THESIS

PRESENTED TO

THE SCIENCE FACULTY OF PARIS

IN ORDER TO OBTAIN

THE DEGREE OF DOCTOR OF MATHEMATICAL SCIENCES

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1st THESIS – CALCULUS OF TRI-QUATERNIONS
2ND THESIS – PROPOSALS GIVEN BY THE FACULTY.

Submitted on April 1902 before the Examining Committee.

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PARIS

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

OF THE BUREAU OF LONGITUDES OF L'ÉCOLE POLYTECHNIQUE,

Quai de Grands-Augustins, 55.

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1902

TO MY FATHER

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FIRST THESIS.

CALCULUS OF TRI-QUATERNIONS

NEW GEOMETRIC ANALYSIS.

INTRODUCTION.

Objective of the memoir. – The objective of this memoir is to establish a geometric analysis that is true in any reference system.

The procedures that were described in Grassmann's *Ausdehnungslehre* realized this objective, although they are capable of being formulated in an absolutely systematic way, which is a state of affairs that was not reached, in our view.

On the contrary, the calculus of tri-quaternions constitutes a geometric analysis in the true sense of the term. It is capable of assimilating, while simplifying, all of the methods of analytic geometry.

It takes place in coordinate axes, but it has need of an origin, which will further persist in some formulas for elements that are foreign to the questions treated, except, however, in the questions that pertain to differential geometry, where the origin disappears completely and the analytical apparatus will take on the greatest possible simplicity.

It is in the complex numerical systems that consist of the quaternions that we have demanded the realization of our goal, and the solution will be provided to us by the numerical system of the tri-quaternions.

By reason of the specialization of the geometric calculi in mathematical science, it seems useful to present the results that have been achieved up to now very succinctly in an *Introduction*, the knowledge of which constitutes the point of departure for the present effort.

This *Introduction* will then have the advantage of neatly separating our own personal research from that which was done before us (to our knowledge).

Complex numerical systems. – A complex number is an expression of the form:

$$x \equiv x_1\, e_1 + x_2\, e_2 + \ldots + x_n\, e_n\,,$$

where $e_1, e_2, \ldots, e_n$ are independent complex units, and $x_1, x_2, \ldots, x_n$ are ordinary numerical quantities.

The product x'' of two of these numbers x and x' will be determined if one knows how to express the two products $e_i\, e_k$ and $e_k\, e_i$ that are defined by each of the pairs of units linearly as functions of the n original units.

By setting:

$$x' = x_1' e_1 + x_2' e_2 + \cdots + x_n' e_n ,$$
$$x'' = x_1'' e_1 + x_2'' e_2 + \cdots + x_n'' e_n ,$$

one can also define the product x'' by expressing each of the parameters of x'' as a function of the products of pairs of parameters from x and x', namely:

$$x_1'' = a_{11} x_1 x_1' + a_{12} x_1 x_2' + \cdots + a_{21} x_2 x_1' + \cdots + a_{nn} x_n x_n' ,$$
$$x_2'' = a_{11}' x_1 x_1' + a_{12}' x_1 x_2' + \ldots\ldots\ldots\ldots ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In the two cases, one finds oneself in the presence of $n^2 (n - 1)$ coefficients whose values define the multiplication in the system considered.

One generally reserves the term "complex numerical systems" for the ones for which the multiplication is an associative operation – i.e., the ones for which one always has:

$$x'' (x' x) = (x'' x') x,$$

x, x', x'' being three arbitrary complex quantities of the system.

From that point onward, the relations that define the multiplication will no longer be arbitrary, and it results from Poincaré's study of complex numerical systems ([1]) that every complex numerical system (which is intended in the restricted sense) with n units corresponds to a group of transformations with n parameters, or, more precisely, to a pair of reciprocal parameter groups.

Each complex quantity of the system corresponds to a transformation of the group, and the complex multiplication represents the composition of the transformations of the group.

One can also regard the n numerical quantities that determine the complex quantity as the *homogeneous* parameters of the transformations of a group, and then its order will be $n - 1$.

It is in this manner that the numerical system of the quaternions represents the group of rotations around a point.

Since it is in the complex numerical systems that we shall seek the realization of a geometric analysis, we will first be led to seek a geometric group of transformations such that some of them are capable of representing the points of space, and then to seek a complex numerical system in which a set of quantities is capable of representing said transformation group.

([1]) Poincaré, *Comptes rendus*, 1884, 2nd semester.

The fundamental group of geometry is the Euclidian group or the group of displacements without deformation.

It consists of transformations that are capable of characterizing the lines in space (viz., rotations through an angle π) and others that are capable of characterizing vectors (viz., translations), but nothing that would serve to represent points.

This condition is found to be realized in the discontinuous group that is defined by the displacement and symmetry transformations.

Indeed, the latter can represent points (viz., symmetries with respect to the groups) and planes (viz., symmetries with respect to planes).

However, since the group is not continuous, it does not seem to be capable of being represented by a complex numerical system.

There exists a continuous group of point-like transformations that include the preceding ones, which is the group of transformations by similitude – i.e., transformations that are each composed of a rotation and a homothety.

Such a transformation can be obtained from the sequence of a homothety and a rotation around a line that passes through the center of the homothety, and in turn, is characterized by a fixed point (three parameters), a fixed line that passes through that point (two parameters), an angle of rotation, and a homothety coefficient, and thus depends upon seven parameters.

In general, a transformation by similitude preserves a point, a line, and a plane, namely, the center of the homothety, the axis of rotation, and the plane that passes through the point and is perpendicular to the line.

One finds three types of symmetry transformations among these transformations: Symmetry with respect to points, planes, and lines; they are characterized by the following parameters:

	Angle of rotation	Coefficient of homothety
Symmetry with respect to a point	0	–1
“ line	π	1
“ plane	π	– 1

We thus find the representation of the principal elements of geometry – namely, the point, plane, and line – in the group of transformations by similitude by means of the various symmetries that characterize these elements.

However, the group of transformations by similitude does not seem to be directly representable by a complex numerical system.

We thus have to look for a complex numerical system that includes quantities whose multiplication will be a representation of the composition of the transformations by similitude.

It is natural to look for the solution among the numerical systems that include the quaternions, since the latter represents the group of rotations around a point, which is included in that of the transformations by similitude.

Calculus of quaternions. – A *quaternion* is a complex expression of the form:

$$w + ix + jy + kz,$$

where w, x, y, z are ordinary numerical quantities (i.e., positive, negative, or imaginary), and i, j, k are special symbols or complex units.

By definition, one sets:

$$Sq \equiv w \text{ (i.e., the scalar part of } q),$$
$$Vq \equiv ix + jy + kz \text{ (vectorial part of } q),$$

and, in turn:

$$q \equiv Sq + Vq.$$

One further sets:

$$Tq \equiv \sqrt{w^2 + x^2 + y^2 + z^2} \qquad \text{(tensor of } q).$$

The addition of the quaternions is defined by the addition of the coefficients of each unit:

$$q + q' \equiv w + w' + i\,(x + x') + j\,(y + y') + k\,(z + z').$$

The product of two quaternions is a quaternion that one obtains by proceeding as one does in algebra and then applying the following rules:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

One sees that one is not permitted to change the order of factors in the product of two quaternions; in other words, the multiplication of quaternions is not a commutative operation.

The calculus of quaternions is independent of any geometric interpretation, and one can give it several of them, moreover.

The customary use of that calculus will not be in question here, but only what was developed by its founder.

The principle of that geometric analysis consists of letting the expression:

$$\alpha \equiv ix + jy + kz$$

represent the vector whose components along three rectangular coordinate axes are x, y, z.

One sees immediately that the symbols i, j, k represent the vectors of length equal to one unit that are directed along the coordinate axes, respectively.

The length of x is represented by:

$$T\alpha \equiv \sqrt{x^2 + y^2 + z^2}\,.$$

Let:

$$\alpha \equiv ix + jy + kz,$$
$$\beta \equiv ix' + jy' + kz'$$

be two vectors.

Upon taking the product $\alpha\beta$, while conforming to the rules that were given above, one will find:

$$S \cdot \alpha\beta = -(xx' + yy' + zz'),$$
$$V \cdot \alpha\beta = i\,(y\,z' - z\,y') + j\,(z\,x' - x\,z') + k\,(x\,y' - y\,x').$$

One recalls that one has:

$$S \cdot \alpha\beta = \quad S \cdot \beta\alpha,$$
$$V \cdot \alpha\beta = -V \cdot \beta\alpha.$$

Let θ be the angle between the two directions of α and β. One has:

$$S \cdot \alpha\beta = -T\alpha \cdot T\beta \cdot \cos\,\theta.$$

As for $V \cdot \alpha\beta$, it is a vector that is perpendicular to α and β and has a length equal to:

$$T\alpha \cdot T\beta \cdot \sin\,\theta.$$

The direction of this vector is related to the relative disposition of α and β in the same manner that the vector k is related to the relative disposition of i and j.

If the directions of i, j, k present the customary disposition of coordinate axes then the propulsion in the sense of $V \cdot \alpha\beta$ will be linked to the rotation of α into β by a right-handed helicoidal motion; i.e., inverse to the motion of the common corkscrew.

In order to employ the calculus of quaternions in the cause of geometric analysis, it will suffice to represent each point of space by the vector that goes from some well-defined origin to that point. It is useless to take the coordinate axes, because any geometric property can be expressed directly by means of the vectors as functions of S and V.

These are the essential principles of the calculus of quaternions. They will suffice for the purpose that we have proposed.

Rotations around a fixed point. – We verify that the numerical system of the quaternions represents the group of rotations around a fixed point, among others.

Take the equations of a rotation around the origin in the form that was given by Olinde Rodrigues:

$$(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2)\;x' = (\alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2)\,x + 2\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\,y + 2\,(\alpha_1\alpha_3 + \alpha_0\alpha_2)\,z,$$

$$(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2)\;y' = 2\,(\alpha_2\alpha_1 + \alpha_0\alpha_3)\,x + (\alpha_0^2-\alpha_1^2+\alpha_2^2-\alpha_3^2)\,y + 2\,(\alpha_2\alpha_3 - \alpha_0\alpha_1)\,z,$$

$$(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2)\;z' = 2\,(\alpha_3\alpha_1 - \alpha_0\alpha_2)\,x + 2\,(\alpha_3\alpha_2 + \alpha_0\alpha_1)\,y + (\alpha_0^2-\alpha_1^2-\alpha_2^2+\alpha_3^2)\,z\,.$$

If λ, μ, ν are the angles that the axis of rotation makes with the coordinate axes, and 2θ is the angle of rotation, then one will have the following relations:

$$\cos\lambda = \frac{\alpha_1}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}, \qquad \cos\mu = \frac{\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}, \qquad \cos\nu = \frac{\alpha_3}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}},$$

$$\cos\theta = \frac{\alpha_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}.$$

The sense of rotation is linked to the positive direction of the axis of rotation by a right-handed torsion.

That is the convention that we shall adopt.

The coefficients α_0, α_1, α_2, α_3 can be taken to be the homogeneous parameters of the rotation.

Execute two successive rotations that are represented by the systems of values (α_0, α_1, α_2, α_3) and $(\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3)$. The resultant rotation can be represented by a system of parameters α''_0, α''_1, α''_2, α''_3 that is determined by the relations:

$$\begin{aligned}
\alpha''_0 &= \alpha_0\alpha'_0 - \alpha_1\alpha'_1 - \alpha_3\alpha'_2 - \alpha_3\alpha'_3, \\
\alpha''_1 &= \alpha_0\alpha'_1 + \alpha_1\alpha'_0 - \alpha_2\alpha'_3 + \alpha_3\alpha'_2, \\
\alpha''_2 &= \alpha_0\alpha'_2 + \alpha_1\alpha'_3 + \alpha_2\alpha'_0 - \alpha_3\alpha'_1, \\
\alpha''_3 &= \alpha_0\alpha'_3 - \alpha_1\alpha'_2 + \alpha_2\alpha'_1 + \alpha_3\alpha'_0.
\end{aligned}$$

Now, one sees that if one sets:

$$\begin{aligned}
q &= \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3, \\
q' &= \alpha'_0 + i\alpha'_1 + j\alpha'_2 + k\alpha'_3, \\
q'' &= \alpha''_0 + i\alpha''_1 + j\alpha''_2 + k\alpha''_3
\end{aligned}$$

then one will have the relation:

$$q'' = q'q.$$

Therefore, the multiplication of quaternions represents the composition of rotations around a fixed point, since a quaternion represents a rotation (in homogeneous parameters).

If one sets:

$$\begin{aligned}
\rho &= ix + jy + kz, \\
\rho' &= ix' + jy' + kz'
\end{aligned}$$

then the formulas of Olinde Rodrigues above will be represented by the quaternion equation:

$$\rho' = q\,\rho\,q^{-1},$$

which can be written, upon multiplying on the right by q:

$$\rho'\,q = q\,\rho.$$

One deduces from this that:

$$V \cdot \rho'\,q = V \cdot q\,\rho,$$

which represents the known formulas:

$$\alpha_0\,x' + \alpha_3\,y' - \alpha_2\,z' = \alpha_0\,x - \alpha_3\,y + \alpha_2\,z,$$
$$\alpha_0\,y' + \alpha_1\,z' - \alpha_3\,x' = \alpha_0\,y - \alpha_1\,z + \alpha_3\,x,$$
$$\alpha_0\,z' + \alpha_2\,x' - \alpha_1\,y' = \alpha_0\,z - \alpha_2\,x + \alpha_1\,y.$$

The group of rotations around a point is not the only group of geometric transformations that is represented by the system of quaternions.

This system also represents the group of projective transformations on a line and the group of special linear transformations around a point in the plane.

Indeed, these two groups have the same structure as the group of rotations around a point.

Displacements without deformation and bi-quaternions. – The group of displacements without deformation in space – or *Euclidian group* – can be represented in the manner that we have defined by a complex numerical system, namely, the *bi-quaternions*, but not in precisely the same way that the group of rotations is represented by the system of quaternions.

A bi-quaternion is a complex quantity of the form:

$$q + \omega q_1\,,$$

where q and q_1 are quaternions, and ω is a new complex unit that commutes with the four quaternion units and has square zero. One thus has:

$$\omega i = i\,\omega, \qquad \omega\,j = j\,\omega, \qquad \omega k = k\,\omega, \qquad \omega^2 = 0.$$

Bi-quaternions are thus complex quantities with eight independent units, which are:

$$1,\ i,\ j,\ k,\ \omega,\ \omega i,\ \omega j,\ \omega k.$$

Therefore, this system cannot directly represent the Euclidian group, even in homogeneous parameters, since it consists of six parameters.

However, one can make each transformation of the Euclidian group correspond to ∞^2 bi-quaternions.

A displacement without deformation can be represented by the following equations:

$$(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)\;x' = 2\,(\alpha_2\,\beta_3 - \alpha_3\,\beta_2 + \alpha_0\,\beta_1 - \alpha_1\,\beta_0)$$

$$+ (\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2)\, x + 2\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\, y + 2\,(\alpha_1\alpha_3 + \alpha_0\alpha_2)\, z,$$

$$(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)\; y' = 2\,(\alpha_3\beta_1 - \alpha_1\beta_3 + \alpha_0\beta_2 - \alpha_2\beta_0)$$
$$+ 2\,(\alpha_2\alpha_1 + \alpha_0\alpha_3)\, x + (\alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2)\, y + 2\,(\alpha_2\alpha_3 - \alpha_0\alpha_1)\, z,$$

$$(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)\; z' = 2\,(\alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_0\beta_3 - \alpha_3\beta_0)$$
$$+ 2\,(\alpha_3\alpha_1 - \alpha_0\alpha_2)\, x + 2\,(\alpha_3\alpha_2 + \alpha_0\alpha_1)\, y + (\alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2)\, z\,.$$

These equations contain eight parameters:

$$\alpha_0\,,\ \alpha_1\,,\ \alpha_2\,,\ \alpha_3\,,\ \beta_0\,,\ \beta_1\,,\ \beta_2\,,\ \beta_3\,.$$

However, one sees that the transformation does not change when one multiplies all of the parameters by the same number and one adds some quantities that are proportional to α_0, α_1, α_2, α_3 to:

$$\beta_0\,,\ \beta_1\,,\ \beta_2\,,\ \beta_3\,.$$

Finally, one can verify that if one successively performs two displacements that correspond to two systems of values of the parameters (α, β) and (α', β'), respectively, then the resulting displacement can be obtained from the system of values (α'', β'') by the following relations:

$$(1)\qquad \left\{\begin{aligned}
\alpha_0'' &= \alpha_0\alpha_0' - \alpha_1\alpha_1' - \alpha_2\alpha_2' - \alpha_3\alpha_3',\\
\alpha_1'' &= \alpha_0\alpha_1' + \alpha_1\alpha_0' - \alpha_2\alpha_3' + \alpha_3\alpha_2',\\
\alpha_2'' &= \alpha_0\alpha_2' + \alpha_1\alpha_3' + \alpha_2\alpha_0' - \alpha_3\alpha_1',\\
\alpha_3'' &= \alpha_0\alpha_3' - \alpha_1\alpha_2' + \alpha_2\alpha_1' + \alpha_3\alpha_0',\\
\beta_0'' &= \alpha_0\beta_0' - \alpha_1\beta_1' - \alpha_2\beta_2' - \alpha_3\beta_3' + \beta_0\alpha_0' - \beta_1\alpha_1' - \beta_2\alpha_2' - \beta_3\alpha_3',\\
\beta_1'' &= \alpha_0\beta_1' + \alpha_1\beta_0' - \alpha_2\beta_3' + \alpha_3\beta_2' + \beta_0\alpha_1' + \beta_1\alpha_0' - \beta_2\alpha_3' + \beta_2\alpha_2',\\
\beta_2'' &= \alpha_0\beta_2' + \alpha_2\beta_1' - \alpha_3\beta_2' + \alpha_1\beta_3' + \beta_0\alpha_2' + \beta_2\alpha_0' - \beta_3\alpha_1' + \beta_1\alpha_3',\\
\beta_3'' &= \alpha_0\beta_3' + \alpha_3\beta_1' - \alpha_1\beta_2' + \alpha_2\beta_1' + \beta_0\alpha_3' + \beta_3\alpha_0' - \beta_1\alpha_2' + \beta_2\alpha_1'.
\end{aligned}\right.$$

Now, one has the development of the relation:

$$r'' = r'\,r,$$

where r represents the bi-quaternion:

$$\alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3 + \omega(\beta_0 + i\beta_1 + j\beta_2 + k\beta_3),$$

and r', r'' represent bi-quaternions that are defined in a manner that is analogous to that of the primed parameters.

The composition of Euclidian displacements will thus be represented by the multiplication of bi-quaternions by making ∞^2 bi-quaternions correspond to each displacement that are obtained by multiplying any of them by an expression of the form:

$$a + \omega b^*,$$

where a and b are ordinary quantities.

In order to complete the analogy between the displacements and the rotations, we will give the equations of the displacement here in the following form, which will result very simply from the calculus of the tri-quaternions later on:

$$\begin{aligned}
\alpha_0 x' + \alpha_3 y' - \alpha_2 z' - \beta_1 &= \alpha_0 x - \alpha_3 y + \alpha_2 z + \beta_1 , \\
\alpha_0 y' + \alpha_1 y' - \alpha_3 x' - \beta_2 &= \alpha_0 y - \alpha_0 z + \alpha_1 x + \beta_2 , \\
\alpha_0 z' + \alpha_2 y' - \alpha_1 y' - \beta_3 &= \alpha_0 z - \alpha_2 x + \alpha_1 y + \beta_3 .
\end{aligned}$$

These equations do not contain β_0.

In order to suppress the indeterminacy of the correspondence between the bi-quaternions and the displacements, we impose the condition upon the bi-quaternion:

$$q + \omega q_1,$$

which represents a displacement, that it must satisfy the relation:

$$\alpha_0 \beta_0 + \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_2 = 0,$$

which one can write as:

$$q\, \mathbf{q}_1 + q_1\, \mathbf{q} = 0,$$

in which $\mathbf{q}$ and $\mathbf{q}_1$ represent the quaternions that are conjugate to q and q_1 – i.e., the quaternions that are obtained by changing the sign of the vector parts.

Two bi-quaternions that satisfy this condition will have a product that likewise satisfies it. In other words, that relation will characterize a subgroup.

If the bi-quaternion $q + \omega q_1$ satisfies the preceding relation then it will represent a displacement whose axis has the coordinates:

$$\frac{p_{01}}{\alpha_1} : \frac{p_{02}}{\alpha_2} : \frac{p_{03}}{\alpha_3} : \frac{p_{23}}{\beta_1 + \dfrac{\alpha_0\beta_0\alpha_1}{\alpha_1^2+\alpha_2^2+\alpha_3^2}} : \frac{p_{31}}{\beta_2 + \dfrac{\alpha_0\beta_0\alpha_3}{\alpha_1^2+\alpha_2^2+\alpha_3^2}} : \frac{p_{21}}{\beta_3 + \dfrac{\alpha_0\beta_0\alpha_3}{\alpha_1^2+\alpha_2^2+\alpha_3^2}},$$

where the angle of rotation 2θ and the shift along the axis 2η are given by the relations:

$$\cos\theta = \frac{\alpha_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}, \qquad \eta = -\frac{\beta_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}.$$

According to the system that was presented above, which consists of representing a geometric element by a transformation such that there is a one-to-one correspondence, we represent a line by the bi-quaternion that characterizes the rotation of an angle π around that line, in such a way that a straight line is represented as a bi-quaternion by an expression of the form:

$$\rho + \omega \rho_1 ,$$

where ρ and ρ_1 are vectors that satisfy the condition:

$$S \cdot \rho \rho_1 = 0.$$

The vectors ρ and ρ_1 represent the set of six coordinates of the line, and if the preceding condition is not satisfied then they can be regarded as the coordinates of a linear complex.

