\tau = e\frac{\partial\varphi}{\partial e}+f\frac{\partial\varphi}{\partial f}+g\frac{\partial\varphi}{\partial g}-\varphi \tag{159}$$

cannot reduce to a constant c identically. Otherwise, due to the identity:

$$e\frac{\partial\varphi}{\partial e}+f\frac{\partial\varphi}{\partial f}+g\frac{\partial\varphi}{\partial g}\equiv\varphi-c, \tag{160}$$

the expression $\varphi - e$ would be a homogeneous function of first order of the e, f, g, and the right-hand sides of (158) would be homogeneous functions of order zero; thus, they would be representable as functions of the two quotients $f : e$ and $g : e$. However, this contradicts the assumed solubility of the system (158). If one thinks of the e, f, g in (159) or in:

$$\tau = ex + fg + gz - \varphi$$

as being expressed in terms of the x, y, z and then forms the function $\tau(x, y, z)$ then, from (158), one will have:

$$d\tau(x, y, z) = e\,dx + f\,dy + g\,dz. \tag{161}$$

In the family of surfaces:

$$\tau(x, y, z) = \text{constant},$$

the direction cosines of the normal to the surface that goes through the point $\Pi(x, y, z)$ will then be proportional to the e, f, g.

A μ_1 of rays or a family of them goes through the ray intersection Π in question. Let $\Pi(x_0, y_0, z_0)$ – or, more briefly, Π_0 – be any of these intersection points. If one defines the τ-surface through Π_0 and determines the system of values e_0, f_0, g_0 for the e, f, g from:

$$x_0 = \frac{\partial\varphi}{\partial e}, \qquad y_0 = \frac{\partial\varphi}{\partial f}, \qquad z_0 = \frac{\partial\varphi}{\partial g}$$

then the direction cosines of the surface normal at Π_0 will be proportional to the e_0, f_0, g_0. The family of light rays for which Π_0 is the common intersection point will be determined by the equations:

$$NM - nm = e_0, \qquad NP - np = f_0, \qquad NQ - nq = g_0;$$

i.e., the conjugate rays of the family behave as if refraction were taking place at Π_0 with indices of refraction n, N and the surface normal as the incidence perpendicular. Furthermore, the rays will be arranged symmetrically around the incidence perpendicular, and thus define a circular cone.

By comparison, if the right-hand sides in (158) are independent of each other then the locus of ray intersections Π will reduce to a certain surface Φ, whose equation in x, y, z is obtained from (158) by eliminating the e, f, g; any point Π on this surface is then the common ray intersection point for a sheaf of light rays. This sheaf may be decomposed into circular cones in the following manner: A given point Π_0 in Φ is associated with a μ_1 of systems of values for the e, f, g that satisfy the equations (158). If e_0, f_0, g_0 is an arbitrarily-selected solution of (158) then the equations:

$$NM - nm = e_0, \qquad NP - np = f_0, \qquad NQ - nq = g_0$$

will determine a family of light paths that have the common ray intersection Π_0. The conjugate rays of this family will, in turn, behave as if refraction took place at Π_0 with the indices of refraction n, N and an incidence perpendicular whose direction cosines are proportional to the quantities e_0, f_0, g_0. In general, these incidence perpendiculars are now all different from each other and define a family. However, should the perpendiculars all coincide, as is necessarily the case for the refraction at a single surface, then all of the solutions of (158) that belong to a point Π_0 would have to be proportional to each other – i.e., the right-hand sides of (158) would have to involve only the ratios of the e, f, g – so they would have to be homogeneous of order zero in e, f, g. The function

φ will then be homogeneous of first order, up to an additive constant. Therefore, we have obtained the following theorem:

If $E(p, q, P, Q)$ is the eikonal for refraction at a surface then, if the coordinate axes for the object space and image space coincide, it can be represented in the form:

$$E(p, q, P, Q) = \varphi(e, f, g), \tag{162}$$

where φ, except for an additive constant, means a homogeneous function of first order of the three quantities:

$$e = NM - nm, \qquad f = NP - np, \qquad g = NQ - nq; \tag{163}$$

the refracting surface itself is determined by the equations:

$$x = \frac{\partial\varphi}{\partial e}, \qquad y = \frac{\partial\varphi}{\partial f}, \qquad z = \frac{\partial\varphi}{\partial g}. \tag{164}$$

This converse of this theorem is also true.

As an example, we would like to choose:

$$E(p, q, P, Q) = \varphi(e, f, g) = \alpha\, e + \beta\, J, \tag{165}$$

where the α, β mean constants, and the quantity J is determined by the equation:

$$\left.\begin{aligned} J^2 &= e^2 + f^2 + g^2 \\ &= N^2 + n^2 - 2Nn(Mm + Pp + Qq) \\ &= (N-n)^2\left(1 + 2Nn\frac{1 - Mn - Pp - Qq}{(N-n)^2}\right). \end{aligned}\right\} \tag{166}$$

The sign of J should coincide with that of $N - n$; i.e., in the equation:

$$J = (N-n)\left(1 + 2Nn\frac{1 - Mm - Pp - Qq}{(N-n)^2}\right)^{1/2}, \tag{167}$$

the roots should always be taken to be positive. Since φ is homogeneous of first order in e, f, g, for this eikonal we are then dealing with refraction at a single surface and indices of refraction n, N. We obtain the equation of the surface from (164) in the form:

$$x = a + \frac{\beta e}{J}, \quad y = \frac{\beta f}{J}, \qquad z = \frac{\beta g}{J},$$

from which it follows that:

$$(x - \alpha)^2 + y^2 + z^2 = \beta^2;$$

the refraction happens at a sphere of radius $\pm\beta$ with a center that lies along the x-axis with the abscissa α. Since one deals with only spherical sections for lens surfaces, we assume, in order to present the formula for this case, that the abscissas x should increase in the direction of the light motion; therefore, only positive values for the M, n will come under consideration. For a light path along the x-axis, one then has:

$$M = m = +1, \qquad f = g = 0,$$
$$e = N - n, \qquad J = N - n,$$

from which, it follows that the vertex abscissa must equal $\alpha + \beta$. If one assumes that the curvature radius ρ of the refracting spherical section is positive or negative according to whether the surface has its hollow side in the direction of increasing or decreasing x, resp., then one will have:

$$\alpha = (\alpha + \beta) + \rho, \qquad \rho = -\beta.$$

The eikonal considered may then be written in the form:

$$E(p, q, P, Q) = (a + \rho)\, e - \rho\, J, \tag{167}$$

where a is the vertex abscissa, $a + \rho$ is the center abscissa, and ρ is the radius of curvature.

If one treats the composition of two refractions between three media ω_1, ω_2, ω_3 that are bounded by each other then if the exponent of refraction n and direction cosines m, p, q for the individual spaces are affected with the indices 1, 2, 3, resp., then one must next construct the expression:

$$\Gamma = \varphi(n_2\, m_2 - n_1\, m_1,\, n_2\, p_2 - n_1\, p_1,\, n_2\, q_2 - n_1\, q_1)$$
$$+\ \psi(n_3\, m_3 - n_2\, m_2,\, n_3\, p_3 - n_2\, p_2,\, n_3\, q_3 - n_2\, q_2),$$

where φ, ψ are homogeneous functions of their arguments. The elimination of p_2 and q_2 by means of the conditions:

$$\frac{\partial\Gamma}{\partial p_2} = \frac{\partial\Gamma}{\partial q_2} = 0$$

then converts Γ into the eikonal $E(p_1, q_1, p_3, q_3)$ of the composed map of the two refractions. One proceeds in a similar way when arbitrarily many refractions are composed. As long as nothing more is assumed about the refracting surfaces, one must regard the φ, ψ, … as arbitrary functions. If one now wants to find the common characteristic properties of the maps that consist of two, three, etc., refractions then one must eliminate the arbitrary functions and the variables p, q for the intermediate media, which leads to partial differential equations. The elimination of the arbitrary functions can be performed with no great difficulty, although I have succeeded in bringing the final formulas into a sufficiently tractable form only in the case of two refracting surfaces of revolution with a common axis, and it seems that one learns from this to preserve the intermediate variables in the search for general relations. Moreover, it is recommended that in the case where the refracting surfaces are completely known, one should present the formulas in such a way that the required elimination is not analytic, but must be done numerically. For example, if an ordinary system of lenses with centered spherical surfaces and the individual media $\omega_1, \omega_2, \ldots, \omega_r$ are given then one lays the x-axis in the figure axis of the system and lets the base planes of the individual spaces coincide. The eikonals $E(\omega_1\ \omega_2)$, $E(\omega_2\ \omega_3)$, … of the individual refractions are then to be established by means of (167). If one attaches the quantities n, h, k, p, q to the numbers of their media then one will next obtain the pair of equations:

$$\left.\begin{array}{lll}
\mathrm{I}: & -n_1h_1 = \dfrac{\partial E(\omega_1\omega_2)}{\partial p_1}, & -n_1k_1 = \dfrac{\partial E(\omega_1\omega_2)}{\partial q_1}, \\
\mathrm{II}: & n_2h_2 = \dfrac{\partial E(\omega_1\omega_2)}{\partial p_2}, & n_2k_2 = \dfrac{\partial E(\omega_1\omega_2)}{\partial q_2}, \\
\mathrm{III}: & -n_2h_2 = \dfrac{\partial E(\omega_2\omega_3)}{\partial p_2}, & -n_2k_2 = \dfrac{\partial E(\omega_2\omega_3)}{\partial q_2}, \\
\mathrm{IV}: & n_3h_3 = \dfrac{\partial E(\omega_2\omega_3)}{\partial p_3}, & n_3k_3 = \dfrac{\partial E(\omega_2\omega_3)}{\partial q_3}.
\end{array}\right\} \tag{168}$$

In order to follow the progress of a given ray in the first medium, one must find the quantities p_2, q_2 from the given h_1, k_1, p_1, q_1 using I and then calculate the h_2, k_2 from II; from III and IV, what then follows is a sequence of similar values p_3, q_3, p_4, q_4, etc. If one brings the solutions and substitutions that are required in (168) into a form that is appropriate for numerical computations by the introduction of suitable auxiliary variables then one will arrive at the well-known computational prescription for rays that do not lie in an axis plane. Whether one then chooses the trigonometric form or the purely algebraic form for the calculations is then based on personal tastes. In the algebraic system of formulas that is mentioned below, as is easy to verify, the problem to be solved for the map (167) is to find H, K, P, Q from given h, k, p, q when:

$$E(p, q, P, Q) = (a + \rho)(NM - nm) - \rho J,$$
$$J^2 = N^2 + n^2 - 2Nn(Mm + Pp + Qq).$$

