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General theory of rectilinear ray systems

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Up to now, the systems of straight lines that fill up all of space, or a part of it, in such a way that one, or a certain number, of discrete rays goes through every point have been examined only a little in full generality. In the geometric consideration of ray systems, one has chiefly restricted oneself to those of a special kind, for which all of the rays take the form of normals to one and the same surface, a theory that has the closest relationship with the study of the curvature of surfaces, and whose most distinguished properties were found by *Monge*, who developed them in several chapters of his *Application de l'Analyse à la Géométrie*. Since systems of rays in space have great significance for optics, the theory of them has be treated several times for the sake of physical interest; however, from that standpoint, one likewise rarely gets very far from systems of normals to a surface. Here, one of the most beautiful theorems of optics has hindered the development of the general theory to a remarkable degree, namely, the theorem that was discovered by *Malus* and generalized by *Dupin* that after the light rays that emanate from a point have experienced an arbitrary number of reflections from arbitrarily-shaped mirrors and refractions from passing through arbitrarily-bounded media with various refracting powers, they will always preserve the property that they are the normals to a surface. It is only when light goes through crystals that this property will break down for irregular rays; these define systems of rays that cannot be normal to a surface that will be called *irregular ray systems*, due to that state of affairs. The question of whether crystals also produce only special kinds of such things prompts one to consider the most general ray systems. As far as I know, they were first treated by *Hamilton* in the *Transaction of the Royal Irish Academy*, Bd. XVI, in a Supplement to his great paper: *Theory of Systems of Rays*, into which they did not enter, because that treatise was directed to the goals of optics, so only regular systems and their variations under reflections and refractions were considered, but the irregular systems that were considered were the ones that arose during the passage of light through crystals. In the aforementioned first Supplement to this treatise, *Hamilton* likewise started from physical principles – namely, the principle of least action – but he pursued a main objective of developing the geometric properties of general ray systems of optics from a basic formula that obeyed that principle. In this way, he discovered a series of distinguishing properties of the general, rectilinear ray systems that still seem to be little known, since no recourse was made to them in several later mathematical articles on related subjects,. Giving this theory of general, rectilinear ray systems that was first treated by *Hamilton* a new foundation by appealing to the analytic geometry of space, and likewise completing it at several essential points, shall be the goal of the present treatise.

§ 1.

Preliminary formulas and notations.

Any straight line of a ray system shall be determined by a point through which it goes, whose rectilinear coordinates will be x, y, z, and by the angles that it makes with the three coordinate axes, whose cosines will be denoted by ξ, η, ζ. The law that couples the straight lines into a system will be given by saying that their six determining data: x, y, z, ξ, η, ζ will be determined as continuous functions of two independent variables u and v. The points x, y, z will then lie on a well-defined surface, and the rays of the system will all be regarded as emanating from the individual points of that surface. Any point of a ray will be determined by its distance from the starting point of the ray, and thus by its abscissa, as measured along the ray, which shall be denoted by r.

If one considers two different rays of the system – the one, whose starting point and direction are determined by the quantities x, y, z, ξ, η, ζ, and the other, for which these quantities will have the values $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, $\xi + \Delta\xi$, $\eta + \Delta\eta$, $\zeta + \Delta\zeta$, where Δx, Δy, etc., denote finite differences – then the relationship of both rays to each other will be determined by the following data: First, the angle ε that they make with each other, second, the length p of the line that is perpendicular to both of them – i.e., the shortest distance between them – and third, the direction of that perpendicular, and thus, the cosine of the angle that it makes with the three coordinate axes, which shall be called κ, λ, μ, and fourth, the abscissa r of that point of the first ray at which one finds the shortest distance to the second ray. As is shown in the elements of analytic geometry, these four data will be determined in the following way from the starting point and the directions of the two rays:

$$\cos\varepsilon = \xi(\xi + \Delta\xi) + \eta(\eta + \Delta\eta) + \zeta(\zeta + \Delta\zeta), \tag{1}$$

$$\sin^2\varepsilon = (\eta\,\Delta\zeta - \zeta\,\Delta\eta)^2 + (\zeta\,\Delta\xi - \xi\,\Delta\zeta)^2 + (\xi\,\Delta\eta - \eta\,\Delta\xi)^2, \tag{2}$$

$$p\sin\varepsilon = (\eta\,\Delta\zeta - \zeta\,\Delta\eta)\,\Delta x + (\zeta\,\Delta\xi - \xi\,\Delta\zeta)\,\Delta y + (\xi\,\Delta\eta - \eta\,\Delta\xi)\,\Delta z, \tag{3}$$

$$\kappa = \frac{\eta\,\Delta\zeta - \zeta\,\Delta\eta}{\sin\varepsilon}, \qquad \lambda = \frac{\zeta\,\Delta\xi - \xi\,\Delta\zeta}{\sin\varepsilon}, \qquad \mu = \frac{\xi\,\Delta\eta - \eta\,\Delta\xi}{\sin\varepsilon}, \tag{4}$$

$$p = \kappa\,\Delta x + \lambda\,\Delta y + \mu\,\Delta z, \tag{5}$$

$$\begin{aligned} r\sin\varepsilon = {} & [\mu(\eta + \Delta\eta) - \lambda(\zeta + \Delta\zeta)]\,\Delta x + [\kappa(\zeta + \Delta\zeta) - \mu(\xi + \Delta\xi)]\,\Delta y \\ & + [\lambda(\xi + \Delta\xi) - \kappa(\eta + \Delta\eta)]\,\Delta z. \end{aligned} \tag{6}$$

By means of the two equations:

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

$$(\xi + \Delta\xi)^2 + (\eta + \Delta\eta)^2 + (\zeta + \Delta\zeta)^2 = 1,$$

from which one will get the equation:

$$\xi\,\Delta\xi + \eta\,\Delta\eta + \zeta\,\Delta\zeta = -\tfrac{1}{2}(\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2), \tag{7}$$

one can also express cos ε, sin ε, and r in the following forms:

$$\cos\varepsilon = 1 - \tfrac{1}{2}(\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2), \tag{8}$$

$$\sin^2\varepsilon = \Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2 - \tfrac{1}{4}(\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2)^2, \tag{9}$$

$$\begin{aligned} r\sin^2\varepsilon &= -(\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta) \\ &\quad + \tfrac{1}{2}(\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2)\,[\Delta x\,(\xi + \Delta\xi) + \Delta y\,(\eta + \Delta\eta) + \Delta z\,(\zeta + \Delta\zeta)]. \end{aligned} \tag{10}$$

If one further considers the distance between two straight lines at any of their points that is measured by the length of a line that is drawn from the second ray to the first one in such a way that it is perpendicular to it, and calls the length of that line q, the abscissa of the point at which it is perpendicular to the first line, R, and the cosines of the angles that its direction makes with the three coordinate axes, κ', λ', μ', then analytic geometry will give the following expressions for these quantities:

$$\left\{ \begin{aligned} q\kappa' &= \Delta x - R\xi + \frac{(R-P)(\xi+\Delta\xi)}{\cos\varepsilon}, \\ q\lambda' &= \Delta y - R\eta + \frac{(R-P)(\eta+\Delta\eta}{\cos\varepsilon}, \\ q\mu' &= \Delta z - R\zeta + \frac{(R-P)(\zeta+\Delta\zeta)}{\cos\varepsilon}, \end{aligned} \right. \tag{11}$$

in which, for the sake of brevity, we have set:

$$P = \xi\,\Delta x + \eta\,\Delta y + \zeta\,\Delta z\,.$$

If one lets the second ray approach the first one infinitely closely – so the differences Δx, Δy, Δz, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ become the differentials dx, dy, dz, $d\xi$, $d\eta$, $d\zeta$ – then the distances p and q and the angle ε will become infinitely small, and will then be denoted by dp, dq, $d\varepsilon$; the infinitely small quantities of higher order will then vanish in comparison to the lower-order ones, and one will get:

$$d\varepsilon^2 = d\xi^2 + d\eta^2 + d\zeta^2, \tag{12}$$

$$\kappa = \frac{\eta\,d\zeta - \zeta\,d\eta}{d\varepsilon}, \qquad \lambda = \frac{\zeta\,d\xi - \xi\,d\zeta}{d\varepsilon}, \qquad \mu = \frac{\xi\,d\eta - \eta\,d\xi}{d\varepsilon}, \tag{13}$$

(14) $$dp = \kappa\, dx + \lambda\, dy + \mu\, dz,$$

(15) $$r = -\frac{dx\, d\xi + dy\, d\eta + dz\, d\zeta}{d\xi^2 + d\eta^2 + d\zeta^2},$$

(16) $$\begin{cases} \kappa' dq = dx + R\, d\xi - \xi(\xi\, dx + \eta\, dy + \zeta\, dz), \\ \lambda' dq = dy + R\, d\eta - \eta(\xi\, dx + \eta\, dy + \zeta\, dz), \\ \mu' dq = dz + R\, d\zeta - \zeta(\xi\, dx + \eta\, dy + \zeta\, dz). \end{cases}$$

Since $x, y, z, \xi, \eta, \zeta$ are functions of the two independent variables u and v, their differentials must be expressed in terms of their partial differential quotients with respect to u and v and the differentials du and dv. In that, the same notations shall be chosen for the first partial differential quotients and the expressions that are composed from them that *Gauss* applied in his treatise *Disquisitiones generales circa superficies curvas*, namely:

(17) $$dx = a\, du + a'\, dv, \qquad dy = b\, du + b'\, dv, \qquad dz = c\, du + c'\, dv,$$

(18) $$bc' - b'c = A, \qquad ca' - c'a = B, \qquad ab' - a'b = C,$$

(19) $$a^2 + b^2 + c^2 = E, \qquad aa' + bb' + cc' = F, \qquad a'^2 + b'^2 + c'^2 = G.$$

