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On integral invariants and some points of geometrical optics

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INTRODUCTION

Consider a sequence of media of an arbitrary nature, where the extreme media are isotropic with a constant index. A light ray transforms into another ray, so one defines a transformation of lines into lines that verifies a known relation, namely, Malus's theorem. If we, with Bruns, are content to look for those transformations of lines into lines that satisfy this important theorem then we are led to write down six conditions (viz., the Malus conditions) that the four functions that define that transformation must verify. They express the idea that a certain quantity:

$$n\,(m\,dx + p\,dy + q\,dz) - N(M\,dX + P\,dY + Q\,dZ)$$

must be a total differential. In paragraph I of the present paper, we shall establish this important result by employing the methods of *Das Eikonal* and simplifying it at only two or three points.

The condition that is imposed, independently of the intermediary media, is certainly satisfied for transformations of light rays. However, the converse is not perhaps exact: It is not possible to affirm that a transformation of lines to lines that verifies Malus's theorem is optically realizable. Meanwhile, that necessary condition suffices for the study of the point-by-point aplanatism of two surfaces or two spaces, and permits one to establish this essential result: The point-by-point transformation of two aplanatic volumes is a similitude.

It is interesting to arrive at these conclusions by a different path, namely, by looking for an integral invariant that is attached to light rays, when considered as trajectories, and to transform it into another one geometrically by the methods of Poincaré, in some way that is attached to the trajectories and independent of the motion. This procedure, which was pointed out by Hadamard, will lead to the consideration of the invariant:

$$n^2 \cos\theta\, ds\, d\omega .$$

That invariant gives the ratio of similitude to the field in the case of point-by-point aplanatism, and it then appears that this ratio will not be different from 1 when the extreme media are identical. The quantity $n^2 \cos\theta\, ds\, d\omega$ intervenes, moreover, in an important optical theorem – viz., Straubel's theorem – which is used quite a bit nowadays.

It seems necessary to us to give that theorem a simple and rigorous proof and to point out that it can be effortlessly generalized to the case in which the rays are replaced with the bicharacteristics of certain partial differential equations that are analogous to the equation:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}-n^2\frac{\partial^2 v}{\partial t^2}=0.$$

I.

THE MALUS CONDITIONS AND "DAS EIKONAL."

Malus's theorem (and, more generally, that of Thomson and Tait for trajectories in dynamics) expresses the idea that light rays that are normal to one surface will again be normal to another surface after refraction. One can, with Bruns ([1]), propose to study the transformation of lines into lines, such that a congruence of normals transforms into another congruence of normals. The problem, thus posed in full generality, admits a very simple solution that was presented in the paper cited.

Let x, y, z be the coordinates of a point on a surface S, when the are referred to three rectangular axes, and let m, p, q be the direction parameters of a line D that passes through that point. The necessary and sufficient condition for the lines D to form a congruence of normals is that m, p, q must be three functions of two parameters that define the position of the point (x, y, z) on the surface, such that:

$$m\,dx + p\,dy + q\,dz$$

is a total differential ([2]). Bruns chose the surface (S) to be the yz-plane, and $y = k$, $z = k$ to be the parameters. Moreover, he supposed that h and k were functions of p, q. The condition is then that $h\,dp + k\,dq$ must be a total differential ([2]); i.e.:

$$\frac{\partial h}{\partial q}=\frac{\partial k}{\partial p}.$$

Let dF be the differential of a function of four variables h, k, p, q; with him, we agree to denote:

$$dF = F_1\,dh + F_2\,dk + F_3\,dp + F_1\,dq,$$

and we let the symbol $(F\,G)_{ij}$ denote the determinant:

([1]) *Das Eikonal*, Abhandlungen der Sächs. Gesellsch, v. **21**, 1895.

([2]) DARBOUX, *Théorie des surfaces*, t. II, pp. 274.

$$(F\,G)_{ij} = \begin{vmatrix} F_i & F_j \\ G_i & G_j \end{vmatrix}.$$

We call any transformation of lines to lines that preserves normal congruences a *Malus transformation*. Then, let there be three rectangular axes $Oxyz$ in a first space that are referred to the lines (h, k, p, q) and three other $O'XYZ$ in a second space that are referred to the lines (H, K, P, Q).

Suppose, then, that the transformation is defined by the equations:

$$H = A(h, k, p, q), \qquad P = C(h, k, p, q),$$
$$K = B(h, k, p, q), \qquad Q = D(h, k, p, q),$$

and is reversible, so $\dfrac{D(H,K,P,Q)}{D(h,k,p,q)} \neq 0$.

When h and k are functions of p and q, H, K, P, Q will be functions of two parameters p and q. Under that hypothesis, we seek the condition for $H\,dP + K\,dQ$ to be a total differential by a direct calculation. Upon using the notations that we just agreed upon, and setting:

$$dh = h_1\,dp + h_2\,dq,$$
$$dk = k_1\,dp + k_2\,dq,$$

we get:

$$\begin{aligned} H\,dP + K\,dQ = &\ [(AC_1 + BD_1)\,h_1 + (AC_2 + BD_2)\,h_2 + (AC_3 + BD_3)\,h_3]\,dp \\ &+ [(AC_1 + BD_1)\,k_1 + (AC_2 + BD_2)\,k_2 + (AC_3 + BD_3)\,k_3]\,dq, \\ = &\ a\,dp + b\,dq. \end{aligned}$$

That quantity will be an exact differential if:

$$\frac{\partial a}{\partial q} = \frac{\partial b}{\partial p},$$

which gives, after some reductions:

$$\begin{aligned} (1) \qquad 0 = &\ (h_1\,k_2 - h_2\,k_1)\,[(AC)_{12} + (BD)_{12}] + h_1[(AC)_{14} + (BD)_{14}] + h_2\,[(AC)_{31} + (BD)_{31}] \\ &+ k_1\,[(AC)_{24} + (BD)_{24}] + k_2[(AC)_{32} + (BD)_{32}] + [(AC)_{34} + (BD)_{34}]\,. \end{aligned}$$

We propose to seek the transformation that will make the condition:

$$\frac{\partial h}{\partial q} = \frac{\partial k}{\partial p}$$

imply that. We therefore take:

$$h = \frac{\partial \theta}{\partial p}, \qquad k = \frac{\partial \theta}{\partial q},$$

$$h_1 = \frac{\partial^2 \theta}{\partial p^2}, \qquad k_2 = \frac{\partial^2 \theta}{\partial q^2}, \qquad h_2 = k_1 = \frac{\partial^2 \theta}{\partial q \, \partial p}.$$

Since θ is an absolutely arbitrary function of p and q, the quantities $h_1 k_2 - h_2 k_1$, h_1, h_2, k_2 can be considered to be independent variables. In order for $H\,dP + K\,dQ$ to continue to be a total differential, it is then necessary and sufficient that:

$$(2) \qquad \begin{cases} (AC)_{12} + (BD)_{12} = 0, \\ (AC)_{14} + (BD)_{14} = 0, \\ (AC)_{23} + (BD)_{23} = 0, \\ (AC)_{34} + (BD)_{34} = 0, \\ (AC)_{13} + (BD)_{13} = (AC)_{24} + (BD)_{24} = E. \end{cases}$$

The six conditions thus determined are called *the first Malus conditions.* They express the idea that the expression (1) reduces to:

$$(k_1 - h_2)\, E = 0;$$

i.e., that if $h\,dp + k\,dq$ is a total differential then $H\,dP + K\,dQ$ is another one, and conversely, if we suppose that $E \neq 0$ and that $H\,dP + K\,dQ$ is a total differential then $h\,dp + k\,dq$ will be another one. The latter property is expressed by six new conditions that are called the *second Malus conditions*, which are, in turn, consequences of the ones that we already wrote down. We propose to look for them: In order to do that, consider h, k, p, q to be functions of H, K, P, Q that are defined by:

$$H = A(h, k, p, q),$$
$$K = B(h, k, p, q),$$
$$\dots\dots\dots\dots,$$

$$(3) \qquad \begin{cases} dH = A_1 dh + A_2 dk + A_3 dp + A_4 dq, \\ dK = B_1 dh + B_2 dk + B_3 dp + B_4 dq, \\ dP = C_1 dh + C_2 dk + C_3 dp + C_4 dq, \\ dQ = D_1 dh + D_2 dk + D_3 dp + D_4 dq. \end{cases}$$

Solve this system of equations for dh, dk, dp, dq. It is good to remark that by virtue of identities (2):

$$(BCD)_{234} = EC_3 .$$

Indeed:

$$(BCD)_{234} = - C_2(BD)_{34} + C_2(BD)_{24} - C_4(BD)_{23} ;$$

i.e.:

$$\begin{aligned} (BCD)_{234} &= E_2(AC)_{34} - C_3(AC)_{24} + C_4(AC)_{23} + EC_3 , \\ &= (ACC)_{234} + EC_3 . \end{aligned}$$

Likewise, one will have:

$$(ACD)_{234} = -ED_3\,, \quad (ABD)_{234} = -EA_3\,, \quad (ABC)_{234} = -EB_3\,, \quad \ldots,$$

and finally:

$$\begin{aligned}(ABCD)_{1234} &= E\,(A_1C_3 + B_1D_3 - A_3C_1 - B_3D_1)\\ &= E\,[(AC)_{13} + (BD)_{13}]\\ &= E^2.\end{aligned}$$

In particular, one sees that since E is the square root of the functional determinant of the transformation, it is never zero. We can therefore always solve the linear equations (3) for dh, dk, dp, dq, which gives:

$$\begin{aligned}dh &= \frac{1}{E}(C_3\,dH + D_3\,dK - A_3\,dP - B_3\,dQ),\\ dk &= \frac{1}{E}(C_4\,dH + D_4\,dK - A_4\,dP - B_4\,dQ),\\ dp &= \frac{1}{E}(C_1\,dH + D_1\,dK - A_1\,dP - B_1\,dQ),\\ dq &= \frac{1}{E}(C_2\,dH + D_2\,dK - A_2\,dP - B_2\,dQ).\end{aligned}$$

A_1 , B_1 , …, D_4 are functions of H, K, P, Q, by the intermediary of h, k, p, k. Writing down the first four Malus conditions, we get:

$$\begin{aligned}(AB)_{13} + (AB)_{24} &= 0,\\ (AD)_{13} + (AD)_{24} &= 0,\\ (BC)_{13} + (BC)_{24} &= 0,\\ (CD)_{13} + (CD)_{24} &= 0.\end{aligned}$$

The last two:

$$(AC)_{13} + (BD)_{13} = (AC)_{24} + (BD)_{24} = \sqrt{\frac{D(H,K,P,Q)}{D(h,k,p,q)}}$$

become:

$$\frac{1}{E^2}[(AC)_{13} + (AC)_{24}] = \frac{1}{E^2}[(BD)_{13} + (BD)_{24}] = \sqrt{\frac{D(h,k,p,q)}{D(H,K,P,Q)}} = \pm\,\frac{1}{E}\,.$$

Therefore:

$$(AC)_{13} + (AC)_{24} = (BD)_{13} + (BD)_{24} = E.$$

The question of sign introduces no ambiguity, because the identity must still be true when $A = h$, $K = k$, $P = p$, $Q = q$, and E are never zero.