The equations are then manipulated in such a way that the largest possible family of final values is attained with a prescribed logarithmic number of digits, such that the loss of precision by which one arrives at simple formulas is minimized; the final equations give the changes that the h, k, p, q suffer under refraction directly. Thus, given h, k, p, q, one computes the sequence from:

$$m^2 = 1 - p^2 - q^2,$$

$$N\rho x = h + \frac{a+\rho}{m}p, \qquad N\rho h = k + \frac{a+\rho}{m}q, \qquad \zeta = \frac{\xi q - \eta p}{m},$$

$$\zeta_1 = m^2(\xi^2 + \eta^2 + \zeta^2), \qquad \zeta_2 = +\sqrt{1-\zeta_1 N^2}, \qquad J = N - n + \zeta_1,$$

$$\zeta_5 = \frac{(N-n)J\zeta_1}{\zeta_2+\zeta_3}, \qquad \zeta_6 = \zeta_5 - J(\xi p + \eta q),$$

$$\frac{M}{m} = 1 - \zeta_6, \qquad P - p = -p\zeta_6 - \xi J, \qquad Q - q = -q\zeta_6 - \eta J,$$

$$\zeta_7 = \frac{aJ}{M} + \rho\frac{N-n}{M}\frac{p^2+q^2}{1+m} + \rho\frac{N-n}{M}\zeta_1 + \rho N\frac{\zeta_6}{1-\zeta_6},$$

$$H - h = \xi\zeta_7, \qquad K - k = \eta\zeta_7 .$$

In this, the following "control equation" comes about:

$$n(hp - kp) = N(HQ - KP).$$

If one desires to determine the focal lines for a previously-computed light path then the calculations will take the following form by the application of the eikonal: One differentiates the equations (168), when one consider all coordinates to be variable. The coefficients of the differentials are then known numerical quantities, and due to the special form of the equations, the elimination of the dh, dk, dp, dq that belong to the intermediate media presents no significant difficulties. After the final elimination, one will obtain equations of the following form for the dh, dk, dp, dq of the two end media ω_1 and ω_r :

$$\begin{aligned}
-n_1\,dh_1 &= E_{11}\,dp_1 + E_{12}\,dq_1 + E_{13}\,dp_r + E_{14}\,dq_r,\\
-n_1\,dk_1 &= E_{21}\,dp_1 + E_{22}\,dq_1 + E_{23}\,dp_r + E_{24}\,dq_r,\\
n_r\,dh_r &= E_{31}\,dp_1 + E_{32}\,dq_1 + E_{33}\,dp_r + E_{34}\,dq_r,\\
n_r\,dk_r &= E_{41}\,dp_1 + E_{42}\,dq_1 + E_{43}\,dp_r + E_{44}\,dq_r,
\end{aligned}$$

where the $E_{\alpha\beta}$ are likewise the second-order partial derivatives of the eikonal $E(p_1, q_1, p_r, q_r)$ of the composed map. However, the quantities that one must substitute in the equations for the focal lines (122) are given by these derivatives.

If one wishes to find the changes in the focal lines that come about under the displacement of a light path then one must then know the third-order derivatives of the eikonal. When one follows through with this manner of reasoning, it leads to the problem of presenting the eikonal as a power series development in its variables. Such power series developments are, up to now, used primarily to represent the so-called *aberrations* in systems of spherical lenses in a purely analytical way. The shortcomings that come with this process are well-known. The groups of terms of a particular order rapidly becomes exceptionally unwieldy with increasing order; furthermore, one cannot always infer any conclusion regarding the numerical order of magnitude of a term from the analytical order number. The former flaw lies in the nature of things. The connection between the rays in the first and last medium already leads to a complicated algebraic picture after even a few refractions. Correspondingly, the caustic that a homocentric σ-sheaf generates in image space is an algebraic surface with developable singularities, and one can practically say that the art of the optician consists in condensing such singularities into as small a space as possible. If one poses the problem of finding concise forms for the development then this must be adapted to the special nature of the relations being represented; however, this comes down to a complete insight into the general and essential properties of these relations, as we understand them at the moment. The use of power series developments and their immediate progeny is therefore restricted to the cases in which the changes in a light path that are being investigated remain inside a limited scope. Since the consideration of these cases is, in any event, an approximation, one might still show the form that things take for centered maps when one employs the eikonal, after one has developed it in a power series and used the initial terms.

XIII.

Centered maps. Series development up to fourth order. Aberration curve. Theoretical minimum error for symmetric systems.

As before, let the two spaces that get mapped to each other be ω, Ω, and let each of them be related to its proper system of axes (x, y, z) and (X, Y, Z), resp. A light path with the two components σ, Σ will imply a certain motion that takes the two rays to the positions s', S', and indeed the motion of σ shall consist of a rotation around the x-axis with the rotational angle φ. Now, should the corresponding motion of Σ likewise be a rotation, and indeed a rotation around the X-axis and with the same rotational angle φ then it would be self-explanatory that certain conditions should be fulfilled for the map, as well as the position of the coordinate axes. If these conditions are fulfilled in general – i.e., for any ray σ and for any value of φ – then we would like to call the map "centered;" the x-axis and the X-axis would then be the "centering axes." The simplest – but not the only – examples of such maps are systems of lenses in which refraction takes place on coaxial surfaces of revolution.

For the next considerations about centered maps, we shall always make the x and X axes the centering axes. If the light path $(\sigma' \Sigma')$ arises from the light path $(\sigma \Sigma)$ by the previously-considered rotation through the quantity φ, and we now further denote the coordinates of σ', Σ' by primes, then we will get the equations:

$$
\begin{aligned}
h' &= h \cos\varphi - k \sin\varphi, & p' &= p \cos\varphi - q \sin\varphi, \\
k' &= h \sin\varphi + k \cos\varphi, & q' &= p \sin\varphi + q \cos\varphi, \\
H' &= H \cos\varphi - K \sin\varphi, & P' &= P \cos\varphi - Q \sin\varphi, \\
K' &= H \sin\varphi + K \cos\varphi, & Q' &= P \sin\varphi + Q \cos\varphi,
\end{aligned}
$$

assuming that the positive directions of the lateral axes in the image space were chosen suitably. If one were to switch the direction $+ Y$, $- Y$, or $+ Z$, $- Z$ in image space with each other then one would have to write the opposite value $- j$ in place of j in the equations for H', K', P', Q'. Due to the centering of the map, the mapping equations must again assume their original form under the introduction of h', k', …, in place of h, k, … The same thing is true for the eikonal $E(p, q, P, Q)$, which then contains the four variables only in those combinations that do not change under the transformation above. If we now assume – as is always appropriate for optical applications – that the map behaves regularly in the neighborhood of the light path:

$$p = q = P = Q = 0,$$

so no discontinuities will arise, then $E(p, q, P, Q)$ will be developable in powers of p, q, P, Q. If we write, accordingly:

$$E = G_0 + G_1 + G_2 + \ldots,$$

where all terms of dimension α are combined into G_α , then each individual G must behave just like E; i.e., it will take on the original form under the transition to h', k', … This property can also be expressed as follows: If the substitution:

$$p = r \cos s, \qquad q = r \sin s, \qquad P = R \cos S, \quad Q = R \sin S$$

is carried out in G_α then G will contain the s, S only in the combination $s - S$, in addition to the r, R. From that, one proves with no difficulty that the G can be represented as entire homogeneous functions of the four combinations:

$$\alpha = p^2 + q^2, \quad \beta = p\,P + q\,Q, \quad \gamma = P^2 + Q^2, \quad \delta = p\,Q - q\,P,$$

and as a result, E can be developed into an ordinary power series in α, β, γ, δ. All terms in E are thus of even dimension in p, q, P, Q, and furthermore the expressions for h, k, H, K contain only terms of odd dimension. Therefore, h, k, H, K vanish simultaneously for the light path:

$$p = q = P = Q = 0;$$

i.e., this light path falls along the centering axes.

If, for the moment, one writes:

$$E(p, q, P, Q) = E_1\,d\alpha + E_2\,d\beta + E_3\,d\gamma + E_4\,d\delta$$

then the mapping equations will become:

$$\left.\begin{aligned} -nh &= 2pE_1 + PE_2 + QE_4, \qquad & NH &= pE_2 + 2PE_3 - qE_4, \\ -nk &= 2qE_1 + QE_2 - PE_4, \qquad & NK &= qE_2 + 2QE_3 + pE_4, \end{aligned}\right\} \tag{169}$$

from which it will follows that:

$$n\,(pk - hq) = N\,(PK - HQ). \tag{170}$$

If the σ-rays lie in a plane g that goes through the x-axis then the left-hand side of (170) will vanish, so it follows that the right-hand side must also be zero. In any event, the conjugate Σ therefore lie in a plane G that goes through the X-axis. If one rotates g around a certain angle then G rotates by the same angle. If one now directs the lateral axes in such a way that the xy –plane and xz-plane are conjugate to the XY-plane and XZ-plane, resp., then two rays σ and σ' that are symmetric about the xy-plane will give two rays Σ and Σ' in image space that are symmetric about the XY-plane. The two light paths that are associated with them will then switch roles when one simultaneously changes the sign of k, q, K, Q. Since this sign change leaves the eikonal unchanged, due to the defining equation:

$$E = \int (-\,nh\,dp - nk\,dq + NH\,dP + NK\,dQ),$$

and of the four quantities α, β, γ, δ, only the sign of δ changes, the series development of E in α, β, γ, δ will contain only even powers of δ. However, due to the identity:

$$\delta^2 = \alpha\gamma - \beta^2,$$

they can be eliminated. Therefore, when the lateral axes in object space and image space are erected in conjugate planes g, G, the eikonal can be written in the form:

$$E(p, q, P, Q) = f(\alpha, \beta, \gamma) \tag{171}$$

and developed into an ordinary power series in α, β, γ. We will always think of the lateral axes as having been chosen in the way that was just given, so E will be assumed to have the form (171).

Precisely the same considerations can be applied to the other three eikonals:

$$E(h, k, P, Q), \quad E(h, k, H, K), \quad E(p, q, H, K).$$

The combinations:

$$\begin{array}{lll} h^2 + k^2, & h\,P + k\,Q, & P^2 + Q^2, \\ h^2 + k^2, & h\,H + k\,K, & H^2 + K^2, \\ p^2 + q^2, & p\,H + q\,K, & H^2 + K^2 \end{array}$$

then enter in place of α, β, γ.

