"Sur le problème de Pfaff," Bull. sci. math. astron. (2) **6** (1882), 49-62.

On the Pfaff problem

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(cont.)

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PART TWO

VIII.

The proposition relating to the invariance properties of the system (10), which was useful to us in the first part of this work, is susceptible to a generalization that we shall now present.

Along with the form:

$$\Theta_d = X_1\, dx_1 + \ldots + X_n\, dx_n ,$$

consider some other forms Θ_d^1, Θ_d^2, …, Θ_d^{2p} that are defined by the equations:

$$\Theta_d^k = X_1^k dx_1 + \ldots + X_n^k dx_n .$$

Require that the variables x_i and the variables t_i must satisfy the differential equations:

$$(1)\qquad \begin{cases} a_{11}dx_1 + \cdots + a_{n1}dx_n = X_1^1 dt_1 + X_1^2 dt_2 + \cdots + X_1^p dt_p , \\ \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \\ a_{1n}dx_1 + \cdots + a_{nn}dx_n = X_n^1 dt_1 + X_n^2 dt_2 + \cdots + X_n^p dt_p , \end{cases}$$

$$(1)\qquad \begin{cases} X_1^{p+1} dt_1 + \cdots + X_n^{p+1} dt_n = 0, \\ \qquad\qquad \cdots , \\ X_n^{2p-1} dt_1 + \cdots + X_n^{2p-1} dt_n = 0, \end{cases}$$

which are $n + p - 1$ in number, and which, consequently, form a determinate system. One can write these equations in the abbreviated form:

$$(2)\qquad \begin{cases} \delta\Theta_d - d\Theta_\delta = \Theta_\delta^1 dt_1 + \cdots + \Theta_\delta^p dt_p , \\ \Theta_d^{p+1} = 0,\ \cdots,\ \Theta_d^{2p-1} = 0, \end{cases}$$

upon assuming that the first one is true for all of the values that are attributed to the auxiliary differentials δ.

When the system (1) is written in the form (2), one immediately recognizes that it expresses some properties that are independent of any choice of variables, and consequently, it will have invariance properties of system (10) in our Part One.

If one replaces the variables x_i with n variables y_i and the form Θ_d^h becomes:

$$\Theta_d^h = Y_1^h dy_1 + \ldots + Y_n^h dy_n$$

then the system (1) takes on the form:

$$(3) \qquad \begin{cases} b_{11} dy_1 + \cdots + b_{n1} dy_n = Y_1^1 dt_1 + \cdots + Y_1^p dt_p, \\ \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ b_{1n} dy_1 + \cdots + b_{nn} dy_n = Y_n^1 dt_1 + \cdots + Y_n^p dt_p, \\ \\ Y_n^{p+1} dy_1 + \cdots + Y_n^{p+1} dy_n = 0, \\ \qquad\qquad \ldots\ldots, \\ Y_n^{2p-1} dy_1 + \cdots + Y_n^{2p-1} dy_n = 0, \end{cases}$$

in which the quantities b_{ik} have the significance that was given before.

If one now considers a new form Θ_d^{2p} then the quotient:

$$(4) \qquad \frac{\Theta_d^{2p}}{dt_q} = X_1^{2p} \frac{dx_1}{dt_q} + \cdots + X_n^{2p} \frac{dx_n}{dt_q},$$

in which q indexes any of the variables $t_1, \ldots, t_n$, will transform into the expression:

$$Y_1^{2p} \frac{dy_1}{dt_q} + \cdots + Y_n^{2p} \frac{dy_n}{dt_q},$$

and it will be defined in the same manner, either by means of the old variables and system (1) or by means of the new ones and system (3). In other words, this quotient will be an absolute invariant for any change of variables. Moreover, there is no difficulty in calculating it. It suffices to eliminate the differentials dx_i, dt_α from equations (3) and (4), and one obtains the following result:

To abbreviate, set:

(5) $$\begin{Bmatrix} \Theta_d^1 & \Theta_d^2 & \cdots & \Theta_d^p \\ \Theta_d^{q+1} & \Theta_d^{q+2} & \cdots & \Theta_d^{q+p} \end{Bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{n1} & X_1^1 & \cdots & X_1^p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{nn} & X_n^1 & \cdots & X_n^p \\ X_1^{q+1} & \cdots & X_n^{q+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_1^{q+p} & \cdots & X_n^{q+p} & 0 & \cdots & 0 \end{vmatrix}.$$

One finds, for example, that:

(6) $$\frac{\Theta_d^{2p}}{dt_p} = -\frac{\begin{Bmatrix} \Theta_d^1 & \cdots & \Theta_d^p \\ \Theta_d^{p+1} & \cdots & \Theta_d^{2p} \end{Bmatrix}}{\begin{Bmatrix} \Theta_d^1 & \cdots & \Theta_d^{p-1} \\ \Theta_d^{p+1} & \cdots & \Theta_d^{2p-1} \end{Bmatrix}}.$$