These six conditions are the second Malus conditions. Let the symbol $(u\,,v)$ denote the operation $(u, v)_{13} + (u, v)_{24}$ that is performed on the function u, v, h, k, p, q. With that system of notation, the conditions become:

(4) $(A\ B) = 0,\qquad (A\ D) = 0,\qquad (B\ C) = 0,\qquad (C\ D) = 0,\qquad (A\ C) = (B\ D) = E.$

If v is a function that is composed from h, k, p, q by the intermediary of $\varphi, \psi, \theta, \ldots$ then one will have:

$$(u\ v) = \frac{\partial v}{\partial \varphi}(u,\varphi) + \frac{\partial v}{\partial \psi}(u,\psi) + \ldots$$

Having posed that, start with the identity:

$$[u\ (v\ w)] + [v\ (w\ u)] + [w\ (u\ v)] = 0,$$

and make, for example:

$$u = A, \qquad v = B, \qquad w = C.$$

By virtue of relations (4), we get:

$$(A, 0) + (B, E) + (C, 0) = 0;$$

i.e., $(B, E) = 0$. One likewise finds that:

$$(A, E) = (B, E) = (C, E) = (D, E) = 0.$$

Due to the identity:

$$(A, E) = \frac{\partial E}{\partial A}(A,A) + \frac{\partial E}{\partial B}(A,B) + \frac{\partial E}{\partial C}(A,C) + \frac{\partial E}{\partial D}(A,D),$$

the condition $(A, E) = 0$ can be written:

$$\frac{\partial E}{\partial C} = 0.$$

Likewise:

$$\frac{\partial E}{\partial A} = \frac{\partial E}{\partial B} = \frac{\partial E}{\partial C} = \frac{\partial E}{\partial D} = 0.$$

The function E of h, k, p, q, when considered to be a function of H, K, P, Q, is independent of these variables; i.e., it is therefore a constant in H, K, P, Q, and in turn, in h, k, p, q.

Under a first transformation that makes the lines of a medium Ω_1 correspond to those of a medium Ω_2, in each of which rectangular axes have been chosen, one gets, upon expressing the Malus conditions, a constant E_{12} . Likewise, one will get a constant E_{23} as a result of the passage from Ω_2 to another medium Ω_3 . That double transformation is obviously equivalent to a transformation that makes a line in the medium Ω_1 go to a line in the medium Ω_3 . That transformation will give us a new constant E_{13} , and since, in general:

$$E^2 = \frac{D(H, K, P, Q)}{D(h, k, p, q)},$$

the rule for the multiplication of functional determinants will give:

$$E_{13}^2 = E_{12}^2 E_{23}^2,$$

and as a result, with no ambiguity, for the reasons that were given already, one will get:

$$E_{13} = E_{12}\, E_{23}\,.$$

It then results that the constant E is independent of the choice of axes. To change the axes is to perform a certain displacement of space with respect to the original axes. Now, any displacement can be obtained from two symmetries with respect to a plane; i.e., two reflections. One will then pass from the space in which one starts to the space that is referred to the new axes by first passing to the same space, referred to the old ones, which gives the constant E, and then to the space that is symmetric with respect to a certain plane that one can choose to be the yz-plane, which gives the constant – 1, … One will obtain, by definition:

$$E(-1)(-1) = E.$$

The constant E is therefore characteristic of the transformation. We agree to assign a number n_α to each space Ω_α, in such a way that the passage from the space Ω_α to the space Ω_β is characterized by n_α / n_β:

$$E(\Omega_\alpha, \Omega_\beta) = \frac{n_\alpha}{n_\beta}\,.$$

That notation exhibits the property of the number E that:

$$E(\Omega_\alpha, \Omega_\beta)\, E(\Omega_\beta, \Omega_\gamma) = E(\Omega_\alpha, \Omega_\gamma).$$

If one considers the six Malus conditions to be partial differential equations that define the unknown functions H, K, P, Q then physics will give a solution with arbitrary functions (e.g., refraction from a sequence of arbitrary surfaces). It is, moreover, easy to solve the problem of the search for the functions H, K, P, Q completely. Indeed, the Malus conditions express the idea that the quantity:

$$n\,(p\,dh + q\,dk) + N\,(H\,dP + Q\,dK)$$

is the total differential of a function S of h, k, p, q. Suppose that the equations:

$$\text{(5)} \qquad \begin{cases} P = C(h, k, p, q), \\ Q = D(h, k, p, q), \end{cases}$$

are soluble for C and D, while $(CD)_{23} \neq 0$. If one replaces p and q in $S(h, k, p, q)$ with their values that one infers from the two equations (5) then S will become a function $E(h, k, p, q)$, and one will obviously have:

$$dE = n\,(p\,dh + q\,dk) + N\,(H\,dP + Q\,dK)$$

and as a result:

$$np = \frac{\partial E}{\partial h}, \qquad nq = \frac{\partial E}{\partial k}, \qquad NH = \frac{\partial E}{\partial P}, \qquad NK = \frac{\partial E}{\partial Q}.$$

Conversely, let E be a function of h, k, P, Q that is chosen in such a way that these equations are soluble for H, K, P, Q. The functions $H = (h, k, p, q)$, … that one infers define a transformation, and if $(CD)_{34} \neq 0$ then that transformation will answer to the Malus conditions.

That solution to the problem has been, in some way, known for some time: Indeed, consider h, k to be variables that represent two of the coordinates x and y of a point, while p, q, -1 represent the parameters of a plane. If one takes X, Y, Z, P, Q to be the coordinates of a point and a plane that passes through that point, which are functions of x, y, z, p, q:

$$\text{(6)} \qquad \begin{cases} X = A(x, y, p, q), & P = C(x, y, p, q), \\ Y = B(x, y, p, q), & Q = D(x, y, p, q), \\ Z = +\dfrac{n}{N}z - E(x, y, p, q), & \end{cases}$$

that are chosen in such a manner that:

$$N\,(P\,dx + Q\,dy) + dE\,(x, y, p\,q) = n\,(p\,dx + q\,dy)$$

then one defines a contact transformation of (x, p) by formulas (6) ([1]), and it is obvious that if $p\,dx + q\,dy$ is a total differential then $P\,dX + Q\,dY$ will be another one. A well-known theorem that was proved by Sophus Lie tells us that the functions A, B, P, Q must verify the partial differential equations:

$$(A, B) = (A, D) = (B, C) = (C, D) = 0,$$

$$(C, A) = (D, B) = -\frac{n}{N}.$$

These are indeed the six sufficient conditions. The preceding proof shows that they are necessary.

The function $E(h, k, P, Q)$ that generates the transformation is called the *eikonal.* It is obvious that one can obtain 16 different eikonals, because the replacement of $H\,dP$ with $P\,dH$, for example, does not alter the integrability of the quantity that was originally considered. Bruns has proved (but this will depart from the context that we have imposed) that for a given transformation there will always exist at least four eikonals; i.e., that of the four quantities $(CD)_{34}$, $(CD)_{14}$, $(CD)_{23}$, $(CD)_{12}$, for example, at most three of them can be zero simultaneously.

([1]) GOURSAT, *Equations aux dérivées partielles,* pp. 281.

II.

THE NECESSARY CONDITIONS FOR APLANATISM

The Malus transformations are therefore such that the quantity:

$$n\,(p\,dh + q\,dk) - N\,(P\,dH + Q\,dK)$$

is an exact differential, or further that:

$$n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ)$$

is one.

Instead of pursuing, as Bruns did, the search for the conditions of aplanatism by the brute-force application of the Malus conditions, it seems easier to us to simply express the idea that the quantity:

$$n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ) \tag{7}$$

is an exact differential. That method has the advantage of permitting us to begin the problem of aplanatism, without having to elaborate upon the methods of the eikonal when one limits one's study to the optically-realizable transformations. For these transformations, the difference (7) is the differential of the optical path, as is easy to verify.

For the moment, refer the various points of space to a single rectangular system of axes. One knows that a ray (m, p, q) that refracts at a point of a surface where the normal has the direction parameters α, β, γ will take on a new direction MPQ that is defined:

$$\begin{aligned} nm - NM &= \lambda\alpha, \\ np - NP &= \lambda\beta, \\ nq - NQ &= \lambda\gamma, \end{aligned}$$

(n, N are the indices of the successive media, and the positive sense of each ray is, for example, opposite to the sense of propagation of the wave).