If one goes up to the terms of order four in the series development for $E(p, q, P, Q)$ then one will always get sufficiently clear relations. We next remain in the first approximation for the terms of second order and assume that:

$$E = a\,(p^2 + q^2) + b\,(p\,P + q\,Q) + c\,(P^2 + Q^2). \tag{172}$$

The mapping equations become:

$$\left.\begin{array}{ll} -nh = 2ap + bP, & NH = bp + 2cP, \\ -nk = 2aq + bQ, & NK = bq + 2cQ. \end{array}\right\} \tag{173}$$

If we again introduce points $\pi(x, y, z)$, $\Pi(X, Y, Z)$ that lie on the conjugate rays σ, Σ then we will have:

$$\left.\begin{array}{ll} h = y - \dfrac{xp}{m}, & H = Y - \dfrac{XP}{M}, \\ k = z - \dfrac{xq}{m}, & K = Z - \dfrac{XQ}{M}. \end{array}\right\} \tag{174}$$

Since third-order terms were neglected in (173), the same will be true for (174); i.e., we can set $m = M = 1$ and get:

$$\left.\begin{array}{ll} -n(y - xp) = 2ap + bP, & N(X - XP) = bp + 2cP, \\ -n(z - xq) = 2aq + bQ, & N(Z - XQ) = bq + 2cQ. \end{array}\right\} \tag{174a}$$

The quotients:

$$\frac{ny}{NY} = \frac{(nx-2a)p-bP}{bp+(NX+2c)P}, \qquad \frac{nz}{NZ} = \frac{(nx-2a)q-bQ}{bq+(NX+2c)Q} \tag{175}$$

will both be independent of the light path, as long as one chooses x, X such that:

$$(nx-2a)\,(NX+2c)+b^2=0.$$

In this, one will have:

$$\frac{ny}{NY} = \frac{nz}{NZ} = \frac{nx-2a}{b} = \frac{-b}{NX+2c}. \tag{176}$$

These are the known equations for the collineation between conjugate points π, Π. The collineation that contradicts MALUS's theorem will obviously come about by setting the quantities $1-m$ and $1-M$ equal to zero in (174). The abscissas of the focal points in object space and image space will become:

$$f = \frac{2a}{n}, \qquad F = -\frac{2c}{N}.$$

The coefficient b is the reduced focal length, and one gets from (175) that:

$$\frac{y}{Y} = \frac{z}{Z} = N\frac{x-f}{b} = -\frac{b}{n(X-F)}, \qquad (x-f)(X-F) = -\frac{b}{n}\frac{b}{N}. \tag{177}$$

The abscissas of the centers are given by the condition:

$$y = Y, \qquad z = Z,$$

and indeed, one gets:

$$x = f + \frac{b}{N}, \qquad X = F - \frac{b}{n}.$$

The cusps or the conjugate points of equal ray divergence follow from (174a) by the conditions:

$$y = z = Y = Z = 0, \qquad p = P, \quad q = Q.$$

The abscissas of the cusps then become:

$$x = f + \frac{b}{n}, \qquad X = F - \frac{b}{N}.$$

The foregoing definition of the two focal points and the reduced focal length obviously touch upon the behavior of the elementary sheaf that lies infinitely close to the centering axes. With the quantities f, F, b, one then has:

$$E(p, q, P, Q) = nf\,\frac{p^2+q^2}{2} + b\,(p\,P + q\,Q) - NF\,\frac{P^2+Q^2}{2}, \tag{178}$$

$$\left.\begin{aligned} -nh &= nfp + bP, \qquad & NH &= bp - NFP, \\ -nk &= nfQ + bQ, \qquad & NK &= bq - NFQ. \end{aligned}\right\} \tag{179}$$

In order to obtain the meaning of the other coefficients in the eikonal that comes under consideration here, we assume:

$$\begin{aligned} E(h, k, P, Q) &= a_1(h^2 + k^2) + a_2(h\,P + k\,Q) + a_3(P^2 + Q^2), \\ E(h, k, H, K) &= b_1(h^2 + k^2) + b_2(h\,H + k\,K) + b_3(H^2 + K^2), \\ E(p, q, H, K) &= c_1(p^2 + q^2) + c_2(p\,H + q\,K) + c_3(H^2 + K^2), \end{aligned}$$

and add the mapping equations to them:

$$\begin{aligned} np &= 2a_1h + a_2\,P, \qquad & NH &= a_2\,h + 2a_3\,P, \\ np &= 2b_1h + b_2\,H, \qquad & -NP &= b_2\,h + 2b_3\,H, \\ -np &= 2c_1h + c_2\,H, \qquad & -NP &= c_2\,h + 2c_3\,H, \end{aligned}$$

which must be equivalent to the system (173). By comparing them with (173), one obtains the eikonals:

$$E(h, k, P, Q) = -\,n\frac{h^2+k^2}{2f} - b\frac{hP+kQ}{f} - \frac{n\,N\,f\,F+b^2}{n}\cdot\frac{P^2+Q^2}{2f},$$

$$E(h, k, H, K) = \frac{(nN)^2}{n\,N\,f\,F+b^2}\left(-F\,\frac{h^2+k^2}{2N} + b\frac{hH+kK}{nN} + f\,\frac{H^2+K^2}{2n}\right),$$

$$E(p, q, H, K) = \frac{n\,N\,f\,F+b^2}{N}\cdot\frac{p^2+q^2}{2F} - b\frac{pH+qK}{F} + N\,\frac{H^2+K^2}{2F}.$$

In these forms, one recognizes that the series development breaks down for certain positions of the base planes. For $E(h, k, P, Q)$, the ω-base plane cannot go through the focal point of the object space, because f is then equal to zero; the corresponding statement is true for $E(p, q, H, K)$. Finally, for $E(h, k, H, K)$, the base planes cannot be conjugate. These restrictions in the position of the base plane are not present for the eikonal $E(p, q, P, Q)$.

If one now includes the fourth-order terms in the series development then the sheaf of rays that appears will generally be astigmatic. The discussion of the basic equations (81) or:

$$0 = \frac{\partial\Theta}{\partial t} = \frac{\partial\Theta}{\partial u} = \frac{\partial\Theta}{\partial T} = \frac{\partial\Theta}{\partial U},$$

which now appear in rational form with certain temporarily undetermined coefficients, can now be carried out in very different ways, according to whether one desires to examine them or some property of the map. A convenient means for the visualization of the ray evolution consists in looking for the intersection of a family of planes that are perpendicular to the centering axis. With the inclusion of fourth-order terms, one sets:

$$p^2 + q^2 = u_1 , \qquad p\,P + q\,Q = u_2 , \qquad P^2 + Q^2 = u_3 ,$$

$$E(p, q, P, Q) = \tfrac{1}{2} n\, f\, u_1 + b\, u_2 - \tfrac{1}{2}\, N\, F\, u_3 + G,$$

$$2G = \sum_{\alpha,\beta} a_{\alpha\beta} u_\alpha u_\beta \qquad (\alpha, \beta = 1, 2, 3),$$

$$dG = G_1\, du_1 + G_2\, du_2 + G_3\, du_3 ,$$

$$\Theta = \tfrac{1}{2} n\, f\, u_1 + b\, u_2 - \tfrac{1}{2}\, N\, F\, u_3 + G + n\,(xm + yp + zq) - N\,(XM + YP + ZQ). \tag{180}$$

The differentiation of Θ with respect to p and P next delivers two conditions, namely:

$$\left.\begin{aligned} 0 &= p(uf + 2G_1) + P(b + G_2) + n\left(y - \frac{px}{m}\right), \\ 0 &= p(b + G_2) + P(-NF + 2G_2) - N\left(Y - \frac{PX}{M}\right), \end{aligned}\right\} \tag{181}$$

while the other two are obtained by switching the lateral axes. With the use of the relation:

$$y - h = \frac{px}{m},$$

we can bring the conditions (181) into the form:

$$\begin{aligned} P &= -\frac{ym}{x}\frac{nf + 2G_1}{b + G_2} - \frac{h}{x}\frac{n(x - mf) - 2mG_1}{b + G_2}, \\ NY &= \frac{y - h}{x} m(b + G_2) + P\left\{\frac{NX}{M} - NF + 2G_3\right\}. \end{aligned} \tag{182}$$

The elimination of P gives:

$$NY = Ay + Bh,$$

where

$$xA = mb + G_2 - \frac{nf + 2G_1}{b + G_2}\left(\frac{NmX}{M} - NmF + 2G_3\right),$$

$$xB = -mb - G_2 - \frac{n(x-mf)-2mG_1}{b+G_2}\left(\frac{NX}{M} - NF + 2G_3\right).$$

These equations are still correct when one includes the terms of all order in G. If one now develops them in u, further performs the allowed omissions, and expresses the P, Q in u in terms of y, z, h, k using (182) then one will obtain A and B in the form:

$$A = A_0 + A_1 (y^2 + z^2) + A_2 (yh + zk) + A_3 (h^2 + k^2),$$
$$B = B_0 + A_1 (y^2 + z^2) + B_2 (yh + zk) + B_3 (h^2 + k^2),$$

where the A_0, B_0, … depend upon only the constants of the eikonal and the abscissas x, X.

One now thinks of an aperture as being given in the base plane of the object space whose opening shall be a circle with a radius D that is described around the x-axis. Furthermore, one thinks of the cone of σ-rays that has the given point $p(x, y, z)$ in object space as its vertex and possesses the boundary of the aperture for its base. This cone generates a Σ-family in image space whose intersection with the plane:

$$X = \text{constant}$$

is determined by the equations:

$$NY = Ay + Bh, \qquad NZ = Az + Bk.$$

If one assumes that $z = 0$, which is no essential restriction, and sets:

$$h = D \cos \varphi, \quad k = D \cos \varphi$$

then one will obtain a representation of Y and Z that takes the form:

$$\left.\begin{aligned} Y &= \alpha + \beta \cos \varphi + \gamma \cos \varphi^2, \\ Z &= \qquad \sin \varphi(\delta + \varepsilon \cos \varphi). \end{aligned}\right\} \tag{183}$$

The intersection of the Σ-family considered, which was referred to as the "aberration curve" by CHARLIER ([1]), is of degree four, and, as one says, *unicursal*, or of rank zero; i.e., the coordinates can be represented as rational functions of one parameter t by the substitution:

$$\cos y = \frac{1-t^2}{1+t^2}, \qquad \sin \varphi = \frac{2t}{1+t^2},$$

and, for that reason, the curve consists of a single line that turns back upon itself.