We remark that if one has $p = 1$ then the denominator will be replaced with:

$$\Delta = \sum \pm a_{11} \ldots a_{nn} .$$

From this, if one considers $2n$ forms and, for the moment, one denotes the determinant:

$$\begin{Bmatrix} \Theta_d^1 & \cdots & \Theta_d^k \\ \Theta_d^{n+1} & \cdots & \Theta_d^{n+k} \end{Bmatrix}$$

by A_k then the quotients:

$$\frac{A_n}{A_{n-1}}, \frac{A_{n-1}}{A_{n-2}}, \ldots, \frac{A_1}{\Delta}$$

will be absolute invariants. However, one has:

$$(-1)^n A_n = \begin{vmatrix} X_1^1 & \cdots & X_1^n \\ \cdots & \cdots & \cdots \\ X_n^1 & \cdots & X_n^n \end{vmatrix} \times \begin{vmatrix} X_1^{n+1} & \cdots & X_n^{n+1} \\ \cdots & \cdots & \cdots \\ X_1^{2n} & \cdots & X_n^{2n} \end{vmatrix},$$

and it is easy to see that if one replaces the variables x_i with other variables y_i then each of the determinants that appear in the right-hand side of that equation are reproduced, but multiplied by the functional determinant:

$$\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)},$$

which is the determinant of the substitution. Therefore, A_n, and consequently A_{n-1}, …, A_1, Δ is reproduced, but multiplied by the square of that determinant.

As a result, all of the functions:

$$\begin{Bmatrix} \Theta_d^1 & \Theta_d^2 & \cdots & \Theta_d^q \\ \Theta_d^{p+1} & \Theta_d^{p+2} & \cdots & \Theta_d^{p+q} \end{Bmatrix}$$

are relative invariants *that one transforms into absolute invariants by dividing by one of the others – for example,* Δ.

I will not stop to show how one can express all of the functions by simpler means in terms of the $\begin{Bmatrix} \Theta_d^i \\ \Theta_d^k \end{Bmatrix}$, and to that end, I will content myself by referring to my paper "Sur la théorie algébrique des formes quadratiques, où se trouve résolue une question analogue." However, there is a property that I will establish at the conclusion of this article: *Whenever these invariants contain the form* Θ_d *itself on both sides, they will be expressed by:*

$$A = \begin{Bmatrix} \Theta_d & \Theta_d^1 & \cdots & \Theta_d^h \\ \Theta_d & \Theta_d^{h+1} & \cdots & \Theta_d^{2h} \end{Bmatrix},$$

so they will enjoy the property of being reproduced, but multiplied by a power of ρ *when one replaces the form* Θ_d *with* $\rho\,\Theta_d$, *where* ρ *is, moreover, an arbitrary function of the independent variables.*

Indeed, consider the expression for A in the form of the determinant:

$$A = \begin{vmatrix} a_{11} & \cdots & a_{n1} & X_1 & X_1^1 & \cdots & X_1^h \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{nn} & X_n & X_n^1 & \cdots & X_n^h \\ X_1 & \cdots & X_n & 0 & 0 & \cdots & 0 \\ X_1^{h+1} & \cdots & X_n^{h+1} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_1^{2h} & \cdots & X_n^{2h} & 0 & 0 & \cdots & 0 \end{vmatrix}.$$

If one multiplies Θ_d by ρ then one must replace X_i with ρX_i and a_{ik} with $\rho\, a_{ik} + X_i \dfrac{\partial\rho}{\partial x_k} - X_k \dfrac{\partial\rho}{\partial x_i}$ in the preceding determinant. After performing this substitution, add the $(n + 1)^{\text{th}}$ row, multiplied by $-\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x_k}$, to the k^{th} one, and the $(n + 1)^{\text{th}}$ column, multiplied by $\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x_i}$, to the i^{th} one. We then obtain the old expression for A, where any element that is included in the square that is formed from the first $n + 1$ rows and columns will have been multiplied by ρ. The determinant A will thus be multiplied by ρ^{n+1-h}.

IX.

We shall apply the preceding propositions, but while considering only the most general forms. In article VII, we saw, moreover, that all of the cases can be converted almost immediately into the ones that intend to study.

First, suppose that n is even and equal to $2m$. The reduced form can then be written:

$$\Theta_d = p_1\, dx_1 + \ldots + p_m\, dx_m\,;$$

I will consider only the following two invariants.

The first one is obtained from the fundamental form and the differential of an arbitrary function φ; its general expression is:

$$\left\{\begin{matrix}\Theta_d \\ d\varphi\end{matrix}\right\} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} & X_1 \\ a_{12} & \cdots & \cdots & a_{n2} & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} & X_n \\ \dfrac{\partial\varphi}{\partial x_1} & \dfrac{\partial\varphi}{\partial x_2} & \cdots & \dfrac{\partial\varphi}{\partial x_n} & 0 \end{vmatrix}. \tag{7}$$

With Clebsch, we employ the symbol (φ) in order to denote the quotient:

$$\varphi = \frac{1}{\Delta}\left\{\begin{matrix}\Theta_d \\ d\varphi\end{matrix}\right\}, \tag{8}$$

which will be an absolute invariant.

