“Sur les invariants intégraux de l’optique,” Bull. Soc. math. France **42** (1914), 91-95.

ON THE INTEGRAL INVARIANTS OF OPTICS

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The very interesting proof of Straubel’s theorem that Dontot gave ([1]) can be simplified, thanks to two theorems that I published in 1913 ([2]), and for which I have pointed out various applications to mathematical physics.

THEOREM I. – *If* ρ *is an invariant and if:*

$$I_m = \int M\ \delta x_1\ \delta x_2 \ldots \delta x_m \tag{1}$$

is an m-tuple integral invariant of the equations:

$$\frac{dx_1}{X_1} = \ldots = \frac{dx_m}{X_m} = dt$$

(X_1, X_2, …, X_m *are continuous and uniform functions of* x_1, …, x_m , *and* t) *then the quantity:*

$$A_{m-1} = \int \frac{M}{\dfrac{\partial\rho}{\partial x_1}}\ \delta x_1\ \delta x_2 \ldots \delta x_m \tag{2}$$

is an $(m - 1)$*-tuple invariant on the manifold* $\rho = \rho_0$.

THEOREM II. – *If* ρ *and* X_1, …, X_m *do not refer to t explicitly then one can deduce the* $(m - 2)$*-uple integral invariant:*

$$A_{m-1} \equiv \frac{M}{\dfrac{\partial\rho}{\partial x_1}} \sum_k (-1)^k\ X_k\ \delta x_1 \ldots \delta x_{k-1}\ ,\ \delta x_{k+1}, \ldots \delta x_m \qquad (k = 2, \ldots, m). \tag{3}$$

([1]) Bull. Soc. math. de France **42** (1914), fasc. 1.

([2]) “Sur un théorème de Jacobi,” C. R. Acad. Sci., 10 February 1913; “Sur la répartition ergodique,” Bull. Acad. royale de Belgique: Cl. des Sciences **3** (1913), 211-221.

on the invariant manifold $\rho = \rho_0$ *from the invariant* (2).

Proof. – If one multiplies A_{m-2} by $\delta\rho$ symbolically ([1]) then one will find the $(m - 1)$-tuple integral form:

$$\frac{M}{\frac{\partial\rho}{\partial x_1}}\Big[X_2\,\delta x_3\,\delta x_4\cdots\delta x_m\,\frac{\partial\rho}{\partial x_1}\,\delta x_1 + X_2\,\delta x_3\delta x_4\cdots\delta x_m\,\frac{\partial\rho}{\partial x_2}\,\delta x_2$$

$$- X_3\,\delta x_2\delta x_4\cdots\delta x_m\,\frac{\partial\rho}{\partial x_1}\,\delta x_1 - X_3\,\delta x_2\delta x_4\cdots\delta x_m\,\frac{\partial\rho}{\partial x_3}\,\delta x_3$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ (-1)^m\,X_m\,\delta x_2\delta x_3\cdots\delta x_{m-1}\,\frac{\partial\rho}{\partial x_1}\,\delta x_1 + (-1)^m X_m\,\delta x_2\cdots\delta x_{m-1}\,\frac{\partial\rho}{\partial x_m}\,\delta x_m\Bigg].$$

Now:

$$\frac{\partial\rho}{\partial x_1}X_1 + \frac{\partial\rho}{\partial x_2}X_2 + \cdots + \frac{\partial\rho}{\partial x_m}X_m = 0,$$

so:

$$X_1 = -\frac{1}{\frac{\partial\rho}{\partial x_1}}\left[\frac{\partial\rho}{\partial x_2}X_2 + \cdots + \frac{\partial\rho}{\partial x_m}X_m\right].$$

Upon substituting this in the preceding integral form, it reduces (up to sign) to:

$$\sum_i (-1)^i\,M\,X_i\,\delta x_1\,\ldots\,\delta x_{i-1}\,\delta x_{i+1}\,\ldots\,\delta x_m \qquad (i = 1, \ldots, m).$$

Now, the latter provides an integral invariant of equations (1).

Q. E. D.

Now consider, with Dontot, the extremals of:

$$\delta\int n\sqrt{x'^2 + y'^2 + z'^2}\,dt = 0,$$

where $x' = dx / dt$, $y' = dy / dt$, $z' = dz / dt$. The study of these extremals amounts to that of the differential equations:

([1]) "Introduction à la théorie des invariants intégraux," Bull. Acad. royale de Belgique: Cl. des Sciences **12** (1913).

$$(4)\qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial u}, & \dfrac{dy}{dt} = \dfrac{\partial H}{\partial v}, & \dfrac{dz}{dt} = \dfrac{\partial H}{\partial w}, \\[2mm] \dfrac{du}{dt} = -\dfrac{\partial H}{\partial x}, & \dfrac{dv}{dt} = -\dfrac{\partial H}{\partial y}, & \dfrac{dw}{dt} = -\dfrac{\partial H}{\partial z} \end{cases}$$

on the invariant manifold:

$$(5)\qquad H = \frac{1}{2n^2}(u^2 + v^2 + w^2) = \frac{1}{2} .$$

One has set:

$$u = \frac{\partial n\sqrt{x'^2 + y'^2 + z'^2}}{\partial x'} = \frac{nx'}{\sqrt{x'^2 + y'^2 + z'^2}},$$

and similarly for v and w.