When one includes the terms up to order $2r$ in the development of the eikonal, one will get the following representation for the aberration curve:

([1]) "Ueber den Gang des Lichtes durch ein System von sphärischen Linsen," Der k. Ges. d. Wiss. zu Upsala, presented on 20 June 1893.

$$\left.\begin{aligned} Y &= \alpha_0 + \alpha_1 \cos\varphi + \cdots + \alpha_r \cos\varphi^r , \\ Z &= \sin\varphi(\beta_0 + \beta_1 \cos\varphi + \cdots + \beta_{r-1} \cos\varphi^{r-1}), \end{aligned}\right\} \tag{184}$$

and indeed, except for the limiting cases, the terms with the coefficients α_r and β_{r-1} actually occur. The curve is then once more unicursal, but of degree $2r$. Now, for a line system that is composed of given algebraic surfaces, the Σ-family that is considered here is an algebraic ruled surface of a completely determined finite degree, and the same thing is true of the intersection curve, while the series development that is used leads to arbitrarily high degrees for successively higher approximations. This contradiction can be resolved by the fact that the true form of the aberration curve is not unicursal, in general.

In order to clarify these remarks, one can perform the calculations in the case of a refracted sphere. As before, we assume that the eikonal has the form:

$$E(p, q, P, Q) = (a + r)(NM- nm) - rJ, \tag{185}$$
$$J^2 = N^2 + n^2 - 2Nn\,(Mm + Pp + Qq), \tag{186}$$

and construct the mapping equations from it:

$$-h = \frac{a+\rho}{m}\,p + \frac{N\rho}{J}\left(P - M\,\frac{p}{m}\right), \qquad -k = \frac{a+\rho}{m}\,q + \frac{N\rho}{J}\left(Q - M\,\frac{q}{m}\right), \tag{187}$$

$$H = -\frac{a+\rho}{M}\,P + \frac{n\rho}{J}\left(p - m\,\frac{P}{M}\right), \qquad K = -\frac{a+\rho}{M}\,Q + \frac{m\rho}{J}\left(q - m\,\frac{Q}{M}\right), \tag{188}$$

the first pair of which, with the abbreviation:

$$h + \frac{a+\rho}{m}\,p = N\rho\xi , \quad k + \frac{a+\rho}{m}\,q = N\rho\eta ,$$

can be brought into the form:

$$0 = J\xi + P - M\,\frac{p}{m}, \qquad 0 = Jh + Q - M\,\frac{q}{m}. \tag{189}$$

We further introduce the three auxiliary quantities α, γ, Γ, which are connected by the equations:

$$\gamma = \sqrt{1-\alpha n^2} , \qquad \Gamma = \sqrt{1-\alpha N^2} , \tag{190}$$

and set:

$$J = N\gamma - n\Gamma, \tag{191}$$

by which the relationship between J and α is determined. It follows from (186) and (191) that:

$$
\begin{aligned}
(N\gamma - n\Gamma)^2 &= N^2 + n^2 - 2Nn(Mm + Pp + Qq), \\
Mm + Pp + Qq &= Mn\alpha + \Gamma\gamma.
\end{aligned}
\tag{192}
$$

Furthermore, the combination of (189) and (192) gives:

$$
\left.
\begin{aligned}
&\frac{M}{m} = Mn\alpha + \gamma\Gamma + J(p\xi + q\eta), \\
&P = p\frac{M}{m} - J\xi, \qquad Q = q\frac{M}{m} - J\eta,
\end{aligned}
\right\}
\tag{193}
$$

with which, the M, P, Q are expressed in terms of h, k, m, p, q, as well as the constants and α. If one forms the square-sum of the last two equations then, after an appropriate reduction, one will get:

$$\alpha = \xi^2 + \eta^2 - (p\xi + q\eta)^2. \tag{194}$$

As above, we again imagine the circular aperture of radius D, place the base plane of the eikonal in the plane of the aperture, and correspondingly set:

$$h = D\cos\varphi, \quad k = D\sin\varphi.$$

If we further restrict ourselves to a parallel σ-family that goes through the aperture then m, p, q will be constant. Finally, if the "plane of the screen," in which the aberration curve will intersect the Σ-family, possesses the abscissa X then the coordinates Y, Z of a point on the curve will be determined from (188) and (193) in the form:

$$
\left.
\begin{aligned}
Y &= \frac{p}{m}(X - a - \rho) - \frac{J\xi}{M}\left(X - a - \rho - \frac{nm\rho}{J}\right), \\
Z &= \frac{q}{m}(X - a - \rho) - \frac{J\eta}{M}\left(X - a - \rho - \frac{nm\rho}{J}\right).
\end{aligned}
\right\}
\tag{195}
$$

If we take the special value $a + \rho$ for X, which is justified for our purposes, and further make the allowed simplification $q = 0$ then we will get:

$$
\left.
\begin{aligned}
(N\rho)^2\alpha &= D^2 + p^2(a+\rho)^2 + 2mp(a+\rho)D\cos\varphi - p^2D^2\cos\varphi^2, \\
Y &= \xi\frac{nm\rho}{M}, \qquad Z = \eta\frac{nm\rho}{M}.
\end{aligned}
\right\}
\tag{196}
$$

The expression α is of second degree in $\cos\varphi$, and furthermore the quotients $Y : \xi$ and $Z : \eta$ include the irrationalities γ, Γ, which cannot be simultaneously eliminated by rational substitutions for cos φ. For that reason, the rank of the curve in question is certainly greater than zero.

The goal that the introduction of the aberration curve shall serve to facilitate can be achieved, moreover, in a much simpler way, as long as the series development of the eikonal up to a certain order exists. If we next assume that we have a parallel sheaf in object space that is bounded in some way – e.g., by circular apertures – then a middle light path will belong to any sheaf of light paths whose coordinates in image space might be H_0, K_0, P_0, Q_0 . If one sets:

$$H = H_0 + \delta H, \quad K = K_0 + \delta K, \quad P = P_0 + \delta P, \quad Q = Q_0 + \delta Q$$

then one will have for the eikonal $E(p, q, P, Q)$:

$$N\ \delta N = \frac{\partial E}{\partial(\delta P)}, \qquad N\ \delta K = \frac{\partial E}{\partial(\delta Q)},$$

and the expressions on the right-hand sides can be constructed immediately, as long as one has derived the development of E in the δP, δQ from the original series. If the plane of the screen has the abscissa X then for the points Y, Z in the plane of the screen, one will get:

$$\delta Y = \delta H - X\ \frac{\partial M}{\partial(\delta P)}, \qquad \delta Z = \delta K - X\ \frac{\partial M}{\partial(\delta Q)}.$$

If one now subjects – e.g., δP, δQ – to the condition:

$$\delta P^2 + \delta Q^2 = \text{constant} = D^2$$

then the Σ-sheaf will decompose into families that fall sufficiently close to the boundary of a circular aperture and generate the aberration curve in the plane of the screen.

Should the vertex of the homocentric σ-sheaf lie at a finite point, then the development would be performed in the same way, except that the eikonal $E(h, k, P, Q)$ would then be used.

As an example of the foregoing, we would like to look for the theoretical minimum for the residual error for a symmetric objective. As was already mentioned previously, it is not possible to simultaneously fulfill the requirements of anastigmatism and a correct image for a symmetric map of the focal plane. In order to fix the presentation, one imagines a photographic objective that is well-defined for infinitely-distant subjects, and for which two congruent line systems are arranged asymmetrically in the plane of the aperture, and are thus united into a single centered system. With the allowed simplification $n = N = 1$, we assume that the eikonal has the form:

$$E(p, q, P, Q) = f(1 - m) + bu - F(1 - M) + G(u, v, w),$$

$$u = p^2 + q^2, \quad v = pP + qQ, \quad w = P^2 + Q^2,$$

where b means the reduced focal length, and f, F means the focal point abscissas. The development of G in u, v, w begins with terms of second order. Due to the assumed symmetry, one has:

$$f = -F, \qquad G(u, v, w) = G(w, v, u).$$

An arbitrary light path cuts the plane of the screen in image space that is perpendicular to the figure axis and has the abscissa X at a point whose coordinates are:

$$Y = (X - F)\frac{P}{M} + bp + \frac{\partial G}{\partial P}, \quad Z = (X - F)\frac{Q}{M} + bq + \frac{\partial G}{\partial Q}.$$

Because the system of lenses is well-defined for infinitely-distant objects, only those sheaves of light paths will come under consideration that consist of parallel rays in object space.

For the middle rays, due to symmetry, the elementary sheaf through the center of the plane of the aperture satisfies the relation:

$$p = P, \qquad q = Q.$$

We next demand that this elementary sheaf in the focal plane must generate an anastigmatic image. One will then have for the middle rays:

$$X = F, \qquad Y = bp + \frac{\partial G}{\partial P}, \qquad Z = bq + \frac{\partial G}{\partial Q}.$$

In order to get the condition for the focal lines to coincide on the middle rays, we must vary the light path, so we must introduce the expressions:

$$P + \delta P, \qquad Q + \delta Q,$$

in place of P, Q, while the p, q remain unchanged. The corresponding variations of Y and Z become:

$$\delta Y = \frac{\partial^2 G}{\partial P^2}\delta P + \frac{\partial^2 G}{\partial P\,\partial Q}\delta Q + \frac{1}{2}\frac{\partial^3 G}{\partial P^3}\delta P^2 + \ldots,$$

$$\delta Z = \frac{\partial^2 G}{\partial P\,\partial Q}\delta P + \frac{\partial^2 G}{\partial P^2}\delta Q + \frac{1}{2}\frac{\partial^3 G}{\partial P^2\,\partial Q}\delta P^2 + \ldots$$

With the allowed simplification $q = 0$, $Q = 0$, one will get, up to terms of second order in δP, δQ:

$$\delta Y = \frac{\partial^2 G}{\partial P^2}\delta P + \frac{1}{2}\frac{\partial^3 G}{\partial P^3}\delta P^2 + \frac{1}{2}\frac{\partial^2 G}{\partial P\,\partial Q^2}\delta Q^2,$$

$$\delta Z = \frac{\partial^2 G}{\partial Q^2}\,\delta Q + \frac{\partial^3 G}{\partial P\,\partial Q^2}\,\delta P \cdot \delta Q\,,$$

and the concurrence of the focal lines require that the two equations:

$$\frac{\partial^2 G}{\partial P^2} = 0, \qquad\qquad \frac{\partial^2 G}{\partial Q^2} = 0$$

be fulfilled for $q = Q = 0$, $P = p$.