The second invariant that we consider will be the following one:

$$\left\{\begin{matrix}d\varphi \\ d\psi\end{matrix}\right\} = \begin{vmatrix} a_{11} & \cdots & a_{n1} & \dfrac{\partial\varphi}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{nn} & \dfrac{\partial\varphi}{\partial x_n} \\ \dfrac{\partial\psi}{\partial x_1} & \cdots & \dfrac{\partial\psi}{\partial x_n} & 0 \end{vmatrix},$$

and we set:

$$(\varphi\ \psi) = \frac{-1}{\Delta}\left\{\begin{matrix}d\varphi \\ d\psi\end{matrix}\right\}, \tag{9}$$

in such a way that $(\varphi\ \psi)$ will again be an absolute invariant.

If one calculates the two symbols (φ), $(\varphi\ \psi)$ with the variables of the reduced form then one effortlessly obtains, by some combinations of rows and columns:

(10)
$$\begin{cases} (\varphi) = p_1 \dfrac{\partial \varphi}{\partial p_1} + \cdots + p_m \dfrac{\partial \varphi}{\partial p_m}, \\ (\varphi\psi) = \dfrac{\partial \varphi}{\partial p_1}\dfrac{\partial \psi}{\partial x_1} - \dfrac{\partial \varphi}{\partial x_1}\dfrac{\partial \psi}{\partial p_1} + \cdots + \dfrac{\partial \varphi}{\partial p_m}\dfrac{\partial \psi}{\partial x_m} - \dfrac{\partial \varphi}{\partial x_m}\dfrac{\partial \psi}{\partial p_m}. \end{cases}$$

The two symbols that we just defined are particular cases of the following one, which plays a fundamental role in the theory of partial differential equations when it is applied to functions of $2m + 1$ variables z, x_i, p_k, and which is defined by the equation:

(11)
$$[\varphi\ \psi] = \frac{\partial \varphi}{\partial p_1}\left(\frac{\partial \varphi}{\partial x_1} + p_1 \frac{\partial \psi}{\partial z}\right) - \frac{\partial \psi}{\partial p_1}\left(\frac{\partial \varphi}{\partial x_1} + p_1 \frac{\partial \varphi}{\partial z}\right) + \ldots$$

Here, our functions do not depend upon z. One thus has:

$$(\varphi\ \psi) = [\varphi\ \psi].$$

However, it is clear that one also has:

(12)
$$(\varphi) = [\varphi\ \psi].$$

By virtue of this remark, the relations that were established by Clebsch between the symbols (φ), $(\varphi\ \psi)$ can all be deduced from one general equation that was given by Mayer (*Mathematische Annalen*, t. IX, pp. 370). Mayer has shown that if one considers three functions φ, ψ, χ of $2m + 1$ variables z, x_i, p_k then one has:

(13)
$$[\varphi\,[\psi\,\chi]] + [\psi\,[\chi\,\varphi]] + [\chi\,[\varphi\,\psi]] = \frac{\partial \varphi}{\partial z}[\psi\,\chi] + \frac{\partial \psi}{\partial z}[\chi\,\varphi] + \frac{\partial \chi}{\partial z}[\varphi\psi].$$

If one applies this relation to three functions that do not contain z then one deduces the Jacobi relation:

(14)
$$(\varphi\,(\psi\,\chi)) + (\psi\,(\chi\,\varphi)) + (\chi\,(\varphi\,\psi)) = 0$$

between the symbols $(\varphi\ \psi)$.

If one sets $\chi = z$, and if one supposes that the functions φ, ψ are independent of z then one likewise finds that:

(15)
$$(\varphi(\psi)) - (\psi(\varphi)) = (\varphi\ \psi) + ((\varphi\ \psi)).$$

These are the two relations that serve as the basis for the Clebsch method of integration.

X.

I will make an application of the preceding results to the study of relations between two different reductions of the same form.

Consider a differential expression Θ_d, and let:

$$p_1\,dx_1 + \ldots + p_m\,dx_m$$

be an initial reduced form; I first state that whenever one can find m functions $X_1, \ldots, X_m$ that give rise to an identity of the form:

$$p_1\,dx_1 + \ldots + p_m\,dx_m = P_1\,dX_1 + \ldots + P_m\,dX_m\,, \tag{16}$$

the right-hand side of that equality will be a new reduced form. In order for this to be true, it will suffice to prove that the functions X_i, P_k are independent, and this is almost obvious. Because there are one or more relations between the variables X_i, P_k, once can express some of these functions in terms of the other ones by means of these relations, and consequently convert:

$$\Theta_d = P_1\,dX_1 + \ldots + P_m\,dX_m$$

into a normal form that contains less than $2m$ functions. One knows that this is impossible, and one can conclude that if m functions X_i satisfy equation (16) then the right-hand side of that equation will certainly be a new reduced form of Θ_d. In other words, the functions X_i, P_k will be independent.