Let u_i , v_i , w_i be the coordinates of a point that passes from the medium with index n_i to the medium with index n_{i+1} , while m_i , p_i , q_i are the direction parameters of a ray that begins in the medium of index n_i . Let ρ_i denote the distance between the point $(u_{i-1}, v_{i-1}, w_{i-1})$ and the point (u_i , v_i , w_i), so:

$$u_i = u_{i-1} + m_i\,\rho_i\,, \qquad \ldots$$

and in turn:

$$du_i = du_{i-1} + dm_i\,\rho_i + + m_i\,d\rho_i\,.$$

It then results that:

$$n_i\,(m_i\,du_i + p_i\,dv_i + q_i\,dw_i) - n_i\,(m_i\,du_{i-1} + p_i\,dv_{i-1} + q_i\,dw_{i-1}) = n_i\,d\rho_i\,.$$

If we then let (x, y, z), (m, p, q) be a point and a ray in the first medium n, while (X, Y, Z), (H, P, Q) is the refracted ray in the final medium N then one will have, upon summing all of the equalities that were obtained by making $i = 1, 2, \ldots$:

$$n\,(m\,du + p\,dv + q\,dw) - N\,(M\,dX + P\,dY + Q\,dZ) = \sum dn_i\,\rho_i\,.$$

One can interpret this formula geometrically: Let A, B be the extremities of the optical path, let AA', BB' be the segments whose projections onto the axes are dx, dy, dz, dX, dY, dZ, and let $ASBS'$ be the light ray. The equality is equivalent to:

$$n\,AA'\cos(AA', AS) - N \cdot BB'\cos(BB', BS') = d(ns).$$

Therefore, if one refers the elements of the first medium to three arbitrary rectangular axes and the elements of the second one to three others then one will always have:

$$n\,(m\,du + p\,dv + q\,dw) - N\,(M\,dX + P\,dY + Q\,dZ) = d\sum ns. \tag{7}$$

The right-hand side is the differential of the optical path. The eikonal function is therefore nothing but the optical path between the point (x, y, z) and the point (X, Y, Z), or, with the notations of the first chapter, between the point $(0, h, k)$ and $(0, H, K)$.

Conversely, if that quantity is a total differential then the transformation will be a Malus transformation. We limit ourselves to showing that one indeed finds the six conditions (4) by expressing the idea that:

$$n\,(p\,dh + q\,dk) - N\,(P\,dH + Q\,dK) = d\sum ns;$$

i.e., that:

$$[np - N\,(CA_1 + DB_1)]\,dh + [nq - N\,(CA_2 + DB_2)]\,dk$$
$$- N\,(CA_3 + DB_3)\,dp - N\,(CA_4 + DB_4)]\,dq = d\sum ns\,.$$

In order for the left-hand side to be a differential, it is necessary and sufficient that:

$$\begin{gathered}(AC)_{12} + (BD)_{12} = 0,\\ n - N\,[(AC)_{13} + (BD)_{13}] = 0,\\ (AC)_{14} + (BD)_{14} = 0,\\ (AC)_{23} + (BD)_{23} = 0,\\ n - N\,[(AC)_{24} + (BD)_{24}] = 0,\\ (AC)_{34} + (BD)_{34} = 0.\end{gathered}$$

These are the first Malus conditions. Moreover, we perceive the value of the quantity E that we were led to consider to be the quotient of two numbers that characterize the extreme media. These numbers, which were denoted by n_α in the first paragraph, are proportional to the indices of refraction, and E is the index of the passage from the first medium to the extreme medium.

We thus obtain the Malus conditions by a process that sheds light upon the fundamental result of the synthesis of Bruns for an optically-realizable transformation, namely, that:

$$n\,(m\,dx + p\,dy + q\,ds) - N\,(M\,dX + P\,dY + Q\,dZ) \tag{8}$$

is a total differential. The converse is quite easy to establish: If the quantity (8) is an exact differential then the transformation is will be Malus transformation. The advantage of the proofs in the paper that was cited above consists in the fact that it shows that one indeed has a necessary condition for the transformation – whether optically-realizable or not – to preserve the normal congruences.

For the study that we shall carry out, which is the search for the conditions that insure point-by-point aplanatism between two manifolds that are two or three-dimensional, it seems more convenient to us to express the idea that the quantity (8) is a total differential without any recourse to the Malus conditions explicitly.

For example, suppose that the points of a space ω correspond aplanatically to those of the space Ω. The point-by-point transformation thus defined is obviously a homographic transformation. We distinguish two cases, according to whether it transforms the plane at infinity in one of the media to a plane at a finite distance in the other – i.e., that it is general – or whether it transforms that plane to the plane at infinity.

In the latter case, we say ([1]) that it is *affine* (from the German *affine*, which is currently employed in the preceding sense). Therefore, first suppose that the points of a certain plane (p) in the space ω correspond to points at infinity in Ω, then take the plane in ω to be the zy-plane, and similarly choose the plane (P) that corresponds to the plane at infinity in (ω) to be the YZ-plane. The point at infinity in the perpendicular direction to (p) corresponds to a point O' in Ω that is situated at (P), and likewise there is a point O of (p) whose correspondent in Ω is at infinity in the direction perpendicular to (P). Choose the z-axes to be the perpendiculars Oz, $O'Z$ to the planes p and P, resp. These two lines will correspond under the following transformation: Two rectangular planes that pass through $O'Z$ correspond to two planes that pass through Oz, and when the first two turn around $O'Z$, the other two form the pairs of an involution around Oz. Choose the xz and yz-planes to be the pair of two rectangular planes to that involution, so their correspondents will be two rectangular planes in the space (Ω). We take the XZ-plane to be the correspondent to xz and the YZ-plane to the correspondent to yz.

The equations of the transformation, when referred to these axes, will take the following form:

$$X = ax, \qquad Y = by, \qquad Z = ct, \qquad T = z.$$

We take x, y, z to be the coordinates x, y, 0, 1 of the point that is situated in the xy-plane. In the space Ω, it will correspond to the point (ax, by, c¸ 0) with:

$$\frac{M}{ax_1} = \frac{P}{by_1} = \frac{Q}{C} = \lambda\,,$$

with:

([1]) D'OCAGNE, *Cours de l'École Polytechnique*, 1912-1913.

$$\lambda = \frac{1}{\sqrt{a^2 x_1^2 + b^2 y_1^2 + c^2}}.$$

The point (m, p, q, 0) corresponds to the point (am, bp, 0, q) in the space Ω. We take X, Y, Z to be am / q, bp / q, 0, resp. It is necessary that:

$$n(m\,dx_1 + p\,dy_1) - N\left[\lambda a\,x_1\,d\left(\frac{am}{q}\right) + \lambda b\,y_1\,d\left(\frac{bp}{q}\right)\right]$$

must be a total differential in x, y, m, p. Set $m = qu$, $p = qv$, and in turn:

$$q = \frac{1}{\sqrt{1+u^2+v^2}}.$$

The quantity:

$$n(m\,dx_1 + p\,dy_1) - N\,\lambda\,(a^2\,x_1\,du + b^2\,y_1 1\,dv)$$

will also be a total differential in x, y, u, v. In order for that to be true, it is necessary that certain conditions must be satisfied; for example:

$$n\frac{\partial q}{\partial v}u = -N\,b^2\,y_1\,\frac{\partial \lambda}{\partial x_1},$$

$$n\frac{\partial q}{\partial u}v = -N\,a^2\,x_1\,\frac{\partial \lambda}{\partial y_1}.$$

Since the quantities in the left-hand side contain neither x_1 nor y_1, while those on the right-hand side contain neither u nor v, there can be no identity unless each of the sides is constant:

$$n\frac{\partial q}{\partial v}u = c_1,$$

$$n\frac{\partial q}{\partial u}v = c_2,$$

which is clearly impossible.

Therefore, a general transformation cannot be realized by a sequence of refractions. Suppose then that the points at infinity correspond to each other in the media (ω) and (Ω); i.e., that the transformation is affine. Choose the origins O and O' to be two corresponding points, and draw three rectangular planes through O' that determine a trihedron $O'\alpha'\beta'\gamma'$. The points at infinity α', β', γ' on each of these edges for a triangle that is conjugate to the umbilical I'. Let α, b, γ denote the points in the first medium that correspond to α', β', γ'; these points are conjugate with respect to the transformation (I) of I'. In order for the trihedron $O\alpha\beta\gamma$ to be rectangular, it is necessary that they also be conjugate to the umbilical J in that medium. These are the summits of the common conjugate triangle to these two conics. One thus sees that if one is given two corresponding points O and O' then one can find two tri-rectangular trihedra that have these points for their summits and whose edges will correspond to each other point-by-point. The first one $Oxyz$ will be chosen to be the coordinate trihedron in the first medium, and the second one, to be $OXYZ$ in the other one.

[In the case in which the two umbilicals correspond (by similitude), one of these two trihedra can be chosen arbitrarily.]

