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On the Bruns eikonal

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In volume 21 of the math. phys. Adhandlungen der Kgl. sächsischen Gesellschaft der Wissenschaften (1895), Bruns published a remarkable article on ray optics, in which he represented the path of a light ray through an arbitrary optical instrument with the help of a function of four variables that he called the *eikonal*. I shall roughly reproduce his basic formulas here.

One denotes the point at which the part of the light ray (if one thinks of it as rectilinearly extended) that passes through the object space intersects the *XY*-plane of the object space by ξ, η, and the direction cosines that it defines with the coordinate axes (in object space) by p, q, r; the notations ξ', η' (p', q', r', resp.) have the corresponding meanings in image space. The eikonal, in its original form (which is all that will be considered here), is then a function of ξ, η, ξ', η':

$$E(\xi, \eta \mid \xi', \eta'),$$

by whose use the path of light rays in the object space and image space are represented by means of the following formulas:

$$\text{(1)} \qquad \begin{cases} p = -c \cdot \dfrac{\partial E}{\partial \xi}, & p' = +c' \cdot \dfrac{\partial E}{\partial \xi'}, \\[2ex] q = -c \cdot \dfrac{\partial E}{\partial \eta}, & q' = +c' \cdot \dfrac{\partial E}{\partial \eta'}; \end{cases}$$

by c (c', resp.), one should understand this to mean the velocity of light in object space (image space, resp.). I will briefly summarize these formulas in the following way:

$$\text{(2)} \qquad dE = -\frac{1}{c}(p\, d\xi + q\, d\eta) + \frac{1}{c'}(p'\, d\xi' + q'\, d\eta').$$

One would now like to compare this with the developments that Hamilton (1828, et. seq.) based his investigations on ray optics upon [1]). Hamilton began by basing the path of a light ray through an instrument on the presently general demand that it be a

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[1]) [For the exact references, cf., e.g., footnote [2]) on pp. 601 of the present volume (viz., Gesammelten Abhandlungen)].

minimum (maximum, resp.), in the well-known way that originated with Johann Bernouli (Fermat, resp.).

Let x, y, z be the initial point of the light ray (in object space), x', y', z', its end point (in image space), c, c_1, c_2, …, c', the light velocities in the successive media through which the light ray passes, and Δl, Δl_1, Δl_2, …, $\Delta l'$, the path lengths that it describes in these media, respectively. The specification of a light ray then results when one requires that it shall give the sum:

$$\sum_{xyz}^{x'y'z'} \frac{\Delta l_i}{c_i}$$

a vanishing first variation for fixed initial points and end points. This was what Johann Bernoulli contributed.

The innovation of Hamilton is that he further proposed the notion that the light ray came about by regarding the latter sum as a function of its two endpoints:

$$\sum_{xyz}^{x'y'z'} \frac{\Delta l_i}{c_i} = \Omega(x, y, z \mid x', y', z'). \tag{3}$$

This Ω is what Hamilton called the *characteristic function* for the instrument. It simply means the *time* that the light ray (as one imagines it in undulation theory) takes in order to go through the instrument from x, y, z to x', y', z'. Thus, one deduces that one can represent the progress of the light ray by this Ω in a simple way; in this regard, one has the formulas:

$$\begin{cases} p = -c \cdot \dfrac{\partial\Omega}{\partial x}, & p' = +c' \cdot \dfrac{\partial\Omega}{\partial x'}, \\ q = -c \cdot \dfrac{\partial\Omega}{\partial y}, & q' = +c' \cdot \dfrac{\partial\Omega}{\partial y'}, \\ r = -c \cdot \dfrac{\partial\Omega}{\partial z}, & r' = +c' \cdot \dfrac{\partial\Omega}{\partial z'}, \end{cases} \tag{4}$$

which I would like to summarize as follows:

$$d\Omega = -\frac{1}{c}(p\, dx + q\, dy + r\, dz) + \frac{1}{c'}(p'\, dx' + q'\, dy' + r'\, dz'). \tag{5}$$

It then follows easily from (4) that Ω satisfies the two partial differential equations:

$$\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial\Omega}{\partial y}\right)^2 + \left(\frac{\partial\Omega}{\partial z}\right)^2 = \frac{1}{c^2}, \qquad \left(\frac{\partial\Omega}{\partial x'}\right)^2 + \left(\frac{\partial\Omega}{\partial y'}\right)^2 + \left(\frac{\partial\Omega}{\partial z'}\right)^2 = \frac{1}{c'^2}. \tag{6}$$

The similarity of these formulas with those of Bruns resides in the fact that, on the one hand, and this seems to be more important, one is given the transition from one system of formulas to another, as the eikonal formulas were presented by Bruns himself in a very complicated way – by the appealing to the theory of contact transformations, which are founded on the theorem of Malus – whereas Hamilton's derivation follows immediately from the definition of Ω, and leaves nothing to be desired in terms of its simplicity. It is precisely this transition that is the objective of the present little article.