The calculus of bi-quaternions permits one to treat a large number of questions that relate to displacements and linear complexes with remarkable simplicity.

However, if it constitutes an analytical procedure that is convenient to line geometry then, in revenge, it does not permit one to introduce symbols into the calculation that represent points, and from that standpoint it is very inferior to the calculus of quaternions, which is an absolutely completely analytical system in itself.

We have seen that we will have the possibility of introducing symbols that represent points if we can find a complex numerical system that contains quantities that are capable of representing the transformations by similitude.

The search for these systems is considerably facilitated by the following proposition, which is due to Scheffers ([1]):

Any quaternionic system (i.e., one that includes the system of quaternions) whose unity agrees with that of the system of quaternions is obtained by multiplying the four quaternionic units by $\mu + 1$ *units* $1, \omega_1, \omega_2, \ldots, \omega_\mu$ *that commute with the former and themselves define a complex numerical system.*

One sees that the system of bi-quaternions indeed satisfies that condition, since the units 1 and ω a define numerical system.

Moreover, there exist only two essentially different binary numerical systems. They are characterized by:

$$\omega^2 = 0 \qquad \text{and} \qquad \omega^2 = 1,$$

respectively.

There thus exist two quaternionic systems with eight units. Whereas the first one, as we have seen, can serve to represent the transformations of the Euclidian group, the second one plays the same role relative to the displacements with deformation of non-Euclidian geometry.

It is therefore among the quaternionic systems with more than eight units that we will look for the one that realizes the desired conditions.

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([1]) SCHEFFERS, *Complexe-Zahlen Systeme* (Math. Ann., Bd. XXXIX, 1891).

CHAPTER I.

PRINCIPLES OF THE CALCULUS OF TRI-QUATERNIONS.

Complex numerical system of the tri-quaternions. – We have confirmed that the quaternionic numerical systems with eight units do not include complex quantities that are capable of representing the point-like transformations of space that we have called *transformations by similitude.* It is thus in the quaternionic systems with twelve units that we shall demand the realization of that condition.

From Scheffers's theorem that was given above, each of these systems is composed of complex quantities of the form:

$$q + \omega q_1 + \mu q_1 ,$$

where q, q_1, q_2 are quaternions and 1, ω, μ are units that commute with the quaternionic units and form a numerical systems among themselves.

One of these systems is characterized by the following multiplication rules:

(1) $$\omega^2 = 0, \qquad \mu^2 = 1, \qquad \omega\mu = - \mu\omega = \omega,$$

which are rules that allow the units 1, ω, μ to determine a complex numerical system, as one can verify directly.

It is this complex numerical system with twelve units to which we give the name of the *system of tri-quaternions*.

We shall first prove that this system includes quantities that are capable of representing the symmetry transformations. That proof is based upon a remark of Study that was the point of departure for the present study.

We have seen that the symmetry transformations, together with the displacements, form a group that is composed of transformations by similitude in which the homothety coefficient is equal to ± 1.

Study ([1]) has shown that the composition of the transformations of that discontinuous group is capable of being represented by formulas (1) (*Introduction*, pp. 8) by means of the following conventions:

If one intends that the bi-quaternion:

$$q + \omega q_1 \equiv \alpha_0 + i\,\alpha_1 + j\,\alpha_2 + k\,\alpha_3 + \omega(\beta_0 + i\,\beta_1 + j\,\beta_2 + k\,\beta_3)$$

should represent a transformation with a homothety coefficient equal to – 1 then the fixed point of the transformation must have the coordinates:

$$\frac{\beta_1}{\alpha_0}, \frac{\beta_2}{\alpha_0}, \frac{\beta_3}{\alpha_0},$$

([1]) STUDY, "Parameter-Darstellung der Bewegungen und Umlegungen," (Math. Ann. Bd. XXXIX, 1891).

and the rotation (i.e., the direction of its axis and the angle of rotation) must be represented in the usual manner by the quaternion:

$$\alpha_0 + i\,\alpha_1 + j\,\alpha_2 + k\,\alpha_3 .$$

Under these conditions, formulas (1) of the *Introduction* will represent the product of two transformations of the discontinuous group considered, *provided that one changes the signs of the terms on the right-hand sides that contain the parameters* β *if the transformation that acts upon the second one (i.e., the primed parameters) is a transformation with coefficient* – 1. Moreover, the result must be considered to be a transformation with coefficient – 1 if one and only one of the composed transformations has coefficient – 1, and is considered to be a displacement in any other case.

We translate this rule into formulas.

We distinguish the transformations with coefficient – 1 by a vertical line that is placed in front of the bi-quaternions.

Study's rule gives the following results:

$$(q' + \omega q_1')(q + \omega q_1) = q'q + \omega(q'\,q_1' + q_1'\,q),$$
$$(q' + \omega q_1')\,|\,(q + \omega q_1) = |\,[q'q + \omega(q'\,q_1' + q_1'\,q)],$$
$$|\,(q' + \omega q_1')(q + \omega q_1) = |\,[q'q + \omega(\,q_1'\,q' - q'q_1)],$$
$$|\,(q' + \omega q_1')\,|\,(q + \omega q_1) = q'q + \omega(\,q_1'\,q' - q'q_1).$$

Now, that rule can be realized in a purely algebraic manner. For that, it will suffice to represent a transformation with coefficient – 1 by the expression:

$$\mu\,q + \omega q_1 ,$$

if one subjects μ and ω to the rules of multiplication (1) that were given at the beginning of the chapter. Indeed, one will have the following formulas in place of the last three of the formulas above, although they agree with them:

$$(q' + \omega q_1')(\mu\,q + \omega q_1) = \mu\,q'q + \omega(q'\,q_1' + q_1'\,q),$$
$$(\mu\,q' + \omega q_1')(\mu\,q + \omega q_1) = \mu\,q'q + \omega(\,q_1'\,q' - q'q_1),$$
$$(\mu\,q' + \omega q_1')(\mu\,q + \omega q_1) = \quad q'q + \omega(\,q_1'\,q' - q'q_1),$$

Since the quantities of the form:

$$q + \omega q_1 + \mu\,q_2$$

indeed define a complex numerical system, moreover, the problem that we have posed is solved.

We subject the tri-quaternions of the form:

$$\mu\,q_2 + \omega q_1 ,$$

which represent transformations with coefficient – 1, to the condition:

$$q_2\, \mathbf{q}_1 + q_1\, \mathbf{q}_2 = 0.$$

We are in possession of a complex numerical system with twelve units that includes quantities that are capable of representing the various symmetry transformations, and in turn, points, planes, and lines.

In the introduction, we already saw that lines are represented by bi-quaternions of the form:

$$\rho + \omega \rho_1 ,$$

where ρ and ρ_1 are vectors that satisfy the condition:

$$S\, \rho\, \rho_1 = 0.$$

From what was said about the various types of symmetries, points and planes will be represented by expressions of the form:

$$\mu\, q + \omega q_1 ,$$

where one equates the angle of rotation to zero for points and to π for planes, which annuls the vectorial part of q in the former case and the scalar part in the latter case. Moreover, the quaternions q and q_1 must satisfy the condition:

$$q\, \mathbf{q}_1 + q_1\, \mathbf{q} = 0,$$

which annuls the scalar part of q_1 in the case of a point and the vectorial part in the case of a plane, because in the latter case it is necessary to take the center of the transformation to be a point at a finite distance.

Therefore, the primordial elements of geometry will find a representation in terms of tri-quaternions, namely:

A point with coordinates x_1 / x_0 , x_2 / x_0 , x_3 / x_0 , by:

$$\mu\, x_0 + \omega(i\, x_1 + j\, x_2 + k\, x_3).$$

A plane whose equation is:

$$\beta_0\, x_0 + \alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3\, x_3 = 0,$$

by

$$\omega \beta_0 + \mu\, (i\, \alpha_1 + j\, \alpha_2 + k\, \alpha_3).$$

A line whose homogeneous coordinates are:

$$\frac{p_{01}}{\alpha_1} : \frac{p_{02}}{\alpha_2} : \frac{p_{03}}{\alpha_3} : \frac{p_{23}}{\beta_1} : \frac{p_{31}}{\beta_2} : \frac{p_{12}}{\beta_3} ,$$

by

$$i\,\alpha_1 + j\,\alpha_2 + k\,\alpha_3 + \omega(i\,\beta_1 + j\,\beta_2 + k\,\beta_3),$$

with the condition that:

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0.$$

We thus find quantities that are capable of representing points, lines, and planes among the tri-quaternions.

That will suffice to establish the rules of the geometric calculus, and it is to the search for such things that we shall first apply ourselves, while returning at the end of the chapter to the proof of the fact that all of the transformations by similitude will find their representation among the tri-quaternions.

Fundamental rules of calculation. – In order to fulfill the condition of making any consideration of coordinates disappear, we must give the rules of calculation that were already posed a form that is independent of the choice of axes.

In the calculus of quaternions, that goal is attained by the introduction of the functions S and V, by means of which, a quaternion will be decomposed in a manner that is independent of the choice of axes.

We shall decompose a tri-quaternion r in an analogous manner.

In order to do that, it will suffice to write:

$$\begin{aligned} r &= w + \rho + \omega(w_1 + \rho_1) + \mu\,(w_2 + \rho_2), \\ &= w + (\omega\,w_1 + \mu\,\rho_2) + (\mu\,w_2 + \rho + \omega\,\rho_1), \\ &= w + p + l, \end{aligned}$$

where w represents an ordinary quantity, p, a plane, and l, an expression of the form:

$$l = \mu\,w_2 + \rho + \omega\,\rho_1\,,$$

whose geometric significance we must seek.

One first sees that l can be put into the form of the sum of a point and a line in an infinitude of ways.

Some particular cases:

If $\rho = 0$ then l will represent a point.

If $w_2 = 0$ then l will represent a linear complex, which will become a line in the case where $S\,\rho\,\rho_1 = 0$.

If one has $\rho = 0$, $w_2 = 0$ then the expression l will reduce to $\omega\rho_1$, and will represent both the point at infinity in the direction that indicated by ρ_1 and the line at infinity that is common to the planes that are perpendicular to ρ_1 – i.e., the polar to that point at infinity with respect to the imaginary circle at infinity. Our calculus does not therefore distinguish between points and lines at infinity, and those are the elements that we shall call *vectors*.

Finally, in a general manner, *the expression l can be written in the form of the sum of a point and a line that passes through that point, and in only one manner.*

One can see this geometrically.

This also results from the fact that one can write l in the form:

$$l = (\mu\, w_2 + \omega\beta) + (\rho + \omega\rho_1 - \omega\beta),$$

where β is a vector (in the quaternionic sense) that is determined by the linear vectorial equation:

$$V\,\rho\beta + w_2\,(\rho_1 - \beta) = 0,$$

which expresses the idea that the line:

$$\rho + \omega(\rho_1 - \beta)$$

must pass through the point:

$$\mu\, w_2 + \omega\beta\,.$$

Upon solving the equation above in β by the procedures of the calculus of tri-quaternions, one will find the value of β from:

$$(w_2^2 - \rho^2)\beta = w_2^2\rho + w_2\,V\rho\rho_1 - \rho S\rho\rho_1\ .$$

With this value of β, the expression for l will take on the form:

$$l = m + d,$$

where m represents a point and d, a line that passes through the point.

We shall call the set of geometric notions that are represented by such an expression a *linear element.*

In summary, a linear element depends upon a point, a segment (direction and length), and a coefficient (a mass or tensor).

Finally, we can define the geometric significance of a tri-quaternion r by agreeing that it should represent an ordinary quantity w, a linear element l, and a plane p, which is a definition that includes the ones that we encountered up to now and which relate uniquely to the displacements without deformations and the ones that accompany symmetry.

Since we have based the geometric significance of a tri-quaternion r upon its decomposition into three parts w, l, and p, it results that this decomposition is independent of the choice of coordinate axes, because the effect of a change of coordinates on the tri-quaternion is defined only by its geometric significance.

We thus set:

$$w = Gr, \qquad l = Lr, \qquad p = Pr.$$

The functions G, L, P play roles in the calculus of tri-quaternions that are analogous to the ones that the functions S and V play in the calculus of quaternions.

They permit us to make any reference system disappear, and no longer appeal to the rules of multiplication that were given by formulas (1) on page 11.

The only rules that are employed are given by the following formulas, which one can easily verify:

$$\text{(I)}\qquad \begin{cases} Gll' = Gl'l, & Lll' = -Ll'l, & Pll' = Pl'l, \\ Glp = 0, & Llp = \ \ Lpl, & Plp = -Ppl, \\ Gpp' = Gp'p, & Lpp' = -Lp'p, & Ppp' = 0, \end{cases}$$

to which, one must adjoin the obvious rule that an ordinary quantity commutes with any tri-quaternion.

If one does not take this last case into account, where one of the factors is a numerical quantity, the preceding results can be summarized in the following manner:

The function G admits the inversion of the order of factors, and is annulled when the factors have different types.

The function L admits the inversion of the order of factors when the factors are of different types and changes the sign in the contrary case.

The function P behaves in an inverse manner, and is annulled, moreover, when the two factors are planes.

We introduce some notations that will be useful for us in the applications.

Given a linear element:

$$l = m + d,$$

where d is a line that passes through the point m, we set:

$$\boldsymbol{l} = m - d.$$

One can qualify the linear elements l and $\boldsymbol{l}$ as *conjugate.*

One sees that one has:

$$\begin{aligned} l\,\boldsymbol{l} &= m^2 - d^2 - md + dm \\ &= m^2 - d^2 - Gmd - Pmd - Lmd + Gdm + Pdm + Ldm. \end{aligned}$$

One verifies directly that the product of a point with a line will have a zero numerical part; i.e., that one will have:

$$Gmd = Gdm = 0.$$

Moreover, one has, by virtue of formulas (1):

$$Pmd = Pdm.$$

Finally, one has, by virtue of the same formulas:

$$Lmd = -\,Ldm,$$

and one verifies that when the point m is situated on the line d, one will have:

$$Lmd = 0.$$

At the end of it all, one thus has:

$$l\,\boldsymbol{l} = m^2 - d^2,$$

where m^2 is a positive, ordinary quantity and d^2 is a negative, ordinary quantity.

One deduces the expression for the inverse of a linear element from the preceding formula:

$$l^{-1} = \frac{1}{m^2 - d^2}\,\boldsymbol{l}.$$

By analogy with the calculus of quaternions, we call the expression:

$$Tr = \sqrt{w^2 + l\,\boldsymbol{l} - p^2}$$

the *tensor of a tri-quaternion r*, and denote it by Tr.

Finally, we very often encounter tri-quaternions r in which the planar part Pr reduces to the symbol ω of the plane at infinity, multiplied by an ordinary quantity w.

Since it is sometimes advantageous to consider w directly, in that case, we set:

$$Pr = w;$$

i.e.:

(2) $$\omega Pr = Pr = \omega w.$$

That notation obviously makes no sense in the case where Pr represents an arbitrary plane.

Before we will be able to appeal to formulas (I) in a unique and useful way, it will be necessary for us to determine the significance of a certain number of simple expressions.

That preliminary interpretation corresponds to the establishment of fundamental formulas in analytic geometry.

What has to be interpreted here is the product of two tri-quaternions.

In truth, the number of results to be addressed is very large, but this is compensated by their simplicity.

Moreover, nothing prevents us from having them before our eyes, rather than appealing to memory.

We have deferred the search for the significance of the product of two tri-quaternions to an appendix.

Here, we shall point out some of the most common results.

We let μ, δ, ϖ denote a point, a line, and a plane, respectively, that all have a tensor equal to unity.

The expression $P\mu\varpi$ represents the symbol ω of the plane at infinity, multiplied by the distance from the point μ to the plane ϖ, taken with a positive or negative sign according to the region in which μ is found, in such a way that the point-like equation of the plane ϖ is:

$$P\mu\varpi = 0,$$

or, if we appeal to the notation that was introduced by formula (2) in this chapter:

$$P\mu\varpi = 0.$$

The expression $L\mu\delta$ represents the vector that is perpendicular to the plane that contains the point μ and the line δ, taken with a tensor that is equal to the distance from the point to the line, in such a way that the formula:

$$L\mu\delta = 0$$

expresses the idea that the point μ is situated on the line δ.

The equation above represents three ordinary equations, since $L\mu\delta$ is a vector.

If one considers δ in it as a fixed line and μ as a variable point then these three linear and homogeneous equations in μ will not be independent, since $L\mu\delta$ is not an arbitrary vector. Indeed, it satisfies the condition of being rectangular with δ.

The equation considered thus represents only two independent, linear, homogeneous equations, and in turn, the point μ indeed preserves one degree of freedom apart from the independence of the tensor.

On the contrary, if one takes δ to be variable then $L\mu\delta$ will be an arbitrary vector, and the equation will represent three independent, homogenous, linear equations.

These three equations express the ideas that δ is a line (and not a complex) and that the line passes through the point μ.

The expression $L\delta\varpi$ represents the point of intersection of the line δ and the plane ϖ, taken with a tensor that equals the sine of the angle between the line and the plane.

The formula:

$$L\delta\varpi = 0$$

expresses the idea that the line δ is situated in the plane ϖ.

That formula represents four ordinary formulas.

If one considers one of the terms δ and ϖ in it as being variable and the other one as being constant then it will represent four homogeneous, linear equations. However, in any case, these equations will not be independent.

That is because if δ is constant then the point $L\delta\varpi$ will satisfy the relation:

$$L \cdot \delta L\delta\varpi = 0,$$

and if ϖ is constant then it will satisfy the relation:

$$P \cdot \delta L\delta\varpi = 0.$$

Of course, these relations results easily from the rules of calculation (I).

The formula:

$$TL\delta\varpi = 0$$

expresses the idea that the point of intersection $L\delta\varpi$ becomes a vector – i.e., that the line δ is parallel to the plane ϖ.

Finally, the expression $L\omega\delta$ – or, more simply, $\omega\delta$ – represents the point at infinity in δ.

The expression $L\varpi\varpi'$ represents the line of intersection of the planes ϖ and ϖ', taken with a tensor that equals the sine of the angle between the two planes.

The formula:

$$L\varpi\varpi' = 0$$

expresses the idea that the planes ϖ and ϖ' coincide.

When considered as an equation in ϖ, it will give rise to some remarks that are analogous to the ones that were made already.

The expression $P\delta\delta'$ represents the symbol ω of the plane at infinity, multiplied by the shortest distance between the two lines δ and δ' and the sine of the angle between them.

The formula:

$$P\delta\delta' = 0$$

thus expresses the idea that the two lines are situated in the same plane.

$P\delta\delta'$ represents the moment of the two lines.

Let c be a complex with an arbitrary tensor.

The equation in d:

$$Pc\delta = 0$$

is the ruled equation of the complex c.

It is, moreover, the most general form for the linear equation to which one can subject the line δ.

If γ and γ' are two complexes with tensors equal to unity then the expression $P\gamma\gamma'$ will represent the *moment* of the two complexes, in such a way that the formula:

$$P\gamma\gamma' = 0$$

will express the idea that the two complexes γ and γ' are *in involution*.

The expression $P\gamma^2$ represents the *auto-moment* – or *parameter* – of the complex γ, and the formula:

$$P\gamma^2 = 0$$

expresses the idea that γ is a special complex, or, according to our terminology, a line.

The expression $G\delta\delta'$ represents the cosine of the angle defined between the positive directions of the two lines δ and δ', with the sign changed.

The formula:

$$G\delta\delta' = 0$$

expresses the orthogonality of the lines δ and δ'.

Since $G\delta\delta'$ is a linear and homogeneous function of d, it can be written in the form:

$$P\delta c,$$

and, in fact, one will easily verify that one has:

$$G\delta\delta' = P \cdot \delta\omega\delta'.$$

The equation in δ:

$$G\delta\delta' = 0$$

is therefore the equation of the special complex $\omega\delta'$.

The expression $G\varpi\varpi'$ represents the sine of the angle between the planes ϖ and ϖ', with the signs changed.

The expression $L\mu\mu'$ represents the vector from μ' to μ, and the formula:

$$L\mu\mu' = 0$$

expresses the coincidence of the two points.

The expression $G\mu\mu'$ is equal to unity.

The expression $L\mu\varpi$ represents the perpendicular that is drawn from the point μ on the plane ϖ, taken in the positive sense of the plane.

The expression $P\mu\delta$ represents the plane that is drawn through μ perpendicular to δ, where the positive side of the plane is determined by the positive sense of δ.

The expression $P\delta\varpi$ represents the plane that is drawn through δ perpendicular to ϖ, taken with a tensor that is equal to the sine of the angle between the line and the positive direction of the normal to the plane.

We finally point out the following formulas:

$$G\mu\delta \equiv 0, \qquad P\mu\mu' \equiv 0, \qquad P\varpi\varpi' \equiv 0.$$

Inverse of a tri-quaternion. – Let:

$$r = w + l + p$$

be a tri-quaternion.

In order to know the inverse of r, it will suffice for us to find a tri-quaternion whose product with r is an ordinary numerical quantity.