The equations of the transformation then become:

$$X = ax, \qquad Y = by, \qquad Z = cz.$$

The light ray that has the direction m, p, q after refraction is parallel to the direction am, bp, cq:

$$\frac{M}{am} = \frac{P}{bp} = \frac{Q}{cq} = \lambda,$$

with:

$$\lambda = \frac{1}{\sqrt{a^2 m^2 + b^2 p^2 + c^2 q^2}}.$$

We take X, Y, Z to be ax, by, cz, resp. The quantity:

$$n\,(m\,dx + p\,dy + q\,dz) - N\lambda\,(a^2\,m\,dx + b^2\,p\,dy + c^2\,q\,dz)$$

is a total differential in x, y¸ z, p, q. This can happen only if:

$$n - N\lambda\,a^2 = 0,$$
$$n - N\lambda\,b^2 = 0,$$
$$n - N\lambda\,c^2 = 0,$$

so

$$a = b = c = \frac{n}{N}.$$

The only transformation that makes a unique point-image correspond to any point-object is a similitude for which the ratio of similitude is equal to the inverse of the index of passage from one medium to the other; i.e., since these media are, in general, identical, to *one*. Bruns ([1]) arrived at the same conclusions by applying the Malus conditions to the transformation. He does not seem to have remarked (although this result is contained in *Das Eikonal*, in principle) that the magnification can generally be different from *one*. Indeed, having examined the preceding case, he said: "Due to its simplicity, it is not necessary to pursue the study any further, especially since in practical optics it will not produce geometrical representations that are similar to the body." As Hadamard ([2])

([1]) *Das Eikonal*, Abhandl. der Sächs. Gesellsch., v. XXI, pp. 370.
([2]) C. R. Acad. Sc., 14 March 1898.

observed, if this result could be obtained with a magnification that is different from *one* then that would constitute the more satisfying solution to the problem of dioptrics, and he also took care to observe that this would not be true in general.

It is from this result that many others are contained, in substance, in *Das Eikonal*, but the author neglected to exhibit them.

For example, seek the condition for a transformation to make the points of two surfaces s and S correspond astigmatically. Suppose that the coordinates of the points of each of the two surface are functions of the same parameters α, β, in such a way that a value of α, β will correspond to two conjugate points.

We take the x, y, z, X, Y, Z in the quantity:

$$n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ) \tag{8}$$

to be the coordinates x, y, z, X, Y, Z – which are functions of α, β – of the points where the light ray meets the conjugate surfaces. The difference (8) will of the form:

$$A\,d\alpha + B\,d\beta,$$

where A is a function of p, q, α, β, and likewise for B. However, since $A\,d\alpha + B\,d\beta$ is a total differential, A and B will be functions of only α and β, since:

$$\frac{\partial A}{\partial p} = 0, \qquad \frac{\partial A}{\partial q} = 0, \qquad \frac{\partial B}{\partial p} = 0, \qquad \frac{\partial B}{\partial q} = 0;$$

as a result:

$$n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ) = d\psi(\alpha, \beta). \tag{9}$$

It is therefore necessary that M, P, Q must verify the equations:

$$\left\{\begin{aligned} M^2 + P^2 + Q^2 &= 1, \\ n\left(m\frac{\partial x}{\partial \alpha} + p\frac{\partial y}{\partial \alpha} + q\frac{\partial z}{\partial \alpha}\right) - N\left(M\frac{\partial X}{\partial \alpha} + P\frac{\partial Y}{\partial \alpha} + Q\frac{\partial Z}{\partial \alpha}\right) &= \frac{\partial \psi}{\partial \alpha}, \\ n\left(m\frac{\partial x}{\partial \beta} + p\frac{\partial y}{\partial \beta} + q\frac{\partial z}{\partial \beta}\right) - N\left(M\frac{\partial X}{\partial \beta} + P\frac{\partial Y}{\partial \beta} + Q\frac{\partial Z}{\partial \beta}\right) &= \frac{\partial \psi}{\partial \beta}. \end{aligned}\right. \tag{10}$$

One can take the function ψ arbitrarily, and the three equations (10) will then determine M, P, Q as functions of p, q, α, β. The transformation thus defined will be a Malus transformation that answers the question.

One then sees, with Bruns, that the problem of the point-by-point correspondence between two surfaces involves an infinitude of solutions, even though the correspondence between the two surfaces is given. We shall now interpret the result that we found geometrically.

Let m, M be two conjugate points, while mt, MT are tangents to two conjugate curves whose arc lengths are s, S:

$$dx = ds\cos(Ox, mt), \qquad \ldots,$$
$$dX = dS\cos(OX, MT), \qquad \ldots$$

Let $maMA$ be a ray that passes through m and M:

$$m\,dx + p\,dy + q\,dz = ds\cos(mt, ma).$$

Consequently, the identity (9) expresses the idea that:

$$n\,ds\cos(mt, ma) - N\,dS\cos(MT, MA) = d\psi(\alpha, \beta).$$

This is the theorem of Thiesen ([1]), which was established by Fatou in the case of approximate aplanatism. The cosine of the angle between the incident ray and a curve in the surface s is linearly related to the cosine of the angle between the refracted angle and the conjugate curve. Moreover, we see that the surfaces s and S are given in this relation, as well as the correspondence between the various points of them, so the coefficients of the cosines are known, and the constant term is a linear function of $d\alpha$ and $d\beta$:

$$\frac{\partial\psi}{\partial\alpha}d\alpha + \frac{\partial\psi}{\partial\beta}d\beta,$$

in which ψ is an arbitrary function of α and β.

Bruns arrived at this theorem without stating it by choosing curves $\psi(\alpha, \beta)$ = const. on the surface, when the function ψ is imposed, for example, by a given optical system.

In order for two given surfaces s and S to be aplanatic, it is necessary that one must be able to determine a family of optically-conjugate curves on each of them; i.e., they are images of each other such that the cosine of the angle between the light ray that passes through a point of one of them is proportional to the cosine of the angle between the refracted ray and the conjugate; the proportionality ratio is $\frac{N}{n}\cdot\frac{dS}{ds}$.

The condition is necessary. Indeed, apply Thiesen's theorem to the curves $\psi(\alpha, \beta)$ = const., $d\psi = 0$, and as a result:

$$n\,ds\cos(mt, ma) - N\,dS\cos(MT, MA) = 0.$$

It is obviously not sufficient.

As Fatou ([2]) has justifiably remarked, the curves ψ cannot be arbitrary when the transformation is given; it is easy to give examples of this situation.

([1]) CZAPSKI, *Grundzüge der Theorie der optischen Instrumenten*, pp. 127.
([2]) Bulletin astronomique, t. XXX, May 1913, pp. 246.

III.

INTEGRAL INVARIANTS AND THE NECESSARY CONDITIONS FOR APLANATISM IN A MEDIUM WITH VARIABLE INDEX.

The fact that the quantity:

$$n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ)$$

is a total differential has a very general character. In a medium with a variable index $n = \varphi(x, y, z)$, the curves that are analogous to the light rays are the extremals of a certain integral:

$$I = \int_A^B n\,ds\,,$$

and if x, y, z, m, p, q denote the coordinates of a point A and the tangent at A to the extremal, and X, Y, Z, M, P, Q are those of B and the tangent to B then its variation is, as one knows, equal to:

$$\delta I = n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ).$$

As one knows, the value of the integral I is a function of $x, y¸ z, X, Y, Z$.

If an extremal path is refracted then δI does not take on a value that is less than the one that is given by the equation; the proof of this is what permits us to write formula (7). It then results immediately that if the object and the image are embedded in media with constant indices (which is always true in practice) then no matter whether the intermediary media do or do not have variable indices, the point-by-point aplanatism of the two multiplicities of dimension two or three will be possible only under the conditions that were already found. Indeed, in Part II, we used only the property of the quantity:

$$n\,(m\,dx + p\,dy + q\,dz) - N\,(M\,dX + P\,dY + Q\,dZ)$$

that it must remain a total differential.

The identity that exists between refraction – i.e., the passage from one medium to a medium with a different index across a discontinuity – and the extremals that give the transit of light for a passage that is similarly effected with no discontinuity, invites one to seek whether certain properties of these curves are not preserved under refraction, and do not extend to systems of light rays. For example, the extremals are defined by canonical equations that possess integral invariants. One can propose to seek the ones that are preserved under refraction. We limit our study to the ones that were pointed out by Hadamard ([1]), which seem to be the simplest and most important ones.

Consider the function:

([1]) C. R. Acad. Sc., 14 March 1898.

$$H = \frac{1}{2}\frac{u^2+v^2+w^2}{n^2},$$

where u¸ v, w are three independent variables, and n is a given function of x¸ y, z. The equations:

$$(8) \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial u}, & \dfrac{dy}{dt} = \dfrac{\partial H}{\partial v}, & \dfrac{dz}{dt} = \dfrac{\partial H}{\partial w}, \\ \dfrac{du}{dt} = -\dfrac{\partial H}{\partial x}, & \dfrac{dv}{dt} = -\dfrac{\partial H}{\partial y}, & \dfrac{dw}{dt} = -\dfrac{\partial H}{\partial z}, \end{cases}$$

where t represents time, are the differential equations of motion of a point (u, v, w, x, y, z) that moves in six-dimensional space. The initial position M_0 of that moving point defines its trajectory completely. The coordinates x¸ y, z, u, v, w, and any function $f(u, v, w, x, y, z)$ of them are functions of time. In particular, H is constant. Indeed:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt} + \frac{\partial H}{\partial u}\frac{du}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt} + \frac{\partial H}{\partial w}\frac{dw}{dt} = 0.$$

Therefore, if one chooses the initial coordinates in such a way that from the start one has:

$$n^2 + v^2 + w^2 = n^2$$

then that equality will persist at any point of the trajectory. It is then an extremal of the integral:

$$\int n\sqrt{x'^2 + y'^2 + z'^2}\,dt = 0.$$

Indeed, the equations of the latter are:

$$(8') \qquad \begin{cases} \dfrac{d}{dt}\left(n\dfrac{x'}{\sqrt{x'^2+y'^2+z'^2}}\right) - \dfrac{\partial n}{\partial x}\sqrt{x'^2+y'^2+z'^2} = 0, \\ \dfrac{d}{dt}\left(n\dfrac{y'}{\sqrt{x'^2+y'^2+z'^2}}\right) - \dfrac{\partial n}{\partial y}\sqrt{x'^2+y'^2+z'^2} = 0, \\ \dotfill \end{cases}$$

Now, we have:

$$x' = \frac{u}{n^2}, \qquad y' = \frac{v}{n^2}, \qquad z' = \frac{w}{n^2},$$

$$x'^2 + y'^2 + z'^2 = \frac{u^2+v^2+w^2}{n^2} = \frac{1}{n^2},$$

so equations (8′) then become:

$$\frac{du}{dt} = + \frac{\partial n}{\partial x}\frac{1}{n} = - \frac{\partial H}{\partial x},$$

$$\frac{dv}{dt} = - \frac{\partial H}{\partial y},$$

$$\frac{dw}{dt} = - \frac{\partial H}{\partial z},$$

which proves the stated proposition.