Having said this, the two symbols (φ), $(\varphi\,\psi)$, being absolute invariants, preserve the same value when one forms them by considering φ, ψ to be either functions of X_i, P_k or functions of x_i, p_k.

One will thus have:

$$\begin{cases} \sum p_i \dfrac{\partial \varphi}{\partial p_i} = \sum P_i \dfrac{\partial \varphi}{\partial P_i}, \\ \sum \dfrac{\partial \varphi}{\partial p_i}\dfrac{\partial \psi}{\partial x_i} - \dfrac{\partial \psi}{\partial p_i}\dfrac{\partial \varphi}{\partial x_i} = \sum \dfrac{\partial \varphi}{\partial P_i}\dfrac{\partial \psi}{\partial X_i} - \dfrac{\partial \varphi}{\partial X_i}\dfrac{\partial \psi}{\partial P_i}. \end{cases} \tag{17}$$

Applying these general equations to the functions X_i, P_k itself, we effortlessly obtain the following equations:

$$\begin{cases} (P_i) = P_i, \quad (X_i) = 0, \\ (P_i X_i) = 1, \quad (P_i X_k) = 0, \; (X_i X_k) = 0, \; (P_i P_k) = 0. \end{cases} \tag{18}$$

We can thus state the following proposition:

If m functions X_i of the $2m$ variables x_i, p_k satisfy a differential identity of the form:

$$P_1\,dX_1 + \ldots + P_m\,dX_m = p_1\,dx_1 + \ldots + p_m\,dx_m$$

then the $2m$ functions X_i, P_k are independent and satisfy the relations:

$$(P_i) = P_i\,, \qquad (X_i) = 0,$$
$$(P_i\,X_i) = 1, \qquad (P_i\,X_k) = 0, \qquad (X_i\,X_k) = 0, \qquad (P_i\,P_k) = 0.$$

The first two equations express the idea that P_i is a homogeneous function of degree one and X_i is a homogeneous function of degree 0 in the variables p_k . This is exhibited by the *finite* equations that were given by Clebsch, which allow one to pass from one normal form to another. I shall not elaborate upon this point, as it is well-known.

I will now establish a fundamental proposition that Lie made the most felicitous use of in his theory of groups: *If one has k independent functions* $X_1, X_2, \ldots, X_k$ *that satisfy the equations:*

$$(X_i) = 0, \qquad (X_i\, X_k) = 0$$

then it will be possible to find a normal form that include the k functions:

$$P_1\, dX_1 + \ldots + P_k\, dX_k + P_{k+1}\, dX_{k+1} + P_m\, dX_m = p_1\, dx_1 + \ldots + p_m\, dx_m\, .$$

I will commence by proving this proposition in the case where one has just one function X_1 . Then, I will determine a function P_1 by the two equations:

$$(P_1) = P_1, \qquad (P_1\, X_1) = 1. \tag{19}$$

It is easy to see that these equations are not incompatible.

The first one shows us that one will have:

$$P_1 = p_1\, \varphi\left(x_1, \cdots, x_m, \frac{p_2}{p_1}, \cdots, \frac{p_m}{p_1}\right),$$

and if we recall that by virtue of the equation:

$$(X_1) = 0$$

that X_1 satisfies, that function is homogeneous of degree zero with respect to the variables p_i then we recognize with no difficulty that the equation:

$$(P_1\, X_1) = 1$$

reduces to a relation between the derivatives of φ and the variables x_i, p_i / p_1 that they depend upon. Therefore, it is always possible, and in an infinitude of ways, to determine a function P_1 that satisfies the two equations (19). It will suffice to take an integral of one linear equation in $2m - 1$ independent variables.

Therefore, suppose that P_1 is determinate. Consider the form:

$$U_d = p_1\, dx_1 + \ldots + p_m\, dx_m - P_1\, dX_1\, .$$

We shall see that it belongs to the type:

$$P_1\, dX_1 + \ldots + P_m\, dX_m\, , \tag{20}$$

which proves the proposition that we have in mind.

In order to do this, I write the system of Pfaff differential equations that relate to the form Θ_d . One has:

$$\delta U_d - dU_\delta = \delta p_1\, dx_1 - dp_1\, \delta x_1 + \ldots + dP_1\, \delta X_1 - dX_1\, \delta P_1,$$

which allows us to construct the desired differential equations in the following form:

$$(21)\qquad \begin{cases} dx_i - \dfrac{\partial P_1}{\partial p_i} dX_1 + \dfrac{\partial X_1}{\partial p_i} dP_1 = -P_1 \dfrac{\partial X_1}{\partial p_i} \lambda dt, \\ dp_i - \dfrac{\partial P_1}{\partial x_i} dX_1 + \dfrac{\partial X_1}{\partial x_i} dP_1 = \lambda dt \left(p_i - P_1 \dfrac{\partial X_1}{\partial p_i} \right). \end{cases}$$