Furthermore, the following analogous notations shall be applied for the partial differential quotients of the quantities ξ, η, ζ, and the expressions that are composed from them:

(20) $$d\xi = \mathrm{a}\, du + \mathrm{a}'\, dv, \qquad d\eta = \mathrm{b}\, du + \mathrm{b}'\, dv, \qquad d\zeta = \mathrm{c}\, du + \mathrm{c}'\, dv,$$

(21) $$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = \mathrm{A}, \qquad \mathrm{ca}' - \mathrm{c}'\mathrm{a} = \mathrm{B}, \qquad \mathrm{ab}' - \mathrm{a}'\mathrm{b} = \mathrm{C},$$

(22) $$\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 = \mathrm{E}, \qquad \mathrm{aa}' + \mathrm{bb}' + \mathrm{cc}' = \mathrm{F}, \qquad \mathrm{a}'^2 + \mathrm{b}'^2 + \mathrm{c}'^2 = \mathrm{G},$$

and, in addition:

(23) $$\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = \mathrm{EG} - \mathrm{F}^2 = \Delta^2.$$

Furthermore, the following four expressions that are composed of partial differential quotients of x, y, z, and ξ, η, ζ will be denoted by simple symbols:

(24) $$\begin{cases} a\mathrm{a} + b\mathrm{b} + c\mathrm{c} = \mathrm{e}, \\ a'\mathrm{a} + b'\mathrm{b} + c'\mathrm{c} = \mathrm{f}, \\ a\mathrm{a}' + b\mathrm{b}' + c\mathrm{c}' = \mathrm{f}', \\ a'\mathrm{a}' + b'\mathrm{b}' + c'\mathrm{c}' = \mathrm{g}. \end{cases}$$

The quotient of the differentials of the two independent variables du and dv shall be denoted simply by t, so:

(25) $$\frac{dv}{du} = t.$$

From the equation:

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

which gives the following equations under differentiation with respect to *u* and *v*:

(26) $$\begin{cases} \xi \mathrm{a} + \eta \mathrm{b} + \zeta \mathrm{c} = 0, \\ \xi \mathrm{a}' + \eta \mathrm{b}' + \zeta \mathrm{c}' = 0, \end{cases}$$

one will also obtain the following expressions for ξ, η, ζ in terms of their partial differential quotients:

(27) $$\xi = \frac{\mathrm{A}}{\Delta}, \qquad \eta = \frac{\mathrm{B}}{\Delta}, \qquad \zeta = \frac{\mathrm{C}}{\Delta},$$

which will be applied to great advantage, but which are undetermined in the case where $\Delta = 0$. The condition $\Delta = 0$, from which it follows that A = 0, B = 0, C = 0, is valid for only a special kind of ray system that will require some slight modifications of the general method for its treatment, but which will not be done in what follows, because this ray system can also be regarded as a limiting case of the general one.

§ 2.

The limit point of the shortest distance from a ray to an infinitely close ray.

If the differentials dx, dy, dz, $d\xi$, $d\eta$, $d\zeta$ are expressed in terms of their partial differential quotients and the differentials du and dv then the expression (15) for the abscissa of the points on the first ray at which it one gets the shortest distance to an infinitely close ray will give:

(1) $$r = -\frac{\mathrm{e} + (\mathrm{f} + \mathrm{f}')t + \mathrm{g}t^2}{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2},$$

when one fixes the sign. For a well-defined value of $t = dv / du$, this expression will give the abscissa *r* of the shortest distance from the first ray to *one* well-defined infinitely-close ray; one gets the value of *r* in question for all of the infinitely-close rays around a ray when one gives *t* all possible values from $t = -\infty$ to $t = +\infty$ in succession. The denominator of that expression can be zero for none of these values of *t*, because $\mathrm{EG} - \mathrm{F}^2 = \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2$ is never negative, and because the special choice for which $\mathrm{EF} - \mathrm{G}^2$ equals zero was excluded. Therefore, the value of *r* can never be infinitely large, and it must then be always contained within certain finite limits that will be given by a maximum and a minimum of *r*. One thus has the following theorem:

The shortest distance from a ray to all of its infinitely-close rays lies on a bounded part of that ray that is bounded by two well-defined points.

When the differential quotient of r with respect to t is set to zero, that will yield the following equation for the value of t for which the two limit points correspond to the shortest distance from the ray to the infinitely-close rays:

$$(2)\qquad (\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2)(\mathrm{f} + \mathrm{f}' + 2\mathrm{g}t) - (\mathrm{e} + (\mathrm{f} + \mathrm{f}')\,t + \mathrm{g}t^2)(2\mathrm{F} + 2\mathrm{G}t) = 0,$$

or, when simplified:

$$(3)\qquad (\mathrm{E} + \mathrm{F}t)\,(\tfrac{1}{2}(\mathrm{f} + \mathrm{f}') + \mathrm{g}t) - (\mathrm{F} + \mathrm{G}t)\,(\mathrm{e} + \tfrac{1}{2}(\mathrm{f} + \mathrm{f}')\,t) = 0,$$

and when arranged in powers of t:

$$(4)\qquad (\mathrm{gF} - \tfrac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{G})\,t^2 - (\mathrm{eG} - \mathrm{gE})\,t + (\tfrac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{E} - \mathrm{eF}) = 0.$$

Let the two roots of this quadratic equation – which, as was shown above, must always be real – be t_1 and t_2, so one will have:

$$(5)\qquad t_1 + t_2 = \frac{\mathrm{eG} - \mathrm{gE}}{\mathrm{gF} - \frac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{G}}, \qquad t_1\,t_2 = \frac{\frac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{E} - \mathrm{gF}}{\mathrm{gF} - \frac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{G}},$$

from which, one will get the noteworthy equations:

$$(6)\qquad \mathrm{E} + \mathrm{F}(t_1 + t_2) + \mathrm{G}\,t_1\,t_2 = 0,$$

$$(7)\qquad \mathrm{e} + \tfrac{1}{2}(\mathrm{f} + \mathrm{f}')(t_1 + t_2) + \mathrm{g}\,t_1\,t_2 = 0,$$

and to which, one might add the following equations, which are easily derived from them:

$$(8)\qquad \mathrm{E} + 2\mathrm{F}\,t_1 + \mathrm{G}\,t_1^2 = (t_1 - t_2)(\mathrm{F} + \mathrm{G}\,t_1),$$

$$(9)\qquad \mathrm{E} + 2\mathrm{F}\,t_2 + \mathrm{G}\,t_2^2 = (t_1 - t_2)(\mathrm{F} + \mathrm{G}\,t_2),$$

$$(10)\qquad (\mathrm{F} + \mathrm{G}\,t_1)(\mathrm{F} + \mathrm{G}\,t_2) = -\Delta^2,$$

$$(11)\qquad (\mathrm{E} + 2\mathrm{F}t_1 + \mathrm{G}\,t_1^2)(\mathrm{E} + 2\mathrm{F}t_2 + \mathrm{G}\,t_2^2) = \Delta^2(t_1 - t_2)^2.$$

If one now denotes the two extreme values of the abscissa r that belong to the values $t = t_1$ and $t = t_2$ by r_1 and r_2, resp., then one will have:

(12) $$r_1 = -\frac{\mathrm{e} + (\mathrm{f} + \mathrm{f}')t_1 + \mathrm{g}t_1^2}{\mathrm{E} + 2\mathrm{F}t_1 + \mathrm{G}t_1^2},$$

(13) $$r_2 = -\frac{\mathrm{e} + (\mathrm{f} + \mathrm{f}')t_2 + \mathrm{g}t_2^2}{\mathrm{E} + 2\mathrm{F}t_2 + \mathrm{G}t_2^2},$$

which are expressions that can assume the following simpler forms by means of equations (2) and (3):

(14) $$r_1 = -\frac{\mathrm{e} + \frac{1}{2}(\mathrm{f} + \mathrm{f}')t_1}{\mathrm{E} + \mathrm{F}t_1} = \frac{\frac{1}{2}(\mathrm{f} + \mathrm{f}') + \mathrm{g}t_1}{\mathrm{F} + \mathrm{G}t_1},$$

(15) $$r_2 = -\frac{\mathrm{e} + \frac{1}{2}(\mathrm{f} + \mathrm{f}')t_2}{\mathrm{E} + \mathrm{F}t_2} = \frac{\frac{1}{2}(\mathrm{f} + \mathrm{f}') + \mathrm{g}t_2}{\mathrm{F} + \mathrm{G}t_2}.$$

If one eliminates t_1 or t_2 from these equations then one will obtain the following quadratic equation, whose roots r_1 and r_2 – which are always real – will be the abscissas of the limit points for the shortest distance from a ray to all of the infinitely-close rays:

(16) $$(\mathrm{EF} - \mathrm{F}^2)\, r^2 + (\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\,\mathrm{F} + \mathrm{eG})\, r + \mathrm{eg} - \tfrac{1}{4}(\mathrm{f} + \mathrm{f}')^2 = 0,$$

from which, it will follow that:

(17) $$r_1 + r_2 = -\frac{\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\,\mathrm{F} + \mathrm{eG}}{\Delta^2}, \qquad r_1\, r_2 = \frac{\mathrm{eg} - \frac{1}{4}(\mathrm{f} + \mathrm{f}')^2}{\Delta^2}.$$

The extent of the interval in which the shortest distance from a ray to the rays that are infinitely close to it will lie is equal to the difference between the abscissas of the two points themselves, so it is equal to $r_2 - r_1$. If one denotes this length by $2d$ and the abscissa of the midpoint of these two limit points by m then one will have:

(18) $$d = \frac{r_2 - r_1}{2}, \qquad m = \frac{r_2 + r_1}{2}.$$

§ 3.

The directions of the shortest distances and the principal planes.