The condition for the correct image would give the equations:

$$Y = \frac{bp}{m} = bp + \frac{\partial G}{\partial P}, \qquad\qquad Z = \frac{bq}{m} = bq + \frac{\partial G}{\partial Q},$$

which reduce to the equation:

$$pb\left(\frac{1}{m} - 1\right) = \frac{\partial G}{\partial P} \qquad\qquad (p = P)$$

for $q = Q = 0$. We would like to temporarily write this condition in the form:

$$pb(\varphi(m) - 1) = \frac{\partial G}{\partial P},$$

in which $\varphi(m)$ means a – for the time being – arbitrary function of m. In other words, we would next like to admit a distortion of the image whose magnitude depends upon the value of the expression:

$$m\,\varphi(m) - 1,$$

and vanishes along with it. If, to abbreviate, one denotes the partial derivatives with respect to u, v, w by indices according to the schema:

$$d\psi(u, v, w) = \psi_1\,du + \psi_2\,dv + \psi_3\,dw$$

then for $q = Q = 0$, one will get:

$$\frac{\partial G}{\partial P} = p\,G_2 + 2PG_3\,,$$

$$\frac{\partial^2 G}{\partial P^2} = 2\,G_3 + p^2\,G_{22} + 4p\,P\,G_{21} + 4\,P^2\,G_{33}\,,$$

$$\frac{\partial^2 G}{\partial Q^2} = 2\,G_3,$$

$$\frac{\partial^3 G}{\partial P^3} = 6\,pG_{21} + 12\,PG_{31} + p^2\,G_{222} + 6p^2\,PG_{221} + 12pP^2\,G_{231} + 8P^3\,G_{333}\,,$$

$$\frac{\partial^3 G}{\partial P\,\partial Q^2} = 2pG_{23} + 4PG_{33}\,,$$

where ultimately p is to be written for P. The anastigmatism of the elementary sheaf through the middle of the aperture and the equation of distortion then produce the three identities:

$$0 = G_3(p^2, p^2, p^2),$$

$$b(\varphi(m) - 1) = \frac{1}{p}\frac{\partial G}{\partial P} = G_2 + 2\,G_3 = G_2\,(p^2, p^2, p^2),$$

$$0 = G_{22}\,(p^2, p^2, p^2) + 4\,G_{23}\,(p^2, p^2, p^2) + 4\,G_{33}\,(p^2, p^2, p^2),$$

in which one has set $m = \sqrt{1 - p^2}$. If one differentiates these equations with respect to p repeatedly and observes that the numbers 1 and 3 can be permuted in the indices of G then one will obtain, after suppressing the intermediate calculations:

$$G_{11} = G_{33} = G_{13} + \frac{b\varphi'}{4m}, \qquad G_{12} = G_{23} = -\,2G_{13} - \frac{b\varphi'}{4m}, \quad G_{22} = 4G_{13}\,,$$

$$G_{222} = 12\,G_{123} + 18\,G_{131} - 2\,G_{333} + b\,\frac{\varphi' - m\varphi''}{2m^3},$$

$$G_{223} = -\,4\,G_{123} - 5\,G_{133} + G_{333} - b\,\frac{\varphi' - m\varphi''}{4m^3},$$

$$G_{223} = G_{123} + G_{133} - G_{333} + b\,\frac{\varphi' - m\varphi''}{8m^3},$$

where φ' and φ'' mean the first and second derivatives of $\varphi(m)$, resp. With hindsight of this, the expressions above for δY and δZ go to:

$$\delta Y = \frac{bp}{4m^3}(3\varphi' - p^2(m\varphi)'')\,\delta P^2 + \frac{bp\varphi'}{4m}\,\delta Q^2\,,$$

$$\delta Z = \frac{bp\varphi'}{2m}\delta P\,\delta Q\,.$$

These formulas give the main terms in the development in δP, δQ, and are true for arbitrary p or arbitrary image angle, since no series development in p, q, P, Q was, in fact, carried out. Any image point that generates the elementary sheaf by the middle of the aperture in the focal plane, is therefore associated with a coma or mane of light (Lichtmähne) for sheaves of finite opening, whose form depends upon only the distortion function $m\varphi$ for a given focal length in its main terms. Should no distortion be present, one would then have $m\varphi = 1$, and one would obtain:

$$\delta Y = -\frac{bp}{4m^3}\left(\frac{3\delta P^2}{m^2} + \delta Q^2\right), \qquad \delta Z = -\frac{bp}{2m^3}\,\delta P\,\delta Q,$$

so the coma would have one and the same form in its main terms for all symmetric systems that produce a correct anastigmatic image in the focal plane with the elementary sheaves through the aperture plane. This form can be changed somewhat when one allows a small astigmatism or a small distortion for the elementary sheaf, which comes about in such a way that one superposes a small variation δE of likewise symmetric form with the eikonal E that is used. Therefore, one cannot arrive at an appreciable reduction in the assumed main coefficients in this way without severe distortion if the vanishing of these coefficients leads to the condition:

$$\varphi = \text{constant},$$

and in this case, infinitely distant lines will be mapped to ellipses whose midpoints lie on the optical axis, and whose semi-major axis is equal to the focal length.

Therefore, the symmetric systems are endowed with a principal deficiency: If, for a given opening, one desires sufficient correctness in the image inside of a given image angle then the sharpness of the image can be increased beyond a well-defined limit that is established theoretically in advance for no choice of refracting surfaces. The fact that one already comes very close to this limit for current photographic objectives can be proved with no difficulty by a small rough calculation using several STEINHEIL constructions on the values of the opening and the usable image angle. Since the remainder of the blurring is necessarily connected with the nature of the symmetry, its further reduction can be achieved only by abandoning the symmetry. It might not be superfluous to emphasize this point expressly, especially because a concise proof of the stated theorem would be hard to accomplish with the tools that have been used in geometrical optics up to now.

XIV.

Anastigmatism on the axis. Aplanacity. Image concavity. Distortion.

In the following, it shall be shown how the conditions for an ideal objective can be defined using the lowest terms in the eikonal series for a centered map. The fact that one must arrive at known relations concerning these matters is self-explanatory. For the sake of brevity, we would now like to refer to the appearance of an anastigmatic surface-pair as "aplanacity," and also apply this expression when we are dealing an infinitely small piece of a surface. The requirements that one ordinarily imposes upon an objective are first, aplanacity, second, the planarity of the image for a planar object, and third the correctness of the image. The second and third conditions assume the first one, since otherwise one could not speak a point-wise map of two surfaces onto each other. By contrast, the aplanacity still does not imply the image planarity and the correctness of the image, and likewise, aplanacity and image planarity are compatible with a distortion.

Similarly, as above in (180), we assume:

$$\left.\begin{aligned}\Theta(p,q,P,Q) &= \tfrac{1}{2}nfu_1 + bu_2 - \tfrac{1}{2}NF + D(u_1,u_2,u_3)\\ &\quad + n(xm+yp+zq) - N(XM+YP+ZQ),\end{aligned}\right\} \tag{197}$$

where D is imagined to be developed in powers of the quantities:

$$u_1 = p^2 + q^2, \quad u_2 = p\,P + q\,Q, \qquad u_3 = P^2 + Q^2.$$

If we decompose D into:

$$D = G + G' + G'' + \dots,$$

in which the terms of equal order have been combined, then G, G', … will be homogeneous of fourth, sixth, etc. order, resp., in p, q, P, Q. The four fundamental equations will be:

$$0 = \frac{\partial\Theta}{\partial p}, \qquad 0 = \frac{\partial\Theta}{\partial P}, \qquad 0 = \frac{\partial\Theta}{\partial q}, \qquad 0 = \frac{\partial\Theta}{\partial Q}, \tag{198}$$

of which, the first two, when written out, will assume the form:

$$\left.\begin{aligned}0 &= p(nf+2D_1) + P(b+D_2) + n\left(y - z\frac{p}{m}\right),\\ 0 &= p(b+D_2) + P(-NF+2D_3) - N\left(Y - X\frac{p}{m}\right),\end{aligned}\right\} \tag{199}$$

if the partial derivatives are once more denoted by the schema:

$$d\varphi(u_1, u_2, u_3) = \varphi_1\,du_1 + \varphi_2\,du_2 + \varphi_3\,du_3\,.$$

The still-missing two equations follow from (199) by switching p, P with q, Q, resp.

Since the map is centered, in the case of aplanacity the conjugate surfaces must be surfaces of revolution around the centering axes. With hindsight of the case of systems of lenses, if we assume that the surfaces behave regularly at the axes then their equations can be written in the form:

$$\left.\begin{aligned} x &= x_0 + x'(y^2+z^2) + x''(y^2+z^2)^2 + \cdots, \\ X &= X_0 + X'(Y^2+Z^2) + X''(Y^2+Z^2)^2 + \cdots \end{aligned}\right\} \tag{199a}$$

The initial terms x_0, X_0 are the abscissas of the "surface vertices;" i.e., the surface points on the centering axes. These axis points are conjugate for the collinear map through paraxial elementary sheaves that was treated in the previous section, so:

$$0 = b^2 + n\,N\,(x_0 - f)(X_0 - F).$$

The lateral expansion that is associated with it is then:

$$V = \frac{b}{N(x_0 - f)} = -\frac{n(X_0 - F)}{b}. \tag{200}$$

For the pair of elementary sheaves that runs between the points x_0, X_0 along the centering axis, one has, from (199):

$$0 = n\,(f - x_0)\,p + b\,P, \qquad 0 = b\,p + N\,(X_0 - F)\,P,$$

so

$$\kappa = \frac{p}{P} = \frac{b}{n(x_0 - f)} = -\frac{N(X_0 - F)}{b} = \frac{NV}{n}; \tag{201}$$

i.e., in the elementary sheaf between x_0 and X_0 in question, κ is the sine of the inclination of the conjugate axes with respect to the associated axes.

In order to obtain the desired conditions directly and at one blow, one must express two of the quantities in two of the fundamental equations (198) in terms of the remaining ones and substitute them in the other two equations (198); e.g., the p, q in:

$$0 = \frac{\partial\Theta}{\partial p}, \qquad 0 = \frac{\partial\Theta}{\partial q}.$$

The resulting pair of equations must then be identical in the case of aplanacity; i.e., it must be fulfilled for arbitrary P, Q. This leads to two types of conditions: ones in which only the form of the eikonals is important, and ones that relate to the form of the two aplanatic surfaces. In order to be able to better understand the meaning of the individual conditions, it is preferable to first treat a special case, namely, the relationship between the vertices of the two surfaces. I begin with the "aplanacity on the axis;" i.e., the surface elements at the vertices x_0, X_0 shall form an aplanatic pair. In this case, not only must the point-pair x_0, X_0 be anastigmatic, but the sine theorem must also be true. The sine conjugate inclinations can then depend upon at most the azimuth, although in the present case, due to the symmetry about the centering axis, they are independent of the azimuth, and are thus constant. Its value that is valid for infinitely-small inclinations is, from (201), equal to κ; and this value is then also substituted for finite inclinations. Since the conjugate rays have equal azimuths for the pair of sheaves in question, it is permissible to consider only the light path that runs in the xy-plane and XY-plane; i.e., one can set the quantities q, Q, z, Z equal to zero in the fundamental equations (198). Two of these equations are then fulfilled, while one obtains the other two from (199) for $q = 0$, $Q = 0$. If one sets:

$$y = Y = 0, \qquad x = x_0\,, \qquad X = X_0\,, \qquad p = k\,P, \tag{202}$$

in (199) then the desired conditions will thus be found, and their number will be infinitely large, since one is not dealing with relations between the numerical values of a finite number of quantities, but with relations between functions, for the time being.