I will prove that these $2m$ equations can be verified without setting $\lambda = 0$ and that two of them are consequences of the other ones. Introduce the unknown variables dX_1, dP_1 that the differentials dx_i, dp_i will be determined as functions of, and attempt to determine dX_1, dP_1 by substituting the values of dx_i, dp_k into the developed expressions for dX_1, dP_1:

$$dX_1 = \sum \frac{\partial X_1}{\partial x_i} dx_i + \sum \frac{\partial X_1}{\partial p_i} dp_i \,,$$

$$dP_1 = \sum \frac{\partial P_1}{\partial x_i} dx_i + \sum \frac{\partial P_1}{\partial p_i} dp_i \,,$$

we thus obtain the two equations:

$$[(P_1\, X_1) - 1]\, (dP_1 + \lambda\, P_1\, dt) = \lambda\, dt[(P_1) - P_1],$$
$$[(P_1\, X_1) - 1]\, dX_1 = \lambda\, dt\, (X_1),$$

which are verified identically. Therefore, equations (21) can be verified without one having to set $\lambda = 0$. They admit a second-order indeterminacy, and consequently the form U_d belongs to the type (20), as we will establish.

It remains for us to prove in a general manner that if one has k independent function $X_1, \ldots, X_k$ that satisfy the equations:

$$(X_h) = 0, \qquad (X_h\, X_{h'}) = 0$$

then it will be possible to find a normal form that they belong to. Since we have proved the theorem for a function, it will suffice to prove that if it is true for $k - 1$ functions $X_1, \ldots, X_{k-1}$ then it will be true for another function V under the condition that this function V must satisfy the equations:

$$(22)\qquad (V) = 0, \qquad (V\, X_i) = 0,$$

and that it is not coupled to the latter functions by any relation and is independent of the variables.

Let:

$$P_1\, dX_1 + \ldots + P_{k-1}\, dX_{k-1} + P_k\, dX_k + \ldots + P_n\, dX_n$$

be one of the normal forms that the $k - 1$ functions $X_1, \ldots, X_{k-1}$ enter into. If one expresses V by means of variables X_i, P_k then by virtue of the invariance properties of the symbols (φ), $(\varphi\ \psi)$ equations (22) become:

$$(23) \qquad P_k \frac{\partial V}{\partial P_k} + \ldots + P_n \frac{\partial V}{\partial P_n} = 0, \qquad \frac{\partial V}{\partial P_1} = 0, \qquad \ldots, \qquad \frac{\partial V}{\partial P_{k-1}} = 0.$$

The function V is therefore independent of $P_1, \ldots, P_{k-1}$, but it is not necessarily independent of $X_1, \ldots, X_{k-1}$. For the moment, make these latter variables constants. Since, by hypothesis, the function V does not depend solely upon them, it remains variable, and since it satisfies the first of equations (23), one sees, from the proposition that was proved to begin with, that one can convert:

$$P_k\, dX_k + \ldots + P_m\, dX_m$$

into the normal form:

$$P'_k dV + P'_{k+1} dX'_{k+1} + \ldots + P'_m dX'_m ,$$

which will contain V. However, one has regarded $X_1, \ldots, X_{k-1}$ as constants; if one lets them be variables then the preceding expression will be augmented with terms in $dX_1, \ldots, dX_{k-1}$ and one will have, consequently:

$$P_k\, dX_k + \ldots + P_m\, dX_m = P'_k dV + P'_{k+1} dX'_{k+1} + \ldots + P'_m dX'_m$$
$$+ A_1\, dX_1 + A_2\, dX_2 + \ldots + A_{k-1}\, dX_{k-1} .$$

Therefore, the original normal form:

$$P_1\, dX_1 + \ldots + P_{k-1}\, dX_{k-1} + P_k\, dX_k + \ldots + P_n\, dX_n$$

will be changed into the following one:

$$(P_1 + A_1)\, dX_1 + \ldots + (P_{k-1} + A_{k-1})\, dX_{k-1} + P'_k dV + P'_{k+1} dX'_{k+1} + \ldots + P'_m dX'_m ,$$

which indeed contains the k functions:

$$X_1, \ldots, X_{k-1}, V;$$

the theorem is thus proved in general.

In summation, we can state the following proposition:

Whenever one has independent functions $X_1, \ldots, X_k$ of the variables x_i, p_k that are homogeneous of degree zero in the variables p_i and satisfy, in addition, the equations:

$$(X_\alpha\, X_\beta) = 0,$$

it will be possible to append $2m - r$ *other functions to them that give rise to the differential identity:*

$$p_1\,dx_1 + \ldots + x_m\,dx_m = P_1\,dX_1 + \ldots + P_m\,dX_m\,.$$

The case where $r = m$ *is not excluded. The functions* X_i , P_i *will all be homogeneous in the variables* p_i *, where the former are of degree* 0 *and the latter are of degree* 1. *They will have They will have an arbitrary form with respect to the variables* X_i .