Now, we shall also direct our attention to the direction that the shortest distance from a ray to the infinitely close rays will have, which is determined by the cosines of the angles that it makes with the three coordinate axes, and which were denoted by λ, μ, ν, above. If one replaces the differentials dx, dy, dz, $d\xi$, $d\eta$, $d\zeta$ in the expressions for these quantities that are given by (13), § 1 with their partial differential quotients and the differentials of the independent variables *du* and *dv*, whose quotient was denoted by *t*, then one will get:

$$
(1) \qquad \begin{cases}
\kappa = \dfrac{\eta \mathrm{c} - \zeta \mathrm{b} + (\eta \mathrm{c}' - \zeta \mathrm{b}')t}{\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}, \\[2ex]
\lambda = \dfrac{\zeta \mathrm{a} - \xi \mathrm{c} + (\zeta \mathrm{a}' - \zeta \mathrm{c}')t}{\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}, \\[2ex]
\mu = \dfrac{\xi \mathrm{b} - \eta \mathrm{a} + (\xi \mathrm{b}' - \eta \mathrm{a}')t}{\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}.
\end{cases}
$$

If one takes the expressions that were given above (17), § 1 for ξ, η, ζ, namely:

$$\xi = \frac{\mathrm{A}}{\Delta}, \qquad \eta = \frac{\mathrm{B}}{\Delta}, \qquad \zeta = \frac{\mathrm{C}}{\Delta},$$

and observes that:

$$
\begin{aligned}
&\mathrm{Bc} - \mathrm{Cb} = \mathrm{a}'\mathrm{E} - \mathrm{aF}, \qquad && \mathrm{Bc}' - \mathrm{Cb}' = \mathrm{a}'\mathrm{F} - \mathrm{aG}, \\
&\mathrm{Ca} - \mathrm{Ac} = \mathrm{b}'\mathrm{E} - \mathrm{bF}, && \mathrm{Ca}' - \mathrm{Ac}' = \mathrm{b}'\mathrm{F} - \mathrm{bG}, \\
&\mathrm{Ab} - \mathrm{Ba} = \mathrm{c}'\mathrm{E} - \mathrm{cF}, && \mathrm{Ab}' - \mathrm{Ba}' = \mathrm{c}'\mathrm{F} - \mathrm{cG}
\end{aligned}
$$

then one will get:

$$
(2) \qquad \begin{cases}
\kappa = \dfrac{\mathrm{a}'(\mathrm{E} + \mathrm{F}t) - \mathrm{a}(\mathrm{F} + \mathrm{G}t)}{\Delta\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}, \\[2ex]
\lambda = \dfrac{\mathrm{b}'(\mathrm{E} + \mathrm{F}t) - \mathrm{b}(\mathrm{F} + \mathrm{G}t)}{\Delta\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}, \\[2ex]
\mu = \dfrac{\mathrm{b}'(\mathrm{E} + \mathrm{F}t) - \mathrm{b}(\mathrm{F} + \mathrm{G}t)}{\Delta\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}.
\end{cases}
$$

If one now considers, in particular, the directions of those two shortest distances that exist at the two limit points – thus, for $t = t_1$ and $t = t_2$, for which the special values of κ, λ, μ will be denoted by κ_1, λ_1, μ_1 and κ_2, λ_2, μ_2 – then by means of equation (6), § 2, which shows that $\mathrm{E} + \mathrm{F}t_1 = - t_2 (\mathrm{F} + \mathrm{G}t_2)$, one will get the following expressions:

$$
(3) \qquad \begin{cases}
\kappa = -\dfrac{(\mathrm{a} + \mathrm{a}'t_2)(\mathrm{F} + \mathrm{G}t_1)}{\Delta V_1}, \\[2ex]
\lambda = -\dfrac{(\mathrm{b} + \mathrm{b}'t_2)(\mathrm{F} + \mathrm{G}t_1)}{\Delta V_1}, \\[2ex]
\mu = -\dfrac{(\mathrm{c} + \mathrm{c}'t_2)(\mathrm{F} + \mathrm{G}t_1)}{\Delta V_1},
\end{cases}
$$

where we have set:

$$\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2} = V_1,$$

for brevity. If one similarly denotes the corresponding root by:

$$\sqrt{\mathrm{E}+2\mathrm{F}t+\mathrm{G}t^2} = V_2$$

then one will have, from (11) and (8), § 2:

$$(4) \qquad V_1\,V_2 = \Delta(t_2 - t_1), \qquad \frac{\Delta V_1}{\mathrm{F}+\mathrm{G}t_1} = V_2, \qquad \frac{\Delta V_2}{\mathrm{F}+\mathrm{G}t_2} = -V_1,$$

and thus:

$$(5) \qquad \kappa_1 = -\frac{\mathrm{a}+\mathrm{a}'t_2}{V_2}, \qquad \lambda_1 = -\frac{\mathrm{b}+\mathrm{b}'t_2}{V_2}, \qquad \mu_1 = -\frac{\mathrm{c}+\mathrm{c}'t_2}{V_2},$$

from which, one will get the values of κ_2, λ_2, μ_2 by switching t_2 and t_1, which makes V_2 go to $-V_1$:

$$(6) \qquad \kappa_2 = \frac{\mathrm{a}+\mathrm{a}'t_1}{V_1}, \qquad \lambda_2 = \frac{\mathrm{b}+\mathrm{b}'t_1}{V_1}, \qquad \mu_2 = \frac{\mathrm{c}+\mathrm{c}'t_1}{V_1}.$$

The cosine of the angle between the directions of the shortest distance from a ray to its infinitely-close rays at the two limit points has the value:

$$\kappa_1\,\kappa_2 + \lambda_1\,\lambda_2 + \mu_1\,\mu_2 = -\frac{(\mathrm{a}+\mathrm{a}'t_2)(\mathrm{a}+\mathrm{a}'t_1)+(\mathrm{b}+\mathrm{b}'t_2)(\mathrm{b}+\mathrm{b}'t_1)+(\mathrm{c}+\mathrm{c}'t_2)(\mathrm{c}+\mathrm{c}'t_1)}{V_1V_2},$$

which will give, after carrying out the multiplications in the individual terms:

$$\kappa_1\,\kappa_2 + \lambda_1\,\lambda_2 + \mu_1\,\mu_2 = -\frac{\mathrm{E}+2\mathrm{F}(t_1+t_2)+\mathrm{G}\,t_1t_2}{V_1V_2},$$

and from equation (6), § 2 will therefore by zero, from which, it will follow that this angle is a right angle. One will then have the following theorem:

The shortest distances from a ray to the infinitely-close rays, which will lie at the two limit points, will be perpendicular to each other.

Those two planes that go through a ray that are perpendicular to the directions of the shortest distances at the two limit points shall be called the *principal planes* of that ray. These two principal planes, which, from the theorem that was just proved, will be mutually perpendicular, or the directions of the shortest distances at the two limit points that are perpendicular to them, will be chosen most conveniently from the ones that lie around a ray, along which those perpendicular directions will be measured by angles.

Let ω be the angle between the direction of the shortest distance from the first ray to an arbitrary, infinitely-close ray and the direction of the shortest distance at one of the limit points, whose abscissa equals r_1, or what amounts to the same thing, the angle of inclination between that direction and the second principal plane of the ray, so one will have:

(7) $$\cos\omega = \kappa_1\,\kappa + \lambda_1\,\lambda + \mu_1\,\mu,$$

and if κ, λ, μ for κ_1, λ_1, μ_1 are replaced with the expressions that were given by (2) and (6) then:

(8) $$\cos\omega = -\frac{\mathrm{E} + \mathrm{F}t_1 + t(\mathrm{F} + \mathrm{G}t_1)}{V_1\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}},$$

or by using equation (6), § 2:

(9) $$\cos\omega = \frac{(\mathrm{F} + \mathrm{G}t_1)(t_2 - t)}{V_1\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}.$$

From this, one will get:

(10) $$\sin\omega = \frac{\Delta(t - t_1)}{V_1\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}.,$$

(11) $$\tan\omega = \frac{\Delta(t - t_1)}{(\mathrm{F} + \mathrm{G}t_1)(t_2 - t)},$$

and as a result:

(12) $$t = \frac{\Delta t_1 \cos\omega + (\mathrm{F} + \mathrm{G}t_1)\,t_2 \sin\omega}{\Delta\cos\omega + (\mathrm{F} + \mathrm{G}t_1)\sin\omega},$$

which is a formula that allows one to replace the quotient $t = dv / du$ everywhere with the angle ω, which expresses the geometric relationship between a neighboring ray and the original ray immediately as a quotient. If one performs that substitution in the expression:

$$r = -\frac{\mathrm{e} + (\mathrm{f} + \mathrm{f}')t + \mathrm{g}t^2}{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}$$

for the abscissa of that point of a given ray at which one of the infinitely-close rays has the shortest distance from it then one will get:

(13) $$\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2 = \frac{\Delta^2 V_1^2}{(\Delta\cos\omega + (\mathrm{F} + \mathrm{G}t_1)\sin\omega)^2},$$

and one will then get:

(14) $$\mathrm{e} + (\mathrm{f} + \mathrm{f}')\,t + \mathrm{g}t^2$$
$$= \frac{\Delta^2(\mathrm{e} + (\mathrm{f} + \mathrm{f}')t_1 + \mathrm{g}t_1^2)\cos^2\omega + (\mathrm{F} + \mathrm{G}\,t_1)(\mathrm{e} + (\mathrm{f} + \mathrm{f}')t_2 + \mathrm{g}t_2^2)\sin^2\omega}{(\Delta\cos\omega + (\mathrm{F} + \mathrm{G}t_1)\sin\omega)^2}.$$

By dividing these expressions, when one makes use of formula (4), from which, one will have $\Delta^2 V_1^2 = (\mathrm{F} + \mathrm{G}t_1)^2\,V_2^2$, one will have:

(15) $$r = -\frac{\mathrm{e}+(\mathrm{f}+\mathrm{f}')t_1+\mathrm{g}t_1^2}{\mathrm{E}+2\mathrm{F}t_1+\mathrm{G}t_1^2}\cos^2\omega - \frac{\mathrm{e}+(\mathrm{f}+\mathrm{f}')t_2+\mathrm{g}t_2^2}{\mathrm{E}+2\mathrm{F}t_2+\mathrm{G}t_2^2}\sin^2\omega,$$

and by means of the expressions for r_1 and r_2 that were given by (12) and (13):

(16) $$r = r_1 \cos^2\omega + r_2 \sin^2\omega.$$

This elegant formula, which expresses a very simple relationship between the limit points of the shortest distance to a ray and the shortest distance to an arbitrary infinitely-close ray, was found by *Hamilton* in the aforementioned Supplement to his treatise *On the Theory of Systems of Rays*, in which he treated the points at which two infinitely-close rays realize the shortest distance under the name of *virtual foci*. He also was the first to establish the limits points of the shortest distance on a ray and the two mutually-perpendicular principal planes of any ray.