One simply denotes the distance from the point x, y, z of the object space to the point ξ, η, 0 itself by ρ, and likewise denotes the distance from x', y', z' to ξ', η', 0 by ρ'. One then has:

$$\begin{cases} x = \xi + \rho p, & x' = \xi' + \rho' p', \\ y = \eta + \rho q, & y' = \eta' + \rho' q', \\ z = \quad \rho r, & z' = \quad \rho' r'. \end{cases} \tag{7}$$

If one substitutes the values of the differentials that this yields:

$$dx = d\xi + p \cdot d\rho + \rho \cdot dp, \qquad \text{etc.}$$

in (5) then one obtains, after a brief intermediate computation:

$$d\Omega = -\frac{1}{c}(d\rho + p\, d\xi + q\, d\eta) + \frac{1}{c'}(d\rho' + p'\, d\xi' + q'\, d\eta'). \tag{8}$$

Comparison with (2) then gives (when I include the arbitrary integration constant in the eikonal):

$$\Omega = -\frac{\rho}{c} + \frac{\rho'}{c'} + E. \tag{9}$$

Therefore: *The eikonal is equal to the characteristic function for* $\rho = 0$, $\rho' = 0$; *it simply refers to the time that the motion of the light takes to propagate from the object point* ξ, η, 0 *to the image point* ξ', η', 0 *along the ray that passes through the instrument.* – Likewise, one obtains it as a result of the fact that the interests of simplicity point to the eikonal (the general characteristic function, resp.). By means of the substitution (7), the two partial differential equations (6) are converted into the following ones:

$$\frac{\partial \Omega}{\partial \rho} = -\frac{1}{c}, \qquad \frac{\partial \Omega}{\partial \rho'} = \frac{1}{c'};$$

the eikonal E is then no longer linked to any partial differential equation.

I cannot conclude this little note without emphatically commenting upon the particular nature of my interest in Hamilton's investigations into ray optics. The method of characteristic functions leads, on the one hand, to a far-reaching treatment of instrumental questions (whose numerous results anticipated the work of later authors), and, on the other hand, to the discovery of conical refraction in biaxial crystals. However, more than this, it is based in the fact that I already discussed the topic in a lecture [2]) that I presented ten years ago to the Naturforscher-Versammlung in Halle (1891), which unfortunately did not succeed in the general objective that I posed at the outset of planting the true roots of Hamilton's discovery in the ground of general dynamics! I can only express the wish that the largely inaccessible and variegated optical treatises of Hamilton will likewise be made more accessible to the greater public; such a publication would not only have historical interest, but also, without a doubt, it would influence and foster the growth of our present culture of ideas in many directions [3]).

[I am pleased to add that Prange will expound in an essay that will appear in the Nova acta Leopoldina, Bd. 107, "W. R. Hamiltons Arbeiten zur Strahlenoptik und Mechanik," as Hamilton anticipated, on a number of ways of looking at modern variational calculus, in particular, the theory of contact transformations, as well. Hamilton has also derived a large number of optical results that were rediscovered by later authors in a more or less complete form. See also a communication of Prange that appeared in the interim in the Jahresberichten der Deutschen Mathematiker-Vereinigung, Bd. 30, 1921. K]

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[2]) See the Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 1 (1891/92) [cf., the present volume, no. LXX.] I have repeatedly developed the subject in detail since the Summer of 1891 in my lectures on mechanics.

[3]) Bruns wrote the following remarks to me on the development of the text: "The connection between the characteristic function and the eikonal comes about when one assumes that under any refraction a light corpuscle experiences a certain delay that depends upon the location of the point of refraction, which is equivalent to saying that, as usual, the points of refraction fill up a surface, or perhaps a physical space. – Moreover, the path that I pursued delivered as a payoff for the elaborate derivation of the proof the fact that most of the theorems of geometrical optics are not actually optical in nature, but properly belong to pure line geometry."