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Application of Cayley geometry to the geometric study of the displacement of a solid around a fixed point

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INTRODUCTION

Chapter I of this paper is dedicated to some notions of infinitesimal Cayley geometry.

I assume that the principles of that geometry are known, as Darboux presented them in his *Leçons sur la théorie des surfaces*. After specifying the notations (no. 1), I will define the curvature (no. 2) and torsion (no. 3) of the Frenet formulas. The coordinates of the summits of a certain tetrahedron that is associated with a point of skew curve enter into that generalization at the basic level in the Frenet formulas as the direction cosines of the edges of the fundamental triehdron. I will finally generalize the theorems of Meusnier and Euler that relate to the curves that are traced on a surface ([1]) (no. 6, 7).

At the beginning of Chapter II, I will consider two tri-rectangular triangles T_1, T in ordinary Euclidian space that have the same summit O, and I will define (no. 8) the coordinates of T relative to T_1. These coordinates are a special system of parameters that are due to O. Rodrigues. In the following paragraphs (no. 9 to 12), I will investigate the effect of substituting one of the trihedra T or T_1 for another trihedron T' or T_1' that is invariably linked to the first one. I will establish these formulas for coordinate changes by a direct calculation; one can likewise derive the formulas for the composition of rotations.

The geometric interpretation of these formulas (no. 11) will constitute the fundamental idea of this paper.

I will consider the coordinates of T, relative to T_1, as defining a point of a three-dimensional multiplicity – viz., the *image point of the system* T_1, T. The trihedron T displaces relative to T_1 in a continuous fashion, so the image point will describe an image figure of the displacement. It will be a line or a surface according to whether the displacement has one or two parameters, respectively.

([1]) A more extended generalization was given by Bianchi in the German edition (Teubner) of his *Leçons de Géométrie différentielle.*

The analytical study of the displacement of a solid that has a fixed point O reduces to that of the relative displacement of two trihedra with their summits at O. One of them T is invariably coupled with the solid, while the other one T_1 is coupled to fixed space. The choice of these two trihedra, and in turn, the image figure of the displacement is possible in an infinitude of ways; in general, it involves six arbitrary quantities.

The formulas for the change of coordinates lead us to define a fundamental quadric in the multiplicity that is swept out by their image points, and to envision that multiplicity as a Cayley space.

The various image figures of the same displacement are deduced from each other by motions in that Cayley space.

The consequences of that proposition will be developed at the end of Chapter II.

I will pass very rapidly (no. 13) over the algebraic displacements, in order to begin (no. 14 and 16) the infinitesimal study of a one-parameter displacement. The Cayley curvature and torsion of an image curve under a one-parameter displacement will be attached to the curvatures of fixed and moving rolling spheres by simple relations. The Cayley curvature, for example, is twice the parameter k that enters into a formula that is analogous to that of Savary.

Finally (no. 18 to 20), I will apply the study of the curves that are traced on a surface in Cayley space to the search for the infinitesimal properties of one-parameter displacements that are part of a two-parameter displacement.

It is natural to look for a method of classifying those linear total differential equations that lead to the use of the moving trihedra of geometry in the preceding theory. One can also generalize that theory by considering other groups of transformations than that of the rotations around a fixed point. Here, I will content myself by pointing out these questions, to which I hope to ultimately return.

CHAPTER I.

NOTIONS FROM INFINITESIMAL CAYLEY GEOMETRY ([1]).

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1. In a three-dimensional Cayley space, a fundamental – or *absolute* – quadric serves to determine angles and distances. The equation:

$$\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 0 \tag{1}$$

will represent the absolute in tetrahedral coordinates.

([1]) *See* DARBOUX, *Leçons sur la théorie des surfaces*, Livre VII, Chap. XIV, t. III.

The homogeneity of the tetrahedral coordinates will be awkward in what follows; we will make it disappear by means of the following convention:

In the rest of this paper, the *coordinates of a point* ([1]) *will denote the tetrahedral coordinates* λ, μ, ν, ρ *of the point such that one will have:*

(2) $$\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1.$$

One can obviously substitute $-\lambda, -\mu, -\nu, -\rho$ for λ, μ, ν, ρ, so the coordinates of real points are well-defined.

The distance $\overline{mm'}$ between two points m and m' is given as a function of their coordinates λ, μ, ν, ρ and $\lambda', \mu', \nu', \rho'$ by the formula:

(3) $$\cos \overline{mm'} = \lambda\lambda' + \mu\mu' + \nu\nu' + \rho\rho'.$$

The linear element of space is given by:

(4) $$ds^2 = d\lambda^2 + d\mu^2 + d\nu^2 + d\rho^2.$$

Let D', D'' be two lines that have a common point m. If m' is the point of D' that is conjugate to m with respect to the absolute, while m'' is the analogous point relative to D'', then the angle between D' and D'' will be equal to the distance $\overline{m'm''}$.

2. If the linear element of space is known then one can define the length of an arc of a curve immediately.

Make a choice of origin and a sense of traversal on a curve (C); let s be the arc length of that curve that ends at a point m. The tangent to (C) at m is the line that joins the point m to the point m_1 whose coordinates are:

(5) $$\lambda_1 = \frac{d\lambda}{ds}, \qquad \mu_1 = \frac{d\mu}{ds}, \qquad \nu_1 = \frac{d\nu}{ds}, \qquad \rho_1 = \frac{d\rho}{ds}.$$

Upon differentiating the two sides of the identity (2) with respect to s, one sees that m and m_1 are conjugate with respect to the absolute. We call m_1 the *director point of the tangent at m.*

When m describes (C), m_1 will describe a curve (C_1). Take a sense of traversal on (C_1) such that the derivative $d\sigma / ds$ of the arc length σ of (C_1) with respect to the arc length s of (C) is positive. One will see that the equation:

(6) $$\frac{1}{\sin\delta} = \frac{d\sigma}{ds}$$

([1]) The advantages of this particular coordinate system are explained by Darboux in the cited place. Its use amount to considering the Cayley geometry of a three-dimensional space as the geometry on a hypersphere in a four-dimensional Euclidian space.

will admit a real root δ that is between 0 and $\pi / 2$. (One supposes that the curve (C) is real). We give that root δ the name of *radius of curvature*, and we call the expression $1 / \mathcal{R} = \cot \delta$ the *curvature*.

In order to verify the preceding assertion, one first establishes the formula:

$$\left(\frac{d\sigma}{ds}\right)^2 = \frac{\sum (d^2\lambda\, d\mu - d^2\mu\, d\lambda)}{ds^2}, \tag{7}$$

where the summation is extended over the six combinations of letters λ, μ, ν, ρ, when taken two at a time.