The trajectories whose equations are (8) are attached to integral invariants. Let M be a multiplier for these equations; i.e., a function of x¸ y, z¸ u, v, w that satisfies the linear differential equation:

$$\frac{\partial M}{\partial x}\frac{\partial H}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial H}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial H}{\partial w} - \frac{\partial M}{\partial u}\frac{\partial H}{\partial x} - \frac{\partial M}{\partial v}\frac{\partial H}{\partial y} - \frac{\partial M}{\partial w}\frac{\partial H}{\partial z} = 0.$$

The integral:

$$\int_{E_6} M(x, y, z, u, v, w)\ dx\ dy\ dz\ du\ dv\ dw$$

keeps a constant value, whether one extends it over the points of a six-dimensional space E_6 or the points that are deduced from it by starting on the trajectories that correspond to the arcs that are described during a certain time. We consider the particular invariant $\int_{E_6} dx\ dy\ dz\ du\ dv\ dw$ that is obtained by making $M = 1$, and from it, we deduce another one that is attached to the extremals of the integral that was cited already. Make the change of variables:

$$u = nm\ \alpha,$$
$$v = np\ \alpha,$$
$$w = nq\ \alpha,$$

in which p, q, α are three independent variables, and m is a quantity such that:

$$m^2 + p^2 + q^2 = 1.$$

Equations (8) become:

$$(9)\qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{m}{n}\alpha, & \dfrac{dy}{dt} = \dfrac{p}{n}\alpha, \qquad \dfrac{dz}{dt} = \dfrac{q}{n}\alpha, \\[2ex] \dfrac{d\alpha}{dt} = 0, & \dfrac{dp}{dt} = \dfrac{\alpha}{n^2}\dfrac{\partial n}{\partial y} = \dfrac{p}{n}\dfrac{dn}{dt}, \qquad \dfrac{dq}{dt} = \dfrac{\alpha}{n^2}\dfrac{\partial n}{\partial z} = \dfrac{q}{n}\dfrac{dn}{dt}, \end{cases}$$

where dn / dt is written in place of $\left(\frac{\partial n}{\partial x} m + \frac{\partial n}{\partial y} p + \frac{\partial n}{\partial z} q\right)\frac{\alpha}{n}$, and the integral invariant transforms into:

$$\int_{E_6} \frac{D(u,v,w)}{D(\alpha,\beta,\gamma)}\, dx\, dy\, dz\, d\alpha\, d\beta\, d\gamma.$$

Now:

$$\frac{D(u,v,w)}{D(\alpha,\beta,\gamma)} = n^2\, a^2 \begin{vmatrix} m & p & q \\ \frac{\partial m}{\partial p} & 1 & 0 \\ \frac{\partial m}{\partial p} & 0 & 1 \end{vmatrix}$$

$$= \frac{n^2\alpha^2}{m},$$

by virtue of the relations $m^2 + p^2 + q^2 = 1$, so:

$$m\, \frac{\partial m}{\partial p} + p = 0,$$

$$m\, \frac{\partial m}{\partial q} + q = 0.$$

Under the previously-cited conditions, the integral:

$$\int_{E_6} n^2\alpha^2 dx\, dy\, dz\, d\alpha \frac{dp\, dq}{m}$$

will keep a constant value. The quantity $\frac{dp\, dq}{m}$, which appears here for the first time, represents the elementary surface portion that is cut out from the sphere of radius 1 at the point (m, p, q), so in the sequel we shall denote it by:

$$d\omega = \frac{dp\, dq}{m}.$$

Choose the multiplicity E_6 to be a cylinder whose base is E_5 and whose height is $a < \alpha < b$, where a and b are two constants, so:

$$\int_{E_6} n^2\alpha^2\, dx\, dy\, dz\, d\alpha = \int_{E_5} n^2\, dx\, dy\, dz\, d\omega \int_a^b \alpha^2 d\alpha.$$

The quantity $\int \alpha^2 \, d\alpha$ is a constant, and the integral:

$$\int_{E_5} n^2 \, dx\, dy\, dz\, d\omega$$

is invariant for a system of extremals of the integral $\int n \, ds$.

One can, by a method that was taught for the first time by Poincaré ([1]), and then employed to great profit by Hadamard ([2]), deduce another one from it that extends to the points of a surface in two-dimensional space and to the two-parameter sheaf of rays that issue from these points, which preserves its value when one replaces each original point with the point that is obtained by cutting the corresponding trajectory with a surface in two-dimensional space.

Indeed, consider for the moment, the multiplicity E_5 that is composed of trajectories that issue from the points of an arbitrary four-parameter multiplicity E_4 ; for example, suppose that x, y, z, p, q are functions of four parameter $\alpha, \beta, \gamma, \delta,$ and time. The invariant becomes:

$$(10)\quad \int_{E_5} \left| \frac{D(y,z,p,q)}{D(\alpha,\beta,\gamma,\delta)} \frac{dx}{dt} + \frac{D(z,p,q,x)}{D(\alpha,\beta,\gamma,\delta)} \frac{dy}{dt} + \cdots + \frac{D(x,y,z,p)}{D(\alpha,\beta,\gamma,\delta)} \frac{dq}{dt} \right| \frac{n^2}{m} d\alpha\, d\beta\, d\gamma\, d\delta\, dt \ ,$$

where the notations employed are the usual notations of functional determinants, and $\frac{dx}{dt}$, $\frac{dy}{dt}, \ldots, \frac{dq}{dt}$ must be replaced with the functions of x, y, z, p, q that are given by equations (9). Let I represent the integral:

$$I = \int_{E_4} \left| \frac{D(y,z,p,q)}{D(\alpha,\beta,\gamma,\delta)} \frac{dx}{dt} \right| \frac{n^2}{m} d\alpha\, d\beta\, d\gamma\, d\delta$$

$$= \int_{E_4} \left| \frac{dx}{dt} dy\, dz\, dp\, dq + \frac{dy}{dt} dz\, dp\, dq\, dx + \cdots \right| \frac{n^2}{m} .$$

I is an integral invariant that keeps the same value when one extends it over an *arbitrary* multiplicity E_4 that cuts a sheaf of trajectories.

Indeed, let S and S' be two arbitrary four-parameter multiplicities that bound the sheaf of trajectories E_5 . One passes from the multiplicity E_5 to an infinitely close multiplicity by replacing the small 5-volume that lies between the 4-surface S and 4-surface that is derived from it by moving along the trajectories that start with it that are arcs traversed in the time dt, and by the small 5-volume that is bounded by the 4-surface S' and the infinitely close surface that is obtained in the same fashion. When the integral (10) is

([1]) "Mémoire des trois corps," Acta mathematica, t. XIII, pp. 66.

([2]) "Sur certaines propriétés des trajectoires en Dynamique," Journal de Mathématiques (5), t. III, fasc. 4, 1897.

taken over these two volumes, it will keep the same value. Now, if I and I' are the particular expressions for the integral I that correspond to the multiplicities E_4 – viz., S or S' – then that unique value will be either $I\,dt$ or $I'\,dt$. It will then results that:

$$I = I'.$$

I is therefore an integral invariant.

We deduce another particularly interesting invariant from it by choosing the E_4 multiplicity to be the one that is composed of the various points of a two-dimensional surface and a two-parameter family of rays that issue from it.

The coordinates of a point of the surface S are functions of the two independent variables u and v. A sheaf of rays emanates from each point (u, v) of the surface that is defined by the variables p and q, which are also independent.

When u, v vary slightly, the point (u, v) describes an surface element $d\sigma$ around an arbitrary point M; likewise, when p and q vary slightly, the light ray describes an elementary brush with a summit angle $d\omega$ that one can call the "elementary brush at the point M." If one considers an arbitrary surface S' whose coordinates are functions of two parameters u', v' then a ray (u, v, p, q) cuts that surface at a point (u', v'), and the angle between the tangent to that ray and the tangent to the point (u, v) has the parameters p', q'. There is then a portion of the surface $d\sigma'$ on S' where an elementary brush of summit angle $d\omega'$ begins that corresponds to a portion of the surface $d\sigma$ and the elementary brush $d\omega$.. That surface portion and that brush will be said to *correspond* to the same elements of S. The integral:

$$K = \int n^2 \left[m \frac{D(y,z)}{D(u,v)} + p \frac{D(z,x)}{D(u,v)} + q \frac{D(y,x)}{D(u,v)} \right] du\, dv\, d\omega$$

keeps the same value whether one extends it over the points of a surface S and the rays that emanate from it or over the corresponding elements of an arbitrary surface S'.