One verifies that this condition is satisfied by the tri-quaternion:

$$(w + \boldsymbol{l} - p)(w - l + p)(w - \boldsymbol{l} - p).$$

Indeed, one has, while always setting:

$$l = m + d,$$

that

$$(w + \boldsymbol{l} + p)(w + \boldsymbol{l} - p)(w - l + p)(w - \boldsymbol{l} - p)$$

$$= [w^2 + 2wm + (l + p)(\boldsymbol{l} - p)][w2 - 2wm + (l - p)(\boldsymbol{l} + p)]$$

$$= w^4 + w^2 [(l - p)(l + p) + (l + p)(\boldsymbol{l} - p) - 4m^2]$$

$$+ 2w [m (l - p)(\boldsymbol{l} + p) - (l + p)(\boldsymbol{l} - p) m] + (l + p)(\boldsymbol{l} - p)(l - p)(\boldsymbol{l} + p)$$

$$= w^4 + 2w^2 (l\,\boldsymbol{l} - p^2 - 2m^2)$$

$$+ 2w [m (l\,\boldsymbol{l} - p^2 + mp - pm + dp + pd) - (l\,\boldsymbol{l} - p^2 - pm + pm - dp - pl) m)]$$

$+ (l + p)(\boldsymbol{l} - p)(l - p)(\boldsymbol{l} + p)$

and

$$(w + l + p)(w + \boldsymbol{l} - p)(w - l + p)(w - \boldsymbol{l} - p)$$

$= w^4 + 2w^2 (l\,\boldsymbol{l} - p^2 - 2m^2) + 8wGmLpd$

$\quad + (l\,\boldsymbol{l} - p^2 + 2\omega Ppm - 2Lpd)(l\,\boldsymbol{l} - p^2 - 2\omega Ppm + 2Lpd)$

$= (w^2 + l\,\boldsymbol{l} - p^2)^2 - 4(w^2 m^2 - 2wTm \cdot TLpd + L^2 pd)$

$= (w^2 + l\,\boldsymbol{l} - p^2)^2 - 4(wTm - TLpd)^2.$

One thus has the following formula for the inverse of the tri-quaternion r:

$$[w^2 + l\,\boldsymbol{l} - p^2)^2 - 4(wTm - TLpd)^2]\, r^{-1} = (w + \boldsymbol{l} - p)(w - l + p)(w - \boldsymbol{l} - p).$$

We point out the simpler special formulas:

$$(w + m)(w - m) = w^2 - m^2,$$
$$(w + d)(w - d) = w^2 - d^2,$$
$$(w + p)(w - p) = w^2 - p^2.$$

One can systematically determine the inverse of a tri-quaternion by utilizing the calculus of quaternions.

Let:

$$r = q + \omega q_1 + \mu q_2 ,$$
$$r' = q' + \omega q_1' + \mu q_2' .$$

Take the product rr' and write down that this product reduces to a numerical quantity. One has:

$$rr' = qq' + q_2 q_2' + \mu(q q_2' + q_2 q') + \omega[(q - q_2) q_1' + q_1 (q' + q_2')] .$$

One must first have:

$$Vqq' + Vq_2\, q_2' = 0,$$
$$Vq\, q_2' + Vq_2\, q' = 0.$$

Hence, upon adding and subtracting:

$$V(q + q_2)(q' + q_2') = 0,$$
$$V(q - q_2)(q' - q_2') = 0,$$

and, in turn:

$$q' + q_2' = x\,(\mathbf{q} + \mathbf{q}_2),$$
$$q' - q_2' = y\,(\mathbf{q} - \mathbf{q}_2),$$

where x and y are the usual quantities, and $\mathbf{q}$ and $\mathbf{q}_2$ are the quaternions that are conjugate to q and q_2 .

Finally, one infers from these last two relations that:

$$q' = a\mathbf{q} + b\mathbf{q}_2 ,$$
$$q_2' = b\mathbf{q} + a\mathbf{q}_2 ,$$

by setting:

$$a = \frac{x+y}{2}, \qquad b = \frac{x-y}{2}.$$

We have written that the coefficient of μ has zero vector part. We write that this coefficient itself is zero.

Upon replacing q' and q_2' with their values, one finds that:

$$b\,(q\mathbf{q} + q_2\,\mathbf{q}_2) + a\,(q\mathbf{q}_2 + q_2\,\mathbf{q}) = 0.$$

Set:

$$\mathbf{A} = q\mathbf{q} + q_2\,\mathbf{q}_2 ,$$
$$\mathbf{B} = q\mathbf{q}_2 + q_2\,\mathbf{q} .$$

The preceding relation becomes:

$$\mathbf{A}b + \mathbf{B}a = 0.$$

Take:

$$a = \mathbf{A}, \qquad b = -\,\mathbf{B}.$$

The expressions for q' and q_2' become:

$$q' = \mathbf{A\,q} - \mathbf{B\,q}_2 ,$$
$$q_2' = -\,\mathbf{A\,q} + \mathbf{B\,q}_2 .$$

It remains for us to determine q_1' by annulling the coefficient of ω in the expression for rr':

$$(q - q_2)\;q_1' + q_1\,(q' + q_2') = 0.$$

Replace q' and q_2' with their values. One gets:

$$(q - q_2)\;q_1' + q_1\,(\mathbf{A} - \mathbf{B})(\mathbf{q} + \mathbf{q}_2) = 0,$$

or, upon multiplying on the left by $\mathbf{q} - \mathbf{q}_2$ and remarking that:

$$(q - q_2)(\mathbf{q} - \mathbf{q}_2) = T^2\,(q - q_2) = \mathbf{A} - \mathbf{B},$$

one will find that:

$$q_1' = -\,(\mathbf{q} - \mathbf{q}_2)\;q_1\;(\mathbf{q} + \mathbf{q}_2).$$

One will thus finally have:

$$r' = \mathbf{A\,q} - \mathbf{B\,q}_2 + \mu\,(\mathbf{A\,q}_2 - \mathbf{Bq}) - \omega(\mathbf{q} - \mathbf{q}_2)\;q_1\;(\mathbf{q} + \mathbf{q}_2)$$

$$= (\mathbf{A} - \mu\mathbf{B})(\mathbf{q} + \mu\mathbf{q}_2) - \omega(\mathbf{q} - \mathbf{q}_2)\;q_1\;(\mathbf{q} + \mathbf{q}_2)$$

and

$$rr' = \mathbf{A}^2 - \mathbf{B}^2.$$

One will verify that if one sets:

$$r = w + l + p$$

then one will have:

$$\mathbf{A} = w^2 + l\,\boldsymbol{l} - p^2, \qquad \mathbf{B} = 2\,(wTm - TLpd)$$

and that the result above is in agreement with the one that was found already.

Transformations by similitude. – In order to make the calculus of tri-quaternions into a geometric analysis, it will suffice for us to confirm that it includes quantities that are capable of representing displacements and displacements that are accompanied by symmetry, namely, the quantities of the form:

$$q + \omega q_1 \qquad \text{with} \qquad q\,\mathbf{q}_1 + q_1\,\mathbf{q} = 0$$

and

$$\mu q_2 + \omega q_1 \qquad \text{with} \qquad q\,\mathbf{q}_2 + q_2\,\mathbf{q} = 0.$$

If r is one of these expressions then the equation of the point-like transformation that it represents will be written:

$$\mu' = r\,\mu\,r^{-1},$$

where μ represents a point, and μ' is its transform.

There is good reason to demand that there exist other tri-quaternions r such that the expression rmr^{-1} represents a point – i.e., such that the equation above represents a point-like transformation of space.

One finds, as a condition for this, that the tri-quaternion r must be of the form:

$$r = (\lambda + \mu\,\lambda')\,q + \omega q_1 ,$$

where λ and λ' are numerical quantities, and the quaternions q and q_1 satisfy the condition:

$$q\,\mathbf{q}_1 + q_1\,\mathbf{q}\ .$$

If one puts the tri-quaternion into the form $w + l + p$ then the condition will be written:

$$2wp - P\,l^2 = 0.$$

It remains for us to determine the nature of the point-like transformations that are represented by tri-quaternions of this form.

In that case, one will have:

$$wr = w^2 + wl + \tfrac{1}{2}P\,l^2,$$

or, upon setting:

$$l = m + d$$

one will have:

$$wr = w^2 + wm + wd + md,$$

because one has:

$$P\,l^2 = md + dm = 2md,$$

by reason of the fact that:

$$Lmd = 0.$$

One can thus write:

$$wr = (w + m)(w + d),$$

which is a formula that decomposes the transformation that is represented by r into two other ones, one of which is represented by:

$$w + d,$$

which is a rotation around the line d, and the other of which, which is represented by:

$$w + m,$$

has the equation:

$$(w^2 - m^2)\,\mu' = (w + m)\,\mu\,(w - m)$$

or

$$\begin{aligned}(w^2 - m^2)\,\mu' &= w^2\mu + w\,(m\mu - \mu m) - m\mu m \\ &= w^2\mu + 2wLm\mu - m^2 m + 2mLm\mu \\ &= (w^2 - m^2)\,\mu + 2(w - Tm)\,Lm\mu\,,\end{aligned}$$

or furthermore:

$$(\mu' - \mu) = \frac{2}{w + Tm} Lm\mu\,.$$

This is the equation of a homothety transformation that has its center at the point m and the coefficient:

$$\frac{w - Tm}{w + Tm}.$$

The transformation that is represented by r can thus be obtained from the sequence of a rotation around the line d and a homothety with respect to the point m that is situated on that line. That will be, in turn, a transformation by similitude.

In summary, one then sees that the point-like transformations that are capable of being represented by tri-quaternions in the mode of representation that we have been studying are the transformations by similitude.

If one applies the operation:

$$r \cdot r^{-1}$$

to a point m, where r represents an arbitrary tri-quaternion, then one will obtain an expression of the form:

$$m' + \omega y,$$

where m' is a point, and y is a numerical quantity.

Moreover, one obtains an expression of the same form by applying the same operation to an expression of that form:

$$m + \omega x,$$

in such a way that one can consider the system of tri-quaternions as representing the group of transformations whose equation will be:

$$m' + \omega y = r\,(m + \omega x)\,r^{-1}.$$

The variable element is characterized by a point and a numerical quantity.

We do not see the simple geometric significance of these transformations.

Linear elements. – The notion of line element, as we have defined it, is independent of the calculus of tri-quaternions.

It is interesting in itself, and is capable of giving rise to a geometry that is analogous to line geometry and the geometry of spheres.

We shall sketch out the basic principle of this geometry whose element is the linear element.

Let:

$$l = \mu\, x_0 + \rho + \omega\,\rho_1$$

be a linear element.

It depends upon seven homogeneous coordinates that we can collectively represent by:

$$x_0,\ \rho,\ \rho_1,$$

or also six ordinary coordinates that are obtained by dividing the components of ρ and ρ_1 by x_0 .

Linear complexes, when considered to be space elements, define a variety whose equation is:

$$x_0 = 0.$$

In this linear variety, the lines (or special complexes) define a quadric whose equation is:

$$S\,\rho\,\rho_1 = 0.$$

If one considers the homogeneous, linear transformations:

$$l' = \varphi(l)$$

that preserve the linear variety of linear complexes and the quadric of lines in that variety then these transformations will define a group that is completely analogous to the Euclidian group, where the linear elements correspond to the points of the point-like space, the variety of complexes, to the plane at infinity, and the quadric of lines, to the imaginary circle at infinity.

From the work of Sophus Lie ([1]), one knows that such a group will preserve a quadratic differential expression, which will be:

$$S \cdot d\rho \, d\rho_1$$

in the case that we are occupied with.

Moreover, this expression is only the differential invariant that corresponds to the simultaneous invariant of the group that is presented by two arbitrary linear elements, namely:

$$S \cdot (\rho - \rho')(\rho_1 - \rho_1'),$$

where ρ and ρ_1 are (inhomogeneous) coordinates of one of the elements, while ρ' and ρ_1' are those of the other element.

That invariant is the analogue of the distance between two points in the point-like space.

In order to have an expression for that invariant that exhibits its geometric significance, replace ρ_1 and ρ_1' by their values as functions of the vectors β and β' of the points of the linear elements l and l', namely:

$$\rho_1 = \beta - V \rho \beta, \qquad \rho_1' = \beta' - V \rho' \beta'.$$

The expression for the invariant becomes:

$$S(\rho - \rho')(\beta - \beta') + S \cdot \rho\rho'(\beta' - \beta),$$

or

$$S(\rho - \rho' - V\rho\rho')(\beta - \beta').$$

The first factor depends upon only the axes of the two elements, and the second factor is the vector that takes the point β' to the point β.

Contrary to what happens for point-like space, the vanishing of that invariant will correspond to real conditions, namely, the orthogonality of the directions $\beta - \beta'$ and $\rho - \rho' - V\rho\rho'$.

In particular, the invariant will be zero for two linear elements that have the same point ($\beta = \beta'$) or the same direction for their axes ($\rho = \rho'$).

Given a linear element ρ, β, each given direction ρ' will correspond to a plane that passes through the point β and contains the points β' that give rise to linear elements ρ', β' for which the invariant will be zero.

The normal α to that plane is related to the direction ρ' by the relation:

$$\rho - \rho' - V\rho\rho' = \alpha.$$

Two linear complexes c and c' – i.e., two linear elements that are situated in the variety:

([1]) S. LIE. *Theorie der Transformationsgruppen*, Section III, Teubner, Leipzig.

$$x_0 = 0,$$

are also an invariant with respect to the group considered.

That invariant, which has the expression:

$$\frac{S\rho\rho_1' + S\rho_1\rho'}{\sqrt{\rho^2\rho'^2}} \qquad \text{or} \qquad \frac{Pcc'}{TcTc'}$$

is the *moment* of the two complexes.

In point-space, it corresponds to the cosine of the angle between the two directions.

The invariant of the two linear elements l and l' is expressed simply if one gives masses equal to unity to the points of these elements; i.e., if one supposes that:

$$x_0 = x_0' = 1.$$

Indeed, that invariant is then represented by:

$$P(l' - l)^2,$$

where $l' - l$ is a linear complex.

$$P(l' - l)^2 = 0,$$

and expresses the idea that $l' - l$ is a straight line, which must be the case, since straight lines are the analogues of cyclic points.

A linear element is determined by a linear complex by means of the use of an element that is taken to be the origin (which one can simply choose to be a point), just as a point is determined by a vector under the same conditions.

We shall not further dwell upon the principles of the geometry of linear elements, which can undoubtedly give rise to some interesting developments, notably in their application to the theory of contact transformations.

In conclusion, we remark that a linear element l is characterized by a one-parameter group of similitude transformations, just as a linear complex is characterized by a one-parameter group of displacements without deformation.

Let μ be the center of the similitude, δ, the axis, 2θ, the angle of rotation, and let 2φ be the logarithm of the homothety coefficient. From what we have seen, the tri-quaternion r that represents the transformation can be written:

$$\begin{aligned} r &= \cos\theta\cosh\varphi + \delta\sin\theta\cosh\varphi - \mu\sinh\varphi(\cos\theta + \delta\sin\theta) \\ &= (\cosh\varphi - \mu\sinh\varphi)(\cos\theta + \delta\sin\theta). \end{aligned}$$

If the transformation is infinitesimal then if one neglects the second-order infinitesimals, one will have:

$$r = 1 + \delta\theta - \mu\varphi.$$

In this case, the equation of the transformation will be:

$$m' = m + 2L(\delta\theta - \mu\varphi)\, m.$$

The infinitesimal transformation, and in turn, the one-parameter group that it determines, is characterized by the linear element:

$$m - d\,\frac{\theta}{\varphi}.$$

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CHAPTER II.

DISPLACEMENTS WITHOUT DEFORMATION.

Parameters and equations of a displacement. – By reason of the origin of the calculus of tri-quaternions, it is natural to apply it to the study of motions of undeformable systems.

A displacement without deformation is represented by a bi-quaternion:

$$r \equiv q + \omega q_1 \equiv \alpha_0 + i\,\alpha_1 + j\,\alpha_2 + k\,\alpha_3 + \omega(\beta_0 + i\,\beta_1 + j\,\beta_2 + k\,\beta_3)$$

that satisfies the conditions:

$$q\,\mathbf{q}_1 + q_1\,\mathbf{q} = 0, \qquad P[r^2 - (Lr)^2] = 0,$$

or

$$\alpha_0\,\beta_0 + \alpha_1\,\beta_1 + \alpha_2\,\beta_2 + \alpha_3\,\beta_3 = 0.$$

The line that remains invariant under the motion is the axis of the complex:

$$Lr \equiv i\,\alpha_1 + j\,\alpha_2 + k\,\alpha_3 + \omega(i\,\beta_1 + j\,\beta_2 + k\,\beta_3).$$

The angle of rotation 2θ and the magnitude of the shift 2η are given by the formulas:

$$\tan\theta = \frac{\alpha_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}, \qquad \eta = -\frac{\beta_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}.$$

Suppose:

$$Tr = 1;$$

i.e.:

$$\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2 = 1,$$

and let δ denote the axis of motion, taken with a tensor that is equal to 1.

We can then write r in the form:

$$(1) \qquad r \equiv \cos\theta + \delta\sin\theta + \omega\,\delta\,\eta\cos\theta - \omega\,\eta\sin\theta.$$

By reason of the relation:

$$\delta^2 = -1,$$

that expression can be written:

$$r \equiv (\cos\theta + \delta\sin\theta)(1 + \omega\,\delta\,\eta).$$

That formula realizes the decomposition of the displacement into two other ones that consist of the rotation around the axis and the translation along that axis.

The two factors commute.

If one takes the product of two similar expressions that have the same axis δ then one will see that the angles add together, as well as the shifts η.

It then results that if the displacements considered present the same ratio η / θ then that ratio will be preserved in the product.

One can then realize the displacement r by a continuous helicoidal motion that has δ for its axis and η / θ for its step.

This helicoidal motion is characterized by the complex:

$$\delta + \omega\delta \frac{\eta}{\theta}.$$

An infinitesimal displacement will correspond to infinitely small values of θ and η.

In this case, the expression for r becomes, upon neglecting second-order infinitesimals:

$$1 + \delta\theta + \omega\,\delta\,\eta,$$

or

$$1 + \left(\delta + \omega\delta \frac{\eta}{\theta}\right)\theta .$$

The infinitesimal displacement is characterized by the complex $\delta + \omega\,\delta\,\eta / \theta$, and conversely, any complex characterizes an infinitesimal displacement.

We now examine the expressions for the effects of the displacements on points, lines, and planes.

One will have the three equations:

$$m' = r\,m\,r^{-1}, \qquad d' = r\,d\,r^{-1}, \qquad p' = r\,p\,r^{-1}.$$

Upon applying the formula that gives the inverse of a tri-quaternion to the bi-quaternion r, one will find:

$$r^{-1} \equiv \cos\theta - \delta \sin\theta - \omega\,\eta\,(\sin\theta + \delta\cos\theta),$$

which amounts to changing the sign of θ and η in r.

Substitute these expressions for r and r^{-1} into the equation:

$$m' = r\,m\,r^{-1}. \tag{2}$$

Upon performing the calculations, one will find:

$$m' = m + 2\delta L\,\delta m \sin^2\theta + L\,\delta m \sin 2\theta + 2\omega\,\delta\,\eta\,T\,m. \tag{2^a}$$

Direct one's attention to the projection $-\delta P \delta\, m$ of the point m onto the axis δ. In order to do this, decompose m in the following manner:

$$m = -\,\delta^2 m = -\,\delta P \delta m - \delta L \delta m.$$

The preceding formula can then be written:

$$m' = -\,\delta P\ \delta m \cos 2\theta + L\ \delta m \sin 2\theta + 2\omega\,\delta\,\eta\,T\,m,$$

or, upon multiplying the term $L\ \delta m \sin 2\theta$ by $-\,\delta^2$ or $\boldsymbol{I}$:

$$m' = -\,\delta P\ \delta m + 2\omega\,\delta\,\eta\,T\,m - (\cos 2\theta + \delta \sin 2\theta)\ \delta L \delta m. \tag{2^b}$$

One easily sees the significance of each term in this formula.

The first term is the projection of m, and in turn of m', onto the axis.

The last term is the vector that takes that projection to the point m, after a rotation around δ through an angle of 2θ, since the expressions $-\delta L \delta m$ or $L \delta m \delta$ represent the vector that takes that projection to the point m itself, and the bi-quaternion $\cos 2\theta + \delta \sin 2\theta$, when it is applied as a multiplier to a vector that is perpendicular to δ, makes it turn around that line through an angle 2θ.

Finally, the second term on the right-hand side is the vector that represents the trajectory that results from the translation along the axis.

The lines and plane give rise to analogous formulas. That is why one will have:

$$d' = r\,d\,r^{-1}, \tag{3}$$

or, upon first supposing that η is zero:

$$d' = d + 2\delta L \delta d \sin^2 \theta + L \delta d \sin 2\theta. \tag{3^a}$$

If we remark that one has:

$$d = -\,\delta^2\,d = -\,\delta(G \delta d + P \delta d + L \delta d)$$

then this will become:

$$d' = -\,\delta G \delta d - \delta P \delta d - (\delta \cos 2\theta - \sin 2\theta)\,L\ \delta d,$$

and finally, upon once more appealing to $\delta^2 = -1$:

$$d' = -\,\delta G \delta d - \delta P \delta d - (\cos 2\theta + \delta \sin 2\theta)\ \delta L \delta d. \tag{3^b}$$

This formula (which represents only the rotation) is easy to interpret.

Indeed, the decomposition of d with respect to d gives rise to three terms:

$-\,\delta G \delta d$, which is a line whose position coincides with δ.

$-\,\delta P \delta d$, which is a vector that has the same direction as δ.

$-\,\delta L \delta d$, which is a complex whose axis meets δ and is perpendicular to it.

The line and the vector that is directed along δ remain invariant during the motion, and the rotation acts uniquely upon the complex $-\delta L \delta d$, as indicated in formula $(3)^b$.

If η is not zero then one must add the expression:

$$2\omega L\, \delta d'\eta$$

to the right-hand sides of formulas $(3)^a$ and $(3)^b$.