Now, instead of performing the substitutions (202) immediately, we would like to take a small detour. For the moment, we start with the assumption that planar elements at the points x_0, X_0 on the axes that are perpendicular to the axes should define an aplanatic pair, and instead of them, consider the family of light paths that are determined by the conditions:

$$q = Q = 0, \qquad p = \kappa P, \tag{203}$$

where κ is calculated from x_0 or X_0 using (201). These light planes cut the normal planes:

$$x = x_0\,, \qquad X = X_0$$

with certain "lateral deviations," i.e., with well-defined displacements from the axes or the points x_0 , X_0 . These lateral deviations are equal to the values of y, Y that follow from (199), when one makes the substitution (203) in it. We next write, with hindsight of (201):

$$-\,ny = -\frac{b}{\kappa}(p-\kappa P)+2pD_1+PD_2+nx_0\left(p-\frac{p}{m}\right),$$

$$NY = b(p - kP) + pD_2 + 2PD_2 - NX_0\left(P-\frac{P}{M}\right),$$

where the value zero is to be used for q, Q in D, m, M. If one introduces, to abbreviate:

$$\mathfrak{D}(p, P) = D(p^2, pP, P^2) + nx_0(\sqrt{1-p^2}+\tfrac{1}{2}p^2) - NX_0(\sqrt{1-P^2}+\tfrac{1}{2}P^2)\,,$$

and splits the terms of equal dimension $\mathfrak{D}$ into:

$$\mathfrak{D}(p, P) = \mathfrak{A}(p, P) + \mathfrak{A}'(p, P) + \mathfrak{A}''(p, P) + \dots$$

then A, A′, A″, … will be of order 4, 6, 8, …, resp., in p, P. If one further writes:

$$d\mathfrak{D} = \mathfrak{D}_1\,dp + \mathfrak{D}_2\,dP, \qquad d\mathfrak{A} = \mathfrak{A}_1\,dp + \mathfrak{A}_2\,dP, \qquad \text{etc.}$$

then one can set the lateral deviations equal to:

$$\left.\begin{aligned} ny &= \frac{b}{\kappa}(p-\kappa P) - \mathfrak{D}_1(p, P), \\ NY &= b(p-\kappa P) + \mathfrak{D}_2(p, P). \end{aligned}\right\} \tag{203a}$$

From this, it follows that with $p = \kappa P$:

$$\left.\begin{aligned} ny &= -P^3\mathfrak{A}_1(\kappa, 1) - P^5\mathfrak{A}_1'(\kappa, 1) - \cdots, \\ NY &= P^3\mathfrak{A}_2(\kappa, 1) + P^5\mathfrak{A}_2'(\kappa, 1) + \cdots, \end{aligned}\right\} \tag{204}$$

since the $\mathfrak{A}$ are homogeneous in p, P.

The $\mathfrak{A}_1$, $\mathfrak{A}_2$, … are then – except for the factors of – n and N – the coefficients in the developments of the lateral deviations in P. Now, should aplanacity be present in the axis then y, Y would have to vanish independently of P, so one would have:

$$0 = \mathfrak{A}_1 = \mathfrak{A}_2, \qquad 0 = \mathfrak{A}_1' = \mathfrak{A}_2', \qquad \dots \tag{205}$$

in (204). One now has:

$$\left.\begin{aligned} p\mathfrak{A}_1(p, P) + P\mathfrak{A}_2(p, P) &= 4\mathfrak{A}(p, P), \\ p\mathfrak{A}_1'(p, P) + P\mathfrak{A}_2'(p, P) &= 6\mathfrak{A}'(p, P), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{206}$$

Therefore, due to (205), the expressions:

$$\mathfrak{A}(k, 1), \qquad \mathfrak{A}'(k, 1), \qquad \dots$$

also vanish, and one can say that κ is a double root for the equations:

$$0 = \mathfrak{A}(k, 1) = \mathfrak{A}'(k, 1) = \dots$$

The expression $\mathfrak{D}$ arises from Θ when one makes the substitutions:

$$x = x_0, \qquad X = X_0, \qquad y = Y = z = Z = 0, \qquad q = Q = 0 \tag{207}$$

and suppresses the terms of second order. Now, since the terms of second order possess the form:

$$\tfrac{1}{2}n\,(f-x_0)\,p^2 + bpP + \tfrac{1}{2}N\,(X_0-F)\,P^{\,2} = -\,\frac{b}{2\kappa}(p-kP)^2,$$

one can also say that the condition for aplanacity in the axis consists in the idea that the Θ must be divisible by:

$$(p-\kappa P)^2,$$

after performing the substitutions (207).

The conditions (205) are simple enough in their external form. In fact, the nuisance that is associated with their application would only consist in the fact that the number of eikonal coefficients to be calculated increases very quickly with increasing order of the affected terms.

When aplanacity on the axis is present, naturally, anastigmatism also exists between the two axis points x_0 , X_0 . Now, in order to be able to present the conditions for just the last property, we next imagine a homocentric σ-sheaf with its vertex at the axis point x_0 . The conjugate Σ-rays cut the normal plane $X = X_0$ with well-defined lateral deviations that one obtains from (203a) just like Y when one makes y equal to zero in it and eliminates one of the quantities p, P. We next write (203a) in the form:

$$ny = \frac{b}{\kappa}(p-\kappa P) - \mathfrak{D}_1, \qquad NY = b(p-\kappa P) + \mathfrak{D}_2 \tag{208}$$

once more and construct from it:

$$N\,(Y - Vy) = \kappa\mathfrak{D}_1 + \mathfrak{D}_2\,.$$

In this, when y has been set equal to zero, we assume the conditions:

$$p = \kappa P + \frac{\kappa}{b}\mathfrak{D}_1(p, P), \qquad NY = \kappa\mathfrak{D}_1(p, P) + \mathfrak{D}_2(p, P), \tag{209}$$

where the right-hand side of the second equation begins with terms of third order. Solving for p can be carried out with the help of LAGRANGE's inversion formula, so the conditions become not as simple as before, by far, and for that reason, I shall restrict myself to the two initial terms, namely:

$$NY = 4P^2\,\mathfrak{A}(\kappa;\,1) + 6P^5\left(\mathfrak{A}'(\kappa,1) + \frac{\kappa}{2b}\mathfrak{A}_1(\kappa,1)^2\right) + \ldots \tag{210}$$

Anastigmatism then leads to the conditions:

$$0 = \mathfrak{A}(\kappa;\,1), \quad 0 = \mathfrak{A}'(\kappa,1) + \frac{\kappa}{2b}\mathfrak{A}_1(\kappa,1)^2, \quad \text{etc.}$$

For the treatment of the aplanacity outside the axis, we go back to equations (199), write them, with consideration given to (201), in the form:

$$
\begin{aligned}
-ny &= -\frac{b}{\kappa}(p-\kappa P) + 2pD_1 + PD_2 - npx_0\frac{1-m}{m} - np\frac{x-x_0}{m}, \\
NY &= b(p-\kappa P) + pD_2 + 2PD_3 + NPX_0\frac{1-M}{M} + NP\frac{X-X_0}{M},
\end{aligned}
\tag{211}
$$

and construct from this:

$$
\left.
\begin{aligned}
N(Y-yV) = 2\kappa pD_1 + (p+\kappa P)D_2 + 2PD_3 - n\kappa px_0\frac{1-m}{m} \\
-n\kappa p\frac{x-x_0}{m} + NPX_0\frac{1-M}{M} + NP\frac{X-X_0}{M}.
\end{aligned}
\right\}
\tag{212}
$$

One then adds to (211) and (212) the associated equations that arise from the ones that were written down by switching the lateral axes. p, q are expressible in terms of the remaining quantities from (211) and the associated equations, and are then substituted into (212) and the associated equations.

For the development, we would to go only up to terms of third order in (211) and (212), so we must think of D as being restricted to its initial term of order four. In (199a) or:

$$
x - x_0 = \frac{y^2+z^2}{2r} + \ldots, \qquad X - X_0 = \frac{Y^2+Z^2}{2R} + \ldots,
$$

the right-hand sides are of second order and r, R are the semi-curvatures of the aplanatic surfaces at the vertices. Accordingly, we next write (212) in the form:

$$
\left.
\begin{aligned}
N(Y-yV) = 2\kappa pD_1 + (p+\kappa P)D_2 + 2PD_3 - \tfrac{1}{2}n\kappa px_0u_1 \\
-n\kappa p(x-x_0) + \tfrac{1}{2}NPX_0u_3 + NP(X-X_0).
\end{aligned}
\right\}
\tag{213}
$$

Since p, q appear in only the terms of third order, it suffices to use the terms of first order in (211), namely:

$$
p = \kappa P + \frac{n\kappa y}{b}, \qquad q = \kappa Q + \frac{n\kappa z}{b}.
$$

Substitution in (213) gives:

$$
N(Y-yV) = yA + PB, \qquad N(Z-zV) = zA + QB,
$$

$$
A = \frac{n\kappa}{b}(2\kappa D_1 + D_2 - \tfrac{1}{2}n\kappa x_0\, u_1 - n\kappa(x-x_0)),
$$

$$B = 2(\kappa^2 D_1 + \kappa D_2 + D_3) - \tfrac{1}{2} n\kappa^2 x_0\, u_1 - n\kappa^2 (x - x_0) + \tfrac{1}{2} NX_0\, u_3 + N(X - X_0),$$

where u_1, u_2, u_3 depend upon the quantities:

$$y^2 + z^2 = \rho, \qquad yP + zQ = \sigma, \quad P^2 + Q^2 = \tau$$

by way of the equations:

$$u_1 = \kappa^2 \tau + \frac{2n\kappa^2}{b}\sigma + \left(\frac{n\kappa}{b}\right)^2 \rho, \qquad u_2 = \kappa\tau + \frac{n\kappa}{b}\sigma, \qquad u_3 = \tau.$$