One then observes that one can choose the coordinate system in such a fashion that m is the point 0, 0, 0, 1, the tangent at m is $\lambda = \mu = 0$, and the osculating plane at the same point is $\lambda = 0$. Upon supposing that the curve is given by expressing the coordinates as functions of one parameter t, such that the value zero corresponds to m, one will have, in a neighborhood of that point ([1]):

$$\left\{\begin{aligned} \lambda &= \dots\dots\dots\dots\dots\dots, \\ \mu &= \qquad\;\; \beta_2 t^2 + \cdots, \\ \nu &= \gamma_1 t + \gamma_2 t^2 + \cdots, \\ \rho &= 1 - \frac{\gamma_1^2}{2} t^2 + \cdots, \end{aligned}\right. \tag{8}$$

where the unwritten terms are of order greater than two.

Formula (7) then gives:

$$\left(\frac{d\sigma}{ds}\right)^2 = \frac{\gamma_1^4 + 4\beta^2}{\gamma_1^4} \tag{9}$$

for the point m; one indeed finds a value for $\sin \delta$ that is between zero and one.

One easily sees, with the aid of formulas (8) and (9), for example, that there exists an osculating Cayley circle at a point m of the curve (C). The radius of the circle will be equal to the radius of curvature, so its center will be called the *center of curvature*.

3. Let b be the pole of the osculating plane to (C) at m with respect to the absolute. We say that mb is the *binormal*, and that b is the director point of the binormal. When m describes (C), b will describe a curve (C'); we will let σ denote the arc length of (C'), when measured from an arbitrary origin, and positively, in a sense that will be specified later on.

One calculates $d\tau / ds$ in the following fashion: One starts with the equation of the osculating plane at m:

$$A\Lambda + B\mathrm{M} + C\mathrm{N} + D\mathrm{P} = 0,$$

(where Λ, M, N, P denote the current coordinates), and one obtains, with no difficulty:

([1]) We confine our study to the real and analytic curves and surfaces.

$$d\tau^2 = \frac{\sum (A\,dB - B\,dA)^2}{(A^2+B^2+C^2+D^2)^2},$$

in which the summation is extended over the pair-wise combinations of A, B, C, D.

In order to transform this expression, one considers the determinant:

$$\Delta = \begin{vmatrix} \lambda & \mu & \nu & \rho \\ d\lambda & d\mu & d\nu & d\rho \\ d^2\lambda & d^2\mu & d^2\nu & d^2\rho \\ d^3\lambda & d^3\mu & d^3\nu & d^3\rho \end{vmatrix},$$

and one takes:

$$A = \frac{\partial\Delta}{\partial d^2\lambda}, \qquad B = \frac{\partial\Delta}{\partial d^2\nu}, \qquad C = \frac{\partial\Delta}{\partial d^2\nu}, \qquad D = \frac{\partial\Delta}{\partial d^2\rho};$$

one will then have:

$$dA = -\frac{\partial\Delta}{\partial d^2\lambda}, \quad dB = -\frac{\partial\Delta}{\partial d^2\nu}, \quad dC = -\frac{\partial\Delta}{\partial d^2\nu}, \quad dD = -\frac{\partial\Delta}{\partial d^2\rho}.$$

One transforms the binomials $A\,dB - B\,dA$, … by means of a known identity that relates to the minors of a determinant, and one then obtains:

$$d\tau^2 = \frac{\Delta^2}{(A^2+B^2+C^2+D^2)^2}\sum (\lambda\,d\mu - \mu\,d\lambda)^2 .$$

The second factor is ds^2, so one gets:

$$\frac{d\tau}{ds} = \pm\frac{\Delta}{A^2+B^2+C^2+D^2}.$$

The sense of traversal, which remains indeterminate, is chosen in such a fashion that the – sign is necessary, so we set:

$$(10) \qquad \frac{1}{\mathcal{T}} = \frac{d\tau}{ds} = -\frac{\Delta}{A^2+B^2+C^2+D^2}.$$

The expression $1 / \mathcal{T}$ will be called the *torsion*.

4. Let n be the pole of the plane mm_1b with respect to the absolute. The tetrahedron mm_1bn that is conjugate with respect to the absolute will be called the *fundamental* *tetrahedron*, relative to the curve (C) and at the point m. We call mn the principal normal and n the director point of the principal normal.

In what follows, λ, μ, ν, ρ ; $\lambda_1, \mu_1, \nu_1, \rho_1$; $\lambda_2, \mu_2, \nu_2, \rho_2$, and $\lambda_3, \mu_3, \nu_3, \rho_3$ will denote the coordinates of m, m_1, n, and b, respectively. However, one will observe that in the formulas that follow one cannot multiply the coordinates of one of these points by – 1 without suitably modifying the other ones.

We shall determine the derivatives of the coordinates with respect to the arc length s of the curve (C).

Formulas (5) give the derivatives of λ, μ, ν, ρ.

The point m_1' whose coordinates are:

$$a = \frac{d\lambda_1}{d\sigma}, \qquad b = \frac{d\mu_1}{d\sigma}, \qquad c = \frac{d\nu_1}{d\sigma}, \qquad d = \frac{d\rho_1}{d\sigma}$$

is the director point of the tangent to the curve (C_1) located at m_1 . One easily verifies that m_1' is situated on the principal normal mn. Upon utilizing the two points m, m_1' on that normal in order to determine the director n, and observing that $a = \frac{d\lambda_1}{ds}\sin\delta$, …, one will verify that one can take ([1]) $\lambda_2 = \left(\lambda + \frac{d\lambda_1}{ds}\right)\tan\delta$, … As a result:

$$\frac{d\lambda_1}{ds} = \frac{\lambda_2}{\mathcal{R}} - \lambda, \quad \frac{d\mu_1}{ds} = \frac{\mu_2}{\mathcal{R}} - \mu, \quad \frac{d\nu_1}{ds} = \frac{\nu_2}{\mathcal{R}} - \nu, \quad \frac{d\rho_1}{ds} = \frac{\rho_2}{\mathcal{R}} - \rho. \tag{11}$$

One determines $\lambda_3, \mu_3, \nu_3, \rho_3$ by the relations:

$$\frac{\lambda_3}{A} = \frac{\mu_3}{B} = \frac{\nu_3}{C} = \frac{\rho_3}{D} = \frac{\varepsilon}{\sqrt{A^2+B^2+C^2+D^2}}, \tag{12}$$

where A, B, C, D have the same significance as in no. 3, and where ε is equal to + 1 or to – 1, and is chosen in such a fashion that the determinant:

$$\begin{vmatrix} \lambda & \mu & \nu & \rho \\ \lambda_1 & \mu_1 & \nu_1 & \rho_1 \\ \lambda_2 & \mu_2 & \nu_2 & \rho_2 \\ \lambda_3 & \mu_3 & \nu_3 & \rho_3 \end{vmatrix}$$

([1]) One can just as well take $\lambda_2 = -\left(\lambda + \frac{d\lambda_1}{ds}\right)\tan\delta$, … The sign is chosen such that the coordinates of our center of curvature are $\lambda_2 \sin\delta + \lambda\cos\delta$, $\mu_2\cos\delta + \mu\cos\delta$,…

If one changes the positive sense on the curve (C) by taking $s' = -s$, instead of s, then one can substitute $-\lambda_1, -\mu_1, -\nu_1, -\rho_1$ for $\lambda_1, \mu_1, \nu_1, \rho_1$, but $\lambda_2, \mu_2, \nu_2, \rho_2$ do not change.

is positive. Upon calculating its square, one verifies that this determinant has the value +1.