By a simple change of notation, this important theorem gives rise to another fundamental proposition that we shall present.

One can give a new form to the identity:

$$p_1\,dx_1 + \ldots + x_m\,dx_m = P_1\,dX_1 + \ldots + P_m\,dX_m\,. \tag{24}$$

Set:

$$\left.\begin{aligned} p_i &= p_m q_i, & x_m &= -z, \\ P_i &= P_m Q_i, & X_m &= -Z, \end{aligned}\right\} \qquad p_m = \rho\,P_m\,.$$

It will become:

$$dZ - Q_1\,dX_1 - \ldots - Q_{m-1}\,dX_{m-1} = \rho(dz - q_1\,dx_1 - \ldots - q_{m-1}\,dx_{m-1}).$$

Consider a function φ of the variables x_i, p_i that is homogeneous and of degree μ in the variables p_i . It takes the form:

$$\varphi = p_m^{\mu}\, f(q_1, \ldots, q_{m-1}, x_1, \ldots, x_{m-1}, z),$$

and one will have:

$$\frac{\partial\varphi}{\partial p_1} = p_m^{\mu-1}\frac{\partial f}{\partial q_1}, \qquad \frac{\partial\varphi}{\partial x_1} = p_m^{\mu}\frac{\partial f}{\partial x_1}, \qquad \frac{\partial\varphi}{\partial z} = p_m^{\mu}\frac{\partial f}{\partial z},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{\partial\varphi}{\partial p_{m-1}} = p_m^{\mu-1}\left[\mu f - q_1\frac{\partial f}{\partial q_1} - \cdots - q_{m-1}\frac{\partial f}{\partial q_{m-1}}\right].$$

If we likewise calculate the derivatives of another function φ_i that is of degree μ with respect to the variables p_i and one substitutes all of these derivatives in the symbol $(\varphi\,\varphi_1)$ then one will have:

$$(\varphi\,\varphi_1) = p_m^{\mu+\mu_1-1}[f\ f_1] - p_m^{\mu+\mu_1-1}\left[\mu f\frac{\partial f}{\partial z} - \mu_1 f_1\frac{\partial f}{\partial z}\right],$$

in which $[f\,f_1]$ denotes the expression:

$$\frac{\partial f}{\partial q_1}\left[\frac{\partial f_1}{\partial x_1} + q_1\frac{\partial f_1}{\partial z}\right] - \frac{\partial f}{\partial q_1}\left[\frac{\partial f}{\partial x_1} + q_1\frac{\partial f}{\partial z}\right] + \ldots$$

For example, suppose that one is dealing with homogeneous functions of degree zero. One will have $\mu = \mu_1 = 0$, so:

(25) $$(\varphi\,\varphi_1) = \frac{[f\ f_1]}{p_m}.$$

If one now likewise operates with the variables Z, Q_i, X_k, and one applies the second equation in (17) then one will have:

$$\frac{[f\ f_1]_z}{p_m} = \frac{[f\ f_1]_Z}{P_m},$$

in which the letters z, Z that are used as indices indicate the system of variables in which one forms the bracket. We can therefore write:

(26) $$[f\,f_1]_z = \rho\,[f\,f_1]_Z .$$

If we apply this equation to all of the functions Z, X_i, Q_k then we can conclude:

$$[X_i\ Z] = 0, \qquad [X_i\ X_k] = 0, \qquad [Q_i\ Q_k] = 0,$$
$$[Z\ Q_k] + \rho Q_k = 0, \qquad [Q_i\ X_i] = \rho.$$

Upon changing the notations, one thus has the following proposition:

Consider $2m + 1$ *functions* Z, X_i, P_k *that satisfy the differential identity:*

(27) $$dZ - P_1\,dX_1 - \ldots - P_m\,dX_m = p\,(dz - p_1\,dx_1 - \ldots - p_m\,dx_m);$$

these functions are necessarily independent. In addition, they satisfy the relations:

(28) $$\begin{cases} [Z\ X_i] = 0, & [X_i X_k] = 0, \\ [P_i X_i] = \rho, & [P_i X_k] = 0, \quad [P_i P_k] = 0, \\ [Z\ X_i] + \rho P_k = 0. \end{cases}$$

Conversely, whenever one has k independent functions Z, X_1, …, X_{k-1} *whose brackets are all zero one can append to them some other functions such that the identity* (27) *is satisfied.*

It is essential to append the following relations to equations (28), which one gets by applying Mayer's formula to three of the functions Z, X_i, P_k:

$$(29)\qquad \begin{cases} [\rho\, Z] = \rho^2 - \rho \dfrac{\partial Z}{\partial z}, \\ [\rho\, X_i] = -\rho \dfrac{\partial X_i}{\partial z}, \\ [\rho\, P_i] = -\rho \dfrac{\partial P_i}{\partial z}. \end{cases}$$

These formulas, which one can prove directly, must be combined with equations (28) if one would like to have the equivalent of relations (18) that relates to the functions that satisfy identity (16).