If one assumes, with hindsight of the desired approximation:

$$x - x_0 = \frac{\rho}{2r}, \qquad X - X_0 = \frac{\rho V^2}{2R},$$

$$D = \tfrac{1}{2}\sum_{\alpha,\beta} D_{\alpha\beta} u_\alpha u_\beta \qquad (\alpha, \beta = 1, 2, 3),$$

$$\left.\begin{array}{l} \mathfrak{E}(u_1, u_2, u_3) = D - \tfrac{1}{8} n x_0 u_1^2 + \tfrac{1}{8} NX_0 u_3^2 = \tfrac{1}{2}\sum_{\alpha,\beta} \mathfrak{E}_{\alpha\beta} u_\alpha u_\beta, \\ \mathfrak{E}_{11} = D_{11} - \tfrac{1}{4} n x_0, \quad \mathfrak{E}_{33} = D_{33} + \tfrac{1}{4} NX_0 \\ \mathfrak{E}_{12} = D_{12}, \quad \mathfrak{E}_{13} = D_{13}, \quad \mathfrak{E}_{22} = D_{22}, \quad \mathfrak{E}_{23} = D_{23}, \end{array}\right\} \tag{214}$$

$$\left.\begin{array}{l} \mathfrak{A}(p, P) = \mathfrak{E}(p^2, pP, P^2) \\ \quad = \tfrac{1}{2}\mathfrak{E}_{11} p^4 + \mathfrak{E}_{12} p^3 P + (\mathfrak{E}_{13} + \tfrac{1}{2}\mathfrak{E}_{22}) p^2 P^2 + \mathfrak{E}_{23} pP^3 + \mathfrak{E}_{33} P^4, \end{array}\right\} \tag{215}$$

$$d\mathfrak{A}(p, P) = \mathfrak{A}_1\, dp + \mathfrak{A}_2\, dP,$$

$$d\mathfrak{E}(u_1, u_2, u_3) = \mathfrak{E}_1\, du_1 + \mathfrak{E}_2\, du_2 + \mathfrak{E}_3\, du_3$$

then one will get:

$$\left.\begin{array}{l} A = \dfrac{n\kappa}{b}\left(2\kappa\mathfrak{E}_1 + \mathfrak{E}_2 - \dfrac{n\kappa\rho}{2r}\right), \\ B = 2(\kappa^2\mathfrak{E}_1 + \kappa\mathfrak{E}_2 + \mathfrak{E}_3) - \dfrac{(n\kappa)^2\rho}{2rn} + \dfrac{(n\kappa)^2\rho}{2RN}, \end{array}\right\} \tag{216}$$

in which A, B represent linear combinations of ρ, σ, τ:

$$A = A_1\rho + A_2\sigma + A_3\tau, \qquad B = B_1\rho + B_2\sigma + B_3\tau. \tag{217}$$

The coefficients in them become:

$$A_1 = \left(\frac{NV}{b}\right)^3 \left(\mathfrak{B} - \frac{b^2}{2NVr}\right), \qquad \mathfrak{B} = 2\kappa\,\mathfrak{E}_{11} + \mathfrak{E}_{12}\,, \tag{218}$$

$$A_2 = \left(\frac{NV}{b}\right)^2 \mathfrak{B}', \qquad \mathfrak{B}' = 4\kappa^2\,\mathfrak{E}_{11} + 4\kappa\,\mathfrak{E}_{12} + \mathfrak{E}_{22}\,, \tag{219}$$

$$\left.\begin{aligned} &A_3 = \frac{NV}{b}\mathfrak{A}_1(\kappa,1), \\ &\mathfrak{A}_1(\kappa,1) = 2\kappa^3\mathfrak{E}_{11} + 3\kappa^2\mathfrak{E}_{12} + \kappa(\mathfrak{E}_{13} + \mathfrak{E}_{22}) + \mathfrak{E}_{23}, \end{aligned}\right\} \tag{220}$$

$$B_1 = 2\left(\frac{NV}{b}\right)^2 \left(\mathfrak{B}'' - \frac{b^2}{4rn} + \frac{b^2}{4RN}\right), \qquad \mathfrak{B}'' = \kappa^2\mathfrak{E}_{11} + \kappa\mathfrak{E}_{12} + \mathfrak{E}_{13}\,, \tag{221}$$

$$B_2 = 2\frac{NV}{b}\mathfrak{A}(\kappa,1)\,, \tag{222}$$

$$B_3 = 4\,\mathfrak{A}(\kappa;\,1). \tag{223}$$

Aplanacity requires the vanishing of the quantities A_2, A_3, B_1, B_2, B_3 , which leads to four conditions, due to the relations $B_2 = 2A_3$. The demand that no aplanacity whatsoever be present, with no consideration for the form of the two conjugate surfaces, leads to the three equations:

$$0 = \mathfrak{A}, \qquad 0 = \mathfrak{A}_1\,, \qquad 0 = \mathfrak{B}'. \tag{224}$$

The first condition leads to anastigmatism on the axis, the second one adds aplanacity on the axis, and the third one, aplanacity outside the axis. The still-remaining fourth equation reads:

$$\mathfrak{B}'' = \left(\frac{b}{2}\right)^2 \left(\frac{1}{nr} - \frac{1}{NR}\right), \tag{225}$$

and gives a relationship between the curvatures of the vertices of the object and image surface, or between the concavities of the object and image. Should one desire a planar image of a planar object then one would need to have:

$$\mathfrak{B}'' = 0. \tag{226}$$

If the four conditions are fulfilled then the equation:

$$N(Y - yV) = \rho A_1 \tag{227}$$

will remain, in which the right-hand side gives the distortion. The correctness of the image requires:

$$b^2 = 2\,NVr\,\mathfrak{B} \qquad \text{or} \qquad 0 = \mathfrak{B}, \tag{228}$$

according to whether r is finite or infinite.

With that, the desired conditions are found, and indeed, independently of the manner of generating the map, under the single assumption that the map is centered. The theorems that were found are thus also true, e.g., for centered systems of lenses with non-spherical surfaces.

If the object lies at infinity then one will have:

$$x_0 = \infty, \qquad \kappa = 0, \qquad \lim n\,x_0\,\kappa = b,$$

and one will get the five conditions:

$$0 = 2D_{12} - b = D_{13} = D_{22} = D_{23} = 4\,D_{33} + NF, \tag{229}$$

which can, moreover, be also derived easily from the explicit representation that was given in (107).

XV.

Composition rule for the fourth-order terms. Concluding remarks.

The application of the series development of the eikonal to a system of lenses assumes that the coefficients are calculated from the defining pieces of the individual refracting surfaces. The path to that is given by the law of composition.

Let the maps $(\omega_1\ \omega_2)$, $(\omega_2\ \omega_3)$, $(\omega_1\ \omega_3)$ be given for the three spaces ω_1, ω_2, ω_3, resp. If one gives the quantities n, h¸ k, p, q indices that refer to their space and forms the three eikonals:

$$A = E(p_1, q_1, p_2, q_2), \qquad B = E(p_2, q_2, p_3, q_3), \qquad C = E(p_1, q_1, p_3, q_3)$$

then, from the compostion rule, the first-order partial derivatives of the expression:

$$C - A - B$$

must each be set to zero. If two of the three maps are centered, and the centering axes of the common space coincide, then the third map will also be centered. If we assume only centered maps then, with the inclusion of the fourth-order terms, we can define the equations:

$$\begin{array}{lll} p_1^2 + q_1^2 = u_1\,, & p_1\,p_2 + q_1\,q_2 = u_2\,, & p_2^2 + q_2^2 = u_3\,, \\ p_2^2 + q_2^2 = v_1\,, & p_2\,p_3 + q_2\,q_3 = v_2\,, & p_3^2 + q_3^2 = v_3\,, \\ p_1^2 + q_1^2 = w_1\,, & p_1\,p_3 + q_1\,q_3 = w_2\,, & p_3^2 + q_3^2 = w_3\,, \end{array}$$

$$\left.\begin{array}{l} A = a_1u_1 + a_2u_2 + a_3u_3 + \tfrac{1}{2}\sum a_{\alpha\beta}u_\alpha u_\beta\,, \\ B = b_1v_1 + b_2v_2 + b_3v_3 + \tfrac{1}{2}\sum b_{\alpha\beta}v_\alpha v_\beta\,, \\ C = c_1w_1 + c_2w_2 + c_3w_3 + \tfrac{1}{2}\sum c_{\alpha\beta}w_\alpha w_\beta\,, \end{array}\right\} \qquad (\alpha, \beta = 1, 2, 3)$$

and set:

$$\begin{aligned} dA &= A_1\,du_1 + A_2\,du_2 + A_3\,du_3\,, \\ dB &= B_1\,dv_1 + B_2\,dv_2 + B_3\,dv_3\,, \\ dC &= C_1\,dw_1 + C_2\,dw_2 + C_3\,dw_3\,. \end{aligned}$$

The differentiation of $C - A - B$ with respect to p_1, p_2, p_3 produces the conditions:

$$0 = 2p_1\,(C_1 - A_1) - p_2\,A_2 + p_3\,C_3\,, \tag{230}$$
$$0 = p_1A_2 + 2\,p_2\,(A_3 + B_1) + p_3\,B_2\,, \tag{231}$$
$$0 = p_1C_2 - p_2\,B_2 + 2\,p_3\,(C_3 - B_3), \tag{232}$$

to which, one must imagine adding three corresponding equations for the derivatives with respect to q_1, q_2, q_3. If one uses the second equation to eliminate p_2 from the first and third one then one will get:

$$p_1\,[4(C_1 - A_1)(A_3 + B_1) + A_2\,A_3] + p_3\,[2C_2\,(A_3 + B_1) + A_2\,B_2] = 0,$$

$$p_1\,[2C_2\,(A_3 + B_1) + A_2\,A_3] + p_3\,[4(C_3 - B_3)(A_3 + B_1) + B_2\,B_2] = 0.$$

From this, and the corresponding equations for q, it follows that:

$$C_1 - A_1 = -\frac{A_2 A_2}{4(A_3 + B_1)}, \qquad C_2 = -\frac{A_2 B_2}{4(A_3 + B_1)}, \qquad C_3 - B_3 = -\frac{B_2 B_2}{4(A_3 + B_1)}. \tag{233}$$