Formulas (12) and the identities that relate to the minors of a given determinant give:

$$d\lambda_3 = \frac{-\Delta}{A^2+B^2+C^2+D^2}\begin{vmatrix} \mu & \nu & \rho \\ d\mu & d\nu & d\rho \\ \mu_3 & \nu_3 & \rho_3 \end{vmatrix};$$

one deduces that $\dfrac{d\lambda_3}{ds} = \dfrac{\lambda_2}{\mathcal{T}}$. One then has:

$$\frac{d\lambda_3}{ds} = \frac{\lambda_2}{\mathcal{T}}, \quad \frac{d\mu_3}{ds} = \frac{\mu_2}{\mathcal{T}}, \quad \frac{d\nu_3}{ds} = \frac{\nu_2}{\mathcal{T}}, \quad \frac{d\rho_3}{ds} = \frac{\rho_2}{\mathcal{T}}. \tag{13}$$

Upon differentiating the identity $\lambda_2^2+\mu_2^2+\nu_2^2+\rho_2^2 = 1$ with respect to s, along with the identities that couple λ_2, μ_2, ν_2, ρ_2 to λ, μ, ν, ρ ; λ_1, μ_1, ν_1, ρ_1 ; λ_3, μ_3, ν_3, ρ_3 , one will have four equations that determine $d\lambda_2 / ds$ and the analogous derivatives. Here is the result:

$$\left\{\begin{aligned} \frac{d\lambda_2}{ds} &= -\frac{\lambda_1}{\mathcal{R}} - \frac{\lambda_3}{\mathcal{T}}, & \frac{d\mu_2}{ds} &= -\frac{\mu_1}{\mathcal{R}} - \frac{\mu_3}{\mathcal{T}}, \\ \frac{d\nu_2}{ds} &= -\frac{\nu_1}{\mathcal{R}} - \frac{\nu_3}{\mathcal{T}}, & \frac{d\rho_2}{ds} &= -\frac{\rho_1}{\mathcal{R}} - \frac{\rho_3}{\mathcal{T}}. \end{aligned}\right. \tag{14}$$

Formulas (5), (11), (13), and (14) are analogous to the Frenet formulas.

5. The preceding result permits one to develop the coordinates of a point of the curve that is close to the point $s = 0$ in powers of s, as soon as one knows the functions $\mathcal{R}$ and $\mathcal{T}$ of the variable s. If one supposes that $s = 0$ gives the point 0, 0, 0, 1, and that the tangent and osculating plane at that point are $\lambda = \mu = 0$ and $\mu = 0$, respectively, then one will get:

$$\left\{\begin{aligned} \lambda &= \frac{s^2}{2\mathcal{R}_0} - \frac{\left(\dfrac{d\mathcal{R}}{ds}\right)_0}{6\mathcal{R}_0^2}s^3 + \cdots, \\ \mu &= \frac{s^3}{2\mathcal{R}_0\mathcal{T}_0} + \cdots, \\ \nu &= s - \frac{\mathcal{R}_0^2+1}{6\mathcal{R}_0^2}s^3 + \cdots, \\ \rho &= 1 - \frac{s^2}{2} + \cdots; \end{aligned}\right. \tag{15}$$

the unwritten terms are of order at least 4; $\mathcal{R}_0$, $\mathcal{T}_0$, $\left(\frac{d\mathcal{R}}{ds}\right)_0$ are the values of $\mathcal{R}$, $\mathcal{T}$, and $\frac{d\mathcal{R}}{ds}$ for $s = 0$.

6. Define a surface (S) by expressing the coordinates λ, μ, ν, ρ of a point as functions of two variable parameters u and v.

Upon denoting the current coordinates by Λ, M, N, P, the equation of the tangent plane at the point λ, μ, ν, ρ will be:

$$(16) \qquad \begin{vmatrix} \Lambda & \mathrm{M} & \mathrm{N} & \mathrm{P} \\ \lambda & \mu & \nu & \rho \\ \frac{\partial\lambda}{\partial u} & \frac{\partial\mu}{\partial u} & \frac{\partial\nu}{\partial u} & \frac{\partial\rho}{\partial u} \\ \frac{\partial\lambda}{\partial v} & \frac{\partial\mu}{\partial v} & \frac{\partial\nu}{\partial v} & \frac{\partial\rho}{\partial v} \end{vmatrix} = 0.$$

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be the coefficients of Λ, M, N, P in the development of that determinant; set:

$$\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 + \mathcal{D}^2 = \mathcal{K}$$

and

$$(17) \qquad l = \frac{\varepsilon\mathcal{A}}{\sqrt{\mathcal{K}}}, \qquad m = \frac{\varepsilon\mathcal{B}}{\sqrt{\mathcal{K}}}, \qquad n = \frac{\varepsilon\mathcal{C}}{\sqrt{\mathcal{K}}}, \qquad r = \frac{\varepsilon\mathcal{D}}{\sqrt{\mathcal{K}}};$$

the determination of the radical is chosen once and for all. However, we take $\varepsilon = +1$ or $\varepsilon = -1$, according to the case. The point is the pole of the tangent plane with respect to the absolute, the line that joins the points λ, μ, ν, ρ ; l, m, n, r is the normal to the surface at the point in question.

If we replace λ, μ, ν, ρ in the linear element (4) of space with the corresponding functions of the variables u and v then we will get the linear element of the surface:

$$(18) \qquad ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

The importance of that expression is well-known.

7. A curve (C) that is traced on (S) is determined by the expressions for u and v as functions of the same parameter; take that parameter to be the arc length s of (C).