Finally, one likewise has:

$$p' = r\, p\, r^{-1}, \tag{4}$$

$$p' = p + 2\delta P\delta p \sin 2\theta + P\delta p \sin 2\theta + 2\omega L\delta p\eta. \tag{4^a}$$

As in the preceding cases, one will find:

$$p' = -\delta L\delta p - (\cos 2\theta + \delta \sin 2\theta)\, \delta P\delta p + 2P\omega\, \delta p\eta. \tag{4^b}$$

The formulas:

$$p = -\delta^2 p = -\delta L\delta p - \delta P\delta p$$

represent the decomposition of the plane p into two planes that pass through the point of intersection of δ and p. The one – namely, $-\delta L\delta p$ – is perpendicular to δ, while the other one, $-\delta P\delta p$, passes through d and the intersection of p with $-\delta L\delta p$.

The plane component that is perpendicular to δ does not move under a rotation by 2θ around δ, and the other one takes on a new position that is represented by:

$$-(\cos 2\theta + \delta \sin 2\theta)\, \delta P\delta p.$$

As for the last term of $(4)^b$ – namely, $2P\omega\, \delta\, p\eta$ – it represents the effect of the translation on the plane p.

These formulas simplify considerably in the case of an infinitesimal displacement.

If one supposes that θ and η are infinitely small in formulas $(2)^a$, $(3)^a$, $(4)^a$ then if one neglects second-order infinitesimals and sets:

$$c = 2(\delta\theta + \omega\delta\eta)$$

they will then become:

$$\left\{\begin{aligned} m' &= m + Lcm, \\ d' &= d + Lcd, \\ p' &= p + Pcp. \end{aligned}\right. \tag{5}$$

Therefore, the infinitesimal transformations in the group of displacements are characterized by linear complexes.

One can remark that we have extended the meaning that is generally attributed to the term *linear complex*.

Whereas it usually refers to the set or variety of lines whose coordinates satisfy a linear equation, we shall make it refer to the geometric notion that is characterized by that variety, while thus avoiding the introduction of new terms such as *screw*, *torsor*, *motor*, *dyname*.

In truth, these terms have a significance that adds the notion of tensor to the geometric notion of complex.

If that reason is sufficient to create new terms then one must give special names to the points that are affected with masses and the planes that are affected with numerical coefficients.

It seems to us that the possibility – which is purely rational, moreover – of ambiguity does not justify the use of very different terms to denote notions that are almost identical.

As we have already seen, we have thus simply referred to these diverse geometric elements that are affected with numerical coefficients as *points*, *plans*, *lines*, *complexes*.

The word *line* will just as well refer to the bounded, directed segment that is the support as the special complex that it determines.

Indeed, it does not seem more natural to employ different terms for the lines and special complex that would change the name of the plane according to whether one considered it a set of points, a set of lines or a spatial element.

Return to the infinitesimal displacement that is represented by equations (5).

The first of these equations shows that the displacement $m' - m$ at each point m is represented by a vector Lcm. We confirm that this vector Lcm is perpendicular to the polar plane of the point m with respect to the complex c.

The finite displacements present an analogous property.

Indeed, take the equation $(2)^b$ in the form:

$$m' = -(1 + 2\,\omega\delta\eta)\,\delta P\delta m - (\cos 2\theta + \delta \sin 2\theta)\,\delta L\delta m.$$

We remark that:

$$m = -\,\delta P\delta m - \delta L\delta m\,,$$

so one has:

$$\frac{m+m'}{2} = -(1 + \omega\delta\eta)\,\delta P\delta m - \tfrac{1}{2}(1 + \cos 2\theta + \delta\sin 2\theta)\,\delta L\delta m.$$

Form $L \cdot \gamma \dfrac{m+m'}{2}$, where one sets:

$$\gamma = Lr = \delta \sin\theta + \omega\delta\eta\cos\theta.$$

If we remark that in the development of the product $L \cdot \gamma \dfrac{m+m'}{2}$, $\delta\sin\theta$ gives a result only with the term in $\delta L\delta m$, and $\delta\cos\theta$, with the term in $\delta P\delta m$, then one will find that:

$$\begin{aligned} L\gamma\frac{m+m'}{2} &= -\,\omega\delta\eta\cos\theta\,\delta P\,\delta m - \tfrac{1}{2}L\cdot\delta\sin\theta\,(1 + \cos 2\theta + \delta\sin 2\theta)\,\delta L\delta m \\ &= -\cos\theta\,[\omega\delta\eta\delta P\delta m + \tfrac{1}{2}\cos\theta\,(\cos 2\theta - 1 + \delta\sin 2\theta)\,\delta L\delta m]. \end{aligned}$$

On the other hand, calculate $m' - m$:

$$m' - m = -2\omega\delta\eta\delta P\delta m - (\cos 2\theta - 1 + \delta \sin 2\theta)\, \delta L \delta m$$
$$= \frac{2}{\cos\theta} L \cdot \gamma \frac{m+m'}{2}.$$

One then sees that the direction $m' - m$ of the chord that is determined by the points m and m' is related to the midpoint $\frac{m+m'}{2}$ of that chord in the same manner as a point is related to the direction of its displacement under an infinitesimal displacement. In other words, that chord is perpendicular to the plane polar to $\frac{m+m'}{2}$ with respect to the complex γ.

The preceding relation can be found much more simply by starting with formula (2). That formula can be written:

$$m' r = rm.$$

Take r in the form:

$$w + \gamma + \omega w_1 .$$

The preceding formula becomes:

$$wm' + m'\gamma + \omega w_1 m' = wm + \gamma m + \omega w_1 m,$$

and, upon taking the linear element of the two sides:

$$wm' - L\gamma m' = wm + L\gamma m, \tag{6}$$

or

$$w(m' - m) = L \cdot \gamma(m + m'),$$

which is a formula that agrees with the one that we already found.

Replace w and γ with their values in formula (6):

$$(6)^{cont.} \quad \begin{cases} \cos\theta\, m' - L\delta m' \sin\theta - \omega\delta\eta T m' \cos\theta \\ = \cos\theta\, m + L\delta m \sin\theta + \omega\delta\eta T m \cos\theta. \end{cases}$$

This is the translation of the formulas on page 9 into tri-quaternions.

Various decompositions of a displacement. – The calculus of tri-quaternions permits one to decompose a displacement into rotations around two axes, one of which is given.

Let r be a bi-quaternion that represents a displacement – i.e., it satisfies the usual conditions:

$$\mathbf{q}\, q_1 + \mathbf{q}_1\, q = 0 \qquad \text{or} \qquad P \cdot [r^2 - (Lr)^2] = 0.$$

A rotation around a line d is represented by a bi-quaternion of the form:

$$w + d,$$

in which, w is an ordinary numerical quantity.

Suppose that r and d are given, and attempt to write r in the form:

$$r = n\,(w + d),$$

where n represents a rotation.

One has:

$$n = r\,(w + d)^{-1},$$

or

$$(w^2 - d^2)\,n = r(w - d).$$

Since the bi-quaternions r and $w - d$ satisfy both of the usual conditions, the same will be true for their product, and it will suffice to write that n represents a rotation, which is expressed by:

$$Pn = 0 \qquad \text{or} \qquad wPr - Prd = 0,$$

so

$$w = \frac{Prd}{Pr}.$$

With this value, one will have an expression for n that satisfies the imposed condition.

In the case where:

$$Pr = 0,$$

i.e., if the displacement is a rotation then in order for the solution of the problem to be possible, it is necessary that:

$$Prd = 0 \qquad \text{or} \qquad P \cdot Lr \cdot d = 0,$$

which is a relation that expresses the idea that d must meet the axis of rotation or be parallel to it, because Lr is a line in this case.

It results from the possibility of decomposing the displacement into two rotations, one of whose axes is given, that the displacement of a line d can be obtained by a simple rotation, because formula (3) on page 31 will be written:

$$\begin{aligned}(w^2 - d^2)\,d' &= n\,(w + d)\,d\,(w - d)\,n^{-1} \\ &= (w^2 - d^2)\,ndn^{-1},\end{aligned}$$

or

$$d' = ndn^{-1}.$$

One can also decompose a displacement into two *reversals* – i.e., two rotations through an angle of 2π. In other words, one can decompose a bi-quaternion r that represents a displacement into a product of two lines.

Set:

$$r = dd'.$$

One has:

$$dr = d^2\, d'.$$

Since the right-hand side is a line, we will get the equation for the locus of d by writing that the left-hand side also represents a line.

Set:

$$r = w + \gamma + \omega w_1 .$$

One has:

$$dr = wd + Gd\gamma + Pd\gamma + Ld\gamma + \omega w_1 d.$$

One must first have:

$$Gdg = 0, \qquad Pdg = 0;$$

i.e., that d belongs to the complex γ, and is rectangular with the axis of that complex. From a known property of complexes, the line d will thus belong to the congruence of lines that meet the axis of γ and are perpendicular to that axis.

With these conditions, the bi-quaternion dr will represent a complex. In order for it to represent a line, it will be necessary that one have:

$$P(dr)^2 = 0 \qquad \text{or} \qquad P(wd + d\gamma + \omega w_1 d)^2 = 0,$$

or

$$P[(w + \omega w_1)^2\, d^{\,2} + (d\gamma)^2 + 2(w + \omega w_1)\, d^{\,2}\, \gamma] = 0,$$

$$2ww_1\, d^{\,2} - d^{\,2} P\gamma^2 = 0,$$

which is a condition that is always satisfied, because the relation:

$$2ww_1 - P\gamma^2 = 0$$

is the condition that r must realize in order for it to represent a displacement.

The line d' is determined by the formula:

$$d^{\,2}\, d' = dr = (w + \omega w_1)\, d + d\gamma.$$

This is also a line that meets the axis of γ and is perpendicular to it.

In order to determine the relative position of the lines d and d', suppose that their tensors are equal to unity, and take the bi-quaternion r in the form (1) that was given at the beginning of this chapter. One will have:

$$Gdd' = \cos\theta, \qquad Pdd' = -\eta \sin\theta,$$

or, upon letting θ' denote the angle between the two lines, and letting e denote the shortest distance between them:

$$-\cos\theta' = \cos\theta, \qquad e \sin\theta' = \eta \sin\theta,$$

which are relations that express the ideas that the lines form an angle between them that equals $\varpi - \theta$ and have a distance between them that is equal to η, or furthermore, that one passes from the axis d' of the first reversal to the axis d of the second one by a positive rotation through an angle θ around the axis of r and a translation that is likewise positive with the value η along that axis. (Recall that θ is the half-angle of rotation of the displacement r, and η is its half-shift.)

One can also obtain a rotation by means of two planar symmetry transformations, as the formula:

$$p_2\, p_1 = \cos\theta + \delta \sin\theta$$

shows.

One obtains a translation if the two planes are parallel:

$$p_2 = p_1 + \omega w, \qquad p_2\, p_1 = -1 + \omega w p_1 .$$

Nonetheless, we shall not go further into the geometric theory of displacements, since this chapter and the ones that follow have no other objective than that of giving some idea of the analytical simplicity that the calculus of tri-quaternions introduces into some classical questions.

Continuous motion of a solid body. – We shall apply the calculus of tri-quaternions to the study of the motion of an undeformable system in the course of time, by following the customary order of exposition, but while going through it much more rapidly, as one will notice, thanks to the simplicity and lucidity of the formulas.

Upon denoting the derivation with respect to time by a prime, one will have, from formulas (5) of the present chapter:

$$(7) \qquad m' = Lcm, \qquad d' = Lcd, \qquad p' = Lcp,$$

where c is a linear complex that one can call the *instantaneous complex* of the motion.

The expression for the acceleration of the point m is obviously:

$$(8) \qquad m'' = Lc'm + Lcm' = Lc'm + L \cdot cLcm .$$

One can replace c with its axis δ in the second term.

Indeed, one can set:

$$c = \delta + \omega\delta\eta,$$

where η is a numerical quantity. One will get:

$$\begin{aligned} L \cdot cLcm &= L \cdot (\delta + \omega\delta\eta)\, L\, (\delta + \omega\delta\eta)\, m \\ &= L \cdot \delta L \delta m + \eta L \cdot \omega\delta L \delta m + \eta L \cdot \delta L \omega\delta m + \eta^2 L \cdot \omega\delta L \omega\delta m. \end{aligned}$$

The second and fourth terms will be zero, because the product of a vector (e.g., $L\delta m$ or $L\omega\delta m$) by an expression that contains the factor ω will be zero.

The third term is likewise zero, because it can be written:

$$L \cdot \delta\omega\delta Tm = \eta TmL\omega\delta^2 = 0.$$

One thus has the following expression for the acceleration:

$$(8)^{cont.} \qquad m'' = Lc'm + L\delta L\delta m.$$

One knows that the vector $L \cdot \delta L\delta m$ can be expressed as the derivative of a point function, which is the square of the length of the vector $L\delta m$, as long as we take the tensor of m to equal unity, at the same time.

Let H be that function of m. Upon appealing to the quaternionic operation that is defined by the formula:

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z},$$

one can write the second term on the right-hand side of (8) as:

$$L \cdot \delta L\delta m = \nabla H.$$

Formulas (8) or $(8)^{cont.}$ lead naturally to Coriolis's theorem.

Let v be the velocity that represents the relative velocity of the point m with respect to an undeformable system whose motion has c for its instantaneous complex. One will have:

$$m' = Lcm + v,$$

so

$$m'' = Lc'm + Lcm' + v' = Lc'm + L \cdot cLcm + Lcv + v'.$$

The first two terms collectively represent the acceleration J_e of the point of the undeformable system that coincides with the point m at the instant considered.

Express v' as a function of the relative acceleration J_r of the point m.

The vector $v + v'\,dt$ represents the relative velocity in space of the point m at the epoch $t + dt$, and consequently represents what the vector $v + J_r\,dt$ becomes due to the effect of the infinitesimal displacement $1 + c\,dt$; i.e., that one has:

$$v + v'dt = v + J_r\,dt + Lc\,(v + J_r\,dt)\,dt,$$

so, upon neglecting the second-order infinitesimals:

$$v' = J_r + Lcv.$$

If δ denotes the axis of rotation of the undeformable system then the formula that gives the expression for m'' becomes:

$$m'' = J_e + J_r + 2Lcv = J_e + J_r + 2L\delta v.$$

That formula expresses Coriolis's theorem.

An important problem in the kinematics of undeformable systems consists of the following question:

Given the expression for the instantaneous complex as a function of time, determine the position of the system at a given instant.

Without wanting to go into the question of the integration of the differential equation that presents itself, we shall look for that differential equation, whose form is remarkably simple.

The position of the undeformable system will be determined by the bi-quaternion r that represents the displacement that takes the system from one arbitrarily-chosen fixed position to its present position.

We seek the differential equation that the bi-quaternion r will satisfy.

We first remark that by virtue of the relation that must be satisfied by bi-quaternions that represent displacements without deformation, the bi-quaternion r will have the form:

$$r = w + \gamma + \frac{P\gamma^2}{2w} = w + \gamma + \omega x,$$

where w and x are numerical quantities, and γ is a complex.

If m_0 represents the initial position of a point m then one will have:

$$m = r\, m_0\, r^{-1},$$

so

$$m' = r' m_0\, r^{-1} + r m_0\, (r^{-1})',$$

upon representing the derivative with respect to time by the putting a prime on the symbol of the function.

We seek the expression for $(r^{-1})'$. One has:

$$rr^{-1} = 1,$$

so

$$r' r^{-1} + r\, (r^{-1})' = 0,$$

and upon multiplying on the left by r:

$$(r^{-1})' = -\, r\, r' r^{-1}.$$

If one substitutes this in the expression for m then one will find that:

$$m' = r' m_0\, r^{-1} - r\, m_0\, r^{-1}\, r' r^{-1},$$

and if one replaces m_0 with its expression as a function of m:

$$m' = r'r^{-1}m - m\,r'r^{-1}$$

then one will have:

$$(w^2 - G\gamma^2)\,r'r^{-1} = (w' + \gamma' + \omega x')(w - g\,w\,x)$$
$$= ww' - G\gamma\gamma' + w\gamma' - w'\gamma + wx\gamma - wx'\gamma - P\gamma'\gamma + \omega(wx' + xw').$$

The planar part:

$$\omega(wx' + xw') - P\gamma'\gamma$$

is the derivative of the expression:

$$\omega wx - \tfrac{1}{2}P\gamma^2,$$

which is always zero, by hypothesis.

One thus has:

$$Pr'r^{-1} = 0,$$

and the expression for m' becomes:

$$m' = Lr'r^{-1}\cdot m - m\,L\,r'r^{-1} = 2L\cdot L\,r'r^{-1}\cdot m,$$

a formula that shows that the instantaneous complex of motion c has the expression:

(9) $$c = 2\,L\,r'r^{-1}.$$

One can further simplify this formula by subjecting r to the condition that it must have a tensor that equals unity. That condition is written:

$$w^2 - G\,\gamma^2 = 1,$$

so

$$ww' - G\gamma\gamma' = 0$$

or

$$G\,r'r^{-1} = 0.$$

One can then write:

(10) $$c = 2\,r'r^{-1}.$$

This is the differential equation that r must satisfy.

It has the form of a differential equation whose solution will require only a quadrature.

However, it is easy to assure oneself that, in reality, the right-hand side $2\,r'r^{-1}$ is not an exact differential.

Upon putting equation (10) into the form:

$(10)^{bis}$ $$2\,r' = cr,$$

one sees that if one knows eight solutions to that equation $r_1, r_2, \ldots, r_8$ then the general solution can be written:

$$r = \sum A_i\,r_i\,,$$

where the coefficients A_i represent eight integration constants that are subject to two relations that result from the condition that r must satisfy:

$$P(r^2 - L^2 r) = 0, \qquad T^2 r = 1.$$

Moreover, if these two relations are satisfied by a value of t then that will be true for any other value.

Let r be a solution to equation (10). The general solution will be:

$$r_1 = r\, r_0 ,$$

where r_0 is an arbitrary, constant bi-quaternion that satisfies the two condition that are satisfied by r.

Indeed, one has:

$$r_1' = r' r_0 ,$$
$$r_1^{-1} = r_0^{-1} r^{-1},$$

so

$$2 r_1' r_1^{-1} = 2 r' r^{-1} = c.$$

If the constant c is constant then the motion will be a uniform helicoidal motion, and a solution to (10) will be:

$$r = \cos\theta + \delta \sin\theta + \omega\delta\eta \cos\theta - \omega\eta \sin\theta,$$

where δ is a fixed line, and η and θ are proportional to time and are determined by the relation:

$$c = 2\delta(\theta' + \omega\eta').$$

It can be convenient to generalize the definition of logarithm by setting:

$$\frac{d}{dt}\log r = \delta(\theta' + \omega\eta'),$$
$$\log r = \delta(\theta' + \omega\eta')\, t,$$

or further:

$$r = e^{\delta(\theta' + \omega\eta')\, t} = e^{ct/2},$$

in such a way that for the extremely simple case in which c is constant, the general solution to equation (10) will take the form:

$$r = e^{ct/2}\, r_0 ,$$

where r_0 is the bi-quaternion with tensor one that represents the displacement that takes the solid body to an arbitrarily-chosen position from the one that it occupied at the epoch $t = 0$.

We finally remark that the formula:

$$e^{\delta(\theta' + \omega\eta)} = \delta^{2(\theta' + \omega\eta)/\pi}$$

results from:

$$e^{\delta\pi/2} = \delta.$$

In place of the instantaneous complex of motion c, one can give the complex c_1 that coincides with c after the displacement r. In other words, one can give the position of the instantaneous complex with respect to the moving system.

The differential equation presents a form that is analogous to the one that was obtained already. Indeed, one has:

$$c_1 = r^{-1}\, cr = 2r^{-1}r'.$$

If the axis of c has a constant direction in space – i.e., if ωc is constant – then the same thing will be true for ωc_1 , because upon referring to the developed expression for $r'\, r^{-1}$, one will see that one has:

$$\omega\, r'\, r^{-1} = \omega\, r^{-1} r'.$$

The properties that are represented by the formulas that are successively encountered in this chapter are all well-known. The proof of these properties by the calculus of tri-quaternions has no other objective than that of exhibiting the aptitude of that calculus for being used as an analytical procedure.

We must mention here that in the questions that relate to the motion of undeformable systems, certain formulas that involve only bi-quaternionic expressions have already been given in a form that is very close to our own by some authors that have treated the calculus of bi-quaternions ([1]).

Equilibrium and dynamics of solid bodies. – One must attend to the fundamental equation of equilibrium and dynamics of an undeformable body, which present themselves in a particularly simple form.

We first establish the condition for equilibrium of a system of forces that is applied to a solid body.

We represent the forces by expressions that have used in order to represent lines.

A virtual displacement of a solid body will be characterized by a complex c with an infinitely-small tensor, and the virtual work of the force will have the expression:

$$-P \cdot f\, Lc\, \mu = -P \cdot f\, \mu\, L\mu\, c = -P \cdot (f\, \mu^2\, c - f\, \mu\, P\mu\, c).$$

Since the lines $f\mu$ and $P\mu c$ are concurrent, one will have:

$$P \cdot f\, \mu\, L\mu\, c = 0,$$

and the work that is done by the force f will finally have the expression:

$$-Pcf.$$

([1]) MacAULAY, *Octonions*, London, Clay and Sons, 1899. – BUCKHEIM, “Mémoire sur les biquaternions” (American Journal, 1885).

The virtual work that is done by all of the forces thus has the value:

$$-Pc\sum,$$

and the condition for that work to be zero, for any *c*, is written:

$$\sum f = 0.$$

The condition for the equivalence of two systems of forces is obviously:

$$\sum f = \sum f'.$$

A system of forces that is applied to a solid body is thus characterized by what we have called a *complex.*

The equation of motion for a solid body is obtained by writing the equivalence between the external forces and the system of segments that represent the acceleration of each point, multiplied by its mass, in both magnitude and direction.