If one sets all p, q in this equal to zero, which leads to the light path that runs along the centering axis, then one will get:

$$c_1 - a_1 = -\frac{a_2 a_2}{4(a_3 + b_1)}, \qquad c_2 = -\frac{a_2 b_2}{4(a_3 + b_1)}, \qquad c_3 - b_3 = -\frac{b_2 b_2}{4(a_3 + b_1)}. \tag{233a}$$

If one writes, for the moment:

$$A_\alpha - a_\alpha = \delta a_\alpha, \qquad B_\alpha - b_\alpha = \delta b_\alpha, \qquad C_\alpha - c_\alpha = \delta c_\alpha,$$

and restricts oneself to the terms of lowest order then it will follow from (233) that:

$$\left.\begin{aligned}
\delta c_1 &= \delta a_1 - \frac{a_2\,\delta a_2}{2(a_3 + b_1)} + \frac{a_2^2(\delta a_3 + \delta b_1)}{4(a_3 + b_1)^2},\\
\delta c_2 &= -\frac{a_2\,\delta b_2 + b_2\,\delta a_2}{2(a_3 + b_1)} + \frac{a_2 b_2(\delta a_3 + \delta b_1)}{2(a_3 + b_1)^2},\\
\delta c_3 &= \delta b_3 - \frac{b_2\,\delta b_2}{2(a_3 + b_1)} + \frac{b_2^2(\delta a_3 + \delta b_1)}{4(a_3 + b_1)^2}.
\end{aligned}\right\} \tag{234}$$

p_2, q_2 have removed from them by means of (231), for which it suffices to set down the equations:

$$2p_2\,(a_3 + b_1) = -\,p_1\,a_2 - p_3\,b_2\,, \qquad 2q_2\,(a_3 + b_1) = -\,q_1\,a_2 - q_3\,b_2$$

with the terms of lowest order. This yields:

$$\begin{aligned}
&4v_2\,(a_3 + b_1)^2 = 4u_2\,(a_3 + b_1)^2 = w_1 a_2^2 + 2w_2 a_2 b_2 + w_3 b_2^2,\\
&2u_2\,(a_3 + b_1) = -\,w_1\,a_2 - w_2\,b_2\,,\\
&2v_2\,(a_3 + b_1) = -\,w_2\,a_2 - w_3\,b_2\,,
\end{aligned}$$

and furthermore, with $2c_2\,(a_3 + b_1) = -\,a_2\,b_2$:

$$\delta a_\alpha = w_1\left(a_{\alpha 1} + \frac{c_2}{b_2}a_{\alpha 2} + \left(\frac{c_2}{b_2}\right)^2 a_{\alpha 3}\right) + w_2\left(\frac{c_2}{b_2}a_{\alpha 2} + 2\frac{c_2}{a_2}\frac{c_2}{b_2}a_{\alpha 3}\right) + w_3\left(\frac{c_2}{a_2}\right)^2 a_{\alpha 3},$$

$$\delta b_\alpha = w_1\left(\frac{c_2}{b_2}\right)^2 b_{\alpha 1} + w_2\left(2\frac{c_2}{a_2}\frac{c_2}{b_2}b_{\alpha 1} + \frac{c_2}{b_2}b_{\alpha 2}\right) + w_3\left(\left(\frac{c_2}{a_2}\right)^2 b_{\alpha 1} + \frac{c_2}{a_2}b_{\alpha 2} + b_{\alpha 3}\right).$$

When this is substituted into (234) or:

$$\begin{aligned}
\delta c_1 &= \delta a_1 + \frac{c_2}{b_2}\delta a_2 + \left(\frac{c_2}{b_2}\right)^2(\delta a_3 + \delta b_1), \\
\delta c_2 &= \frac{c_2}{b_2}\delta b_2 + \frac{c_2}{b_2}\delta a_2 + 2\frac{c_2}{a_2}\frac{c_2}{b_2}(\delta a_3 + \delta b_1), \\
\delta c_3 &= \delta b_3 + \frac{c_2}{a_2}\delta b_2 + \left(\frac{c_2}{a_2}\right)^2(\delta a_3 + \delta b_1),
\end{aligned}$$

that will give the $c_{\alpha\beta}$ by splitting up the w. With the abbreviations:

$$c_2 : a_2 = \alpha, \qquad\qquad c_2 : b_2 = \beta,$$

I next pose the two equations:

$$\left.\begin{aligned}
c_{13} &= \alpha^2(a_{13} + a_{23}\beta + a_{33}\beta^2) + \beta^2(b_{11}\alpha^2 + b_{12}\alpha + b_{13}), \\
c_{22} &= \alpha^2(a_{22} + 4a_{23}\beta + 4a_{33}\beta^2) + \beta^2(b_{22} + 4b_{22}\alpha + 4b_{11}\alpha^2).
\end{aligned}\right\} \tag{235}$$

The other equations can then be summarized as follows: One lets the symbol $2F(a, \lambda, \mu)$ denote the bi-quadratic form:

$$\left.\begin{aligned}
2F(a, \lambda, \mu) &= \lambda^2(a_{11}\lambda^2 + a_{12}\lambda\mu + a_{13}\mu^2) \\
&\quad + \lambda\mu(a_{21}\lambda^2 + a_{22}\lambda\mu + a_{23}\mu^2) \\
&\quad + \mu^2(a_{31}\lambda^2 + a_{32}\lambda\mu + a_{33}\mu^2) \\
&= a_{11}\lambda^2 + 2a_{12}\lambda^3\mu + (2a_{13} + a_{22})\lambda^2\mu^2 + 2a_{23}\lambda\mu^3 + a_{33}\mu^4,
\end{aligned}\right\} \tag{236}$$

whose close relationship to the bi-quadratic expression $\mathfrak{A}$ is conspicuous, while the derivatives of F with respect to λ and μ can be related to $\mathfrak{A}_1$ and $\mathfrak{A}_2$. With this symbol, one then gets the equation:

$$F(c, \lambda, \mu) = F(a, \lambda, a\mu + \beta\lambda) + F(b, \alpha\mu + \beta\lambda, \mu) \tag{237}$$

between a, b, c. This formula can be referred to as the rule for the composition of aberrations along the axis.

If one is dealing with a centered system of lenses then the centering axes of the individual maps will coincide along a single line, and one can likewise make the base planes of the individual spaces coincide. For the application of formulas (235) and (236), the eikonal for an individual refraction must then be developed. If:

$$A = a_1\, u_1 + a_2\, u_2 + u_3\, u_3 + \tfrac{1}{2}\sum a_{\alpha\beta} u_\alpha u_\beta$$

is the eikonal for the refraction on a sphere of radius r and vertex abscissa a between two media with the indices n_1 , n_2 then the development of the expressions that areconstructed from (167) gives:

$$a_1 = \tfrac{1}{2} a n_1 - \frac{r}{2}\frac{n_1^2}{n_2 - n_1}, \qquad a_2 = \frac{r\, n_1\, n_2}{n_2 - n_1}, \qquad a_3 = -\tfrac{1}{2} a n_2 - \frac{r}{2}\frac{n_2^2}{n_2 - n_1}, \tag{238}$$

$$a_{22} = r\frac{(n_1\, n_2)^2}{(n_2 - n_1)^3}, \qquad a_{13} = \frac{r}{4}\frac{n_1\, n_2 (n_1^2 - n_1 n_2 + n_2^2)}{(n_2 - n_1)^3}, \tag{239}$$

$$F(a, \lambda, \mu) = \frac{a}{4}(n_1 \lambda^4 - n_2 \mu^4) + \frac{r (n_1 n_2)^2}{4}\frac{(\lambda - \mu)^4}{(n_2 - n_1)^3} - \frac{r}{4}\frac{(n_1 \lambda^2 - n_2 \mu_2)^2}{n_2 - n_1}. \tag{240}$$

If one assumes the eikonal:

$$B = b_1 v_1 + b_2\, v_2 + b_3\, v_3 + \tfrac{1}{2}\sum b_{\alpha\beta} v_\alpha v_\beta$$

in the same way for a second spherical surface with vertex abscissa b, radius s, and indices n_2, n_3 then the composition of (233a), (235), and (236) will give the eikonal for the lens that is defined by the two surfaces.

It is not necessary to carry out these developments any further here, since they must lead to known relationships in their main points, and since, moreover, the content of the last section was sufficient to prove that the ordinary treatment of the so-called spherical aberration includes, for the most part, theorems that are completely independent of the special manner of generating the ray maps that are considered. Thus, e.g., the formulas that were derived above for aplanacity, image concavity, and the correctness of the image were linked to just the one condition that the eikonal in question had to be centered.

If we wish to summarize the essential content of our investigation, in hindsight, then we could state the following things:

In the theory of optical instruments (this word is taken in the ordinary, restricted sense) one actually deals with ray-wise maps of two spaces to each other. The properties of this type of maps subdivide into two classes, according to whether the map is or is not independent of the special manner in which it was generated. The former class of properties owes its essence to geometry, and indeed, to line geometry. The contemporary customary derivation on the basis of optical assumptions carries with it the restriction that is unnecessary for the proof to know whether the purely geometric properties are also valid for maps that are never actually realized by the dioptic (*) media that are of interest to us. One will then first set foot in the realm of true optics when one treats properties that depend essentially upon the manner of generating the map, to which belongs, above all, the study of achromatism, which did not come under consideration in the present investigation for exactly that reason.

All of the various optical instruments have one and only one property in common, namely, the validity of MALUS's theorem, which was expressed in purely geometric form for the present purposes; its validity leads to the existence of the mapping functions that are referred to as *eikonals.* The properties of a map are completely determined by its associated eikonal. From this, one may establish a certain distinction. Namely, the eikonal includes, in addition to the light path coordinates and the indices of the spaces being mapped to each other, a certain parameter that depends upon the way that the eikonal came about. If the eikonal is defined by requiring certain properties – e.g., it must be the solution of a system of equations of condition – then, in essence, the parameter will play the role of a constant that one may make arbitrary or ascribe any value to, and the main thrust of the discussion will lead to geometric properties. By comparison, if the eikonal arises from the consideration of a particular optical system then the parameter that appears will depend upon the indices of the terminal media, as well as on the wave function of the light; the discussion must then treat the eikonal as a function of not only the ray coordinates, but also the parameter, and also correspondingly lead to the true optical properties of the map.

The benefits that the introduction of the eikonal brings with it are initially of a purely formal nature when one takes the position that the essential assumptions for any individual question are neatly separated. Nonetheless, one can go beyond this when one sees that it is now possible to treat problem statements that do not truly have a merely theoretical significance, for which the methods of series development scarcely allow one to define a mathematical Ansatz; it clears the foremost obstacle in the path, if one prefers to say it that way.

(*) [D.H.D.: i.e., refracting]