Upon preserving the recent notations, one will obtain:

$$\lambda_1 = \frac{\partial\lambda}{\partial u}\frac{du}{ds} + \frac{\partial\lambda}{\partial v}\frac{dv}{ds},$$

$$\frac{d\lambda_1}{ds} = \frac{\lambda_2}{\mathcal{R}} - \lambda = \frac{\partial^2\lambda}{\partial u\,\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2\lambda}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + \frac{\partial\lambda}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial\lambda}{\partial v}\frac{d^2v}{ds^2},$$

and analogous formulas. One then gets:

$$\text{(19)} \qquad \frac{\cos\theta}{\mathcal{R}} = \frac{l\lambda_2 + m\mu_2 + n\nu_2 + r\rho_2}{\mathcal{R}} = \frac{\varepsilon}{\sqrt{\mathcal{K}}}\frac{E_1 du^2 + 2F_1 du\,dv + G_1 dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}.$$

In that formula, $E_1\,du^2 + 2\,F_1\,du\,dv + G_1\,dv^2$ is the result of substituting:

$$\frac{\partial^2\lambda}{\partial u^2}du^2 + \frac{\partial^2\lambda}{\partial u\,\partial v}du\,dv + \frac{\partial^2\lambda}{\partial v^2}dv^2, \ldots$$

for Λ, M, N, P in the determinant (16), and θ is the angle (which is found between 0 and $\pi / 2$) between the principal normal to (C) and the normal to (S). One thus chooses $\varepsilon = +1$ or $\varepsilon = -1$, according to whether $\dfrac{E_1 du^2 + 2F_1 du\,dv + G_1 dv^2}{\sqrt{\mathcal{K}}}$ is positive or negative.

One thus sees that all of the curves that are traced on the surface that pass through the same point and have the same osculating plane at that point will also have the same curvature.

Upon denoting the right-hand side of (19) by $1 / \mathcal{R}'$, one will get:

$$\text{(20)} \qquad \frac{\cos\theta}{\mathcal{R}} = \frac{1}{\mathcal{R}'}.$$

This result constitutes what we call a *generalization of Meusnier's theorem*.

One is then reduced to the study of the curvature of normal plane sections.

For such a section, $\cos\theta = 1$, and $\lambda_2, \mu_2, \nu_2, \rho_2$ will be equal to l, m, n, r. One avoids any discussion by considering the center of curvature, instead of the curvature, whose expression again contains ε.

One knows (no. 4, note) that the coordinates of this point are:

$$\lambda\cos\delta + \lambda_2\sin\delta = \lambda\cos\delta + l\sin\delta, \ldots$$

Upon taking $\delta = \varepsilon\varphi$ and:

$$l' = \varepsilon\,l = \frac{\mathcal{A}}{\sqrt{\mathcal{K}}}, \qquad m' = \varepsilon\,m = \frac{\mathcal{B}}{\sqrt{\mathcal{K}}}, \qquad n' = \varepsilon\,n = \frac{\mathcal{C}}{\sqrt{\mathcal{K}}}, \qquad r' = \varepsilon\,r = \frac{\mathcal{D}}{\sqrt{\mathcal{K}}},$$

one will get the coordinates of the center of curvature:

$$\lambda\cos\delta + l'\sin\delta, \qquad \mu\cos\delta + m'\sin\delta, \qquad \nu\cos\delta + n'\sin\delta, \qquad \rho\cos\delta + r'\sin\delta.$$

The angle φ, which is between $-\pi/2$ and $+\pi/2$, is given by:

$$\text{(21)} \qquad \cot\varphi = \frac{1}{\sqrt{\mathcal{K}}}\frac{E_1 du^2 + 2F_1 du\,dv + G_1 dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}.$$

We say that $1 / [\mathcal{R}] = \cot\varphi$ is the *algebraic curvature* of the normal section that is being considered.

If one equates the Jacobian of the quadratic forms that appear in the numerator and the denominator of formula (21), one will get, in general, two rectangular tangents that we call *tangents to the lines of curvature*. They determine the normal sections for which $[\mathcal{R}]$ is a maximum or minimum; let $[\mathcal{R}_1]$ and $[\mathcal{R}_2]$ be the corresponding values of $[\mathcal{R}]$, respectively.

Let α denote the angle between the direction du, dv and the tangent that corresponds to $[\mathcal{R}_1]$. One easily establishes the formula:

$$\text{(22)} \qquad \frac{1}{[\mathcal{R}]} = \frac{\cos^2\alpha}{[\mathcal{R}_1]} + \frac{\sin^2\alpha}{[\mathcal{R}_2]}.$$

This is a *generalization of Euler's theorem.*

CHAPTER II.

DISPLACEMENT OF A SOLID AROUND A FIXED POINT.

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8. First, recall the geometric significance of the parameters that Olinde Rodrigues used to define a change of coordinates or a rotation ([1]).

Let $T_1(O, x_1, y_1, z_1)$ and $T(O, x, y, z)$ be two tri-rectangular trihedra with the same summit. On the axis of rotation, make T_1 coincide with T, choose a direction for OI, and set:

$$\alpha = \cos(IO, x) = \cos(IO, x_1), \quad \beta = \cos(IO, y) = \cos(IO, y_1), \quad \gamma = \cos(IO, z) = \cos(IO, z_1).$$

Let θ denote the angle of rotation.

The numbers:

([1]) *See* the notes of DARBOUX: Note V in his *Leçons sur la théorie générale des surface* and Note I of the *Leçons de Cinématique* of KOENIGS.

(1) $$\lambda = \alpha \sin\frac{\theta}{2}, \quad \mu = \beta \sin\frac{\theta}{2}, \quad \nu = \gamma \sin\frac{\theta}{2}, \quad \rho = \cos\frac{\theta}{2},$$

constitutes a system of parameters of Olinde Rodrigues.

They satisfy the relation:

(2) $$\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1.$$

We say that λ, μ, ν, ρ are the *coordinates of the trihedron T relative to the trihedron* T_1.

It is clear that one can replace θ with $2\pi + \theta$, and in turn, λ, μ, ν, ρ with $-\lambda, -\mu, -\nu, -\rho$.

The coordinates of T_1 relative to T are obviously $-\lambda, -\mu, -\nu, -\rho$, or, what amounts to the same thing $\lambda, \mu, \nu, -\rho$.

Let x, y, z and x_1, y_1, z_1 be the coordinates of the same point M in space relative to T and T_1, resp. Darboux (*loc. cit.*) has established the relations:

(3) $$\begin{cases} \rho x - \nu y + \mu z = \rho x_1 + \nu y_1 - \mu z_1, \\ \nu x + \rho y - \lambda z = -\nu x_1 + \rho y_1 + \lambda z_1, \\ -\mu x + \lambda y + \rho z = \mu x_1 - \lambda y_1 + \rho z_1, \\ \lambda x + \mu y + \nu z = \lambda x_1 + \mu y_1 + \nu z_1. \end{cases}$$

Only three of these relations are independent. Upon solving them for x, y, z, one obtains the formulas of Olinde Rodrigues.

9. An analogous process will permit us to solve the following problem:

Let T, T_1, T_0 (viz., $Ox_1 y_1 z_1$, $Oxyz$, $Ox_0 y_0 z_0$) be three tri-retangular trihedra with the same summit O.