One easily sees that this segment (or line) can be considered to be the derivative with respect to time of a segment *e* that passes through the point considered and represents the quantity of motion of that point, in such a way that the equation of motion will be:

$$\sum \frac{de}{dt} = \sum f$$

or

$$\frac{d}{dt}\sum e = \sum f \,.$$

The complex $\sum f$, which we shall denote by λ, can be called the *motor complex*, and the complex $\sum e$, the *quantity of motion* of the solid body.

It remains for us to express that quantity of motion η as a function of the instantaneous complex ε.

The quantity of motion η is composed of a line (or segment) that passes through the center of gravity and represents the quantity of motion of translation and a vector that represents the moment that results from the quantities of motion with respect to the center of gravity.

One easily expresses these two terms as functions of ε, and one has:

$$\eta = \Phi(\varepsilon),$$

where Φ is a homogeneous, linear function of ε that depends upon the mass of the solid body, the position of its center of gravity, and that of the central ellipsoid of inertia.

One has:

$$\eta' = \Phi(\varepsilon') + \Phi'(\varepsilon).$$

We now seek the expression for $\Phi'(\varepsilon)$.

Apply the infinitesimal displacement $1 + \varepsilon\, dt$ to the transformation:

$$c' = \Phi(c).$$

It becomes:

$$c_1' - L\varepsilon\, c_1'\, dt = \Phi(c_1) - \Phi(L\,\varepsilon\, c_1)\, dt,$$

or, upon neglecting the higher-order infinitesimals:

$$c_1' = \Phi(c_1) + [L \cdot \varepsilon\,\Phi(c_1) - \Phi(L\,\varepsilon\, c_1)]\, dt.$$

One then sees that one will have:

$$\Phi'(c) = L\,\varepsilon\,\Phi(c) - \Phi(L\,\varepsilon\, c)$$

and

$$\Phi'(\varepsilon) = L\,\varepsilon\,\Phi(\varepsilon).$$

One will then have:

$$\eta' = \Phi'(\varepsilon) + L\,\varepsilon\,\Phi(\varepsilon),$$

and if one takes the unknown to be the bi-quaternion r with tensor one that represents the displacement that takes the solid body from an arbitrarily-chosen fixed position to its position at the epoch t then the equations of motion can be written:

$$(11) \qquad 2r' r^{-1} = \varepsilon, \qquad \Phi'(\varepsilon) + L\,\varepsilon\,\Phi(\varepsilon) = \sum f,$$

where the operator Φ is a function of r.

One can also take the auxiliary variable to be η, instead of ε, and the equations of motion can then take the form:

$$2r' r^{-1} = \Phi^{-1}(\eta), \qquad \eta' = \sum f = \lambda.$$

However, Φ is not expressed very simply as a function of r, except in some special cases (a body of revolution, for example).

In order to avoid that difficulty, one is then led to introduce elements into the equations whose positions are determined with respect to the solid body.

If one lets ε_1 and λ_1 denote complexes that will be made to coincide with ε and λ by the displacement r then the equations of motion can be written:

$$(12) \qquad 2r^{-1} r' = \varepsilon_1, \qquad \Phi_1(\varepsilon_1') + L\varepsilon_1\,\Phi_1(\varepsilon_1) = \lambda_1,$$

in which Φ_1 represents the function that Φ will become under the displacement r.

The second of these equations is the extension to the case of a free solid body of Euler's equations that relate to a solid body that is subject to turning around a fixed point.

One will easily find the known integrals in the integrable cases.

In the case of a completely free body, the decomposition of the motion along the center of gravity removes any interest in the intervention of the calculus of tri-quaternions. However, it does not seem to be without some utility for the study of the motion of a solid body that is subject to geometric constraints.

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CHAPTER III.

LINEAR COMPLEXES.

Equation of a complex. Various decompositions. – A linear complex is defined by *a set of lines whose coordinates satisfy a scalar linear equation.*

The most general expression for a scalar quantity that is a homogeneous linear function of a line:

$$d = \rho + \omega \rho_1$$

is obviously:

$$P\gamma d,$$

where γ is an expression of the form:

$$\gamma = \alpha + \omega \beta.$$

The equation of a complex is then:

$$P\gamma d = 0,$$

in such a way that a linear complex is characterized by an expression of the form above, up to an arbitrary factor.

If one has:

$$P\gamma^2 = 0$$

then γ will represent a special complex, which one has no reason to distinguish from the line that it determines in a unique manner. It is for that reason that we have considered the line to be a special case of a complex.

The expression γ represents a system of segments, if one brings its tensor into consideration.

The various modes of decomposition of a system of segments, as they are used in mechanics, are explained very simply by means of tri-quaternions.

That is why one will have, upon letting μ, ϖ, and δ denote a point, plane, and line with a tensor that is equal to unity, respectively:

$$\gamma = \mu^2 \gamma = \mu\, P\, \mu\, \gamma + \mu\, L\, \mu\, \gamma = \mu\, P \mu\, \gamma + L\, \gamma \mu,$$

$$\gamma = -\varpi^2 \gamma = -\varpi\, P\, \varpi\, \gamma - \varpi\, L\, \varpi\, \gamma,$$

$$\gamma = -\delta^2 \gamma = -\delta G\, \delta\, \gamma - \delta(P\, \delta\, \gamma + L\, \delta\, \gamma).$$

These decompositions are generally employed when one has to exhibit the properties of a complex with respect to a point, a plane, or a line.

The formulas above are easy to interpret.

The first one represents the decomposition of γ into a line $\mu\, P\, \mu\, \gamma$ that passes through μ and a vector $L\, \gamma \mu$ that represents the moment of the system of segments with respect to the point *m*, or even the velocity of a point μ in motion whose instantaneous situation will be represented by γ.

The second formula represents the decomposition of γ into two lines, one of which, namely, $-\varpi L\varpi\gamma-$ is perpendicular to *v*, and the other of which – namely, $-\varpi L\varpi\gamma-$ is situated in that plane.

The third formula decomposes γ into the line δ and a complex that is not generally a line.

There is some interest in determining the decomposition of γ into two lines, one of which coincides with δ in position. In order to do that, set:

$$d = x\delta + y\gamma,$$

and write that the right-hand side represents a line:

$$Pd^2 = 0 \qquad \text{or} \qquad 2xP\delta\gamma + yP\gamma^2 = 0.$$

This equation determines the ratio x / y, and one will have an infinitude of solutions for *d* that differ only by their tensors.

One can observe that this decomposition of a complex into two lines, one of which is given, permits us to reduce the infinitesimal displacement of a line, which is given by the second of formulas (5) on page 32, to a rotation.

Two lines are called *conjugate with respect to a complex g* when one can choose their tensors in such a manner that one has:

(1) $$\gamma = d + d'.$$

Any line δ that belongs to the complex γ and meets a line *d* will also meet its conjugate d', since one will have:

$$P\delta d' = P\delta\gamma - P\delta d = 0.$$

Conversely, any line that meets two conjugates will belong to the complex.

Any plane *p* that is parallel to two conjugate lines *d* and d' will be parallel to the axis of the complex, because one has:

$$P \cdot p\omega\gamma = P \cdot p\omega d + P \cdot p\omega d' = 0,$$

which is a formula that expresses the idea that the plane *p* contains the vector $\omega\gamma$ – i.e., it is parallel to the axis of γ.

Any line *d* that belongs to the complex will coincide with its conjugate, and conversely, because one will have:

$$d' = dP\gamma^2 - 2\gamma P\gamma d = dP\gamma^2$$

as the expression for the conjugate d'.

Now, bring the tensors into consideration.

Formula (1) then represents the determination of a system of segments in terms of two segments.

One has:

$$P\gamma^2 = Pd^2 + 2Pd\,d' + Pd'^2 = 2Pdd'.$$

One thus has the theorem:

The volume of the tetrahedron that is determined by the two segments by which one can represent a system of segments is independent of the choice of the pair of segments.

Let d, d' and d_1, d_1' be two pairs of equivalent segments:

$$d + d' = d_1 + d_1'.$$

If three of the lines belong to a complex γ – i.e., if one has:

$$P\gamma d = 0, \qquad P\gamma d' = 0, \qquad P\gamma d_1 = 0,$$

then the fourth line will likewise belong to it. Now, one can find three linearly-independent complexes that have three lines in common. The fourth line will also belong to these complexes, and in turn, to the ruled series that they have in common.

Focal point and focal plane. – We look for the locus of lines d that belong to a complex γ and pass through a point μ.

A line that passes through μ will have the form:

$$d = L\,\mu\,\varpi,$$

in which ϖ will represent an arbitrary plane.

One must have:

$$P\,\gamma d = 0,$$

$$P\,\gamma d = P \cdot \gamma L\,\mu\,\varpi = P \cdot \gamma(\mu\varpi - P\mu\varpi) = P \cdot \gamma\mu\varpi = P(L\gamma\mu \cdot \varpi).$$

One thus has:

$$P(L\gamma\mu \cdot \varpi) = 0,$$

which is a formula that expresses the idea that the plane ϖ contains the vector $L\gamma\mu$, and in turn, that the line d, which is perpendicular to the plane ϖ, is likewise perpendicular to that vector – i.e., its locus is the plane that passes through μ and is perpendicular to the direction of $L\gamma\mu$.

The latter plane will be called the *focal plane* of the point μ.

We likewise seek the locus of the lines d that belong to a plane ϖ and the complex γ.

Any line that is situated in the plane ϖ will have the form:

$$d = L\varpi p,$$

in which p is an arbitrary plane. One must have:

$$P \cdot \gamma L\varpi p = 0,$$

or

$$P \cdot \gamma(\varpi p - G\varpi p) = P \cdot \gamma\varpi p = P \cdot L\,\gamma\omega \cdot p = 0.$$

The last formula expresses the idea that the plane p, and in turn, the line d, will pass through the point $L\gamma\varpi$. That point is called the *focal point* of the plane ϖ.

The most natural means of determining the locus of the lines of a complex that pass through a given point or are situated in a given plane consists of the decomposition that was mentioned already of the complex into the point or plane.

For example, in the case of the plane, one has:

$$\gamma = -\varpi^2\gamma = -\varpi L\,\varpi\gamma - \varpi P\,\varpi\gamma$$

or

(2) $$\gamma = f + d_0\,,$$

where f and d_0 represent lines (which are obviously conjugate), one of which – viz., f – is perpendicular to ϖ and passes through the point $L\varpi\gamma$, and the other of which – viz., d_0 – is situated on the plane ϖ.

The lines d of the complex that belong to the plane ϖ are determined by the equation:

$$P\,(f + d_0)\,d = 0,$$

or

$$P\,f\,d = 0,$$

which is a formula that expresses the idea that the line d in the plane ϖ meets the line f – i.e., it passes through the point $L\varpi\gamma$.

The line d_0 is the locus of points in the plane ϖ such that the focal planes pass through the line f – i.e., they are perpendicular to that plane ϖ.

Since that line d_0 is the intersection of the plane ϖ and the plane $P\varpi\gamma$, which is parallel to the axis of γ, it will be parallel to the projection of that axis onto ϖ.

The line d_0 is called the *characteristic* of the plane ϖ, when one considers the fact that it is, indeed, the characteristic of that plane under the infinitesimal displacement that is represents by the complex γ.

It is easy to confirm that the projections onto the plane ϖ of two conjugate straight lines will cut at a point of the characteristic of that plane.

Indeed, let d and d' be two conjugate lines such that one has:

$$g = d + d'.$$

The planar projections of d and d' onto ϖ are:

$$P\varpi d, \qquad P\varpi d',$$

respectively.

On the other hand, since f is perpendicular to ϖ, one will have:

$$P\,\varpi f = 0,$$

and in turn:

$$P\varpi d + P\varpi d' = P\varpi g = P\varpi (f + d_0) = P\varpi d_0\,,$$

which is a formula that shows that the three planes $P\varpi d$, $P\varpi d'$, and $P\varpi d_0$, which are perpendicular to ϖ, pass through the same straight line, and in turn, that the projections of d and d' cut on d_0.

The lines f and d_0 constitute a pair of mutually-perpendicular conjugate lines.

Lines that are rectangular with their conjugates. – We seek the locus of lines that present the property of being rectangular with their conjugates.

Let d be one of these lines, and let d' be its conjugate.

One must have:

$$Gdd' = 0.$$

The equation for the locus is thus:

$$G\cdot d\,(d\,P\,\gamma^2 - 2\gamma P\,\gamma d) = 0,$$

or

(3) $$d^2\,P\,\gamma^2 - 2Gd\,\gamma P\,\gamma d = 0,$$

which is the equation of a second-degree complex.

We now seek the locus of the lines that belong to that complex and pass through a point μ.

Decompose the complex γ with respect to the point μ:

$$\gamma = \mu^2\,\gamma = \mu\,P\mu\,\gamma + \mu\,L\,\mu\,\gamma$$
$$= \mu\,P\,\mu\,\gamma + L\,\gamma\mu,$$

where $\mu\,P\mu\gamma$ represents a line that passes through the point μ, and $L\gamma\mu$ represents a vector.

We set:

$$L\,\gamma\mu = \omega d_1\,, \qquad \mu\,P\mu\,\gamma = d_2\,,$$

where d_1 and d_2 represent lines that pass through the point μ.

We will thus have:

$$\gamma = d_2 + \omega d_1\,.$$

For a line d that passes through the point μ, one will have:

$$P\,\gamma d = G\,d_1\,d, \qquad G\,\gamma d = G\,d_1\,d,$$

in such a way that equation (3) becomes:

$$d^2\, G\, d_1\, d_2 - G\, d_1\, d\, G\, d_2\, d = 0 \tag{4}$$

for the lines d that pass through the point μ, which is the equation of a second-degree cone that has the lines d_1 and d_2 among its generators, and whose cyclic sections are perpendicular to these lines, because the common generators of the cone (4) and the cone of isotropic lines:

$$d^2 = 0$$

are situated in the planes:

$$G\, d_1\, d = 0, \qquad G\, d_2\, d = 0,$$

which contain the lines d that are perpendicular to d_1 and d_2, respectively.

On each of the generators of the cone (4), there exists a point m whose focal plane is perpendicular to that generator. We seek the locus of these points.

Since the focal plane of any point m is perpendicular to the vector:

$$L\, \gamma m,$$

one must have:

$$L\, \gamma m = x\, L\, \mu\, m,$$

which is a formula that expresses the idea that the vector $L\, \gamma m$ has the same direction as the vector $L\, \mu\, m$ that is determined by the two points μ and m, while x denotes a numerical quantity.

The equation of the locus of the point m will then be:

$$L\, d_2\, m + \omega d_2\, T\, m = x\, L\, \mu\, m, \tag{5}$$

which is a homogeneous, linear vectorial equation in m that determines a position of m for each value of the variable x.

However, since we know one surface that is the locus of the points m, it will suffice to determine another one from it.

Perform the operation:

$$P \cdot d_2$$

on the two sides of equation (5).

One gets:

$$\begin{aligned} P \cdot \omega d_2\, d_1\, T\, m &= x\, P \cdot d_2\, L\, \mu\, m \\ &= x\, P \cdot d_2\, \mu\, m \\ &= x\, P \cdot P\, d_2\, \mu \cdot m \\ &= -\, x\, P\, m\, p_2 \end{aligned}$$

upon setting:

$$P\, d_2\, \mu = p_2\,.$$

Upon likewise performing the operation:

$$P \cdot d_2$$

and setting:

$$P\, d_1\, \mu = p_1\,,$$

one will get:

$$P \cdot d_1\, L\, d_2\, m + d_1^2\, Tm = -\, x\, P\, m\, p_1\,.$$

Finally, upon eliminating x from the two equations thus obtained, one will get the equation of the locus:

$$P \cdot m\, [p_1\, P \cdot \omega d_2\, d_1 T\, m - p_2\, (P \cdot d_1\, L\, d_2\, m + d_1^2\, Tm)] = 0$$

or

(6) $$P \cdot m\, [p_1\, G\, d_2\, d_1 T\, m + p_2\, (P \cdot \mu\, d_1\, L\, d_2\, m - d_1^2\, Tm)] = 0,$$

which is a homogeneous, second-degree equation in m that represents a second-degree surface.

In the following chapter, we shall confirm that the means of determining the elements of a second-degree surface is represented by such an equation.

However, the form of equation (6) directly exhibits the rectilinear generators of the surface. They are determined by the equations:

$$Pmp_1 = xPmp_1\,, \qquad xGd_1d_2Tm = -\, P \cdot Pm\, d_1\, d_2 \cdot m + d_1^2\, Tm$$

and

$$Gd_1\, d_2\, T\, m = y\, P\, m\, p_2\,, \qquad y\, P\, m\, p_1 = -\, P \cdot P\mu\, d_1\, d_2 \cdot m + d_1^2\, Tm\,,$$

where x and y are variable numerical quantities.

One recognizes that this is a hyperbolic paraboloid that has the planes p_2 and $P\mu\, d_1\, d_2$ for its director planes; the latter is the plane that is determined by the concurrent lines d_1 and d_2.

Among the generators of one of the systems, one finds the intersection of the planes p_1 and p_2 – i.e., the perpendicular that goes through the point μ in the plane $P\mu\, d_1\, d_2$.

We now seek the locus of the lines d that belong to the complex (3) and are situated in a plane ϖ.

Decompose the complex with respect to the plane ϖ:

$$\gamma = -\, \varpi^2\, \gamma = -\, \varpi\, L\, \varpi\, \gamma - \varpi\, P\, \varpi\, \gamma = f + d_0\,.$$

For any line d that is situated in the plane ϖ, one has:

$$Pdd_0 = 0, \qquad Gdf = 0,$$

and, in turn:

$$Pd\gamma = Pdf, \qquad Gd\gamma = Gdd_0\,.$$

One has, moreover:

$$P\gamma^2 = 2P\, f\, d_0\,.$$

For the lines in the plane ϖ, equation (3) then becomes:

(7) $$d^2 P f d_0 - G d d_0 P df = 0,$$

which is the tangential equation of a conic.

This conic is obviously a parabola that has the foot of the line f for its focus and the line d_0 for the tangent to the summit, because, on the one hand, equation (7) is satisfied by:

$$d = d_0,$$

and, on the other hand, the isotropic lines:

$$d^2 = 0,$$

which satisfy the equation (7), are determined by the equations:

$$Gdd_0 = 0, \qquad Pdf = 0;$$

i.e., they are either perpendicular to d_0 or they meet the line f.

We once more seek a locus of lines that belong to the quadratic complex (3).

In order to do this, at each point μ of a given line δ, consider the line d whose direction is $L\gamma\mu$ – i.e., it is perpendicular to the focal plane of μ.

The ruled series that is described by the line d when the point μ is displaced along the line δ is represented by some very simple equations.

Decompose the complex γ with respect to the line δ – i.e., set:

$$\gamma = \delta + \delta',$$

by taking δ to have a convenient tensor and while letting δ' denote the line that is conjugate to δ with respect to the complex.

The equations of the ruled series that is described by d are the following ones:

(8) $$P\delta d = 0, \qquad G\,\delta' d = 0, \qquad P \cdot L\,\delta\delta' \cdot d = 0.$$

The first one expresses the idea that the line d meets δ, and the second one, the idea that it is rectangular with δ'. The third one is verified in the following manner, which will likewise serve to establish the first two, if they are not obvious.

Upon letting ϖ denote the plane that is determined by the line δ' and the point m that is situated on the line δ, one will have:

$$d = \varpi L\,\varpi\,\delta,$$

so

$$P(L\,\delta\delta' \cdot d) = P(L\,\delta\delta' \cdot \varpi L\,\varpi\,\delta) = P(L\,\delta\delta' \cdot \varpi^2\,\delta) - P(L\,\delta\delta' \cdot \varpi P\,\varpi\,\delta).$$

The two terms in this expression are zero; indeed, the first one can be written:

$$P(L\,\delta'\delta \cdot \delta) = P \cdot \delta'\delta^2 = 0,$$

and the second one:

$$P\,(\delta'\,\delta\,\varpi\,P\varpi\,\delta) = P\,(\delta'\,P\delta\,\varpi\,P\varpi\,\delta) = P\,[\delta L\,(P\delta'\,\varpi\,P\varpi\,\delta)] = 0,$$

because the expression:

$$L(P\delta'\,\varpi\,P\varpi\,\delta)$$

represents a line in the plane $P\varpi\,\delta$, and in turn, it must meet δ.

Equations (8) express the idea that the line d belongs to the three linear complexes:

$$\delta, \qquad \omega\delta', \quad \text{and} \quad L\delta\delta'.$$

The line d, which remains parallel to the same plane, describes a hyperbolic paraboloid.

When the lines δ and δ' are rectangular – i.e., when the line δ belongs to the quadratic complex (3) – the expression $L\delta\delta'$ will represent the common perpendicular to δ and δ', and in turn, the line d will generate the plane that is drawn through δ and perpendicular to δ'.

The normals along d to the ruled surface (8) likewise form a ruled series that has the equation:

$$P\delta\,d = 0, \qquad G\delta\,d = 0, \qquad P\delta'\,d = 0.$$

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CHAPTER IV.

SECOND-DEGREE SURFACES.

New notations. – As a final example of the use of the calculus of quaternions in the name of geometric analysis, we apply it to the proof of some fundamental properties of quadrics.

Before doing that, it is necessary to complex the calculus by means of some new notations, as well as to present the first principles of a theory of projective transformations in terms of tri-quaternions.

The calculus of tri-quaternions presents a serious gap in that it does not provide the means to represent certain simple projective properties (without the aid of a reference system).