§ 4.

Focal points of rays, their midpoint, and focal planes.

One finds the quantities of the shortest distance dp between two infinitely-close rays and the infinitely small angle $d\varepsilon$ that these rays make with each other from the expressions that were given above by (12) and (14) by introducing the partial differential quotients and the differentials du and dv of the two independent variables, instead of the differentials dx, dy, dz, $d\xi$, $d\eta$, $d\zeta$, and by applying the values found for the quantities κ, λ, μ, which will give the following expressions:

(1) $$d\varepsilon = du\,\sqrt{\mathrm{E}+2\mathrm{F}t+\mathrm{G}t^2}\,,$$

(2) $$dp = \frac{du\,((\mathrm{f}'+\mathrm{g}t)(\mathrm{E}+\mathrm{F}t)-(\mathrm{e}+\mathrm{f}t)(\mathrm{F}+\mathrm{G}t))}{\Delta\sqrt{\mathrm{E}+2\mathrm{F}t+\mathrm{G}t^2}}\,,$$

so

(3) $$\frac{dp}{d\varepsilon} = \frac{(\mathrm{f}'+\mathrm{g}t)(\mathrm{E}+\mathrm{F}t)-(\mathrm{e}+\mathrm{f}t)(\mathrm{F}+\mathrm{G}t)}{\Delta\sqrt{\mathrm{E}+2\mathrm{F}t+\mathrm{G}t^2}}\,.$$

It follows from this that for those values of t that satisfy the equation:

(4) $$(\mathrm{f}' + \mathrm{g}t)(\mathrm{E} + \mathrm{F}t) - (\mathrm{e} + \mathrm{f}t)(\mathrm{F} + \mathrm{G}t) = 0,$$

the ray will be intersected by the infinitely close rays in question; that is, the shortest distance from the ray, which is generally a first-order infinitely small quantity, is a higher-order infinitely small quantity for this special value of t, and thus, for the same associated infinitely-close rays. When this condition equation is developed, that will give:

$$(5)\qquad (\mathrm{gF} - \mathrm{fG})\, t^2 + (\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\, \mathrm{F} - \mathrm{eG})\, t + \mathrm{f}'\, \mathrm{E} - \mathrm{e}\, \mathrm{F} = 0,$$

and when the two roots of this quadratic equation are denoted by τ_1 and τ_2, one will get:

$$(6)\qquad \tau_1 + \tau_2 = \frac{-\mathrm{gE} + (\mathrm{f} - \mathrm{f}')\, \mathrm{F} + \mathrm{eG}}{\mathrm{gF} - \mathrm{fG}}, \qquad \tau_1 \tau_2 = \frac{\mathrm{f}'\, \mathrm{E} - \mathrm{eF}}{\mathrm{gF} - \mathrm{fG}}.$$

This quadratic equation does not have the distinguishing property of the one that was treated above – viz., that its roots τ_1 and τ_2 are always real; they will be real or imaginary, moreover, according to the nature of the laws that couple the lines in space into a system. One will then have two special categories of ray systems to distinguish from each other, namely, the ones in which any ray is intersected by infinitely-close ones, and the ones in which an intersection of infinitely-close rays is nowhere to be found. As a third category of ray systems, we add the ones in which certain parts of the system will belong to one or the other of the two categories.

Those two points of a ray at which it is intersected by infinitely-close rays will be called the *focal points* of that ray. They will be real points only when τ_1 and τ_2 are real.

One finds the abscissas of the two focal points from the general expression for the abscissa of the point at which the ray finds its shortest distance to an infinitely-close ray, which was found above (1), § 2, when one gives the two well-defined values τ_1 and τ_2 to the t there. If one denotes the corresponding abscissas of the focal points by ρ_1 and ρ_2 then one will have:

$$(7)\qquad \begin{cases} \rho_1 = -\dfrac{\mathrm{e} + (\mathrm{f} + \mathrm{f}')\,\tau_1 + \mathrm{g}\,\tau_1^2}{\mathrm{E} + 2\mathrm{F}\,\tau_1 + \mathrm{G}\,\tau_1^2} \\[2ex] \rho_2 = -\dfrac{\mathrm{e} + (\mathrm{f} + \mathrm{f}')\,\tau_2 + \mathrm{g}\,\tau_2^2}{\mathrm{E} + 2\mathrm{F}\,\tau_2 + \mathrm{G}\,\tau_2^2}, \end{cases}$$

and by means of the quadratic equation (4), whose roots are τ_1 and τ_2, this expression will take on the simpler form:

$$(8)\qquad \begin{cases} \rho_1 = -\dfrac{\mathrm{e} + \mathrm{f}\,\tau_1}{\mathrm{E} + \mathrm{F}\,\tau_1} = -\dfrac{\mathrm{f}' + \mathrm{g}\,\tau_1}{\mathrm{F} + \mathrm{G}\,\tau_1}, \\[2ex] \rho_2 = -\dfrac{\mathrm{e} + \mathrm{f}\,\tau_2}{\mathrm{E} + \mathrm{F}\,\tau_2} = -\dfrac{\mathrm{f}' + \mathrm{g}\,\tau_2}{\mathrm{F} + \mathrm{G}\,\tau_2}. \end{cases}$$

If one eliminates τ_1 or τ_2 from these equations then one will obtain the following quadratic equation, whose roots are ρ_1 and ρ_2:

$$(9)\qquad (\mathrm{EG} - \mathrm{F}^2)\, r^2 + (\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\, \mathrm{F} + \mathrm{eG})\, r + \mathrm{eG} - \mathrm{ff}' = 0,$$

so one will have:

$$(10)\qquad \rho_1 + \rho_2 = -\frac{\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\, \mathrm{F} + \mathrm{eG}}{\Delta^2}, \qquad \rho_1\, \rho_2 = \frac{\mathrm{eg} - \mathrm{ff}'}{\Delta^2}.$$

If one compares this quadratic equation, whose roots ρ_1 and ρ_2 are the abscissas of the two focal points, with the one whose roots r_1 and r_2 are the abscissas of the limit points of the shortest distance then one will get:

$$\text{(11)} \qquad \rho_1 + \rho_2 = r_1 + r_2 ,$$

$$\text{(12)} \qquad \rho_1 \, \rho_2 = r_1 \, r_2 + \frac{(\mathrm{f} - \mathrm{f}')^2}{4\Delta^2} .$$

The first of these two equations gives the following theorem:

The midpoint of the two focal points of any ray will coincide with the midpoint of the two limit points of the shortest distances.

The common midpoint of the two focal points and the two limit points shall be called the *midpoint of the ray*. Let the distance from the focal points to the midpoint be equal to δ, so:

$$\text{(13)} \qquad \delta = \frac{\rho_2 - \rho_1}{2} .$$

The four quantities r_1, r_2, ρ_1, ρ_2 can be expressed in terms of the three quantities m, d, and δ; namely, from equation (11) and the two equations (18), § 2, one will get:

$$\text{(14)} \qquad \begin{cases} r_2 = m + d, & r_1 = m - d, \\ \rho_2 = m + \delta, & \rho_1 = m - \delta. \end{cases}$$

Equation (12) will then give:

$$\text{(15)} \qquad d^2 - \delta^2 = \frac{(\mathrm{f} - \mathrm{f}')^2}{4\Delta^2} ,$$

from which, it will then follow that *the distance from the two focal points to the midpoint is never greater than the distance from the two limit points of the shortest distance to the midpoint*, so the focal points can only lie between the limit points of the shortest distance or at most, coincide with them.

The two planes that go through a ray and one of the two infinitely-close rays that intersect the former ray shall be called the *focal planes* of that ray.

Focal planes will exist as real planes only when the focal points are real, so the positions of them with respect to each other and with respect to the principal planes will be determined most simply by the equation $r = r_1 \cos^2 \omega + r_2 \sin^2 \omega$, which will give:

$$\text{(16)} \qquad \cos^2 \omega = \frac{r_2 - r}{r_2 - r_1} , \qquad \sin^2 \omega = \frac{r - r_1}{r_2 - r_1} .$$

Namely, if one takes $r = \rho_1$ then ω will be the angle that the first focal plane makes with the first principal plane, and if takes $r = \rho_2$ then ω will be the angle that the second focal plane makes with the first principal plane. If one denotes these two angles with ω_1 and ω_2 then the difference $\omega_2 - \omega_1$ will give the angle that the two focal planes make with each other, which shall be denoted by γ. One will thus have:

$$(17)\qquad \begin{cases} \cos^2\omega_1 = \dfrac{r_1 - \rho_1}{r_2 - r_1}, & \sin^2\omega_1 = \dfrac{\rho_1 - r_1}{r_2 - r_1}, \\ \cos^2\omega_2 = \dfrac{r_2 - \rho_2}{r_2 - r_1}, & \sin^2\omega_2 = \dfrac{\rho_2 - r_2}{r_2 - r_1}, \end{cases}$$

and when one expresses the abscissas of the focal points and limit points in terms of the three quantities m, d, δ using equations (14), one will get:

$$(18)\qquad \begin{cases} \cos\omega_1 = \sin\omega_2 = \sqrt{\dfrac{d+\delta}{2d}}, \\ \sin\omega_1 = \sin\omega_2 = \sqrt{\dfrac{d-\delta}{2d}}. \end{cases}$$

Therefore, since $\omega_1 = \frac{1}{2}\pi - \omega_2$, and due to the perpendicular orientation of the two principal planes with respect to each other, the angle between the second focal plane and the second principal plane will be equal to $\frac{1}{2}\pi - \omega_2$, and thus equal to the angle ω_1 between the first focal plane and the first principal plane, and it will follow that:

The two focal planes of any ray will lie symmetrically with respect to its two principal planes, in such a way that the bisecting plane of the angle between the focal planes will be the same as the bisecting plane of the right angle that the two principal planes define.