Knowing the coordinates λ, μ, ν, ρ of T relative to T_1 and the coordinates $\lambda_0, \mu_0, \nu_0, \rho_0$ of T_1 relative to T_0, find the coordinates Λ, M, N, P *of T relative to T_0*.

Denote the coordinates of the same point M with respect to the three trihedra by x, y, z; x_1, y_1, z_1; x_0, y_0, z_0, resp.

Write down the system that is analogous to (3) relative to the system T_1, T_0:

(4) $$\begin{cases} \rho_0 x_1 - \nu_0 y_1 + \mu_0 z_1 = \rho_0 x_0 + \nu_0 y_0 - \mu_0 z_0, \\ \nu_0 x_1 + \rho_0 y_1 - \lambda_0 z_1 = -\nu_0 x_0 + \rho_0 y_0 + \lambda_0 z_0, \\ -\mu_0 x_1 + \lambda_0 y_1 + \rho_0 z_1 = \mu_0 x_0 - \lambda_0 y_0 + \rho_0 z_0, \\ \lambda_0 x_1 + \mu_0 y_1 + \nu_0 z_1 = \lambda_0 x_0 + \mu_0 y_0 + \nu_0 z_0. \end{cases}$$

If one multiplies both sides of equations (3) by $\pm\lambda_0$, $\pm\mu_0$, $\pm\nu_0$, $\pm\rho_0$, respectively, and adds them, with the signs chosen in such a fashion that the coefficient of x will be $\rho\rho_0 - \lambda\lambda_0 - \mu\mu_0 - \nu\nu_0$, then one will get:

$$x(\rho\rho_0 - \lambda\lambda_0 - \mu\mu_0 - \nu\nu_0) + y(-\nu\rho_0 - \rho\nu_0 + \lambda\mu_0 - \mu\lambda_0) + z(\mu\rho_0 + \lambda\nu_0 - \mu\mu_0 - \nu\lambda_0)$$

(5)

$$= x_1(\rho\rho_0 + \nu\nu_0 + \mu\mu_0 - \lambda\lambda_0) + y_1(\nu\rho_0 - \rho\nu_0 - \lambda\mu_0 - \mu\lambda_0) + z_1(-\mu\rho_0 - \lambda\nu_0 + \rho\mu_0 - \nu\lambda_0).$$

If one likewise combines equations (4), in such a way that the coefficient x_0 is again $\rho\rho_0 - \lambda\lambda_0 - \mu\mu_0 - \nu\nu_0$ then one will get:

$$x_1(\rho\rho_0 + \nu\nu_0 + \mu\mu_0 - \lambda\lambda_0) + y_1(-\rho\nu_0 + \nu\rho_0 - \mu\lambda_0 - \lambda\mu_0) + z_1(\rho\mu_0 - \nu\lambda_0 - \mu\rho_0 - \lambda\mu_0)$$

(6)

$$= x_0(\rho\rho_0 - \lambda\lambda_0 - \mu\mu_0 - \nu\nu_0) + y_0(\rho\nu_0 + \nu\rho_0 + \mu\lambda_0 - \lambda\mu_0) + z_0(-\rho\mu_0 + \nu\lambda_0 - \mu\rho_0 - \lambda\nu_0).$$

The terms in x_1, y_1, z_1 will be the same in equations (5) and (6), which permits one to eliminate these variables. Upon making y and y_0, and then z and z_0, in turn, play the role of x and x_0 in the preceding combinations, one will obtain three equations that no longer contain x_1, y_1, z_1. These equations can be written:

$$(7)\qquad \left\{\begin{aligned} \mathrm{P}x - \mathrm{N}y + \mathrm{M}z &= \mathrm{P}x_0 + \mathrm{N}y_0 - \mathrm{M}z_0, \\ \mathrm{N}x + \mathrm{P}y - \Lambda z &= -\mathrm{N}x_0 + \mathrm{P}y_0 + \Lambda z_0, \\ -\mathrm{M}x + \Lambda y + \mathrm{P}z &= \mathrm{M}x_0 - \Lambda y_0 + \mathrm{P}z_0, \end{aligned}\right.$$

if one sets:

$$(8)\qquad \left\{\begin{aligned} \Lambda &= \rho\lambda_0 + \lambda\rho_0 - \mu\nu_0 + \nu\mu_0, \\ \mathrm{M} &= \rho\mu_0 + \mu\rho_0 - \nu\lambda_0 + \lambda\nu_0, \\ \mathrm{N} &= \rho\nu_0 + \nu\rho_0 - \lambda\mu_0 + \mu\lambda_0, \\ \mathrm{P} &= \rho\rho_0 - \lambda\lambda_0 - \mu\mu_0 - \nu\nu_0. \end{aligned}\right.$$

One has, moreover:

$$\Lambda^2 + \mathrm{M}^2 + \mathrm{N}^2 + \mathrm{P}^2 = 1.$$

Formulas (8) thus give the required coordinates.

10. Consider T to be moving with respect to T_1; in other words, regard λ, μ, ν, ρ as variables that are coupled by only the relation (2).

One then sees that the replacement of the trihedron T with a trihedron T' that is invariably coupled to T translates analytically into the formulas:

$$
(9) \qquad \begin{cases} \lambda' = r\lambda + l\rho - m\nu + n\mu, \\ \mu' = r\mu + m\rho - n\nu + l\mu, \\ \nu' = r\nu + n\rho - l\nu + m\lambda, \\ \rho' = r\rho - l\lambda - m\mu - n\nu, \end{cases}
$$

where $\lambda', \mu', \nu', \rho'$ are the coordinates of T' relative to T_1, and the constants l, m, n, r denote the coordinates of T' relative to T.

Likewise, the replacement of the trihedron T_1 with a trihedron T_1' that is invariably coupled to T_1 translates into the linear substitution:

$$
(10) \qquad \begin{cases} \lambda_1' = r_1\lambda + l_1\rho + m_1\nu - n_1\mu, \\ \mu_1' = r_1\mu + m_1\rho + n_1\nu - l_1\mu, \\ \nu_1' = r_1\nu + n_1\rho + l_1\nu - m_1\lambda, \\ \rho_1' = r_1\rho - l_1\lambda - m_1\mu - n_1\nu, \end{cases}
$$

where the variables λ_1', μ_1', ν_1', ρ_1' and the constants l_1, m_1, n_1, r_1 denote the coordinates of T relative to T_1' and those of T_1 relative to T_1', respectively.

One recognizes in this the linear substitutions whose composition will give the most general linear substitution that transforms the form $\lambda^2 + \mu^2 + \nu^2 + \rho^2$ into itself ([1]).

11. Let λ, μ, ν, ρ be the coordinates of a trihedron T relative to a trihedron T_1. Consider the point P in the same three-dimensional space whose tetrahedral coordinates are λ, μ, ν, ρ. We say that the *point P* is the *image of the system* (T_1, T).