Let M be an arbitrary point of S, MA, a light ray m, p, q, and let MN be the normal with direction parameters α, β, γ. One has:

$$\frac{D(y,z)}{D(u,v)} = \alpha\, dv, \ldots$$

Furthermore, letting θ be angle between MA and MN, the integral K can be written:

$$K = \int n^2 \cos\,\theta\, d\omega\, d\sigma.$$

The quantity under the integral sign is invariant under a change of axes; the same is true for refraction. Indeed, suppose that some trajectories begin at a point M of the surface S whose tangents are contained in the solid angle $d\omega$. If the index jumps from the value n to the value N under the traversal of the surface then the trajectories will refract, and the new tangents will be contained in the interior of a solid angle $d\Omega$. In order to evaluate the ratio of these angles, we take the axes to be three rectangular axes, one of which Mx is normal to the surface at M, while the other two are in the tangent plane. If p,

q are the parameters of a tangent to a trajectory, and M, P, Q are the parameters for the refracted trajectory then:

$$np - NP = 0,$$
$$nq - NQ = 0.$$

Starting with:

$$d\Omega = \frac{dP\,dQ}{M} = \frac{n^2}{N^2}\frac{dp\,dq}{M},$$

$$M\,d\Omega = \frac{n^2}{N^2} m\,d\omega,$$

and if θ and Θ are the angles between the normal and the incident and refracted trajectories, resp., then:

$$\cos\Theta\,d\Omega = \frac{n^2}{N^2}\cos\theta\,d\omega;$$

therefore, one finally has:

$$n^2\cos\theta\,d\omega = N^2\cos\Theta\,d\Omega.$$

The integral (K) preserves its value when one replaces the elements that relate to the trajectories that begin at a point of the surface S with the ones that correspond to the refracted trajectory on the surface, without changing the surface.

In particular, suppose that the trajectories that issue from the various points of a surface S are refracted at the points where they meet a given surface (Σ); i.e., suppose that the index n is subjected to a brief passage from one value to a different value at each point of the surface. The integral (K) cannot take on less than the constant value that one applies to the points of the surface S and to the rays that emanate from it into the first medium, or to the corresponding elements of a surface in the second one. Indeed, (K) is invariant in the first medium and keeps the same value that one applies to S or to the refringent surface (Σ). (K) is not altered by refracting the trajectories on (Σ), and remains invariant in the second medium, which proves the property.

Therefore, (K) is an integral of a particular type: It keeps the same value whether one extends it over the points of a four-parameter multiplicity that is formed from the points of an arbitrary surface and to the two-parameter sheaf that issues from it or over the points that are deduced from it by moving arbitrary segments along the trajectories of the first one, when these trajectories have been subjected to an arbitrary number of refractions or reflections along the path. In order to simplify the language, we shall say that the integral keeps a constant value when it is extended over the portion of an arbitrary surface that is intersected by the sheaf.

Before pursuing the study that we shall devote ourselves to, it is convenient to investigate whether the new element and the integral invariant that we introduced can give us results that could not be obtained from, for example, the Malus conditions. It seems that since the notion of integral invariant is related to that of trajectory, its existence will depend essentially upon the fact that a light ray is a trajectory that presents some angular points, but whose coordinates x, y, z will vary without discontinuities. Now, that is nothing. We shall show that the integral (K) preserves the same value whether one extends it over a surface from which a sheaf of rays emanates or over a portion of another surface that is, moreover, arbitrary, and is intersected by the sheaf that reduces to the first one by the Malus transformation. It results from this that the introduction of that integral invariant will lead us only to results that we knew already.

Indeed, consider the integral:

$$(F) = \int n^3 \frac{dx\,dy\,dz\,dp\,dq}{m},$$

which is taken over the points of a certain volume and the rays (m, p, q) that issue from these points. One of these rays AB, which issues from the point $B(x, y, z)$, meets the yz-plane at A $(0, h, k)$, and one has:

$$x = ml,$$
$$y = h + pl,$$
$$z = k + ql;$$

l denotes the segment AB.

Consider a point (x, y, z, p, q) in five-dimensional space whose coordinates are defined as functions of time by the equations:

$$\frac{dx}{dt} = mvn, \qquad \frac{dy}{dt} = pvn, \qquad \frac{dz}{dt} = qvn,$$

$$\frac{dp}{dt} = \frac{dq}{dt} = 0,$$

in which n denotes the index of the medium and v is a constant. The point (x, y, z) in three-dimensional space describes the light ray m, n, p. Suppose that at the arbitrary instant t, the motion is replaced by one whose equations are:

$$\frac{dX}{dt} = MvN, \qquad \frac{dY}{dt} = PvN, \qquad \frac{dZ}{dt} = QvN,$$

$$\frac{dP}{dt} = 0, \qquad \frac{dQ}{dt} = 0.$$

Furthermore, M, P, Q, X, Y, Z are deduced from m, p, q, x, y, z by the formulas:

$$\begin{array}{llll} X = ML, & x = ml, & H = A(h, k, p, q), & L = \dfrac{n}{N}t, \\ Y = H + PL, & y = h + pl, & K = B(h, k, p, q), & \\ Z = K + QL, & z = k + ql, & P = C(h, k, p, q), & \\ & & Q = Dh, k, p, q). & \end{array}$$

The initial conditions x_0, y_0, z_0, p_0, q_0 will then correspond to a perfectly-determined trajectory that is composed of two pieces of a line, and the position of the moving point

(x, y, z, p, q) will be fixed at each instant. (F) is an integral invariant for this motion. Indeed, under the motion along the incident ray, one will have:

$$(F)_0 = \int n^3 \, dh \, dk \, dp \, dq \, dl,$$

while under the motion along the transformed ray, one will have:

$$\begin{aligned}(F)_1 &= \int N^3 \, dH \, dK \, dP \, dQ \, dL \,, \\ &= \int n \, N^2 \frac{D(H,K,P,Q)}{D(h,k,p,q)} dh \, dk \, dp \, dq \, dl, \\ &= \int n^3 \, dh \, dk \, dp \, dq \, dl.\end{aligned}$$

Indeed, we have seen that:

$$\frac{D(H,K,P,Q)}{D(h,k,p,q)} = \frac{n^2}{N^2}\,.$$

A transformation that is analogous to the transformation that was already employed will give the invariant:

$$\int n^2 \left[m \frac{D(y,z)}{D(u,v)} + p \frac{D(z,x)}{D(u,v)} + q \frac{D(x,y)}{D(u,v)} \right] \frac{dp \, dq}{m} \,,$$

under the same conditions are before.

One even sees, moreover, that for a transformation of lines to lines such that:

$$\frac{D(H,K,P,Q)}{D(h,k,p,q)} = \frac{n^2}{N^2}$$

the integral (K) will keep the same value when it is taken over two arbitrary surfaces, along with the lines that pass through the points of the first one and the transformed lines that pass through the points of the second one.

Such a transformation is obviously more general than a Malus transformation, and we shall see that it can also be stigmatic only in the case where the correspondence that is established point-by-point is a similitude. Recall the notations that were employed for a telescopic transformation:

$$X = \frac{ax}{z}, \qquad Y = \frac{by}{z}, \qquad Z = \frac{c}{z},$$

$$H = \lambda \, a \, x, \qquad H = \lambda \, b \, y, \qquad H = c \, \lambda \,.$$

Extend the integral (K) in the first medium to a portion of the xy-plane.

The ray m, p, q that passes through the point $(x, y, 0)$ is transformed into a ray $\lambda a x_1$, $\lambda b y_1$, λc, $\lambda = \dfrac{1}{\sqrt{a^2 x_1^2 + b^2 y_1^2 + c^2}}$ that passes through the point $\dfrac{am}{q}, \dfrac{bp}{q}, 0$. We therefore likewise extend the integral (K) over the xy-plane in the second medium. One must then have:

$$\int n^2 \frac{q}{m} dx_1\, dy_1\, dp\, dq = \int N^2 \frac{\lambda c}{\lambda a x_1} \frac{D\left(\frac{am}{q}, \frac{bp}{q}, \lambda b y_1, \lambda c\right)}{D(x_1, y_1, p, q)} dx_1\, dy_1\, dp\, dq\,;$$

in other words:

$$n^2 = N^2 \frac{c}{a x_1} \frac{D\left(\frac{am}{q}, \frac{bp}{q}\right)}{D(p, q)} \frac{D(\lambda b y_1, \lambda c)}{D(x_1, y_1)}.$$

In particular:

$$\frac{D(\lambda b y_1, \lambda c)}{D(x_1, y_1)} \frac{1}{x_1}$$

will be constant; i.e., $\dfrac{\lambda}{x_1} \dfrac{\partial \lambda}{\partial x_1}$ is constant, or finally, λ is constant, which cannot be true if a and b are non-zero. The telescopic transformation is therefore once more impossible in the most general case. Suppose, then, that the affine transformation:

$$X = ax, \qquad Y = by, \qquad Z = cz,$$

$$M = \lambda\, am, \qquad P = \lambda\, bp, \qquad Q = \lambda\, cq, \qquad \lambda = \frac{1}{\sqrt{a^2 m^2 + b^2 p^2 + c^2 q^2}}.$$

When the integral is taken over an arbitrary surface and then over its transform, it will keep the same value:

$$\int n^2 \left[m \frac{D(y, z)}{D(u, v)} + p \frac{D(z, x)}{D(u, v)} + q \frac{D(x, y)}{D(u, v)} \right] \frac{dp\, dq}{m}$$

$$= \int N^2 \lambda abc \left[m \frac{D(y, z)}{D(u, v)} + \cdots \right] \frac{D(\lambda bp, \lambda cq)}{D(p, q)} \frac{dp\, dq}{\lambda am}.$$

It is therefore necessary that one must have:

$$n^2 = N^2\, bc\, \frac{D(\lambda bp, \lambda cq)}{D(p, q)},$$

identically.