For the angle $\gamma = \omega_2 - \omega_1$ between the two focal planes, one will have, since $\omega_1 + \omega_2 = \frac{1}{2}\pi$:

$$(19)\qquad \begin{cases} \gamma = \frac{1}{2}\pi - 2\omega_1 & = 2\omega_2 - \frac{1}{2}\pi, \\ \omega_1 = \frac{1}{4}\pi - \frac{1}{2}\gamma, & \omega_2 = \frac{1}{4}\pi + \frac{1}{2}\gamma, \end{cases}$$

so $\sin\gamma = \cos 2\omega_1 = \cos^2\omega_1 - \sin^2\omega_1$ and $\cos\gamma = \sin 2\omega_1 = 2\sin\omega_1\cos\omega_1$; equations (18) will then give:

$$(20)\qquad \sin\gamma = \frac{\delta}{d}, \qquad \cos\gamma = \frac{\sqrt{d^2 - \delta^2}}{d}.$$

§ 5.

The surfaces that are connected with any ray system.

The five well-defined points for any line of the system, namely, the two limit points of the shortest distance, the two focal points, and the midpoint, have geometric loci over all rays of the system that consist of five surfaces that are determined completely by the ray system, and have a close relationship to it.

The two surfaces on which the limit points of the shortest distance lie are ordinarily represented by only one and the same equation, so they can also be regarded as two different parts or shells of one surface; however, since it is quite necessary to distinguish one from the other, in all of what follows they shall be regarded as two surfaces and will be denoted by F_1' and F_2'. *These two surfaces will divide all of space in such a way that the shortest distances to all infinitely-close rays of the entire system will lie between them, but none of them outside, however.*

If one goes from any ray of the system to the infinitely-close ray whose shortest distance from it lies on the surface F_1, and then goes from that one to the next one whose shortest distance from it lies in F_1, and so forth, then all of these successive rays will collectively define a rectilinear surface O_1 whose intersection a_1 with the surface F_1 will be the curve of the rectilinear surface O_1 on which the shortest distance between any two successive straight lines will lie. The same rectilinear surface O_1 will also intersect the surface F_2 in a certain curve b_2 . If one does the same thing with the surface F_2 then one will obtain a rectilinear surface O_2 for which the shortest distance between two infinitely-close straight lines will lie on F_2 on a curve a_2 , and the rectilinear surface O_2 will also intersect F_1 in a certain curve b_1 . Because all of this will be true for any ray of the system, from which one would like to start, one will have an entire family of rectilinear surfaces O_1 whose curves of shortest distance between any two infinitely-close straight lines will yield a family of curves a_1 in the surface F_1, and which will intersect the surface F_2 in a family of curves b_2 . Likewise, one will have a second family of rectilinear surfaces O_2 that will have their curves a_2 of shortest distance lying between infinitely-close straight lines on F_2 , and which will intersect F_1 in a family of curves b_1 .

If x', y', z' are the coordinates of the first limit point of the shortest distance for the ray that starts at the point x, y, z then one will have:

$$x' = x + r_1\,\xi, \quad y' = y + r_1\,\eta\,, \quad z' = z + r_1\,\zeta$$

as the equations of the surface F_1, in such a form that the coordinates of any point of that surface will be expressed as functions of the two independent variables u and v. In the same way, one will have:

$$x' = x + r_2\,\xi, \quad y' = y + r_2\,\eta\,, \quad z' = z + r_2\,\zeta$$

as the equations of the surface F_2 . In order to find the families of rectilinear surfaces O_1 and O_2, one must integrate the two differential equations:

$$\frac{dv}{du} = t_1 , \qquad \frac{dv}{du} = t_2 .$$

If the complete integrals of them, which will include arbitrary constants, have been found, and one eliminates the quantities u and v for any ray by means of one of these two integral equations from the two equations:

$$\frac{x' - x}{\xi} = \frac{y' - y}{\eta} = \frac{z' - z}{\zeta}$$

then one will obtain an equation for the coordinates x', y', z' that will include an arbitrary constant and will represent the entire family of rectilinear surfaces O_1 or O_2 , according to whether one or the other integral equation was applied. One will obtain the two families of curves a_1 and b_1 on F_1 and a_2 and b_2 in F_2 immediately when one couples the three equations for one of these two surfaces with one of the two integral equations.

The two surfaces on which the focal points of any ray will lie, which will be called the *focal surfaces of the ray system* and which will be briefly denoted by Φ_1 and Φ_2 here, will exist as real surfaces only when the rays have real focal points and the two roots τ_1 and τ_2 of the quadratic equation (5), § 4 have real values.

If one advances from an arbitrary ray to the infinitely-close ray that intersects a focal point that lies in Φ_1, and then proceeds further from that one to the one that cuts the focal point that lies on Φ_1, and so forth, then one will obtain a sequence of rays, each of which will intersect the foregoing ones at a point of the surface Φ_1 that will therefore collectively define a developable surface whose curve of regression will lie on the surface Φ_1, and which will also intersect the surface Φ_2 in a certain curve. This developable surface shall be denoted by Ω_1, its curve of regression by α_1, and its intersection curve with the surface Φ_2 by β_2 . Because one can then start from any arbitrary ray of the system, one will obtain an entire family of developable surfaces Ω_1 whose regression points will define a family of curves α_1 on the surface Φ_1, and which will determine a family of curves β_2 on the surface Φ_2 . Likewise, starting from the focal points of the rays that lie on the surface Φ_2, one will obtain a second family of developable surfaces Ω_2 whose curves of regression α_2 will be a family of curves that lie on the surface Φ_2, and which will determine a family of curves β_1 on the surface. Thus:

Any system of lines that has real focal surfaces can be composed, in two different ways, into a family of developable surfaces whose curves of regression have the two focal surfaces for their geometric loci.

Because the curve of regression α_1 of the developable surface Φ_1, as such, will contact all of the rays that lie in Ω_2, and because it will lie on the surface Φ_1, it will then follow that all rays of any developable surfaces Ω_1, and so all of the rays of the system, will contact the surface Φ_1 . It likewise also follows that all rays of the system must contact the other focal surface Φ_2 . One will then have the following theorem:

All rays of a system that has real focal points will be common tangents to the two focal surfaces.

As an immediate consequence of this theorem, the following theorem deserves to be mentioned:

Any ray system that has real focal surfaces can be defined to be the common tangents of two surfaces or also as the system of all double tangents to one and the same surface.

In order to completely determine a system, one can also think of just one of its two focal surfaces – e.g., Φ_1 – as having been given, and similarly, the family of curves α_1 on it; thus:

Any ray system that has real focal surfaces can be defined geometrically to be the system of all tangents to a family of curves that lie on one surface.

Because the rays that lie on a developable surface Ω_1 will also contact the surface Ω_2, it will then follow that the curve β_2 that it has in common with them must be a contact curve of the two surfaces. It will likewise follow that any developable surface Ω_2 will contact the surface Φ_1 along an entire curve; i.e., it will envelop it. Thus:

Either of the two focal surfaces will be enveloped by one of the two families of developable surfaces into which all of the rays of the system can be composed.

Since, from a well-known theorem, the generating straight lines of a developable surface that contact another surface along an entire curve will be the conjugate tangents to the tangents of that curve, it will then follow that:

The two families of curves that are determined by the two families of developable surfaces on the focal surfaces of a ray system will intersect on either of the two focal surfaces in conjugate directions.

If the two focal surfaces Φ_1 and Φ_2 intersect then any tangent of the intersection curve will be one of the rays of the system, and therefore, a tangent to one of the curves α_1 . The intersection curve and the curve α_1 will thus have a common tangent, and in fact, at the same point. The intersection curve will then contact the curve α_1, and because that will be true for all of the various tangents to the intersection curve, it will then follow that the intersection curve of all curves will contact the family α_1. It will likewise follow that the intersection curve will also contact all curves of the family α_2 on Φ_2 . One will then have the following theorem:

The intersection curve of the two focal surfaces is the enveloping curve – or boundary curve – for all of the curves of regression that line on the two focal surfaces of developable surfaces that can be composed of the rays of the system.

One can obtain the equations of the two focal surfaces in the same way as the equations for the limit surfaces of the shortest distance were obtained above, with the help of the abscissas of the focal points ρ_1 and ρ_2, namely:

$$x' = x + \rho_1\,\xi, \qquad y' = y + \rho_1\,\eta, \qquad z' = z + \rho_1\,\zeta,$$

and

$$x' = x + \rho_2\,\xi, \qquad y' = y + \rho_2\,\eta, \qquad z' = z + \rho_2\,\zeta.$$

One obtains the two families of developable surfaces Ω_1 and Ω_2, and the families of curves α_1, β_1 on Φ_1 and α_2, β_2 on Φ_2 by the complete integration of the differential equations:

$$\frac{dv}{du} = \tau_1, \qquad \frac{dv}{du} = \tau_2,$$

in the same way as was shown above for the surfaces $O_1$and O_2, along with the system of curves a_1, b_1 on F_1 and a_2, b_2 on F_2 .