Replace the trihedra T_1, T with the trihedra T_1' and T' such that the first one is invariably linked to T_1 and the second one to T. The point P' that is the image of the system (T_1', T') is deduced from the point P by a homographic transformation that leaves the quadric:

$$
(11) \qquad \lambda^2 + \mu^2 + \nu^2 + \rho^2 = 0
$$

invariant.

Conversely, such a homographic transformation will correspond to a change of trihedra of the preceding nature.

If one takes the quadric (11) *to be the absolute in the space E then the preceding transformations will be the ones that preserve the angles and distances in the Cayley space thus defined; in other words, they will be the displacements in that Cayley space.*

Two points P, P' have one and only one invariant relative to the group of Cayley displacements, namely, the distance between them. It is easy to interpret this.

([1]) *See* KLEIN's signature course: *Nicht Euklidische Geometrie*, Bd. II, pp. 119 and 120.

In order to do this, consider two trihedra T, T' whose coordinates relative to the same initial trihedron T_1 are λ, μ, ν, ρ ; $\lambda', \mu', \nu', \rho'$. Let P, P' refer to the image points of (T_1, T), (T_1, T'). Observe that the coordinates of T_1 relative to T are $-\lambda, -\mu, -\nu, \rho$; the formulas (8), (1), and (I.3) ([1]) show that *the Cayley distance* $\overline{PP'}$ *represents one-half the angle of rotation that makes T coincide with T′.*

12. Let Σ be a solid ([2]) that has a fixed point O. The usual analytical representation of a continuous displacement of Σ relative to the fixed space Σ_1 consists of choosing two trihedra T_1, T such that the former is invariably coupled to Σ_1 and the latter to Σ, and to consider the coordinates of T relative to T_1 as functions of one or two parameters.

The point P that is the image of the system (T_1, T) will describe a curve (C) or a surface (S) according to whether the displacement has one or two parameters, resp. We say that the figure – whether a curve or surface – is the *image* of the displacement of Σ.

However, the choice of trihedra T_1 and T is possible in an infinitude of ways. There will then be an infinitude of image figures that have the same displacement.

From the foregoing, one can say:

The same displacement of Σ *relative to* Σ_1 *has an image that is any of the figures – viz., curves or surfaces – that are deduced from a given figure by a Cayley displacement* ([3]).

We shall often utilize the arbitrariness in the choice of image figure of the displacement in order to give that figure a simple position relative to the reference tetrahedron. At the same time, we shall obtain a convenient analytic representation of the displacement.

13. We apply this to some examples of algebraic displacements.

All of the lines in Cayley space are equal to the line:

$$\lambda = \mu = 0. \tag{12}$$

A point of the lines (12) is the image of a system (T_1, T) of trihedra that has the axis Oz_1 in common. It then results that *an arbitrary line is the image of a continuous rotation around a fixed axis.*

A Cayley *plane* is equal to the plane $\rho = 0$. Two trihedra that define a system whose image is a point of that plane will be deducible from each other by a reversal. An arbitrary plane will then be the image of a two-parameter displacement that one can define in the following fashion: *The various positions that are occupied by the moving*

([1]) We are thus referring to a formula in Chapter I.

([2]) The word *solid* is employed here in order to facilitate one's intuition of the displacement of two superposed Euclidian space.

([3]) To abbreviate, we say that these figures are mutually equal.

solid Σ *are deduced from a fixed solid* Σ_1 *by reversals around lines that pass through the fixed point O.*

One can likewise say that the various positions of S are deduced from each other by *rotations around the axes that pass through O and are situated in the same plane.*

A *plane curve* in Cayley space is equal to a curve in the plane $\rho = 0$. Let:

$$\text{(13)} \qquad \begin{cases} \rho = 0, \\ \varphi(\lambda, \mu, \nu) = 0, \end{cases}$$

be the equations of such a curve. That curve is the image of the displacement of a trihedron T relative to a trihedron T_1. However, since $\rho = 0$, the systems (T, T_1), (T_1, T) will have the same image point, and the displacement of T_1 relative to T will again have the curve (13) for its image.

One thus arrives at the following result:

A plane curve in Cayley space is the image of a one-parameter displacement where the fixed and moving rolling cones are equal; the rolling takes place in such a way that two homologous generators of these equal cones will agree in the course of displacement.

Moreover, the equation:

$$\text{(14)} \qquad \varphi(x_1, y_1, z_1) = 0$$

represents a cone (the coordinates axes are the edges of T_1). The various positions of T are deduced from T_1 by reversals around the generators of the cone. The combination of two infinitely close reversals gives the infinitely small rotation that permits one to pass from one position of the moving trihedron to an infinitely close one. This shows that *the fixed rolling cone is supplementary to the cone* (14).

When the preceding results are applied to *Cayley circles*, that would show that these lines are the images of displacements where the rolling cones are equal to cones of revolution.

It is easy to multiply the examples. The study of the algebraic displacements around a fixed point would be assuredly interesting. However, that would leave the scope that we have defined here. We shall not pursue that topic, and we go on to the infinitesimal properties of one-parameter displacements.

14. Let (C) be a curve in Cayley space, and let m and m' be two points of that curve. (C) is the image of the one-parameter displacement of a trihedron T relative to a trihedron T_1. Let (Γ) denote that displacement, and let t and t' be the positions of T that correspond to m and m'.

There exists a rotation around a fixed axis that makes a moving trihedron pass through the positions t and t'. The displacement of the second trihedron relative to T_1 will obviously have the line mm' for its image.

Suppose that m' is infinitely close to m.

The differential of the Cayley arc length of the curve (C) is interpreted from the viewpoint of the displacement (Γ) by recalling that the Cayley distance $\overline{mm'}$ is one-half the angle of the rotation that makes t coincide with t'.

We see, in turn, that *the tangent at a point m of the curve* (C) *is the image of the rotation that is tangent to the displacement* (Γ) *for the position t of the moving trihedron*.

Two curves (C), (C') that have a common point m will be the images of two displacements (Γ), (Γ') that are referred to the same fixed trihedron T_1, and in each of them the moving trihedron will coincide with a trihedron t.

In order to interpret the angle between (C) and (C') at the point m, it will suffice to confine oneself to the case where (C) and (C') are lines. Upon supposing that the coordinates of m are 0, 0, 0, 1, and succeeding in determining the lines by their traces on the plane $\rho = 0$, one will see that:

The Cayley angle between two curves at a point m is equal to the angle between the rotational axes that are tangent to the corresponding displacements (for the position t of the moving trihedron that corresponds to m).

15. We shall seek the elements that correspond to the Cayley curvature and torsion of the image (C) under a displacement (Γ).