Equations (4) admit the integral invariant:

$$I_4 \equiv \int \delta x\, \delta y\, \delta z\, \delta u\, \delta v\, \delta w.$$

Applying Theorem I gives the 5-tuple integral invariant:

$$A_5 \equiv \int \frac{n^2}{u}\, \delta x\, \delta y\, \delta z\, \delta v\, \delta w$$

in the manifold (5).

Applying Theorem II gives the 4-tuple integral invariant ([1]):

$$A_4 \equiv \int \frac{n^2}{u}\left[\frac{u}{n^2}\delta y\,\delta z\,\delta v\,\delta w - \frac{v}{n^2}\delta x\,\delta z\,\delta v\,\delta w + \frac{w}{n^2}\delta x\,\delta y\,\delta v\,\delta w \right. \\ \left. + \frac{\partial H}{\partial y}\delta x\,\delta y\,\delta z\,\delta w - \frac{\partial H}{\partial z}\delta x\,\delta y\,\delta z\,\delta v\right].$$

If one considers two arbitrary surfaces σ and σ' in ordinary space and joins the various points of σ to the various points of σ', respectively, by light rays then one will have $\delta x\, \delta y\, \delta z = 0$, and A_4 will reduce to:

$$A_4 \equiv \int (u\,\delta y\,\delta z + v\,\delta z\,\delta x + w\,\delta x\,\delta y)\,\frac{\delta v\,\delta w}{u}.$$

Set, with Dontot:

$$m = \frac{u}{n} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}},$$

([1]) The simplification that is introduced seems important to me – here, especially.

$$p = \frac{v}{n},$$

$$q = \frac{w}{n}.$$

Therefore, *m*, *p*, *q* are the direction cosines of the semi-tangent to the light ray in the sense of the propagation of light. A_4 becomes:

$$A_4 \equiv \int n^2 [m\,\delta y\,\delta z + p\,\delta z\,\delta x + q\,\delta x\,\delta y] \frac{\delta p\,\delta q}{m}.$$

Now, $\dfrac{\delta p\,\delta q}{m} = \delta\omega$ (i.e., the solid angle or elementary surface area of a sphere of radius *one*), so if we let θ be the angle of the semi-normal to a surface element $\delta\sigma$ then:

$$n^2 \cos\theta\,\delta\sigma\,\delta\omega = \text{const.}$$

Q. E. D.

The latter result can be obtained more rapidly by starting with the relative invariant ([1]):

$$\dot{J}_1 \equiv \int \frac{\partial n\sqrt{x'^2 + y'^2 + z'^2}}{\partial x'}\delta x + \frac{\partial n\sqrt{x'^2 + y'^2 + z'^2}}{\partial y'}\delta y + \frac{\partial n\sqrt{x'^2 + y'^2 + z'^2}}{\partial z'}\delta z.$$

Upon representing the direction cosines by *m*, *p*, *q*, one can rewrite:

$$\dot{J}_1 \equiv \int n\,(m\,\delta x + p\,\delta y + q\,\delta z).$$

By symbolic differentiation, one introduces the absolute (2-tuple) integral invariant:

$$\dot{I}_2 \equiv \int \delta n\,(m\,\delta x + p\,\delta y + q\,\delta z) + n\,(\delta m\,\delta x + \delta p\,\delta y + \delta q\,\delta z).$$

Upon multiplying $\dot{I}_2$ by $\dot{I}_2$ symbolically and taking into account the fact that $\delta x\,\delta y\,\delta z$ is zero in this optical theorem, one will obtain:

$$\dot{I}_4^* \equiv \int n^2\,(\delta x\,\delta y\,\delta m\,\delta p + \delta y\,\delta z\,\delta p\,\delta q + \delta z\,\delta x\,\delta q\,\delta n)$$

([1]) "Étude sur les invariants intégraux," Rendiconti Circolo matematico di Palermo, v. XVI, 1902 (Chap. XIII).

$$\equiv \int n^2 \left(q\,\delta x\,\delta y \frac{\delta m\,\delta p}{q} + m\,\delta y\,\delta z \frac{\delta p\,\delta q}{m} + p\,\delta z\,\delta x \frac{\delta q\,\delta m}{p} \right)$$

$$\equiv \int n^2 \cos\theta\,\delta\sigma\,\delta\omega.$$

Q. E. D.