That is why we do not have the expression that represents the line that joins two given points or the plane that passes through a given line and point.

All the same, it is remarkable that if we take the plane to be the element then we will immediately obtain the representation of Grassmann's exterior products, namely: The line that is common to the two planes p and p' is represented by Lpp', the point that is common to the planes p, p', p'', by $L \cdot pp'p''$, and the volume of the tetrahedron that is formed by the planes p, p', p'', p''', by $\frac{1}{6} P \cdot p\, p' p'' p'''$.

It does not seem that the calculus of tri-quaternions permits one to resolve the difficulty that presents itself in the expression of the analogous properties that relate to a point in a satisfying manner.

We thus content ourselves by introducing a notation that will at least present the advantage of simplifying the notation.

We represent the line that joins the points m and m' by $V \cdot m, m'$ and the focal plane to the point m with respect to the complex c by $S \cdot c, m$. Even better, if we set:

$$m = \mu\, x_0 + \omega \rho, \qquad m' = \mu x_0' + \omega \rho', \qquad c = \alpha + \omega \beta$$

then we will define the functions V and S by the formulas:

$$(1) \qquad \begin{cases} V \cdot m, m' = x_0\, \rho' - x_0' \rho + \omega L \rho\rho', \\ S \cdot c, m = \mu(x_0 \beta + G\alpha\beta) + \omega G \beta\rho. \end{cases}$$

Upon applying the functions V and S to the vectors $\omega\rho$ and $\omega\rho'$, in particular, one will get:

$$V \cdot \omega\rho,\ \omega\rho' = \omega L\, \rho\rho',$$
$$S \cdot \omega\rho,\ \omega\rho' = \omega G\, \rho\rho'.$$

One sees that the functions S and V permit us to express the properties of vectors as in the calculus of tri-quaternions, while we cannot do that by means of the functions G, L, and P, since the product of two vectors is zero.

We can subject the functions S and V to the tri-quaternion calculus by establishing some properties that are based upon the defining formulas (1).

We cite the following formulas, which one can easily verify:

$$S \cdot c, m = S \cdot m, c, \qquad V \cdot m, m' = - V \cdot m', m,$$

$$G \cdot cS \cdot c, m = 0, \qquad L \cdot cS \cdot c, m = \tfrac{1}{2} Pc^2 \cdot m, \qquad P \cdot cS \cdot c, m = S \cdot c, Lcm,$$

$$G \cdot S \cdot c, m \cdot m = 0, \qquad L \cdot S \cdot c, m \cdot m = V \cdot m, Lcm, \qquad P \cdot S \cdot c, m \cdot m = 0,$$

$$G \cdot mV \cdot m, m' = 0, \qquad L \cdot mV \cdot m, m' = 0, \qquad P \cdot mV \cdot m, m' = - S \cdot m, Lmm',$$

The equation in p:

$$L \cdot cp = m$$

is solved by the formula:

$$\tfrac{1}{2} Pc^2 \cdot p = S \cdot c, m.$$

Finally, in order to simplify the notation, we set:

$$S \cdot m, m', m'' = S \cdot (V \cdot m, m'), m'' = S \cdot m\,(V \cdot m' m'').$$

In summation, the functions S and V, for which one establishes the preliminaries of some formulas, can be introduced in the calculus like functions G, L, P themselves, and thus fill in the aforementioned gap.

In order to fill another gap, we shall present a means of applying the calculus of tri-quaternions to plane geometry.

One can likewise establish a complex calculus that applies to rectilinear geometry in the following manner:

Let i be a line whose tensor is equal to unity.

Represent each point of that line by the plane that drawn through that point and perpendicular to the line – i.e., by an expression of the form:

$$\mu\, i + \omega x,$$

where x is the abscissa of that point with respect to a point m that is taken to be the origin.

If one adds to these quantities, the ones that one obtains by means of the units 1 and ωi then one will obtain quantities of the form:

$$r = w + \mu\, i + \omega x + \omega\, i\, y = Gr + Pr + Lr,$$

which constitutes a complex numerical system.

The expression:

$$Lr = \omega\, i\, y$$

depends upon just one scalar quantity. It will often be advantageous to set:

$$Lr = iy \qquad \text{or} \qquad \omega L\, r = Lr.$$

The complex numerical system can be especially useful in the determination of the intersection of a given line with a give surface.

We have seen that a complex in space can be decomposed relative to a plane into two lines, one of which is situated in the plane, and the other of which is perpendicular to that plane.

Since we can represent the points of a plane by the lines that are drawn through these points perpendicular to the plane, we see that the set of complexes in space suffices to represent the points and lines in the plane.

If one adds the complex quantities that are obtained by means of the units 1 and ω to these quantities then one will obtain a bi-quaternion precisely:

$$r = w + c + \omega w_1 ,$$

in such a way that plane geometry possesses a complex numerical calculus, just like the geometry of space.

Take an origin that is situated in the plane and three rectangular coordinate axes, two of which are situated in the plane, and the third of which is perpendicular to them. We write r in the form:

$$r = w + ix + jy + kz + \omega(ix_1 + jy_1 + kz_1) + \omega w_1 .$$

We decompose this bi-quaternion into four others:

The numerical part: Gr or ω,

The term in ω: Pr or ωw_1,

The part of the complex Lr or c that represents a line that is perpendicular to the plane, which we denote by Mr:

$$M\, r = k\, z + \omega(ix_1 + jy_1).$$

Finally, the part of the complex c that represents a line in the plane, which we denote by Dr:

$$D\, r = i\, x + jy + \omega kz_1 .$$

The rules in this planar calculation are represented by the following formulas, in which d, d' represent lines in the plane, and n, n' represent lines perpendicular to the plane – i.e., points in the plane:

$$(2) \qquad \left\{ \begin{array}{llll} Gnn' = Gn'n, & Mnn' = -Mn'n, & Dnn' = 0, & Pnn' = 0, \\ Gnd = 0, & Mnd = 0, & Dnd = -Ddn, & Pnd = Pdn, \\ Gdd' = Gd'd, & Mdd' = -Md'd, & Ddd' = 0, & Pdd' = 0. \end{array} \right.$$

We remark that this calculus includes that of equipollences.

It has the same properties relative to plane geometry that the calculus of tri-quaternions has relative to the geometry of space, where the function M plays the role of the function L, and the function D, that of the function P.

The two calculations also present the same gaps.

A remarkable property of these two calculi consists in their capacity to pass from the calculus of tri-quaternions to that of bi-quaternions, relative to a given plane, when that might be necessary.

For example, one can always find the bi-quaternionic equation of the plane section of a surface that is represented by an equation in tri-quaternions. One encounters some examples of this in the study of certain geometric constraints. The procedure is general.

It is not less simple to obtain the bi-quaternionic equation of the projection into a given plane of a curve that is represented by two equations in tri-quaternions.

The passage from geometry in space to plane geometry is based upon the formulas:

$$n = \varpi m \qquad \text{or} \qquad m = -\varpi n,$$

where m represents a point of the plane ϖ, and n, the perpendicular to that plane, and in turn, the point itself in the system of calculation that relates to the plane.

In order to find the equation of the projection on a plane ϖ of the intersection curve of two surfaces:

$$F(m) = 0, \qquad F'(m) = 0,$$

one sets:

$$m = -\varpi n + \omega \varpi x,$$

in which x is a numerical quantity, and $\omega \varpi x$ represents the vector that is determined by the point m and its projection $-\varpi n$ onto the plane ϖ.

One will have the desired equation in n by eliminating x from the two equations:

$$F(-\omega n + \omega \varpi x) = 0, \qquad F'(-\omega n + \omega \varpi x) = 0.$$

Projective transformations. – In order to begin the theory of second-degree surfaces, it is necessary to establish some notions and formulas that relate to the application of the complex calculus to the projective transformations of space.

That is the objective of this paragraph. Moreover, these properties are interesting in their own right.

A projective transformation of space can be presented in three forms:

1. A homogeneous, linear transformation:

$$m' = \varphi(m),$$

where m and m' are points, represents a *homographic transformation.*

2. A homogeneous, linear transformation:

$$p' = \varphi_1(m),$$

where m is a point and p' is a plane, represents a *dualistic projective transformation*.

3. Finally, a homogeneous, linear transformation:

$$c' = \Phi(c),$$

where c and c' are complexes, represent a projective transformation from one space to another, on the condition that it leave invariant the quadratic equation:

$$Pc^2 = 0$$

of the variety that is formed by the lines in the variety of complexes.

We shall say a few words about each of these homogeneous, linear functions φ, φ_1, Φ, notably, on the manner by which one can realize their inverses – i.e., how one can solve the equations above for m and c without appealing to the ordinary systems of equations that represent them.

The inversion of a function of the first kind is treated by a method that was modeled on the one that was presented by Hamilton for the inversion of homogeneous, linear, vectorial functions, and which is based upon the existence of an identity relation that these functions must satisfy.

In the case of point-like functions, the relation is of degree four – i.e., one will have identically:

$$\varphi^4(m) + A\,\varphi^3(m) + B\,\varphi^2(m) + C\,\varphi(m) + Dm = 0,$$

where A, B, C, D are numerical coefficients that one can determine by calculating φ^2, φ^3, φ^4.

Upon replacing $\varphi(m)$ with m' in the identity relation, it will become

$$\varphi^3(m') + A\,\varphi^2(m') + B\,\varphi(m') + C\,m' + Dm = 0,$$

which is a formula that gives the expression for m as a function of m' – i.e., it solves the problem of inverting the function φ.

Moreover, the theory of systems of homogeneous, linear equations or matrices was developed from that viewpoint, notably by Sylvester ([1]).

We complete it by establishing a notion that will be useful to us, namely, that of conjugate transformations.

Hamilton used the term *function conjugate* to a homogeneous, linear, vectorial function φ to mean another function φ' of the same type, such that if ρ and ρ' are two arbitrary vectors then one will always have the relation:

$$S\,\rho\,\varphi(\rho_1) = S\rho \cdot \varphi'(\rho).$$

The definition is reciprocal.

One can observe that this property is metric.

([1]) SYLVESTER, *Comptes rendus*, 1884.

On the contrary, the property by which we shall define the transformation conjugate to a homographic transformation is projective.

Let:

$$\begin{aligned} x' &= ax + by + cz + dt, \\ y' &= a'x + b'y + c'z + d't, \\ z' &= a''x + b''y + c''z + d''t, \\ t' &= a'''x + b'''y + c'''z + d'''t \end{aligned}$$

be a system of equations, where the variables are coordinates of points, and which we represent by:

$$m' = \varphi(m).$$

Let p an arbitrary plane with coordinates u, v, w, n. One can write:

$$ux' + vy' + wz' + nt' = (au + a'v + a''w + a'''n)\, x + (bu + b'v + \dots)\, y + \dots,$$

which is a relation that can be written in tri-quaternions:

$$P \cdot p\, \varphi(m) = P \cdot \varphi'(p)\, m,$$

upon letting φ' denote the planar, homogeneous, linear function:

$$\begin{aligned} u' &= au + a'v + a''w + a'''n, \\ v' &= bu + b'v + b''w + b'''n, \\ w' &= cu + c'v + c''w + b'''n, \\ n' &= du + d'v + d''w + a'''n. \end{aligned}$$

We call φ' the function *conjugate* to φ.

The definition is, moreover, reciprocal.

Consider three points m_1, m_2, m_3, and the plane that passes through these three points:

$$p = S \cdot m_1, m_2, m_3 .$$

The three points:

$$m'_1 = \varphi(m_1), \quad m'_2 = \varphi(m_2), \quad m'_3 = \varphi(m_3)$$

likewise determine a plane:

$$p' = S \cdot m'_1, m'_2, m'_3 ,$$

which is a homogeneous, linear function of the plane p:

$$p' = \psi(p).$$

One obviously has:

$$Pm'p' = \lambda\, P\, m\, p,$$

where λ is a numerical quantity, or:

$$P \cdot m' \psi(p) = P \cdot p\ \psi'(m') = P \cdot p\ \psi'[(\varphi(m')] = \lambda\, P\, m\, p,$$

and, in turn:

$$\psi' \varphi = \lambda \qquad \text{and} \qquad \lambda\, \varphi^{-1} = \psi'.$$

One has an expression for the function that is inverse to φ with this.

That is all that we shall say about homographic transformations.

Let:

(3) $$p' = \varphi_1(m)$$

be a dualistic transformation.

We shall refer to the *conjugate* of the function φ_1 when we mean the function $\varphi_1'(m)$ that is characterized by the identity relation:

$$P \cdot m_1\ \varphi_1(m) = P \cdot m\, \varphi_1'(m_1)\ .$$

If φ_1 is expressed in the following form, which is obviously general:

$$\varphi_1(m) = p_1'\ P\, p_1\, m + p_2'\ P\, p_2\, m + p_3'\ P\, p_3\, m + p_4'\ P\, p_4\, m,$$

then one will have:

$$\varphi_1'(m_1) = p_1\, P\ p_1'\ m + p_2\, P\ p_2'\ m + p_3\, P\ p_3'\ m + p_4\, P\ p_4'\, m,$$

by virtue of the identity:

$$\begin{aligned} P \cdot m_1\ \varphi_1(m) &= P\, m_1\, p_1'\ P\, p_1\, m + P\, m_1\, p_2'\ P\, p_2\, m + P\, m_1\, p_3'\ P\, p_3\, m + P\, m_1\, p_4'\ P\, p_4\, m, \\ &= P\, m\, p_1\, P\ p_1'\ m + P\, m\, p_2\, P\ p_2'\ m + P\, m\, p_3\, P\ p_3'\ m + P\, m\, p_4\, P\ p_4'\, m, \\ &= Pm\ \varphi_1'(m_1)\, . \end{aligned}$$

Since the functions φ_1 and φ_1' have the same type, they can be identical to each other, and in that case, we will say that the function φ_1 is *auto-conjugate*.

One will define the transformation conjugate to a dualistic transformation that transforms a plane to a point in an analogous manner.

The conjugate of the sum of two functions is the sum of their conjugates.

It then results that $\varphi_1 + \varphi_1'$ is auto-conjugate.

One has:

$$\varphi_1 = \frac{\varphi_1 + \varphi_1'}{2} + \frac{\varphi_1 - \varphi_1'}{2}\, .$$

identically.

One thus decomposes a function φ_1 into an auto-conjugate function $\frac{\varphi_1 + \varphi_1'}{2}$ and another function $\frac{\varphi_1 - \varphi_1'}{2}$ that presents a special character.

Indeed, $\varphi_1 - \varphi_1'$ is composed of pairs of terms of the form:

$$p_1' P\, p_1\, m - p_1\, P\, p_1'\, m\,,$$

which is an expression that represents the plane that passes through the intersection of the planes p_1 and p_1' and the point m. Indeed, one has the formula:

$$p_1' P\, p_1\, m - p_1\, P\, p_1'\, m = S \cdot L\, p_1\, p_1',\, m.$$

Each pair of terms of this sort will give a similar reduction, and one will finally have:

$$\varphi_1(m) - \varphi_1'(m) = S \cdot c,\, m,$$

where c is the complex that is the sum of all the lines $L\, p_1\, p_1'$.

The equation:

$$p' = S \cdot cm,$$

is the general form of the dualistic transformations that make any point correspond to a plane that contains it.

The decomposition that we just realized is that of Clebsch ([1]):

$$\begin{aligned} u &= ax + by + cz + dt, \\ &= ax + \frac{b+a'}{2}y + \frac{c+a''}{2}z + \frac{d+a'''}{2}t + \frac{b-a'}{2}y + \frac{c-a''}{2}z + \frac{d-a'''}{2}t\,, \end{aligned}$$

$$\begin{aligned} v &= a'x + b'y + c'z + d't, \\ &= \frac{b+a'}{2}x + b'y + \frac{c+b''}{2}z + \frac{d'+b'''}{2}t + \frac{a'-b}{2}x + \frac{c-b''}{2}z + \frac{d'-b'''}{2}t\,, \end{aligned}$$

$$\begin{aligned} w &= a''x + b''y + c''z + d''t, \\ &= \frac{c+a''}{2}x + \frac{c'+b''}{2}y + c''z + \frac{d'+c'''}{2}t + \frac{a''-b}{2}x + \frac{b''-c'}{2}y + \frac{d'-c'''}{2}t\,, \end{aligned}$$

$$\begin{aligned} n &= a'''x + b'''y + c'''z + d'''t, \\ &= \frac{d+a'''}{2}x + \frac{d'+b'''}{2}y + \frac{d''+c'''}{2}z + d'''t + \frac{a'''-d}{2}x + \frac{b'''-d'}{2}y + \frac{c'''-d''}{2}z\,. \end{aligned}$$

From the analytical viewpoint, this decomposition is the analogue of the Helmholtz decomposition for homogeneous, linear vectorial functions. However, whereas the latter decomposition is simply metric – i.e., it is invariant with respect to the displacements without deformation in space – the decomposition that just presented is projective.

([1]) CLEBSCH, *Vorlesungen über Geometrie*, II Band.

One cannot apply the procedure that is based upon the existence of an identity relation to the solution of equation (3) for m because the iteration of a dualistic transformation has no meaning.

Consider three points m_1, m_2, m_3, and the plane that passes through these three points:

$$p = S\, m_1, m_2, m_3 .$$

The three points m_1, m_2, m_3 correspond to three planes:

$$p'_1 = \varphi_1 (m_1), \qquad p'_2 = \varphi_1 (m_2), \qquad p'_3 = \varphi_1 (m_3).$$

Let their common point be:

$$m' = L \cdot p'_1 p'_2 p'_3 ,$$

which is obviously a homogeneous, linear function of p:

$$m' = \psi_1(p).$$

If a point m is situated in a plane p then the plane p' that corresponds to m will pass through the point m' that corresponds to p – i.e., one will have:

$$Pm'p' = \lambda\, P\, m\, p$$

identically, where λ is a numerical quantity, or:

$$P \cdot \psi_1(p)\ \varphi_1(m) = \lambda\, P\, m\, p,$$

and, in turn:

$$P \cdot m\, \varphi'_1[\psi_1(p)] = \lambda\, P\, m\, p$$

or finally, since one is dealing with an identity relation:

$$\varphi'_1 \psi_1 = \lambda ;$$

i.e., ψ_1 is the inverse of φ'_1, up to a factor, and in turn, the function ψ'_1 that is conjugate to ψ_1 is the inverse of φ_1 .

Knowing λ and ψ'_1, φ_1^{-1} will be given by the formula:

$$\lambda \varphi_1^{-1} = \psi'_1 .$$

In order to obtain ψ_1, it suffices to apply the function φ_1 at four arbitrarily-chosen points m_1, m_2, m_3, m_4 that are not, however, coplanar:

$$p'_1 = \varphi_1 (m_1), \qquad p'_2 = \varphi_1 (m_2), \qquad p'_3 = \varphi_1 (m_3), \qquad p'_4 = \varphi_1 (m_4).$$

One forms the four points:

$$m_1' = L\, p_2'\, p_3'\, p_4', \qquad m_2' = L\, p_4'\, p_1'\, p_3', \qquad m_3' = L\, p_4'\, p_1'\, p_2', \qquad m_4' = L\, p_1'\, p_2'\, p_3'.$$

One obtains the expression for ψ_1 by the formula:

$$P \cdot m_1\, S \cdot m_2, m_3, m_4 \cdot \psi_1(p) \equiv m_1' P \cdot m_1\, p + m_2' P \cdot m_2\, p + m_3' P \cdot m_3\, p + m_4' P \cdot m_4\, p.$$

One finally has:

$$P \cdot m_1\, S \cdot m_2, m_3, m_4 \cdot \psi_1'(p) \equiv m_1 P \cdot\ p_2'\, p_3'\, p_4' p + m_2 P \cdot\ p_3'\, p_4'\, p_1' p$$
$$+\, m_3 P \cdot\ p_4'\, p_1'\, p_2' p + m_4 P \cdot\ p_1'\, p_2'\, p_3' p.$$

One chooses m_1, m_2, m_3, m_4 in such a manner as to simplify the calculations in regard to the given function φ_1 .

As for the value of λ, one sees that upon replacing p with $\varphi_1(m_1) = p_1'$ in the preceding identity, one must have:

$$P \cdot m_1\, S \cdot m_2, m_3, m_4 \cdot \psi_1' \varphi_1(m_1) = m_1 P \cdot\ p_2'\, p_3'\, p_4' p_1.$$

Thus:

$$\lambda = \frac{P \cdot p_1'\, p_2'\, p_3'\, p_4'}{P \cdot m_1 S \cdot m_2, m_3, m_4}.$$

The number λ is a projective invariant of the transformation φ_1 . It is the determinant of the coefficients of that transformation.

If we apply the indicated calculation to the general form of φ_1 that was given at the beginning of this paragraph then we will find:

$$(4)\qquad \left\{ \begin{aligned} & P \cdot p_1 p_2 p_3 p_4\, P\, p_1' p_2' p_3' p_4' \cdot \varphi_1^{-1}(p') \\ & \quad = L \cdot p_2 p_3 p_4\, P\, p' p_2' p_3' p_4' + L \cdot p_3 p_4 p_1\, P\, p' p_3' p_4' p_1' \\ & \qquad + L \cdot p_4 p_1 p_2\, P\, p' p_4' p_1' p_2' + L \cdot p_1 p_2 p_3\, P\, p' p_1' p_2' p_3'. \end{aligned} \right.$$

If λ is zero then the solution will be illusory.