We first determine the fixed and moving rolling cones. The displacement (Γ) and the curve (C) are determined by expressing the coordinates λ, μ, ν, ρ of the moving trihedron $T(O, x, y, z)$ relative to a trihedron $T_1(O, x_1, y_1, z_1)$ as functions of one parameter u. The instantaneous axis of rotation is obtained with the aid for formulas (8) ([1]). One then finds that the director parameters (relative to $Oxyz$) of that axis are:

$$(15)\qquad \left\{\begin{aligned} \xi &= \rho\frac{d\lambda}{du} - \lambda\frac{d\rho}{du} + \nu\frac{d\mu}{du} - \mu\frac{d\nu}{du}, \\ \eta &= \rho\frac{d\mu}{du} - \mu\frac{d\rho}{du} + \lambda\frac{d\nu}{du} - \nu\frac{d\lambda}{du}, \\ \zeta &= \rho\frac{d\nu}{du} - \nu\frac{d\rho}{du} + \lambda\frac{d\mu}{du} - \lambda\frac{d\mu}{du}. \end{aligned}\right.$$

The preceding formulas take on a more elegant form if one utilizes the relations:

$$\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1,$$

$$\lambda\frac{d\lambda}{du} + \mu\frac{d\mu}{du} + \nu\frac{d\nu}{du} + \rho\frac{d\rho}{du} = 0.$$

Indeed, one finds that:

$$\xi^2 + \eta^2 + \zeta^2 = \frac{ds^2}{du^2},$$

([1]) *See* Note V in Darboux's the *Leçons sur la théorie des surfaces*, t. IV.

where s is the Cayley arc length of (C), and in turn:

The direction cosines relative to Oxyz of one of the directions that one can choose on the tangent rotational axis are given by:

$$(16)\qquad \begin{cases} P = \rho\lambda_1 - \lambda\rho_1 + \nu\mu_1 - \mu\nu_1, \\ Q = \rho\mu_1 - \mu\rho_1 + \lambda\nu_1 - \nu\lambda_1, \\ R = \rho\nu_1 - \nu\rho_1 + \mu\lambda_1 - \lambda\mu_1, \end{cases}$$

where, as in no. 2, λ_1, μ_1, ν_1, ρ_1 denote the derivatives $\frac{d\lambda}{ds}, \frac{d\mu}{ds}, \frac{d\nu}{ds}, \frac{d\rho}{ds}$.

Formulas (15) and (16) indeed determine the moving rolling cone.

In order to find the fixed rolling cone, one looks for P_1, Q_1, R_1, which are the direction cosines of the preceding direction relative to $Ox_1\, y_1\, z_1$. To that effect, one employs the relations (16) and the formulas of Olinde Rodrigues. One can also utilize the inverse displacement. Here is the result:

$$(17)\qquad \begin{cases} P_1 = \rho\lambda_1 - \lambda\rho_1 - \nu\mu_1 + \mu\lambda_1, \\ Q_1 = \rho\mu_1 - \mu\rho_1 - \lambda\nu_1 + \nu\lambda_1, \\ R_1 = \rho\nu_1 - \nu\rho_1 - \mu\lambda_1 + \lambda\mu_1. \end{cases}$$

16. We now suppose that the coordinates λ, μ, ν, ρ are given as functions of the arc length s of (C), which is taken to be the parameter in the series (I.15), and that the point in whose neighborhood one studies (C) will correspond to $s = 0$.

In this case, formulas (16) and (17) will give:

$$(18)\qquad \begin{cases} P = \dfrac{s}{\mathcal{R}_0} - \dfrac{\left(\dfrac{d\mathcal{R}}{ds}\right)_0}{2\mathcal{R}_0} s^2 + \cdots, \\[2ex] Q = \dfrac{s^2}{2\mathcal{R}_0}\left(\dfrac{1}{\mathcal{T}_0} - 1\right) + \cdots, \\[2ex] R = 1 - \dfrac{s^2}{2\mathcal{R}_0^2} + \cdots, \end{cases}$$

$$(19)\qquad \begin{cases} P_1 = \dfrac{s}{\mathcal{R}_0} - \dfrac{\left(\dfrac{d\mathcal{R}}{ds}\right)_0}{2\mathcal{R}_0} s^2 + \cdots, \\ Q_1 = \dfrac{s^2}{2\mathcal{R}_0}\left(\dfrac{1}{\mathcal{T}_0} - 1\right) + \cdots, \\ R_1 = 1 - \dfrac{s^2}{2\mathcal{R}_0^2} + \cdots, \end{cases}$$

in which the unwritten terms will be of order higher than two.

When s varies, the point whose coordinates relative to $Oxyz$ are P, Q, R will describe a curve that traced on a sphere of radius equal to unity that has the origin for its center. That curve is the *moving rolling sphere;* one likewise defines the *fixed rolling sphere*.

For $s = 0$, the moving trihedron will coincide with the fixed trihedron, and the two preceding curves will be tangent to the point $A = (0, 0, 1)$.

Let $O\omega_f$, $O\omega_m$ be the axes of the osculating circles at A to the fixed and moving rolling spheres, and let r_f, r_m denote the angles $AO\omega_f$, $AO\omega_m$, when measured positively in the sense of Oz_1 to Oy_1. One easily gets the equations for these axes, and in turn, the *formulas:*

$$(20)\qquad \begin{cases} \cot r_f = \mathcal{R}_0\left(\dfrac{1}{\mathcal{T}_0} + 1\right), \\ \cot r_m = \mathcal{R}_0\left(\dfrac{1}{\mathcal{T}_0} - 1\right), \end{cases}$$

will give the curvatures of the rolling spheres as functions of the Cayley curvature and torsion; the problem that was posed at the beginning of no. 15 is thus resolved.

Formulas (20) give:

$$(21)\qquad 2\mathcal{R}_0 = \cot r_f - \cot r_m .$$

One sees that $2\mathcal{R}_0$ is the constant $1 / k$ that figures in the formula that is analogous to Savary's formulas relative to the displacements of plane figure ([1]) for the displacements considered.

17. By means of formulas (16) and (I.11), one easily finds that:

$$\frac{dP}{ds} = \frac{1}{\mathcal{R}}(\rho\lambda_2 - \lambda\rho_2 + \nu\mu_2 - \mu\nu_2),$$

([1]) *See* KOENIGS, *Leçons de Cinématique,* pp. 190.

and analogous expressions for dQ / ds, dR / ds.

One will observe that these expressions do not change:

1. If one multiplies λ, μ, ν, ρ by – 1 (from the remark in no. 4, one must then also multiply λ_2, μ_2, ν_2, ρ_2 by –1).

2. If one takes $s' = -s$ everywhere, instead of s (*see* no. 4, note). $\mathcal{R}^2 \frac{dP}{ds}$, $\mathcal{R}^2 \frac{dQ}{ds}$, $\mathcal{R}^2 \frac{dR}{ds}$ enjoy the same property.