Calculate:

$$\frac{D(\lambda bp, \lambda cq)}{D(p,q)} = \left(\lambda^2 + \lambda\frac{\partial\lambda}{\partial p}\,p + \lambda\frac{\partial\lambda}{\partial q}\,q\right)bc\,.$$

Now:

$$\lambda^2 = \frac{1}{a^2m^2 + b^2p^2 + c^2q^2}\,, \qquad \lambda\,d\lambda = -\,\frac{a^2m\,dm + b^2p\,dp + c^2q\,dq}{(a^2m^2 + b^2p^2 + c^2q^2)^2}\,,$$

and by virtue of the equality $m\,dm + p\,dp + q\,dq = 0$, one will have:

$$\lambda\,d\lambda = +\,\lambda^4\,(a^2 - b^2)\,p\,dp + \lambda^4\,(a^2 - c^2)\,q\,dq,$$

$$\lambda^2 + \lambda\,\frac{\partial\lambda}{\partial p}\,p + \lambda\,\frac{\partial\lambda}{\partial q}\,q = \lambda^4[a^2\,m^2 + b^2\,p^2 + c^2\,q^2 + (a^2 - b^2)\,p^2 + (a^2 - c^2)\,q^2]$$
$$= \lambda^4a^2\,.$$

$$\frac{D(\lambda bp, \lambda cq)}{D(p,q)} = \frac{a^2bc}{[a^2 + (b^2 - a^2)\,p^2 + (c^2 - a^2)q^2]^2}\,.$$

One must then have:

$$\frac{a^2bc}{[a^2 + (b^2 - a^2)\,p^2 + (c^2 - a^2)q^2]^2} = \frac{n^2}{N^2}\,,$$

identically.

It would be illusory to use an analogous procedure to look for the Malus transformations for which two surfaces correspond stigmatically. Indeed, there exist transformations that do not enjoy the fundamental property of preserving the normal congruences and for which the integral (K) is an invariant.

For example, take the following transformation that was already employed by Fatou, which transforms one plane into another plane:

$$Y = \alpha y, \qquad Z = \beta z,$$

$$P = \frac{n}{N}\frac{p}{\alpha} + \varphi(y, z), \qquad Q = \frac{n}{N}\frac{q}{\beta} + \psi(y, z).$$

It does not preserve normal congruences, because if the lines in the first medium form a congruence then:

$$p\,dy + q\,dz = d\theta(x, y).$$

Now, in the second one:

$$P\,dY + Q\,dZ = \frac{n}{N}\,(p\,dy + q\,dz) + \alpha\,\varphi(y, z)\,dy + \beta\,\psi(y, z)\,dz;$$

$P\ dY + Q\ dZ$ will not be an exact differential, so in other words, the transformed rays form a normal congruence only if:

$$\alpha\,\varphi(y, z)\,dy + \beta\,\psi(y, z)\,dz$$

is an exact differential. One can always arrange for this to not be true. Meanwhile:

$$\int N^2 M\,\frac{D(Y,Z)}{D(y,z)}\,dy\,dz\,\frac{dP\,dQ}{M} = \int n^2 dy\,dz\,\frac{dp\,dq}{m}.$$

Indeed:

$$\frac{D(P,Q,y,z)}{D(p,q,y,z)} = \frac{D(P,Q)}{D(p,q)} = \frac{n^2}{N^2}\frac{1}{\alpha\beta}$$

and

$$\frac{D(Y,Z)}{D(y,z)} = \alpha\beta,$$

which proves the proposition.

IV.

STRAUBEL'S THEOREM.

Due to the very large number itself of the transformations that conserve it, the integral invariant in question cannot give us the solutions to all of the problems of geometrical optics, but its use, when it is convenient, sometimes leads to necessary conditions for the possibility of the problem. In any case, its introduction will not be pointless, because it is the mathematical expression of a theorem of geometrical optics that was stated by Straubel, and whose importance has been shown by Hilbert, and more recently, Langevin.

With the notations of the preceding paragraphs, it expresses the idea that for the corresponding elements of two surfaces, one has:

$$n \cos \theta\,ds\,d\omega = n' \cos \theta'\,ds'\,d\omega',$$

and for a planar sheaf:

$$n' \cos \theta\,ds\,d\omega = n' \cos \theta'\,ds'\,d\omega',$$

The physical interpretation of these equalities is extremely simple: Let dQ be the quantity of light that is emitted normally from a portion of the surface $d\sigma$ into the solid angle $d\omega$. We call the limit of $\dfrac{dQ}{d\omega\,d\sigma}$ when the sheaf reduces to the ray considered the "specific intensity L at a point for a given ray;" L will then be a function of x, y, z, p, q. However, since the flux:

$$dQ = L\,d\omega\,d\sigma$$

is conserved, for an elementary sheaf one will have:

$$L\,d\omega d\sigma = L'\,d\omega'\,d\sigma',$$

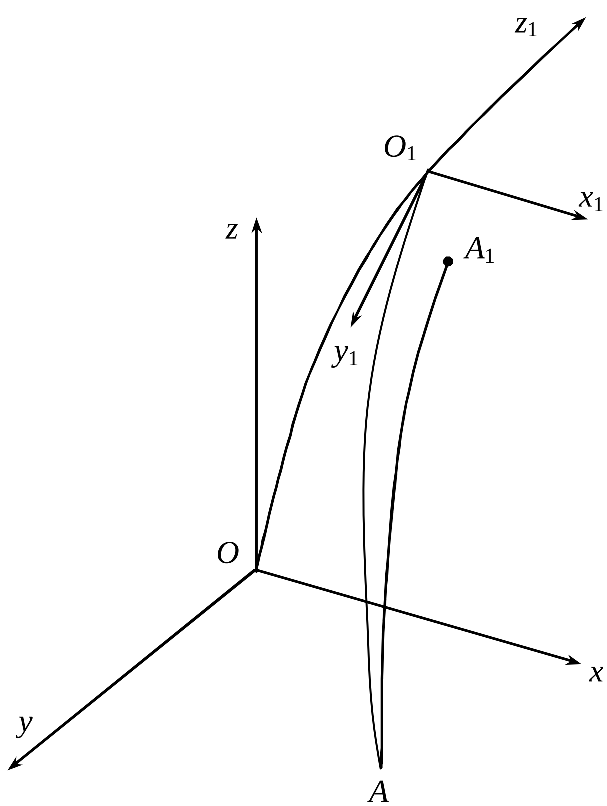
upon letting L, L' denote the specific intensity of the same ray at two points:

$$\frac{L}{n^2} = \frac{L'}{n'^2}.$$

The specific intensities are then proportional to the squares of the indices.

Straubel remarked, moreover, that from the viewpoint of energy this result can be regarded as obvious: Helmholtz and Clausius certainly had knowledge of it before him, although they did not state the result. It came down to him to give a proof that was reprised by Langevin in his course (1913), and which is easy to give in a form that is rigorous and mathematical.

We immediately place ourselves in three-dimensional space. Suppose that the medium is isotropic with a variable index n. In general, a trajectory will pass through two points A, A_1. In any case, there exists an absolute minimum for the integral $\int_A^{A_1} n\,ds$, which is a function $T(A, A_1)$ of the two points A and A_1, and represents the time that is taken by the light in order to go from A to A_1.



Let OO_1 be a ray, let Oz be the tangent at O, let $O_1\,z_1$ be the tangent to that ray at O_1, and let Ox, Oy be two axes that are perpendicular to Oz and to each other. Likewise, suppose that $O_1\,x_1$, $O_1\,y_1$ are perpendicular to each other and to $O_1\,z_1$. Choose points A, A_1 that are situated in the xy and x_1y_1-planes of the two coordinate systems thus determined. The function $T(A, A_1)$ then becomes a function $T(x, y, x_1, y_1)$ of the coordinates x, y, x_1, y_1 of the points A, A_1.

One knows that the variation of the integral $\int_A^{A_1} n\,ds$ is expressed simply as a function of the direction parameters $\alpha, \beta, \alpha_1, \beta_1, \gamma_1$ of the tangents to the trajectory at A and A_1 and the elementary displacements of A and A_1 :

$$\delta \int_A^{A_1} n\,ds = n\,(\alpha\,dx + \beta\,dy) - n_1\,(\alpha_1\,dx_1 + \beta_1\,dy_1).$$

It then results that:

$$\frac{\partial T}{\partial x} = n\alpha, \qquad \frac{\partial T}{\partial y} = n\beta, \qquad \frac{\partial T}{\partial x_1} = -\,n_1\alpha_1, \qquad \frac{\partial T}{\partial y_1} = -\,n_1\beta_1\,.$$

Now, if A_1 is at O_1 and if A describes a portion of the surface surrounding the point O in the xy-plane then the solid angle $d\omega_1$ that is swept out by the tangent at O_1 will have the value:

$$d\omega_1 = \frac{d\alpha_1\,d\beta_1}{\gamma_1};$$

i.e.:

$$d\omega_1 = \frac{1}{n_1^2}\begin{vmatrix} \dfrac{\partial^2 T}{\partial x_1\,\partial x} & \dfrac{\partial^2 T}{\partial x_1\,\partial y} \\ \dfrac{\partial^2 T}{\partial y_1\,\partial x} & \dfrac{\partial^2 T}{\partial y_1\,\partial y} \end{vmatrix} dz\,dy,$$

so

$$n_1^2\;d\omega_1\,dx_1\,dy_1 = \begin{vmatrix} \dfrac{\partial^2 T}{\partial x_1\,\partial x} & \dfrac{\partial^2 T}{\partial x_1\,\partial y} \\ \dfrac{\partial^2 T}{\partial y_1\,\partial x} & \dfrac{\partial^2 T}{\partial y_1\,\partial y} \end{vmatrix} dx\,dy\,dx_1\,dy_1\,.$$

It results from this that since the right-hand side is symmetric in x, y, x_1, y_1, one will have:

$$n_1^2\;d\omega_1\,d\sigma_1 = n^2\,d\omega\,d\sigma.$$

This simple proof has the advantage of showing clearly how the existence of Hadamard's integral invariant results from the property of the quantity:

$$n\,(\alpha\,dx + \beta\,dy + \gamma\,dz) - n_1\,(\alpha_1\,dx_1 + \beta_1\,dy_1 + \gamma_1\,dz_1)$$

that it is an exact total differential.

In the case of a planar sheaf that remains planar as a result of refraction, one finds, by an analogous process, that:

$$n\, d\sigma\, d\theta = n'\, d\sigma'\, d\theta',$$

in which $d\sigma$ denotes an element of arc, this time.