Finally, as far as the (always real) surface on which the midpoints of all rays of the system will lie, and which will be called the *middle surface* of the ray system for that reason, it is especially important that it can be chosen most conveniently to be the surface from which all rays of the system are considered to start. In fact, if one calculates the abscissas of the points on the individual rays of the middle surface then one will have:

$$r_1 = -\,r_2\,, \qquad \rho_1 = -\,\rho_2\,, \qquad \mathrm{gE} - (\mathrm{f} + \mathrm{f}')\,\mathrm{F} + \mathrm{eG} = 0,$$

from which a not-inconsiderable simplification will come about.

One obtains the equations of the middle surface from the expression for the abscissa of the midpoint:

$$m = \frac{r_1 + r_2}{2} = -\,\frac{\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\mathrm{F} + \mathrm{eG}}{2\Delta^2},$$

namely:

$$x' = x + m\,\xi, \quad y' = y + m\,\eta, \quad z' = z + m\,\zeta.$$

All of these surfaces that are closely-linked with the ray system – viz., the limit surfaces of the shortest distance, the focal surfaces, and the middle surface – can degenerate into lines or even points in special cases, and some of these surfaces can also vanish at infinity, or also unite with each other in such a way that they cover themselves. The systems of normals to a surface for which the two focal surfaces coincide with the limit surfaces of shortest distance also belong the various special types of ray systems that appear in this way as the limit surfaces of general ones. The relationship between these special kinds of ray systems to the general case shall be treated thoroughly later on. In addition, the type of ray system deserves a special mention here for which $\Delta = 0$ and, at the same time, A = 0, B = 0, C = 0, which must be excluded from the general examination, because the expressions for ξ, η, ζ in terms of partial differential quotients (27), § 1 will yield indeterminate values for them. For this special kind of ray system, the limit surfaces of shortest distance will both vanish at infinity, and likewise the middle surface will vanish at infinity, but only one of the two focal surfaces will be lost at infinity, while the other one will remain a finitely-determined surface. Of the two families of developable surface that can be composed of the rays of such a system, the one of them whose curve of regression lies on the infinitely-distant focal surface will contain only cylindrical surfaces. As one can infer from this, such a system can be represented geometrically as the system of all those tangents to a surface that are parallel to the tangents of any of those given curves.

§ 6.

The measure of density.

If one considers three quantities ξ, η, ζ that satisfy the equation:

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

to be the rectangular coordinates of a sphere whose radius is equal to one then one will have a point on the sphere that corresponds to each ray of the system, and a continuous curve on the sphere that corresponds to any continuous sequence of rays. If one now draws a plane through any point of a line that is perpendicular to it and draws a curve in that plane then the family of rays that go through that curve will correspond to a curve on the sphere. If one now takes any curve such that its individual points are separated from the base point of the first ray by only infinitely little, and such that it circumscribes an infinitely small surface that lies around that point then one will likewise obtain a closed curve around an infinitely small surface as the corresponding curve on the sphere. The relationship between these two infinitely small surfaces, which in the case where the ray system is a system of normals to a surface and the perpendicular plane is a tangent plane to it was defined by *Gauss* to be the measure of curvature of that surface, also has the same importance for the most general system, not as a measure of the curvature, but as a measure of the *density* of the ray system. When a plane goes through any point of line that is perpendicular to it, and in it, a curve that is infinitely close to the ray is assumed, whose surface is equal to f, and the surface of the corresponding curve on the sphere is equal to φ then φ / f shall be called the *density measure of the ray system at this point*.

Let dq be the infinitely-small distance from a point of the curve f to the base point of the ray that is perpendicular to the plane of that curve, which is given by the quantities x, y, z, ξ, η, ζ, and the abscissa R, and furthermore, let κ', λ', μ' be the cosines of the angles that dq defines with the three coordinate axes. Furthermore, let the ray that goes through the other end point of dq be determined by the quantities $x + dx$, $y + dy$, $z + dz$, $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$. From (16), § 1, one will then have the equations:

$$(1) \qquad \left\{ \begin{aligned} \kappa' dq &= dx + R\, d\xi - \xi(\xi\, dx + \eta\, dy + \zeta\, dz), \\ \lambda' dq &= dy + R\, d\eta - \eta(\xi\, dx + \eta\, dy + \zeta\, dz), \\ \zeta' dq &= dz + R\, d\zeta - \zeta(\xi\, dx + \eta\, dy + \zeta\, dz). \end{aligned} \right.$$

Furthermore, let α be the angle that dq makes with a perpendicular to the first principal plane, and then let $\frac{1}{2}\pi - \alpha$ be the angle that it makes with a perpendicular to the second principal plane, so one will have:

$$(2) \qquad \begin{cases} \cos\alpha = \kappa_1\kappa' + \lambda_1\lambda' + \mu_1\mu', \\ \sin\alpha = \kappa_2\kappa' + \lambda_2\lambda' + \mu_2\mu'. \end{cases}$$

If one multiplies these two equations by dq and inserts the values of $\kappa' dq$, $\lambda' dq$, $\mu' dq$ from (1), when one observes that:

$$\kappa_1\xi + \lambda_1\eta + \mu_1\zeta = 0,$$
$$\kappa_2\xi + \lambda_2\eta + \mu_2\zeta = 0,$$

then one will obtain:

$$(3) \qquad \begin{cases} dq\cos\alpha = \kappa_1 dx + \lambda_1 dy + \mu_1 dz + R(\kappa_1 d\xi + \lambda_1 d\eta + \mu_1 d\zeta), \\ dq\sin\alpha = \kappa_2 dx + \lambda_2 dy + \mu_2 dz + R(\kappa_2 d\xi + \lambda_2 d\eta + \mu_2 d\zeta). \end{cases}$$

If one now replaces κ_1, λ_1, μ_1 and κ_2, λ_2, μ_2 with their values that were found in (5) and (6) of § 3 and expresses the differentials dx, dy, dz, $d\xi$, $d\eta$, $d\zeta$ in terms of their partial differential quotients and the differentials du and dv of the independent variables then one will get:

$$(4) \qquad \begin{cases} dq\cos\alpha = -A_2 du - B_2 dv, \\ dq\sin\alpha = +A_2 du + B_2 dv, \end{cases}$$

where, for the sake of brevity, we have set:

$$A_1 = \frac{\mathrm{e} + \mathrm{f}' t_1 + R(\mathrm{E} + \mathrm{F} t_1)}{V_1} \qquad A_2 = \frac{\mathrm{e} + \mathrm{f}' t_2 + R(\mathrm{E} + \mathrm{F} t_2)}{V_2},$$

$$B_1 = \frac{\mathrm{f} + \mathrm{g} t_1 + R(\mathrm{F} + \mathrm{G} t_1)}{V_1} \qquad B_2 = \frac{\mathrm{f} + \mathrm{g} t_2 + R(\mathrm{F} + \mathrm{G} t_2)}{V_2}.$$

From these two equations, it will follow, by division, that:

$$(5) \qquad \tan\alpha = -\frac{A_1 + B_1 t}{A_2 + B_2 t},$$

and from that:

$$(6) \qquad t = -\frac{A_1\cos\alpha + A_2\sin\alpha}{B_1\cos\alpha + B_2\sin\alpha}.$$

Now let $d\sigma$ be the arc length element on the sphere that corresponds to dq, so since the coordinates of its endpoints are ξ, η, ζ and $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$, one will have:

(7) $$d\sigma = \sqrt{d\xi^2 + d\eta^2 + d\zeta^2} = du\,\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}\,.$$

The cosines of the angles that the element $d\sigma$ on the sphere makes with the three coordinate axes, namely:

$$\frac{d\xi}{d\sigma}, \quad \frac{d\eta}{d\sigma}, \quad \frac{d\zeta}{d\sigma},$$

will thus be equal to:

(8) $$\frac{\mathrm{a} + \mathrm{a}'t}{\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}, \qquad \frac{\mathrm{b} + \mathrm{b}'t}{\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}, \qquad \frac{\mathrm{c} + \mathrm{c}'t}{\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}}\,.$$

Now, if t_0 denotes the value of t for $\alpha = 0$ then from equation (6) one will have:

(9) $$t_0 = -\frac{A_1}{B_1},$$

and if the angle on the sphere that corresponds to the angle α is denoted by α' then, from the known directions of its two sides, one will get:

(10) $$\cos\alpha' = \frac{(\mathrm{a} + \mathrm{a}'t_0)(\mathrm{a} + \mathrm{a}'t) + (\mathrm{b} + \mathrm{b}'t_0)(\mathrm{b} + \mathrm{b}'t) + (\mathrm{c} + \mathrm{c}'t_0)(\mathrm{c} + \mathrm{c}'t)}{\sqrt{\mathrm{E} + 2\mathrm{F}t_0 + \mathrm{G}t_0^2}\,\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}},$$

or, when simplified:

(11) $$\cos\alpha' = \frac{\mathrm{E} + \mathrm{F}t_0 + (\mathrm{F} + \mathrm{G}t_0)\,t}{\sqrt{\mathrm{E} + 2\mathrm{F}t_0 + \mathrm{G}t_0^2}\,\sqrt{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2}},$$

from which, one will derive the following expression for $\tan\alpha'$:

(12) $$\tan\alpha' = \frac{\Delta(t - t_0)}{\mathrm{E} + \mathrm{F}t_0 + (\mathrm{F} + \mathrm{G}t_0)\,t},$$

which gives, when differentiated:

(13) $$d\alpha' = \frac{\Delta\,dt}{\mathrm{E} + 2\mathrm{F}t + \mathrm{G}t^2},$$

and as a result, when multiplied by $d\sigma^2$, from equation (7), one will get:

(14) $$d\sigma^2\,d\alpha' = \Delta\,du^2\,dt.$$

By differentiating equation (6), one will further obtain:

(15) $$dt = \frac{(A_1 B_2 - A_1 B_1)\, d\alpha}{(B_1 \cos\alpha + B_2 \sin\alpha)^2},$$

and from the first of the two equations (4), when $dv / du = t$ will be expressed in terms of α by using equation (6):

(16) $$dq = \frac{(A_1 B_2 - A_1 B_1)\, du}{B_1 \cos\alpha + B_2 \sin\alpha},$$

so:

(17) $$dq^2\, d\alpha = (A_1\, B_2 - A_2\, B_1)\, du^2\, dt,$$

and when this equation is coupled with (14), that will give:

(18) $$d\sigma^2\, d\alpha' = \frac{\Delta}{A_1 B_2 - A_2 B_1} \cdot dq^2\, d\alpha.$$

Since the line dq is the radius vector for the infinitely-small curve f, and α is the associated angle, and for the infinitely-small curve φ on the sphere, $d\sigma$ is the radius vector and α' is the associated angle, one will have:

(19) $$f = \tfrac{1}{2}\int_0^{2\pi} dq^2 d\alpha, \qquad \varphi = \tfrac{1}{2}\int_0^{2\pi} d\sigma^2 d\alpha'.$$

The integration of equation (18) between the limits $\alpha = 0$ to $\alpha = 2\pi$, which correspond to the same limits on α', will then give:

(20) $$\varphi = \frac{\Delta}{A_1 B_2 - A_2 B_1} \cdot f.$$

If one now denotes the density measure by Θ, such that $\Theta = \varphi / f$, then one will have the following expression for it:

(21) $$\Theta = \frac{\Delta}{A_1 B_2 - A_2 B_1}.$$

From the values of the quantities A_1, B_1, A_2, B_2 that are given by (4), one will obtain:

$$A_1\, B_2 - A_2\, B_1 = (\mathrm{eg} - \mathrm{ff}' + (\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\,\mathrm{F} + \mathrm{eG})\, R + \Delta^2 R^2);$$

however, from the values for the abscissas ρ_1 and ρ_2 that were found in § 4, (10) for the two focal points, one will have:

$$\mathrm{eg} - \mathrm{ff}' = \rho_1 \rho_2\, \Delta^2,$$
$$\mathrm{gE} - (\mathrm{f} + \mathrm{f}')\,\mathrm{F} + \mathrm{eG} = -(\rho_1 + \rho_2)\, \Delta^2,$$

and since, from (4), § 3, one will have $V_1 V_2 = \Delta(t_2 - t_1)$, one will have:

$$A_1 B_2 - A_2 B_1 = \Delta(\rho_1\rho_2 - (\rho_1 + \rho_2) R + R^2). \tag{22}$$

The expression for the density measure Θ will thus assume the following simple form:

$$\Theta = \frac{1}{\rho_1\rho_2 - (\rho_1 + \rho_2)R + R^2}, \tag{23}$$

or

$$\Theta = \frac{1}{(\rho_1 - R)(\rho_2 - R)}. \tag{24}$$

The density measure at any point of a ray is then equal to the reciprocal value of the product of the distances from that point to the two focal points of the ray.

The density measure is always real, even when the two focal points are imaginary. For ray systems with real focal surfaces, the density measure will be positive for all points that lie outside of the two focal surfaces, negative for the ones that lie between them, and, as is easy to see from the their expressions, it will attain its largest value at the midpoint of any ray, but it will be infinitely large at the focal points. The density measure is always positive for ray systems with imaginary focal points, and has its maximum at the midpoint of any ray.

If one groups together all of the rays that are infinitely close to a given ray that go through the infinitely-small surface that is perpendicular to the ray and denoted by f then they will define an infinitely-thin *ray bundle* that is bounded by those rectilinear surfaces whose generating straight lines are the rays that go through the curve that circumscribes the surface f. The infinitely-small surface is a *cross-section* of this infinitely-thin ray bundle, and, in fact, the cross-section that belongs to the abscissa R. If one now considers a second perpendicular cross-section f' whose abscissa is equal to R' then the same corresponding infinitely-small surface φ on the sphere that belongs to f will belong to it, because all of the rays that go through the boundary curve of f will also go through the boundary curve of f'. Therefore, when the density measure at the point whose abscissa is R' is denoted by Θ', one will have:

$$\frac{\varphi}{f'} = \Theta', \qquad \frac{\varphi}{f} = \Theta,$$

and it will follow that:

$$\frac{f}{f'} = \frac{\Theta'}{\Theta}. \tag{25}$$

Therefore: *The surface areas of two cross-sections of an infinitely-thin ray bundle will behave inversely to the density measures at these places in the ray bundle*.

If one considers, not merely the density measures, but also the densities themselves that the rays of an infinitely-thin ray bundle have at the various places, then it will be clear that they must have the opposite relationship to the surface areas of the cross-sections of the ray bundles. All of the rays that are contained in the ray bundle will then spread out in the cross-section over the entire surface of it, and must then be denser in the same ratio as the cross-sections are smaller. It follows from this that *the densities at the different places in one and the same infinitely-thin ray bundle will behave like the associated density measures.* The terminology "density measure" will then be justified completely by that fact.

For two points that lie on *different* rays or on an infinitely-thin ray bundle, the ratio of the densities of the rays is not necessarily the same as the ratio of the density measures. One recognizes this most clearly in the simplest system, for which all rays emanate from one and the same point, which can be so arranged that the rays go in all the different directions with equal densities, or also such that the density is a function of the direction. In the first case, the density will be the same for all equally-distant points from the starting point, and will therefore be everywhere proportional to the density measure, but in the second case the density will not be dependent upon just the distance from the starting point, but also on a function of direction. In general, if the ray system – as was assumed above – is determined in such a way that one ray in a well-defined direction goes through each point of a surface that is chosen to be a geometric locus of the starting points of all rays then the density of the rays on this entire surface can be determined in some way as a function of the coordinates of the starting point x, y, z, or what amounts to the same thing, as a function of the two independent variables u and v, and the density of the rays at all points of the entire system to one of them will be first completely determined by that relationship. The density itself will be, in turn, equal to the density measure, multiplied by a function of u and v that does not involve the abscissa R, and for that reason, will be the same for all of the various points of a ray. When this function is constant, and as a result, the density is proportional to the density measure at all points of the system, the ray system can be referred to as *homogeneous* in relation to the density of rays.

All of the points on the various rays of a system that have the same well-defined value of the density measure will lie on a well-defined surface that shall be called a *surface of equal density measure.* Since one can give all possible values to the density measure, it will then follow that this will give an entire family of surfaces of equal density measure on a ray system. All of these surfaces will be determined very simply through the expression for the density measure that is given by (23), from which, one will have:

$$R^2 - (\rho_1 + \rho_2)\, R + \rho_1\, \rho_2 = \frac{1}{\Theta}.$$

If one takes Θ to be constant, and solves this quadratic equation for R, from which, one will get:

$$R = \frac{\rho_1 + \rho_2}{2} \pm \sqrt{\left(\frac{\rho_1 - \rho_2}{2}\right)^2 + \frac{1}{\Theta}}, \tag{26}$$

then for those two values of R:

$$x' = x + R\,\xi, \quad y' = y + R\,\eta, \quad z' = z + R\,\zeta \tag{27}$$

will be the coordinates of all of the points of the system whose density measure has the constant value Θ. They will then give the equations for the surfaces of equal density measure in such a way that the coordinates of any point of these surfaces will be determined as functions of the two independent variables *u* and *v*. In order for these surfaces to be real, it is necessary and sufficient that the constant value of 1 / Θ lie between the limits $-\left(\frac{\rho_1 - \rho_2}{2}\right)^2$ and + ∞. For the value Θ = ∞, *R* will then be real when the two focal points are real, and one will have either $R = \rho_1$ or $R = \rho_2$, from which, it will follow that the two focal surfaces belong to the surface of equal density measure for which it is infinitely large.

When the two focal surfaces are real and are given such that all rays of the system can be considered to be their common tangents, one can construct all of the surfaces of equal density measure very easily, when one constructs a third point to the two contact points of a ray that are its focal points, whose distances from the two contact points have a constant product. When the given value of that product is positive, that point must be taken outside of the two focal points, and inside of them when it is negative.

§ 7.

The rotation angle of infinitely-close rays.

When two straight lines in space are given, and one drops perpendiculars from two different points of the second line to the first line, whose base points on it might lie at *a* and *b*, then the angle between these two perpendiculars shall be called the *rotation angle of the second line around the first one for the line segment from a to b*. The rotation angle for the whole infinite length of the first line is, by this definition, equal to two right angles, while the rotation angle for finite line segments will all be smaller than two right angles. If the two straight lines lie in a plane then the rotation angle for any line segment will be equal to zero or two right angles, according to whether that line segment does or does not include that intersection point of the two lines, resp. If *a*, *b*, *c* are three points of the first straight line then the rotation angle from *b* to *c* will be equal to the difference between the two rotation angles from *a* to *c* and *a* to *b*, so all of the rotation angles for arbitrarily-limited line segments on the first line will be given by the rotation angle that is computed at a particular point.