We call the point B whose coordinates relative to t are $\mathcal{R}^2 \frac{dP}{ds}$, $\mathcal{R}^2 \frac{dQ}{ds}$, $\mathcal{R}^2 \frac{dR}{ds}$ the *point that is associated with the position t of the moving trihedron under the displacement* (Γ).

This point is determined unambiguously as long as (Γ) and t are known; one can define it geometrically in the following fashion: It is situated in the plane that is common to the two rolling cones on the perpendicular to the tangent rotational axis that is drawn through the summit O of these cones at a distance from the summit $OB = \mathcal{R} = 1 / 2k$. Finally, here is how one specifies the sense of OB: Upon displacing the fixed trihedron in such a manner as to bring Ox_1 to OB and Oz_1 to an arbitrarily-chosen direction OA on the tangential axis of rotation, it is necessary that the angles $r_m = AO\omega_m$, $r_f = AO\omega_f$, when evaluated algebraically as in no. 16, give a positive value for the difference $\cot r_f - \cot r_m$.

18. We shall now interpret the infinitesimal properties of surfaces.

Let (S) be a surface, let (C) be a curve that is traced on that surface, and let m be a point of that curve.

The surface (S) is the image of a two-parameter displacement (Σ); the curve (C) is the image of a one-parameter displacement (Γ) that is *incorporated* into the preceding one, such that all of the positions that are occupied by the moving trihedron in the course of the second displacement can be obtained by means of the first one. Finally, m will correspond to a position t of the moving trihedron.

The tangent to (C) at m will be the image of the tangent rotation to (Γ) for the position t of the moving trihedron. Let OA be the axis of that rotation. Since (C) always varies upon passing through m, the preceding tangent will describe a plane. One knows (no. 13), that under these conditions the axis OA of the tangent rotation will describe a plane (that passes through O), which we call the *plane of tangent rotations* relative to t and (Σ).

19. The second-order infinitesimal properties ([1]) are summarized in formulas (I.19) and (I.22). In order to interpret them, consider the surface (S) and the displacement (Σ) that are determined by expressing the coordinates λ, μ, ν, ρ as functions of two parameters u, v; suppose that the point m in the neighborhood of which one studies the surface is 0, 0, 0, 1, that the tangent plane at that point is $\nu = 0$, and that the tangents to

([1]) *See* no. 7.

the lines of curvature are $\lambda = \nu = 0$ and $\mu = \nu = 0$. One can then take $l' = m' = r' = 0$, $n' = 1$.

The instantaneous plane of rotation relative to (Σ) and t (that corresponds to m) is $z = 0$.

The tangent at m to a curve (C) that is traced on (Σ) is situated in the plane $\nu = 0$. The director point of that tangent will be situated in the same plane and in the polar plane to m with respect to the absolute; therefore, $\nu_1 = \rho_1 = 0$. Since $\lambda_1^2 + \mu_1^2 + \nu_1^2 + \rho_1^2 = 1$, one can set:

$$\lambda_1 = \cos \alpha, \qquad \mu_1 = \sin \alpha.$$

Analogous considerations show that $\rho_2 = 0$, and that:

$$\lambda_2 \cos \alpha + \mu_2 \sin \alpha. = 0.$$

Set:

$$\lambda_2 = -\sin \alpha \sin \varphi, \qquad \mu_2 = \cos \alpha \sin \varphi, \qquad \nu_2 = \cos \varphi.$$

The preceding elements suffice to determine the curvature of (C) and m. We denote it by $1 / \mathcal{R}_{\alpha,\varphi}$, and suppress the last index when $\varphi = 0$. The osculating plane to (C) at m is then normal to (S).

Let OA_α denote the tangent axis of rotation of (Γ) for the position t of the moving trihedron; that axis will depend upon only α or the value of du / dv that corresponds to the tangent to (C) at m.

One first has:

$$\cos \theta = \varepsilon \cos \varphi. \tag{22}$$

Now look for the point $B_{\alpha,\varphi}$ that is associated with (Γ) and t (no. 17). It will have the coordinates:

$$-\mathcal{R}_{\alpha,\varphi} \sin \alpha \sin \varphi, \qquad \mathcal{R}_{\alpha,\varphi} \cos \alpha \sin \varphi, \qquad \mathcal{R}_{\alpha,\varphi} \cos \varphi$$

relative to t.

The plane that is perpendicular to $OB_{\alpha,\varphi}$ at $B_{\alpha,\varphi}$ will intersect Oz at a point B_α where z has the value $\dfrac{\mathcal{R}_{\alpha,\varphi}}{\cos\varphi}$. However, if only φ varies then du / dv will be constant, and formulas (I.20) and (22) will show that:

$$\frac{\mathcal{R}_{\alpha,\varphi}}{\cos\varphi} = \varepsilon \mathcal{R}_\alpha = [\mathcal{R}_\alpha],$$

while always denoting the algebraic curvature of the normal plane section to (S) that corresponds to du / dv by $1 / [\mathcal{R}_\alpha]$.

One thus has the following result:

Consider a two-parameter displacement (Σ) and a particular position of the moving trihedron. Let (Γ) be a one-parameter displacement that is incorporated into the preceding one, and that makes the moving trihedron pass through the position t, such that the tangent axis of rotation that corresponds to t is a given straight line OA_α (that is situated in the tangent plane of rotation that corresponds to (Σ) and t).

The fixed and moving rolling cones that correspond to (Γ) will touch along OA_α when the moving trihedron occupies the position t. Their common tangent plane $OA_\alpha B_{\alpha\beta}$ will vary along the displacement (Γ) considered. *The point $B_{\alpha\beta}$ that is associated with* (Γ) *and t is the projection onto that tangent plane of a fixed point B_α* that is situated on the normal OI to the instantaneous plane of rotation that corresponds to (Σ) and t.

The foregoing was the translation of the generalized Meusnier theorem into the language of the geometry of displacements.

20. We now remark that the rotations that correspond to the tangents to the lines of curvature at m ($\lambda = \nu = 0$ and $\mu = \nu = 0$) will have Oy and Ox for their axes, precisely. The angle α of the preceding number can thus be considered to be identical with the angle α of no. 7. Therefore:

The law of variation of the point B_α on the perpendicular to the instantaneous plane of rotation when the plane $OB_\alpha A_\alpha$ turns around that perpendicular will be identical to that of the center of curvature of a normal plane section to a surface in ordinary Euclidian space when the plane of the section turns around a well-defined normal.

One can observe that, by definition, when α and φ vary, $B_{\alpha,\varphi}$ will describe a surface that is the locus of the centers of curvature relative to a given point of the curves that are traced on a surface in Euclidian space.