These two formulas apply to any intermediary media, and even if we are dealing with light rays when the transformation that they are subjected to is either a Malus transformation or perhaps one of the more general ones that we spoke of.

Following another line of inquiry, Straubel's proposition is further true for generalized rays – i.e., for the bicharacteristics of certain partial differential equations.

For example, take the equation:

$$a\frac{\partial^2 V}{\partial x^2} + a'\frac{\partial^2 V}{\partial y^2} + a''\frac{\partial^2 V}{\partial t^2} + 2b\frac{\partial^2 V}{\partial y\,\partial t} + 2b'\frac{\partial^2 V}{\partial x\,\partial t} + 2b''\frac{\partial^2 V}{\partial x\,\partial y} + c = 0,$$

where $a, a', a'', b, b', b'', c$ are arbitrary functions of $x, y, t, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial t}, V$. One knows that one uses the term *characteristic* that corresponds to a given solution $V = \varphi(x, y, t)$ of the partial differential equation to refer to a solution to the first-order equation:

$$H = ap^2 + a'q^2 + a'' - 2bq - 2b'p - 2b''pq = 0,$$

where p and q represent $\frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}$, and in the coefficients of which one has replaced V and its partial derivatives with their values as functions of x, y, t. The *bicharacteristics* are then the characteristics of that first-order equation, which are defined by the equations:

$$\frac{dx}{\dfrac{\partial H}{\partial p}} = \frac{dy}{\dfrac{\partial H}{\partial q}} = \frac{-dp}{\dfrac{\partial H}{\partial x} + p\dfrac{\partial H}{\partial t}} = \frac{-dq}{\dfrac{\partial H}{\partial y} + q\dfrac{\partial H}{\partial t}} = \frac{dt}{p\dfrac{\partial H}{\partial p} + q\dfrac{\partial H}{\partial q}}.$$

We will focus on the single case in which, H being a function of x, y, p, q that is independent of t, these equations become:

$$\frac{dx}{\dfrac{\partial H}{\partial p}} = \frac{dy}{\dfrac{\partial H}{\partial q}} = \frac{-dp}{\dfrac{\partial H}{\partial x}} = \frac{-dq}{\dfrac{\partial H}{\partial y}} = d\tau.$$

When one considers τ to be time, these will be the equations of motion whose trajectories are bicharacteristics if one chooses the initial conditions in such a way that:

$$H(x_0, y_0, p_0, q_0) = 0.$$

If this were not true, x_0, y_0, p_0, q_0 being arbitrary initial values, then the motion that is defined by the preceding equations would be attached to the integral invariant:

$$\int dx\, dy\, dp\, dq,$$

where one would deduce, as before, an invariant:

$$\int \left| \frac{\partial x}{\partial \alpha}\frac{\partial H}{\partial q} - \frac{\partial y}{\partial \alpha}\frac{\partial H}{\partial p} \right| dp\, dq\, dz,$$

that is extended over the various points of a line (i.e., x, y are functions α) – which is arbitrary, moreover – and the two-parameter trajectories that issue from it. For example, take the arc length s of the curve to be the variable. Let v be the velocity $\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}\right)$ at a point of a trajectory that is situated on the given line, and let φ and ψ be the angles that the normal to the line and the tangent to the trajectory make with the x-axis. One has:

$$\frac{\partial x}{\partial \alpha} = \sin \varphi, \qquad \frac{\partial y}{\partial \alpha} = -\cos \varphi,$$

$$\frac{\dfrac{\partial H}{\partial p}}{\cos\psi} = \frac{\dfrac{\partial H}{\partial q}}{\sin\psi} = v.$$

These last two equations define p and q as functions of v and ψ, on the condition that:

$$\frac{D(v,\psi)}{D(p,q)} \neq 0.$$

Suppose that this is true, and choose the arbitrary variables s, v, ψ. The invariant will become:

$$\int v \,|\sin\psi\sin\varphi + \cos\varphi\cos\psi|\left|\frac{D(p,q)}{D(v,\psi)}\right| ds\, dv\, d\psi\,.$$

From the two identities:

$$\frac{\partial H}{\partial p} = v \cos \psi, \qquad \frac{\partial H}{\partial q} = v \sin \psi,$$

one infers $\frac{\partial v}{\partial p}$, $\frac{\partial v}{\partial q}$, $\frac{\partial \psi}{\partial p}$, $\frac{\partial \psi}{\partial q}$, and, in turn, $\frac{D(v,\psi)}{D(p,q)}$.

Indeed:

$$\frac{\partial^2 H}{\partial p^2} = \frac{\partial v}{\partial p}\cos\psi - v\sin\psi\,\frac{\partial \psi}{\partial p},$$

$$\frac{\partial^2 H}{\partial p\,\partial q} = \frac{\partial v}{\partial q}\cos\psi - v\sin\psi\,\frac{\partial \psi}{\partial q} = \frac{\partial v}{\partial p}\sin\psi + v\cos\psi\,\frac{\partial \psi}{\partial p},$$

$$\frac{\partial^2 H}{\partial q^2} = \frac{\partial v}{\partial q}\sin\psi + v\cos\psi\,\frac{\partial \psi}{\partial q},$$

$$\begin{vmatrix} \dfrac{\partial^2 H}{\partial p^2} & \dfrac{\partial^2 H}{\partial p\,\partial q} \\ \dfrac{\partial^2 H}{\partial p\,\partial q} & \dfrac{\partial^2 H}{\partial q^2} \end{vmatrix} = \frac{D(v,\psi)}{D(p,q)}\begin{vmatrix} \cos\psi & -v\sin\psi \\ \sin\psi & v\cos\psi \end{vmatrix},$$

$$\frac{D(v,\psi)}{D(p,q)} = \frac{1}{v}\begin{vmatrix} \dfrac{\partial^2 H}{\partial p^2} & \dfrac{\partial^2 H}{\partial p\,\partial q} \\ \dfrac{\partial^2 H}{\partial p\,\partial q} & \dfrac{\partial^2 H}{\partial q^2} \end{vmatrix}.$$

Therefore, if the determinant $\begin{vmatrix} a & b' \\ b' & a' \end{vmatrix}$ is non-zero then a sheaf of bicharacteristics will admit the invariant:

$$\int \frac{v^2\,|\cos(\varphi-\psi)|}{|b'^2 - aa'|}\,ds\,d\psi\cos\theta.$$

It is not possible to deduce another invariant from this one that would be attached to that of the bicharacteristics by methods that are analogous to the ones that were used for light rays without making some new hypotheses. For example, suppose that:

$$b'' = 0, \qquad a = a',$$

$$v^2 = \left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial q}\right)^2,$$

and

$$H = a(p^2 + q^2) - 2\,bq - 2\,b'p + a',$$

and as a result:

$$v^2 = 4(aH + b'^2 + b^2 - aa'').$$

Therefore, if we make the change of variables:

$$v^2 = 4(au + b'^{\,2} + b^2 - aa'')$$

then u will keep its initial value all along the trajectory, and $u = 0$ will correspond to the bicharacteristics. With these notations, the invariant will become:

$$\int v^2 \frac{\partial v}{\partial u} \frac{\cos\theta}{a^2} d\psi\, ds\, du .$$

Extend the integral over the volume of a cylinder ($0 < u < \alpha$). Any point of an initial cylinder (E_0) will correspond to a trajectory, and at the end of an arbitrary length of time, to a point that is situated in the interior of a cylinder (E). The integral:

$$\int \frac{\cos\theta}{a^2} v^2 \frac{\partial v}{\partial u} d\psi\, ds\, du ,$$

or even better:

$$\frac{1}{\alpha}\int \frac{\cos\theta}{a^2} v^2 \frac{\partial v}{\partial u}\, du\, d\psi\, ds = \frac{1}{\alpha}\int \frac{\cos\theta}{a^2} d\psi\, ds \int_0^\alpha v^2 \frac{\partial v}{\partial u}\, du$$

will keep the same value whether one extends it over E or E_0 .

If we let α tend to 0 then the latter will tend to a limit:

$$\int \frac{\cos\theta}{a} (b'^2 + b^2 - aa'')^{1/2}\, d\psi\, ds ,$$

and this new integral will keep the same value whether one extends it over the points of the base B_0 of (E_0) or the base B of E. Now, the corresponding trajectories are bicharacteristics, so the quantity:

$$\int \frac{\cos\theta}{a} (b'^2 + b^2 - aa'')^{1/2}\, d\psi\, ds$$

will indeed be an invariant for these curves that is analogous to the one that was found in the preceding chapters. It will keep the same value whether one extends it over the points of a curve and the bicharacteristics that emanate from it, or over another curve and the bicharacteristics that begin on it.

Therefore, if one lets n denote the quantity:

$$n = \frac{(b'^2 + b^2 - aa'')^{1/2}}{a}$$

and calls it the *index* then one will have a generalization of Straubel's theorem: The bicharacteristics of the equations:

$$a\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+2b\frac{\partial^2 V}{\partial y\,\partial t}+2b'\frac{\partial^2 V}{\partial x\,\partial t}+2a''\frac{\partial^2 V}{\partial t^2}+c=0$$

are rays, and they satisfy the Straubel relation, namely:

$$n\,ds\,d\psi\cos\theta = n'\,ds'\,d\psi'\cos\theta'.$$

It is obvious that an analogous calculation will generalize the same proposition to characteristics of several dimensions. We shall not insist upon that fact, or upon the consequences that one can infer from that viewpoint; for example, the possible point-by-point aplanatism of volumes or surfaces. We shall be content to remark that it is indeed a simple generalization, namely, that in the case of light rays that are bicharacteristics of the equation:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}-n^2\frac{\partial^2 V}{\partial t^2}=0,$$

the generalized index will be identical to the ordinary index.

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