In that case, $\varphi_1(m)$ will satisfy a homogeneous, linear relation:

$$P \cdot\ m_0'\, \varphi_1(m) \equiv 0.$$

One will deduce from this that:

$$P \cdot m\ \varphi_1'(m_0') \equiv 0,$$

or, since m is arbitrary:

$$\varphi_1'(m_0') = 0.$$

The function φ_1' also has a zero determinant, and there exists a point m_0 such that one has:

$$\varphi_1(m_0) = 0, \qquad P \cdot m_0\ \varphi_1'(m) = 0.$$

φ_1 makes a plane p' that passes through m_0' correspond to each point m in space, and the same plane will correspond to all points that are situated on the same line that passes through m_0, as is shown by the formula:

$$\varphi_1(m + xm_0) = \varphi_1(m) + x\varphi_1(m_0) = \varphi_1(m)\ .$$

In order for the given equation (3) to have a solution, it is necessary that one have:

$$P \cdot\ m_0'\ p' = 0.$$

Once this condition is satisfied, p' will correspond to all of the points of a line that passes through m_0 , in such a way that it will suffice to consider what happens in an arbitrarily-chosen plane ϖ that does not contain either m_0 or m_0'. The function φ_1 makes a point in that plane correspond to a line in that plane, and one is thus led to the study of a dualistic transformation in the plane.

In order to do this, it will suffice to consider the function:

$$L \cdot \varpi\ \varphi_1(m),$$

in place of $\varphi_1(m)$, while supposing that m belongs to the plane ϖ, and one will have to solve the equation:

$$L \cdot \varpi\ \varphi_1(m) = L \cdot \varpi\ p'$$

in the plane ϖ, which one can easily put into the form:

$$\chi(n) = d'.$$

The solution of this equation of plane geometry is achieved by a procedure that is modeled up on the one that we just discussed for space.

Upon calling the solution thus obtained m_1, the general solution to equation (3) will be written:

$$m_1 + x\ m_0\ ,$$

in which x is an arbitrary numerical quantity.

Therefore, in this case, p' will be a function of the line $V \cdot m_0, m$, properly speaking.

Effectively, one can write:

$$\varphi_1(m) \equiv p_1' G\, \delta_1 V \cdot m_0, m + p_2' G \cdot \delta_2 V \cdot m_0, m + p_3' G \cdot \delta_3 V \cdot m_0, m$$

where p_1', p_2', p_3' are three planes that pass through m_0' and where δ_1, δ_2, δ_3 are lines that pass through the point m_0, and if one sets:

$$V \cdot m_0, m = d \qquad \text{and} \qquad \varphi_1(m) = \theta(d),$$

the equation:

$$\theta(d) = p'$$

will generally have a solution in d if p' passes through m_0'.

Whether one employs the auxiliary plane or considers p' to be a function of d, one will be dealing with ternary linear functions, and one can achieve their inversion by a procedure that is identical to the one that we just presented for quaternary functions.

The solution can again become illusory, and it is clear that in that case the planes p' will pass through the same line, and can be considered to be functions of the planes that pass through the same line, which will coincide with the first one if the function φ_1 is auto-conjugate.

In summary, one can achieve the solution to equation (3) in any case, along with the reduction to its simplest form.

We have determined a function ψ_1 that is linked to the function φ_1 and transforms a plane into a point.

One can likewise determine the transformation that makes a line:

$$d = V \cdot m_1, m_2$$

that connects two arbitrary points in space correspond to another line:

$$d = L\, p_1'\, p_2'$$

that is common to the planes p_1', p_2':

$$p_1' = \varphi_1(m_1), \qquad p_2' = \varphi_1(m_2).$$

We are also led to say a few words about the transformations of third type.

Upon forming $L \cdot \varphi_1(m_1)\, \varphi_1(m_2)$, one will obtain a function of $V \cdot m_1, m_2$, and one will thus have a transformation that makes a line correspond to another one or even a complex to another one:

(5) $$c' = \Phi(c).$$

One will obtain a similar function by starting with a homographic transformation.

The function Φ presents the property of preserving the quadratic variety that is composed of the lines – i.e., that one will have the relation:

$$Pc'^2 = \lambda\, P\, c^2.$$

Upon introducing the notion of conjugate function, as we did for the other functions that were encountered already, by means of the relation:

$$P \cdot c\,\Phi(c_1) = P \cdot c_1\,\Phi'(c),$$

we see that the preceding condition can be written:

$$P \cdot [\Phi'(c) - \lambda c] = 0$$

or

$$\Phi'\,\Phi = \lambda.$$

The condition for the transformation Φ' to preserve the quadric of the lines – i.e., to determine a projective transformation – is thus that $\Phi'\,\Phi$ must be numerical quantity.

That property will permit us to easily achieve the inversion of the function Φ.

We can thus assume that we know how to achieve the inversion of homogeneous, linear functions of the point, plane, and line that give rise to projective transformations in terms of tri-quaternions.

That knowledge is indispensable if one is to begin the theory of second-degree surfaces, in which we likewise use the notion of a conjugate, homogeneous, linear function.

Second-degree surfaces. – Some doubts might persist regarding the efficacy of the use of the calculus of tri-quaternions in geometric analysis, if we do not show that it lends itself very naturally to the establishment of some elementary properties of quadrics.

The general form of a homogeneous second-degree equation in m is obviously:

$$P \cdot m\,\varphi(m) = 0, \tag{6}$$

where $\varphi(m)$ represents a homogeneous, linear, planar function of m, which is a function that one can assume to be auto-conjugate, by reason of the identity:

$$P \cdot m\,S \cdot c, m = 0.$$

A quadric is thus determined in a unique manner by an auto-conjugate, homogeneous, linear function φ.

The plane $\varphi(m_0)$ is then the polar of the point m_0 with respect to the quadric.

Indeed, let m be a point in that plane. Any point of the line that joins the points m and m_0 will be of the form:

$$m_0 + x\,m,$$

and the values of x that correspond to the points of intersection of that line with the surface are given by the equation:

$$P \cdot (m_0 + x\,m)\,\varphi\,(m_0 + x\,m) = 0,$$

or

$$x^2\,P \cdot m\,\varphi(m) + 2\,x\,P \cdot m\,\varphi(m_0) + P \cdot m_0\,\varphi(m_0) = 0.$$

The condition for the points of intersection to be harmonically separated with m and m_0 is that the values of x must be equal and of opposite sign, which gives:

$$P \cdot m\ \varphi(m_0) = 0,$$

which indeed expresses the idea that the locus of the point m is the plane $\varphi(m_0)$.

If the point m_0 is situated on the surface then the plane $\varphi(m_0)$ will be the tangent plane to that point. Indeed, upon differentiating equation (6), one has that:

$$P \cdot m\ \varphi(dm) + P \cdot dm\ \varphi(m) = 0,$$

or

$$2P \cdot dm\ \varphi(m) = 0,$$

which is a formula that expresses the idea that the point or vector dm is indeed situated in the plane $\varphi(m)$ – i.e., that $\varphi(m)$ is the tangent plane at m.

Set:

$$\varphi(m) = p \qquad \text{or} \qquad m = \varphi^{-1}(p).$$

Upon substituting this expression for m into equation (6), one will obtain the tangential equation of the quadric:

$$P \cdot p\ \varphi^{-1}(p) = 0,$$

which can be written, upon introducing the function $\psi(p)$ of the preceding paragraph:

(7) $$P \cdot p\ \psi(p) = 0.$$

One can also obtain the condition for a line to be tangent to the quadric in an analogous form.

Upon recalling the function Φ of the preceding paragraph, that condition will be written:

(8) $$P \cdot c\ \Phi(c) = 0,$$

where one can obviously assume that Φ is auto-conjugate – i.e., that it satisfies the condition:

(9) $$\Phi^2 = \lambda.$$

One must add the condition:

$$P\ c^2 = 0$$

to equation (8).

However, without adding that condition, equation (8) suffices to determine the quadric, just as equations (1) or (7) do. That equation (8) then defines the set of complexes that are auto-conjugate with respect to the quadric.

In the case where the projective invariant λ is zero, according to the preceding paragraph, one will have identities of the form:

$$P \cdot m_0\ \varphi(m) = 0, \qquad \varphi(m_0) = 0,$$

in which m_0 is a well-defined point in space.

The surface is then a cone whose summit is the point m_0 .

The inverse function $\varphi^{-1}(p)$ will then make sense only if the plane p passes through the point m_0, and in that case, it will have an infinitude of determinations of the form $m + x\, m_0$.

As for the function:

$$m' = \psi(p)$$

that is obtained by taking:

$$p = S \cdot m_1, m_2, m_3, \qquad m' = L \cdot \varphi(m_1)\, \varphi(m_2)\, \varphi(m_3),$$

it will take the form:

$$\psi(p) \equiv m_0\, P \cdot m_0\, p,$$

in which the tensor of m_0 is assumed to be chosen conveniently.

The tangential equation (7) thus presents itself in the form of a square that is equal to zero:

$$(P \cdot m_0\, p)^2 = 0.$$

The tangent planes to the cone are determined by two equations, namely, the tangential equation at the point m_0:

$$P \cdot m_0\, p = 0$$

and the tangential equation to an arbitrary quadric that is inscribed in the cone. The latter equation will be obtained by starting with a point-like equation of the form:

$$P \cdot m\, \varphi(m) + [P \cdot m\, p_0]^2 = 0,$$

in which p_0 is an arbitrary plane that does not pass through the point m_0.

The function Φ also presents some peculiarities in the case that we are occupied with. One can put it into the form:

$$\Phi(c) = A\, \delta_1\, P \cdot \delta_1\, c + B\, \delta_2\, P \cdot \delta_2\, c + C\, \delta_3\, P \cdot \delta_3\, c,$$

in which δ_1, δ_2, δ_3 are three conveniently-chosen lines that pass through the point m_0, and A, B, C are numerical coefficients.

In reality, $\Phi(c)$ depends upon only the plane $S \cdot m_0, c$, because one has:

$$T\, m_0 \cdot P\, \delta_1\, c = P \cdot \omega\, \delta_1\, S \cdot m_0, c, \qquad T\, m_0 \cdot P\, \delta_2\, c = P \cdot \omega\, \delta_2\, S \cdot m_0, c,$$
$$T\, m_0 \cdot P\, \delta_3\, c = P \cdot \omega\, \delta_3\, S \cdot m_0, c,$$

and in turn:

$$T\, m_0 \cdot \Phi(c) = A\, \delta_1\, P \cdot \omega\, \delta_1\, S \cdot m_0, c + B\, \delta_2\, P \cdot \omega\, \delta_2\, S \cdot m_0, c + C\, \delta_3\, P \cdot \omega\, \delta_3\, S \cdot m_0, c.$$

If one confines oneself to considering lines then one will see that the function Φ makes any line d in space correspond to a line that passes through m_0 and depends upon only the plane $S \cdot m_0, d$.

Equation (8) becomes:

$$A\,[P \cdot \delta_1\, c]^2 + B\,[P \cdot \delta_2\, c]^2 + C\,[P \cdot \delta_3\, c]^2 = 0,$$

or

$$A\,[P \cdot \omega\delta_1\, S \cdot m_0, c]^2 + B\,[P \cdot \omega\delta_2\, S \cdot m_0, c]^2 + C\,[P \cdot \omega\delta_3\, S \cdot m_0, c]^2 = 0.$$

If one considers only lines then this equation will indeed represent the set of lines that are tangent to the cone, because it expresses the idea that the lines d and $\Phi(d)$ are situated in the same plane, namely, $S \cdot m_0\, d$.

Upon setting:

$$S \cdot m_0, d = p,$$

the equation above becomes:

$$A\,[P \cdot \omega\delta_1\, p]^2 + B\,[P \cdot \omega\delta_2\, p]^2 + C\,[P \cdot \omega\delta_3\, p]^2 = 0,$$

which, along with the equation:

$$P \cdot m_0\, p = 0,$$

represents the tangent planes to the cone.

One will treat all of the degenerate cases in an analogous manner.

Now, determine the section of a quadric by a plane ϖ.

We have presented a bi-quaternionic calculus that presents the same properties with respect to plane geometry that the calculus of tri-quaternions does with respect to the geometry of space.

One can thus represent a conic that is situated in a plane ϖ by an equation of the form:

$$Pn\varphi_\varpi(n) = 0,$$

where n represents the line that is perpendicular to the plane ϖ that is drawn through a point of that plane, and $\varphi_\varpi(n)$ is a line in the plane ϖ that is a homogeneous, linear function of n.

We shall now determine an equation of the form above that represents the section of the quadric (1) by the plane ϖ.

Let m be a point of that section, and let n be the line that is drawn through m and perpendicular to the plane ϖ. One has:

$$\begin{aligned} m &= -\,\varpi^2\, m = -\,\varpi\, L\varpi\, m - \varpi\, P\varpi\, m \\ &= \varpi\, n. \end{aligned}$$

Equation (6) thus becomes:

$$P \cdot \varpi\, n\varphi\,(\varpi\, n) = 0$$

for the points of the plane ϖ, or:

$$P \cdot n\varpi\, \varphi\,(\varpi\, n) = 0.$$

The expression $\varpi\,\varphi\,(\varpi\,n)$ represents the intersection of the plane $\varphi\,(m)$ and the plane ϖ, and in turn, will be a line in the latter plane. It is the desired function $\varphi_\varpi(n)$.

The reduction of $\varphi_\varpi(n)$ to a canonical form can always be achieved by exhibiting either an auto-polar triangle, a center and axes, the asymptotes, the focal points, etc., of that function, and doing this by means that are similar to the ones that we pointed out for surfaces.

We shall now determine the equation of the cone that is circumscribed in a quadric and has its summit at a given point m_0 .

This cone is a quadric that passes through the intersection of the quadric (6) and the polar plane $\varphi(m_0)$. Its equation then has the form:

$$x\,P\,m\,\varphi(m) + [P\,m\,\varphi(m_0)]^2 = 0$$

or

$$P \cdot m\,[x\,\varphi(m) + \varphi(m_0)\,P\,m\,\varphi(m_0)] = 0.$$

The function:

$$x\,\varphi(m) + \varphi(m_0)\,P\,m\,\varphi(m_0)$$

must have a zero invariant λ. However, since we already know the summit m_0 of the cone, it will suffice to write:

$$x\,\varphi(m_0) + \varphi(m_0)\,P\,m_0\,\varphi(m_0) = 0,$$

which is a condition that is satisfied by:

$$x = -\,P\,m_0\,\varphi(m_0),$$

thanks to the factor $\varphi(m_0)$.

The homogeneous, linear function relative to the cone will then be:

$$\varphi(m)\,P\,m_0\,\varphi(m_0) - \varphi(m_0)\,P\,m\,\varphi(m_0).$$

We shall now determine the points of intersection of a given line δ with the quadric (6).

That determination can be done in several ways. We choose the one that utilizes the calculus that we presented in the first paragraph of this chapter, and which plays the same role with respect to rectilinear geometry that the calculus of bi-quaternions does with respect to plane geometry.

Represent each point m of the line δ by the plane p that is drawn through that point perpendicular to δ. In order to determine the points of intersection of δ with the quadric (1), we shall seek an equation of the form:

$$P\,p\,\varphi_\delta(p) = 0,$$

where $\varphi_\delta(p)$ represents a plane perpendicular to δ and is a homogeneous, linear function of p.

For any point m of the line δ, one will have:

$$m = -\delta^2 m = -\delta L\, \delta m - \delta P\, \delta m$$
$$= \delta p.$$

The equation of the quadric (6) can thus be written:

$$P \cdot \delta p\, \varphi(\delta p) = 0 \qquad \text{or} \qquad P \cdot \delta L\, p\varphi(\delta p) = 0$$

for any point δp of the line δ.

Decompose the plane $\varphi(\delta p)$ into a plane perpendicular to δ and another one that passes through that line:

$$\varphi(\delta p) = -\delta^2 \varphi(\delta p) = -\delta L\, \delta\, \varphi(\delta p) - \delta P\, \delta\, \varphi(\delta p).$$

The second term will give rise to a zero term in the preceding equation, because if one denotes it by p' then Lpp' will represent a line in the plane p' – i.e., a line that meets δ.

The equation in p will thus become:

$$P\, [\delta L \cdot p\, \delta L\, \delta\, \varphi(\delta p)] = 0,$$

or

$$\omega\, \delta L \cdot p\, \delta L\, \delta\, \varphi(\delta p) = 0,$$

or finally:

$$L \cdot p\, \delta L\, \delta\, \varphi(\delta p) = 0.$$

The desired function $\varphi_\delta(p)$ is then:

$$\varphi_\delta(p) \equiv \delta L\, \delta\, \varphi(\delta p).$$

The expression $L\delta\varphi(\delta p)$ represents the point of intersection of the line δ and the polar plane $\varphi(\delta p)$ of the point δp, in such a way that $\varphi_\delta(p)$ represents the plane through that point of intersection that is perpendicular to the line δ.

The second-degree equation:

$$L\, p\varphi_\delta(p) = 0$$

will give two values for p.

These two values will coincide if there exists a plane p_0 that is perpendicular to d for which one has:

$$\varphi_\delta(p_0) = 0 \qquad \text{or} \qquad L\, \delta\varphi(\delta p_0) = 0,$$

which is a relation that expresses the idea that the plane $\varphi(\delta p_0)$ that is polar to the point δp_0 with respect to the quadric contains the line δ.

Finally, if the equation is satisfied identically then the line δ will be completely on the surface; i.e., if one has:

$$\varphi_\delta(p) \equiv 0 \qquad \text{or} \qquad L\, \delta\varphi(\delta p) \equiv 0,$$

which is a relation that concisely expresses the idea that the line conjugate to δ with respect to the quadric agree with δ.

We shall now determine the second-degree cones that pass through the intersection of the two given quadrics.

If φ and φ_1 represent the linear functions that correspond to two quadrics then the linear function that corresponds to a quadric that passes through the intersection of the latter two will have the form

$$x\,\varphi + y\,\varphi_1 ,$$

in which x and y are numerical coefficients.

One can determine the ratio x / y in such a manner that the function represents a cone. One will thus have four values for x / y. Let m_0 be the summit of one of the cones thus obtained. One will have:

$$x\,\varphi(m_0) + y\,\varphi_1(m_0) = 0.$$

If the point m_0 is situated on the quadric φ – i.e., if one has:

$$P\,m_0\,\varphi(m_0) = 0,$$

then it will result that:

$$P\,m_0\,\varphi_1(m_0) = 0;$$

i.e., that it is likewise situated on the quadric φ_1 .

The tangent planes to m_0, $\varphi(m_0)$, and $\varphi_1(m_0)$ coincide in position, and in turn, the two quadrics will be tangent.

If one chooses p_1, p_2, p_3, p_4 to be four planes that determine an auto-polar tetrahedron then one can put the function φ that relates to one quadric into the form:

$$\varphi(m) = A\,p_1\,P\,p_1\,m + B\,p_2\,P\,p_2\,m + C\,p_3\,P\,p_3\,m + D\,p_4\,P\,p_4\,m.$$

One can also put the equation for the quadric into the following form, which exhibits the rectilinear generators:

$$P \cdot m\,p_1\,P\,m\,p_2 - P\,m\,p_1'\,P\,m\,p_2' = 0,$$

which is an equation that can be written:

$$P\,m\,(p_1\,P\,m\,p_2 - p_1'\,P\,m\,p_2') = 0.$$

The expression in parentheses does not represent an auto-conjugate function.

The function that is conjugate to it has the expression:

$$p_1\,P\,m\,p_2 - p_1'\,P\,m\,p_2' .$$

The sum of these two functions is auto-conjugate, and can be identified with any function φ that is auto-conjugate relative to the quadric considered, up to a numerical factor.

If the function φ is given in an arbitrary form then one will obtain the rectilinear generators by choosing p_1 and p_1' to be two tangent planes and determining p_2 and p_2' by means of the identity:

$$p_1\, P\, m\, p_2 + p_2\, P\, m\, p_1 - p_1' P m p_2' - p_2' P m p_1' \equiv \varphi(m).$$

Up to now, we have envisioned the properties of quadrics that implement only projective notions – namely, intersection, contact, line, and anharmonic ratio.

We have sufficiently shown that from this viewpoint the calculus of tri-quaternions presents no obstacle to following analytic geometry, step-by-step.

In concluding, we shall say some words about the properties of quadrics that involve the plane at infinity and indeed metric properties – i.e., the ones that depend upon the imaginary circle at infinity.

The first ones are coupled to the notions of asymptotes, center, and conjugate direction, which are the elements that we shall first determine.

The asymptotic directions are given by the intersection of the quadric with the plane at infinity ω.

If ρ represents a vector – i.e., a point at infinity – then $\varphi(\rho)$ will represent the polar plane to that point, $\omega\,\varphi(\rho)$ will represent a vectorial function of ρ, and upon setting:

$$\omega\,\varphi(\rho) \equiv \varphi_1(\rho),$$

the equationsfor the asymptotic directions will be the equation:

$$S \cdot \rho,\ \varphi_1(\rho) = 0.$$

The center of the quadric φ is determined by:

(10) $$\varphi(m) = \omega,$$

which is an equation that one knows can be solved as in the preceding paragraph.

The point m_0 thus obtained can reduce to a vector – i.e., it can go to infinity. That is the case for paraboloids.

When the projective invariant λ of φ is zero there will exist a point m_0 such that one has:

$$\varphi(m_0) = 0,$$

and since this point m_0 is the pole of any plane in space, it can be considered to be the pole of the plane at infinity, in particular. The surface is a cone with summit m_0 .

The point m_0 can itself be at infinity, and the plane ω, which is the right-hand side of equation (10), will then contain the point m_0 . In the preceding paragraph, we saw that in this case equation (10) will have an infinitude of solutions that are represented by points that are situated on a line that passes through the point m_0 at infinity. That is the case for cylinders.

The notion of conjugate directions results from that of planes polar to the points at infinity.