We again point out a particular case of the preceding proposition: *One can satisfy equation* (27) *by taking Z arbitrarily, and then p must satisfy just the first of equations* (29).

XI.

Now, suppose that n is odd and equal to $2m + 1$. The determinant $\Delta = \sum a_{11} \ldots a_{nn}$ will be zero; however, if we confine ourselves to the general case then none of the first-order minors will be zero. As for the invariant R, which is defined by:

$$(30)\qquad R^2 = \begin{Bmatrix} \Theta_d \\ -\Theta_d \end{Bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{n1} & X_1 \\ a_{12} & \cdots & a_{n2} & X_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & X_n \\ -X_1 & \cdots & -X_n & 0 \end{vmatrix},$$

it will not be zero, so Θ_d belongs to the indeterminate type, and its reduced form can be written:

$$dz - p_1\, dx_1 - \ldots - p_m\, dx_m .$$

We consider the following two invariants:

The symbol (φ) will defined by the formula:

$$(31)\qquad R^2(\varphi)^2 = \begin{Bmatrix} d\varphi \\ -d\varphi \end{Bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{n1} & \dfrac{\partial \varphi}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{nn} & \dfrac{\partial \varphi}{\partial x_n} \\ -\dfrac{\partial \varphi}{\partial x_1} & \cdots & -\dfrac{\partial \varphi}{\partial x_n} & 0 \end{vmatrix},$$

and the symbol $[\varphi\ \psi]$, by the relation:

(32) $$R^2\,[\varphi\;\psi] = \begin{Bmatrix} \Theta_d & d\varphi \\ \Theta_d & -d\psi \end{Bmatrix}.$$

From the properties of skew-symmetric determinants, all of these invariants are rational.

If one calculates with the reduced form then one will find:

(33) $$\begin{cases} R^2 = 1, \\ (\varphi)^2 = \left(\dfrac{\partial\varphi}{\partial z}\right)^2, \\ [\varphi\psi] = \dfrac{\partial\varphi}{\partial p_1}\left[\dfrac{\partial\psi}{\partial x_1} + p_1\dfrac{\partial\psi}{\partial z}\right] - \dfrac{\partial\psi}{\partial p_1}\left[\dfrac{\partial\varphi}{\partial x_1} + p_1\dfrac{\partial\varphi}{\partial z}\right] + \cdots \end{cases}$$

We take:

$$(\varphi) = \frac{\partial\varphi}{\partial z}.$$

When one takes squares roots in formula (31), it will suffice to choose the sign on the right-hand side in such a manner that the absolute invariant (φ) reduces to $\partial\varphi / \partial z$ when one calculates with the reduced form.

The invariant R belongs to the class that we considered at the end of article VIII, and it is easy to recognize that it will be reproduced, but multiplied by ρ^{n+1}, when one multiples the form Θ_d by an arbitrary function ρ. Therefore, $\rho\,\Theta_d$ belongs to the most general type for any ρ. In particular, consider a normal form for Θ_d. We have the following theorem:

No matter what function ρ of the variables z, x_i, p_k one chooses, it is possible to find functions Z, X_i, P_k that satisfy the identity:

$$dZ - P_1\,dX_1 - \ldots - P_m\,dX_m = \rho(dz - p_1\,dx_1 - \ldots - p_m\,dx_m)$$

that we already considered.

The expressions (33) allow us to develop a method of integration that is similar to the one that Clebsch employed in the case of an even number of variables. Here, I will utilize only their invariance properties in order to study further the relations between the various reduced forms.

XII.

I first say that whenever one has:

$$\Theta_d = dZ - P_1\,dX_1 - \ldots - P_m\,dX_m,$$

the variables Z, X_i, P_k being independent. This proposition is proved as in the preceding case.

Now, consider two different reduced forms that give rise to the identity:

(34) $$dz - p_1\,dx_1 - \dots - p_m\,dx_m = dZ - P_1\,dX_1 - \dots - P_m\,dX_m,$$

and remark that upon applying the invariance properties of the symbols (φ), $[\varphi\ \psi]$ one will have:

(35) $$\begin{cases} \dfrac{\partial\varphi}{\partial z} = \dfrac{\partial\varphi}{\partial Z}, \\ [\varphi\psi]_z = [\varphi\psi]_Z. \end{cases}$$

When the first equation is applied to Z, that will give us:

$$\frac{\partial Z}{\partial z} = 1,$$

and consequently:

$$Z = z + \Pi,$$

where Π depends upon only the variables x_i, p_k. The same equation, when applied to the functions X_i, P_k, shows us that *they are independent of* z. If one then replaces Z with its value in the identity (34) then it becomes:

(36) $$d\Pi = P_1\,dX_1 + \dots + P_m\,dX_m - p_1\,dx_1 - \dots - p_m\,dx_m,$$

and z *is eliminated completely.*

Conversely, for any equality of the form (36), one can return to the equality (34) by replacing Π with $Z - z$. These two equalities must therefore be considered as absolutely equivalent.