Any vector ρ corresponds to a polar plane $\varphi(\rho)$ that obviously passes through the center of the quadric. This polar plane can also be defined as the plane that is diametral to the chords parallel to the direction ρ.

Indeed, let m be the midpoint of one of these chords. Any point of that chord will have the form:

$$m + x\,\rho.$$

The two intersection points of the chord with the quadric will be given by the equation:

$$P \cdot (m + x\,\rho)\ \varphi\,(m + x\,\rho) = 0,$$

or

(11) $$x^2\, P \cdot \rho\ \varphi(\rho) + 2x\, P \cdot m\ \varphi(\rho) + P \cdot m\ \varphi(m) = 0.$$

Upon writing that this equation has roots that are equal and of opposite signs, one will have:

$$P \cdot m\ \varphi(\rho) = 0,$$

which expresses the idea that the desired locus is, in fact, the plane $\varphi(\rho)$.

One will likewise deduce the equation for asymptotic directions:

$$P \cdot \rho\ \varphi(\rho) = 0, \qquad \text{which is equivalent to} \qquad S \cdot \rho,\ \omega\, \varphi(\rho) = 0$$

from equation (11).

We finally write down that equation (11) has equal roots:

$$[P\ m\ \varphi(\rho)]^2 - P\ \rho\ \varphi(\rho)\ P\ m\ \varphi(m) = 0.$$

When one regards ρ to be a variable, this equation will represent the directions of the cone that the quadric circumscribes and has its summit at m.

On the contrary, if one takes m to be the variable then that equation will be that of the cylinder that circumscribes the quadric and has its generators parallel to ρ.

One can also seek the locus of the centers of the parallel sections.

Let m be the center of the section of the surface by a plane ϖ – i.e., the pole of the line at infinity $\omega\varpi$ of the plane ϖ with respect to the conic of intersection.

One sees that the polar of a point m in the plane v with respect to that conic is:

$$L\ \varpi\ \varphi(m).$$

One must then have:

$$L\ \varpi\ \varphi(m) = x\,\omega\varpi,$$

which is a formula that expresses the idea that the planes ϖ and $\varphi(m)$ are parallel, and in turn, that the locus of the center m coincides with the locus of the pole of the plane ϖ.

The metric properties of quadrics are established by analogous prcedures.

The axial directions are obtained from the condition that they must be normal to their conjugate directions.

The cyclic sections are obtained as planes that pass through the common generators of a cone that is asymptotic to the surface and the cone of isotropic lines with the same summit.

Let ρ be a line that passes through an arbitrary fixed point.

The asymptotic cone has an equation of the form:

$$P\,\rho\,\varphi_1(\rho) = 0,$$

where $\varphi_1(\rho)$ is a vector.

The isotropic cone with the same summit has the equation:

$$\rho^2 = 0.$$

The linear function that relates to a second-order cone that has the same summit as the preceding one and passes through their lines of intersection will have the form:

$$x\,\varphi_1(\rho) = y\,\rho.$$

Upon annulling the determinant of that function, one will have three values for x / y, each of which will correspond to a pair of directions with real or imaginary cyclic sections.

As for the focal points – i.e., the points such that the cone that circumscribes the surface and has one of these points for its summit is bitangent to the cone with the same summit that is composed of isotropic directions – one will obtain them by writing that the cone:

$$P\cdot\,\rho\,[\varphi_1(\rho)\,P\,m_0\,\varphi(m_0) - \omega\,\varphi(m_0)\,P\,\rho\,\varphi(m_0)] = 0$$

is bitangent to the cone:

$$\rho^2 = 0.$$

One will then obtain two equations in m_0 that will determine the focal lines.

We shall not go further into that study, so for us it will suffice to present the principal procedures by which the calculus of tri-quaternions can substitute for analytic geometry in its initial questions.

The very elementary character of the questions that were treated has the advantage of not initially requiring a presentation, which would be unavoidably long, of the means by which there is reason to endow this calculus in order to make it satisfy all of the demands of analysis.

In a general fashion, these means are common to all complex calculi.

In these calculi, infinitesimal analysis notably takes on a very special aspect that was inspired by the remarkable work of Grassmann.

We likewise point out the necessity of establishing a theory of the elimination of a complex quantity from several equations, and even from just one complex equation, notably, insofar as linear equations are concerned, whose types are indeed numerous.

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APPENDIX.

GEOMETRICAL SIGNIFICANCE OF THE PRODUCT OF TWO TRI-QUATERNIONS

Product of two linear elements.

$$l = \mu x_0 + \rho + \omega \rho_1 ,$$
$$l' = \mu x_0' + \rho + \omega \rho_1' ,$$

$$ll' = x_0 x_0' + S\rho\rho' + V\rho\rho' + \omega[V\cdot(\rho\rho_1' + \rho_1\rho') + x_0'\rho_1 - x_0\rho_1'] + \mu(x_0\rho' + x_0'\rho) + \omega S\cdot(\rho\rho_1' + \rho_1\rho') .$$

One thus has:

$$Gll' = x_0 x_0' + S\rho\rho' ,$$
$$Lll' = V\rho\rho' + \omega[V\cdot(\rho\rho_1' + \rho_1\rho') + x_0'\rho_1 - x_0\rho_1'] ,$$
$$Pll' = \mu(x_0\rho' + x_0'\rho) + \omega S\cdot(\rho\rho_1' + \rho_1\rho') .$$

$L \cdot ll'$ – One sees that Lll' represents a complex whose axis is perpendicular to that of l and l', because $V\rho\rho'$, which gives the direction of the axis, is perpendicular to ρ and ρ'.

$L \cdot l^2 = 0$. – If one takes $l' = l$ then $L \cdot ll'$ will be annulled.

$L \cdot mm'$ – If the two factors are points:

$$m = \mu x_0 + \omega \rho_1 ,$$
$$m' = \mu x_0' + \omega \rho_1'$$

then one will have:

$$Lmm' = x_0'\rho_1 - x_0\rho_1' = x_0 x_0' \left(\frac{\rho_1}{x_0} - \frac{\rho_1'}{x_0'}\right).$$

This is the vector that takes the point m' to the point m, multiplied by the masses of the two points.

$L \cdot \gamma\gamma'$ – If the two factors are complexes:

$$\gamma = \rho + \omega \rho_1 ,$$
$$\gamma' = \rho' + \omega \rho_1'$$

then one will have:

$$L\gamma\gamma' = V\rho\rho' + \omega V(\rho\rho_1' + \rho_1\rho') ,$$

which is a complex whose axis is rectangular to that of γ and γ', and which is in involution with that of the two complexes. One notably verifies that one will have:

$$P \cdot \gamma L\gamma\gamma' = P \cdot \gamma(\gamma\gamma' - G\,\gamma\gamma' - P\,\gamma\gamma') = P \cdot \gamma^2\,\gamma'$$
$$= P\,(G\,\gamma^2 + L\,\gamma^2 + P \cdot \gamma^2)\,\gamma' = 0.$$

It results from this that $L\gamma\gamma'$ satisfies the four important relations:

$$G \cdot \gamma L\gamma\gamma' = 0, \qquad G \cdot \gamma' L\gamma\gamma' = 0,$$
$$P \cdot \gamma L\gamma\gamma' = 0, \qquad P \cdot \gamma' L\gamma\gamma' = 0.$$

Ldd´ – If the two factors are lines – i.e., if one makes the hypothesis that:

$$S\rho\rho_1 = 0, \qquad S\rho'\rho'_1 = 0,$$

in the preceding case – then the product will be a complex whose axis is the common perpendicular to the two lines, and whose parameter (or auto-moment) will be equal to the distance between the two lines, multiplied by the cotangent of their angle.

In order to see this, let i, j, k be three lines that form a tri-rectangular trihedron. Take i to be one of the lines, and suppose that k is the common perpendicular to the two lines – i.e., that the second line meets k and is parallel to the plane that is determined by i and j. This line will have an expression of the form:

$$i \cos\theta + j \sin\theta + \omega e\,(j \cos\theta - i \sin\theta),$$

in which θ is the angle between the two lines, and e is the distance between them.

We give the two lines tensors that equal unity. Indeed, it will suffice to multiply this result by the tensors of the factors.

One will have:

$$L \cdot i\,[i \cos\theta + j \sin\theta + \omega e\,(j \cos\theta - i \sin\theta)] = k \sin\theta\,(1 + \omega e \cot\theta),$$

which is indeed the stated result.

The tensor of the complex is $\sin\theta$; it will be zero when the two lines are parallel, and in that case, the complex will reduce to a vector:

$$\omega k\, e \cos\theta,$$

which will itself be zero if the two lines coincide.

If d and d' are rectangular then $L \cdot dd'$ will reduce to a line.

Return to the product of two complexes and set:

$$\gamma = \delta + \omega\,\delta\,\eta,$$
$$\gamma' = \delta' + \omega'\,\delta'\,\eta',$$

in which δ and δ' are the axes of the complexes. One will have:

$$L\gamma\gamma' = L\delta\delta' + \omega L\,\delta\delta'(\eta + \eta').$$

This complex has the same axis as $L\delta\delta'$ – i.e., it has the common perpendicular to the axes of the two factors γ and γ' as its axis.

We finally remark that one has:

$$L\gamma^2 = 0, \qquad L\, d^2 = 0.$$

Ldm – Set:

$$d = k, \quad m = \mu + \omega\rho,$$

$$Ldm = \omega V\, k\, \rho.$$

This is the vector that is perpendicular to the plane that is determined by the point and the line, which is taken to have a length that is equal to the distance from the point to the line.

If d represents an axis of rotation in position and magnitude then Ldm will represent the velocity of the point m in magnitude and direction, while the tensor of that point is assumed to be equal to unity.

$L\gamma m$ – Set:

$$\gamma = \delta + \omega\,\delta\,\eta.$$

One will have:

$$L\gamma m = L \cdot \delta m + \omega\,\delta\,\eta\, T\, m.$$

If $Tm = 1$ then $L\gamma m$ will represent the velocity of the point m in the state of motion that is characterized by the complex γ at the instant considered.

We seek the geometric condition that is expressed by:

$$Lll' = 0.$$

Set:

$$l' = \mu\, x_0' + ak.$$

The expression for Lll' will become:

$$Lll' = a\, V\rho\, k + \omega(a\, V\rho_1\, k + x_0'\,\rho_1).$$

One must then have:

$$V\rho\, k = 0, \qquad a\, V\rho_1\, k + x_0'\,\rho_1 = 0.$$

The first condition gives:

$$\rho = x\, k,$$

in which x is an ordinary quantity.

The second condition can be realized only by:

$$\rho_1 = 0,$$

after excluding the case in which one will have both:

$$x_0' = 0, \qquad a = 0;$$

i.e., $l' = 0$.

One thus has:

$$l = \mu\, x_0 + x\, k;$$

i.e., the points of the linear elements will coincide, along with their axes, although the mass and the length can be different for both elements.

If one of the elements is a line – i.e., if its point is zero – then the other element must have its point on that line and its axis directed along that line. In particular, if one of the factors is a point *m*, and the other one is a line *d* then the relation:

$$L\, d\, m = 0,$$

will express the idea that the point is situated on the line, which conforms to the significance that we have found for *Ldm*, moreover.

Pll′ – From the expression that we found for *ll′*, one will have:

$$Pll' = \mu(x_0\rho' + x_0'\rho) + \omega S(\rho\rho_1' + \rho_1\rho').$$

This is a plane perpendicular to the resultant of the two axes ρ / x_0 and ρ' / x_0' of the two linear elements.

If that result is zero – i.e., if the axes define a couple – then the plane will be pushed off to infinity, and it will be zero if one has:

$$S(\rho\rho_1' + \rho_1\rho') = 0$$

or

$$S(x_0\rho\rho_1' - x_0'\rho_1\rho') = S \cdot \rho(x_0\rho_1' - x_0'\rho_1) = 0.$$

This relation expresses the idea that:

$$\frac{\rho_1'}{x_0'} - \frac{\rho_1'}{x_0}$$

is perpendicular to ρ.

A figure will easily show that it is necessary that the points *l* and *l′* are situated in the same plane perpendicular to ρ.

The other case where the plane *Pll′* is pushed to infinity is the one where x_0 and x_0' are zero – i.e., where the two factors are complexes γ and γ'.

$P \cdot \gamma\gamma'$ – In this case, one will have:

$$P\gamma\gamma' = \omega S(\rho\rho_1' + \rho_1\rho') = -\,\omega T\,\gamma T\,\gamma' \mathfrak{M},$$

in which $\mathfrak{M}$ is the moment of the two complexes.

Pdd′ – If the complexes are lines then, as one knows, $\mathfrak{M}$ will be equal to the product of their distances with the sine of the angle between them.

The relations:

$$P \cdot \gamma\gamma' = 0, \qquad Pdd' = 0$$

express the ideas that the complexes *g* and *g′* are in involution, and that the lines *d* and *d′* meet them, respectively.

$P\gamma^2$ – One has:

$$P\gamma^2 = -2\ \omega S\,\rho\,\rho_1\,.$$

This is the product of the square of the tensor of γ and the parameter (or automoment) of that complex by ω.

$Pd^2 = 0$ – The condition:

$$P\gamma^2 = 0$$

expresses the idea that γ is a line.

Pmd – If one of the factors is a point *m* and the other one is a line *d* then one will have:

$$P \cdot md = \mu x_0 \rho' + \omega S \rho \rho_1'\,.$$

This is a plane that is perpendicular to the line *d* that passes through the point *m*.

$Pm\gamma$ – This is the plane through the point *m* that is perpendicular to the axis of γ.

Pl^2 – One has:

$$l^2 = x_0^2 + \rho^2 + 2(x_0\,\mu\,\rho + \omega S\,\rho\,\rho_1).$$

Thus:

$$Gl^2 = x_0^2 + \rho^2,$$
$$Pl^2 = 2(x_0\,\mu\,\rho + \omega S\,\rho\,\rho_1).$$

Pl^2 represents the plane perpendicular to the direction of *l* that passes through the point of *l*.

The condition:

$$Pl^2 = 0$$

demands that one must have:

$$x_0 = 0 \qquad \text{or} \qquad \rho = 0;$$

i.e., that *l* must be a point or a line.

We remark that the tensors of the point and the line of *l* can be expressed as functions of Gl^2 and $(Tl)^2$ (which one will generally write as $T^2 l$). One will have, moreover:

$$Tl^2 = \sqrt{(x_0^2 + \rho^2)^2 - 4x_0^2\rho^2} = x_0^2 - \rho^2$$

$$= T^2 l.$$

Gll′ – One has:

$$Gll' = x_0 x_0' + S\rho\rho'.$$

We point out only the three formulas:

$$Gmd = 0, \qquad Gmm' = TmTm', \qquad Gdd' = -TdTd'\cos(d, d').$$

The vanishing of *Gdd′* will express the idea that the two lines *d* and *d′* are rectangular to each other.

Product of two planes.

Let:

$$p = \mu\rho + \omega w,$$
$$p' = \mu\rho' + \omega w'.$$

Gpp′, *Lpp′* – One has:

$$pp' = TpTp'[-\cos(p, p') + d\sin(p, p')],$$
$$Gpp' = -TpTp'\cos(p, p'),$$
$$Lpp' = TpTp'\delta\sin(p, p'),$$

where δ is the line of intersection of the two planes *p* and *p′* and (*p*, *p′*) is the angle between them – i.e., the angle that is determined by the positive rotation around δ that makes the positive edge of *p* coincide with the positive edge of *p′*.

The sense of positive rotation around a directed line is given by the conventions that were already made, in such a way that if one arbitrarily chooses a positive sense of the line of intersection then it will result that sin(*p*, *p′*) will have a sign that changes with the chosen sense of δ. However, the product $\delta\sin(p, p')$ is well-defined in magnitude and direction. Its direction is, in summary, the one that gives a positive value to sin(*p*, *p′*).

Product of a linear element and a plane.

Let:

$$l = \mu x_0 + \rho + \omega\rho_1,$$
$$p = \mu\alpha + \omega w.$$

One has:

$$lp = \mu S\rho\alpha + x_0\alpha + \omega(w\rho + V\rho_1\alpha) + \mu V\rho\alpha + \omega S\rho_1\alpha - \omega x_0 w.$$

Glp = 0 – One thus has:

$$Glp = 0,$$

$$Llp = \mu \, S \, \rho \, \alpha + x_0 \, \alpha + \omega (w \, \rho + V \, \rho_1 \, \alpha).$$

Llp – The axis of that linear element is $x_0 \, \alpha$, and consequently, it is perpendicular to the plane *p*. Suppose that the plane *p* passes through the origin – i.e., that one has:

$$w = 0.$$

The coefficient of ω reduces to $V \, \rho_1 \, \alpha$, which is a vector that is situated in the plane *p*, in such a way that the point of *Llp* is likewise situated in that plane.

Lmp – If *l* reduces to a point *m* then one will have:

$$Lmp = x_0 \, \alpha + \omega V \, \rho_1 \, \alpha,$$

which is a line that is perpendicular to the plane *p* and passes through the point *m*, because one will verify directly that one has:

$$L \cdot m \, L \, m \, p = 0.$$

Ldp – If *l* reduces to a line *d* then one will have:

$$Ldp = \mu \, \delta \rho \, \alpha + \omega (w \, \rho + V \, \rho_1 \, \alpha).$$

One can suppose that the line *d* and the plane *p* both pass through the origin – i.e., that one has:

$$w = 0, \qquad \rho_1 = 0.$$

The preceding expression becomes:

$$\mu \, S \, \rho \, \alpha,$$

and one sees that if one abstracts from the tensors that are always multipliers in the product then *Ldp* will be the point of intersection of the line and the plane, which is taken to have a negative mass and the sine of the angle between the line and the plane form its absolute value.

If the line is parallel to the plane then one will have:

$$S \, \rho \, \alpha = 0,$$

and the expression for *Ldp* will become:

$$\omega (w \, \rho + V \, \rho_1 \, \alpha).$$

Like α, ρ_1 is rectangular with ρ, so $V\rho_1\alpha$ is of the form $x\rho$, and *Ldp* is a vector whose direction is contained in the plane *p*. As for its length, one sees geometrically that it is equal to the distance from the line to the plane, if one always abstracts from the tensors of the factors.

$L\gamma p$ – This is the focal point of the plane p with respect to the complex γ (*see* pp. 49).

The relation:

$$Ldp = 0$$

obviously expresses the idea that the line d is situated in the plane p.

Plp – One has:

$$Plp = \mu V \rho \alpha + \omega(S \rho_1 \alpha - x_0 w).$$

This is a plane that is perpendicular to the plane p and parallel to the axis of l.

Pdp – If l reduces to a line d – i.e., if $x_0 = 0$ – then one will have:

$$Pdp = \mu V \rho \alpha + \omega S \rho_1 ,$$

and one can verify that:

$$L \cdot dPdp = 0;$$

i.e., that Pdp is the plane through d that is perpendicular to the plane p. Its tensor is equal to the cosine of the angle between the line and the plane, in such a way that:

$$Pdp = 0$$

expresses the idea that the line d is perpendicular to p.

Pmp – If l reduces to a point $m = \mu x_0 + \omega \rho_1$ then one will have:

$$Pmp = \omega(S \rho_1 \alpha - x_0 w).$$

Suppose that the point m is the origin. One will have $Pmp = - \omega x_0 w$, where $- w$ is the product of Tp with the distance from the point m to the plane p.

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FUNDAMENTAL FORMULAS OF THE CALCULUS.

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(1) $$\begin{cases} Gll' = Gl'l, & Lll' = -Ll'l, & Pll' = Pl'l, \\ Glp = 0, & Llp = Lpl, & Plp = -Ppl, \\ Gpp' = Gp'p, & Lpp' = -Lp'p, & Ppp' = 0. \end{cases}$$

GEOMETRIC SIGNIFICANCE OF THE PRODUCT OF TWO TRI-QUATERNIONS

If the tensors are assumed to be equal to unity then:

μ and μ' will represent points
δ and δ' " lines,
ϖ and ϖ' " planes.

$G\mu\mu' \equiv 1$,
$G\mu\delta \equiv 0$,
$G\delta\delta'$ cosine of the angle between the two lines, with the sign changed,
$G\mu\varpi \equiv 0$,
$G\delta\varpi \equiv 0$,
$G\varpi\varpi'$ cosine of the angle between the two planes, with the sign changed,

$L\mu\mu'$ vector from μ' to μ,

$L\mu\delta$ vector that is perpendicular to the plane that contains the point and the line, with the tensor equal to the distance from the point to the line,

$L\delta\delta'$ complex whose axis is the common perpendicular, and whose auto-moment is the product of the shortest distance with the cotangent of the angle,

$L\mu\varpi$ perpendicular to the plane ϖ that goes through μ

$L\delta\varpi$ point of intersection of the line and the plane, with a tensor that equals the sine of the angle between the line and the plane,

$L\varpi\varpi'$ line of intersection, with a tensor that equals the single of the angle,

$P\mu\mu' \equiv 0$,

$P\mu\delta$ plane through *m* and perpendicular to *d*,

$P\delta\delta'$ symbol of the plane at infinity ϖ, multiplied by the shortest distance and the sine of the angle between the two lines,

$P\mu\varpi$ symbol of the plane at infinity, multiplied by the distance from μ to ϖ, and taken with the positive or negative sign, according to whether the point μ is situated on the edge of the positive face or the negative face of the plane ϖ,

$P\delta\varpi$ plane through the line perpendicular to the plane, whose tensor equals the sine of the angle between the line and the positive direction of the normal to the plane,

$P\varpi\varpi' \equiv 0$.

Read and approved:
Paris, 25 October 1901,
DEAN OF THE FACULTY OF THE SCIENCES
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Read and permission to print granted:
Paris, 25 October 1901,
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