Apply the second of formulas (35) to the functions Z, X_i, P_k; we will have:

(37) $$\begin{cases} (X_i\,X_k) = 0,\ (P_i\,P_k) = 0,\ (X_i\,P_k) = 0,\ (P_i\,X_i) = 1, \\ (\Pi\,X_i) = p_1\dfrac{\partial X_i}{\partial p_1} + \dots + p_m\dfrac{\partial X_i}{\partial p_m}, \\ (\Pi\,P_i) = p_1\dfrac{\partial P_i}{\partial p_1} + \dots + p_m\dfrac{\partial P_i}{\partial p_m} - P_i. \end{cases}$$

We are thus led to the following proposition:

If $2m + 1$ *functions* X_i, P_k, Π *of the variables* x_i, p_k *satisfy an equations of the form:*

(38) $$d\Pi = P_1\,dX_1 + \dots + P_m\,dX_m - p_1\,dx_1 - \dots - p_m\,dx_m$$

then the functions X_i, P_k are independent, and when they are combined with the function Π *they satisfy relations* (37).

I will now conclude by proving that if r independent functions $X_1, \ldots, X_r$ of the variables x_i, p_k satisfy the equations:

$$(X_\alpha X_\beta) = 0$$

then one can append functions to them that allow one to satisfy equation (38), or – what amounts to the same thing, as we have proved – equation (34).

The proof is similar to the one that we developed in article X, so I will content myself by pointing that out.

First, consider the case of just one function X_i and determine a function P_1 of the variables x_i, p_k by the equation:

$$(P_1 X_1) = 1;$$

it is easy to see that if one considers the form:

$$U_d = dz - p_1\, dx_1 - \ldots - p_m\, dx_m + P_1\, dX_1$$

then the Pfaff equations that relate to this form and are summarized in the single equation:

$$\delta U_d - dU_\delta = 0$$

are indeterminate. Moreover, as a result of the presence of the differential dz, U_d can only belong to the indeterminate type. One will thus necessarily have:

$$U_d = dZ - P_2\, dX_2 - \ldots - P_m\, dX_m ,$$

and consequently:

$$dz - p_1\, dx_1 - \ldots - p_m\, dx_m = dZ - P_1\, dX_1 - \ldots - P_m\, dX_m ,$$

or furthermore:

$$d\Pi = P_1\, dX_1 + \ldots + P_m\, dX_m - p_1\, dx_1 - \ldots - p_m\, dx_m .$$

The theorem is therefore proved in the case of just one function.

When there are several of them, it will suffice to repeat, almost word-for-word, the proof of article X. We shall dispense with that reproduction.

We have now made known the three propositions of Lie that relate to the identities:

$$\begin{aligned} p_1\, dx_1 - \ldots - p_m\, dx_m &= P_1\, dX_1 - \ldots - P_m\, dX_m , \\ \rho(dz - p_1\, dx_1 - \ldots - p_m\, dx_m) &= dZ - P_1\, dX_1 - \ldots - P_m\, dX_m , \\ p_1\, dx_1 - \ldots - p_m\, dx_m &= P_1\, dX_1 - \ldots - P_m\, dX_m + d\Pi. \end{aligned}$$

Since they have numerous applications, we would like to prove then by the most elementary process. The only proposition that we have borrowed from the theory of partial differential equations is the following one: *Any first-order equation admits at least* *one solution.* Moreover, this proposition is likewise proved by arguments that are given in article VII.

We remark that the proposition in article X – namely, that one can satisfy the equation:

$$\rho(dz - p_1\, dx_1 - \ldots - p_m\, dx_m) = dZ - P_1\, dX_1 - \ldots - P_m\, dX_m$$

by taking Z to be an arbitrary function – provides a means of attaching the theory of partial differential equations to the solution of the Pfaff problem that is different from the one in article VII.

That is because if:

$$Z = 0$$

is the equation to be integrated then one can propose to convert the differential expression in an *odd* number of variables:

$$dz - p_1\, dx_1 - \ldots - p_m\, dx_m ,$$

to the form:

$$\frac{1}{\rho}(dZ - P_1\, dX_1 - \ldots - P_m\, dX_m),$$

and once that problem is solved, the equations:

$$X_1 = C_1 , \qquad \ldots, X_m = C_m$$

will give a complete integral to the proposed one. In truth, this method seems less direct than the one in article VII, and it seems that it augments the difficulty in the problem, since it leads to the solution, not only of the equation:

$$Z = 0,$$

but also of:

$$Z = C.$$

However, as one knows, it is easy to introduce a constant into a partial differential equation. For example, one replaces x_i with $x_i + C$, z with $z + C$ or $z + C_k\, x_k$, and upon solving with respect to that constant one can make the objection that we just pointed out disappear.