In order to now investigate the rotation that a certain ray will make relative to infinitely-close rays of the system, the rotation angle shall be calculated at the starting point of the ray, whose abscissa will be equal to zero. Let *dq* be the length of a perpendicular that goes from one of the infinitely-close rays to a given ray, which meets it at the point whose abscissa is *R*, and *a* is the angle that this perpendicular makes with a perpendicular to the first principal plane. Furthermore, let dq_0 be the length, and let α_0 be the corresponding angle of the perpendicular that meets the given ray at the starting point, whose abscissa is zero, so one will have, as was shown in § 6, (4), the equations:

(1) $$\begin{cases} dq\cos\alpha = -A_2\,du - B_2\,dv, \\ dq\sin\alpha = +A_1\,du + B_1\,dv, \end{cases}$$

where:

$$A_1 = \frac{\mathrm{e} + \mathrm{f}'t_1 + R(\mathrm{E} + \mathrm{F}t_1)}{V_1}, \qquad A_2 = \frac{\mathrm{e} + \mathrm{f}'t_2 + R(\mathrm{E} + \mathrm{F}t_2)}{V_2},$$

$$B_1 = \frac{\mathrm{f} + \mathrm{g}\,t_1 + R(\mathrm{F} + \mathrm{G}t_1)}{V_1}, \qquad B_2 = \frac{\mathrm{f} + \mathrm{g}\,t_2 + R(\mathrm{F} + \mathrm{G}t_2)}{V_2},$$

and, in turn, for $R = 0$:

(2) $$\begin{cases} dq_0\cos\alpha_0 = -\dfrac{\mathrm{e} + \mathrm{f}'t_2}{V_2}\,du - \dfrac{\mathrm{f} + \mathrm{g}\,t_2}{V_2}\,dv, \\[2ex] dq_0\sin\alpha_0 = +\dfrac{\mathrm{e} + \mathrm{f}'t_1}{V_1}\,du + \dfrac{\mathrm{f} + \mathrm{g}\,t_1}{V_1}\,dv, \end{cases}$$

and as a result:

(3) $$\begin{cases} dq\cos\alpha - dq_0\cos\alpha_0 = -\dfrac{R(\mathrm{E} + \mathrm{F}t_2)}{V_2}\,du - \dfrac{R(\mathrm{F} + \mathrm{G}\,t_2)}{V_2}\,dv, \\[2ex] dq\sin\alpha - dq_0\sin\alpha_0 = +\dfrac{R(\mathrm{E} + \mathrm{F}t_1)}{V_1}\,du + \dfrac{R(\mathrm{F} + \mathrm{G}\,t_1)}{V_1}\,dv. \end{cases}$$

One will get the following values for the differentials du and dv from these two equations:

(4) $$\begin{cases} du = \dfrac{dq\sin\alpha - dq_0\sin\alpha_0}{RV_1} - \dfrac{dq\cos\alpha - dq_0\cos\alpha_0}{RV_2}, \\[2ex] dv = \dfrac{(dq\sin\alpha - dq_0\sin\alpha_0)\,t_1}{RV_1} - \dfrac{(dq\cos\alpha - dq_0\cos\alpha_0)\,t_2}{RV_2} \end{cases}$$

and if one substitutes these two values in the two equations (2), when one observes that:

$$\mathrm{e} + (\mathrm{f} + \mathrm{f}')\,t_1 + \mathrm{g}\,t_1^2 = -\,r_1V_1^2, \qquad \mathrm{e} + (\mathrm{f} + \mathrm{f}')\,t_2 + \mathrm{g}\,t_2^2 = -\,r_1V_2^2,$$

$$\mathrm{e} + \tfrac{1}{2}(\mathrm{f} + \mathrm{f}')(t_1 + t_2) + \mathrm{g}\,t_1t_2 = -\,r_1V_1^2, \qquad V_1\,V_2 = \Delta(t_2 - t_1),$$

then one will obtain:

(5) $$\begin{cases} R\,dq_0\cos\alpha_0 = -r_2(dq\cos\alpha - dq_0\cos\alpha_0) + \left(\dfrac{\mathrm{f} - \mathrm{f}'}{2\Delta}\right)(dq\sin\alpha - dq_0\sin\alpha_0), \\[2ex] R\,dq_0\sin\alpha_0 = -\left(\dfrac{\mathrm{f} - \mathrm{f}'}{2\Delta}\right)(dq\cos\alpha - dq_0\cos\alpha_0) - r_1(dq\sin\alpha - dq_0\sin\alpha_0), \end{cases}$$

and because, from (15), § 4:

$$\left(\frac{\mathrm{f}-\mathrm{f}'}{2\Delta}\right)^2 = d^2 - \delta^2 = \rho_1\,\rho_2 - r_1\,r_2\,,$$

one will then obtain the following expressions for $dq \cos \alpha$ and $dq_0 \sin \alpha$ from this:

$$\text{(6)}\qquad \begin{cases} dq\cos\alpha = \left(1-\dfrac{Rr_1}{\rho_1\rho_2}\right)dq_0\cos\alpha_0 - \dfrac{R\sqrt{d^2-\delta^2}}{\rho_1\rho_2}dq_0\sin\alpha_0, \\ dq\sin\alpha = \dfrac{R\sqrt{d^2-\delta^2}}{\rho_1\rho_2}dq_0\cos\alpha_0 + \left(1-\dfrac{Rr_2}{\rho_1\rho_2}\right)dq_0\sin\alpha_0. \end{cases}$$

When these two equations, which show that the perpendicular dq and the associated angle α can be determined for any point on a ray from the corresponding data at its starting point, are divided, that will give:

$$\text{(7)}\qquad \tan\alpha = \frac{R\sqrt{d^2-\delta^2}\cos\alpha_0 + (\rho_1\rho_2 - Rr_2)\sin\alpha_0}{(\rho_1\rho_2 - Rr_2)\cos\alpha_0 - R\sqrt{d^2-\delta^2}\sin\alpha_0}.$$

The rotation angle of the rays that are infinitely-close to the first ray for the line segment with abscissa R is, as was shown above, equal to $\alpha - \alpha_0$. If one denotes it by β, such that $\beta = \alpha - \alpha_0$ and one will then obtain the following expression for the rotation angle from equation (7):

$$\text{(8)}\qquad \tan\alpha = \frac{R(\sqrt{d^2-\delta^2} - d\sin 2\alpha_0)}{\rho_1\rho_2 - R(r_1\cos^2\alpha_0 + r_1\sin^2\alpha_0)}.$$

It would follow from this that the tangent of the rotation angle is equal to zero for any arbitrary value of the abscissa R – so the rotation angle itself will be equal to zero or two right angles – when one has:

$$\text{(9)}\qquad \sqrt{d^2-\delta^2} = d\sin 2\alpha_0\,,$$

so, from (20), § 4, if $\sin 2\alpha_0 = \cos\gamma$ – or $\alpha_0 = \frac{1}{4}\pi \pm \frac{1}{2}\gamma$ – then $\alpha_0 = \omega_1$ or $\alpha_0 = \omega_2$, where ω_1 and ω_2 are the angles that the two focal planes make with the first principal plane. The rotation angle will then be everywhere equal to zero or equal to right angles for the two infinitely-close rays that lie in the focal plane. This would also follow immediately from the fact that each of these infinitely-close rays, along with the first ray, lie in one and the same plane, namely, the focal plane.

In those ray systems that have imaginary focal surfaces, and therefore also no focal planes, the rotation angle can be nowhere equal to zero for finite line segments, so the rotation of the rays around each other will never change its sense. When any two infinitely-close rays of such a system lie such that the rotation of one of them with respect to the other one can be referred to as a clockwise rotation, any two mutually infinitely-close rays of the whole system must necessarily have the same relationship to each other of a clockwise rotation. The ray systems with imaginary focal surfaces will then divide into two distinct classes: ray systems with clockwise rotation of all rays and rays systems with counter-clockwise rotations. However, to any ray system, which might have real or imaginary focal surfaces, there is another one that is, in a sense, symmetrically or improperly equivalent to it, such that the single difference between them consists only in the opposite sense of rotation of all rays compared to each other, a difference that is expressed analytically by only the difference in signs of the quadratic roots.

If the focal points are real, and one examines the rotation angle from the starting point of a ray up to the focal points then for $R = \rho_1$ and $R = \rho_2$ – which might be denoted by β_1 and β_2, respectively – one will obtain, with the help of equations (14), § 4, from which, it follows that $r_1 \cos^2 \alpha_0 + r_1 \sin^2 \alpha_0 = m - d \cos 2\alpha_0$:

$$\tan \beta_1 = \frac{\sqrt{d^2-\delta^2} - d\sin 2\alpha_0}{\delta + d\cos 2\alpha_0}, \qquad \tan \beta_2 = \frac{\sqrt{d^2-\delta^2} - d\sin 2\alpha_0}{-\delta + d\cos 2\alpha_0}, \tag{10}$$

and from this, with the help of the expressions that were given in (18), § 4 for the angles ω_1 and ω_2 that the focal planes make with the principal planes:

$$\begin{cases} \tan \beta_1 = \dfrac{\sin 2\omega_1 - \sin 2\alpha_0}{\cos 2\omega_1 - \cos 2\alpha_0} = \tan(\omega_1 - \alpha_0), \\[2ex] \tan \beta_2 = \dfrac{\sin 2\omega_2 - \sin 2\alpha_0}{\cos 2\omega_2 - \cos 2\alpha_0} = \tan(\omega_2 - \alpha_0); \end{cases} \tag{11}$$

one will then have:

$$\beta_1 = \omega_1 - \alpha_0, \qquad \beta_2 = \omega_2 - \alpha_0 . \tag{12}$$

From these simple expressions for the rotation angles that are computed from the starting point of a ray to the focal points of its infinitely-close rays, one will get:

$$\beta_2 - \beta_1 = \omega_2 - \omega_1 = \gamma. \tag{13}$$

Thus: *The rotation angles from a focal point of a ray to the other focal point have the same values for all rays that are infinitely-close to it, and are equal to the inclination of the two focal planes.*

If one takes the rotation angle β to be a given quantity then one can determine the length of the abscissa R for which the rotation angle from the first ray to an infinitely-close ray will be that given quantity. Equation (8) will give the following expression for R:

$$R = \frac{\rho_1 \rho_2 \sin \beta}{(r_1 \cos^2 \alpha_0 + r_2 \sin^2 \alpha_0)\sin \beta + \sqrt{d^2-\delta^2}\cos \beta - d \sin 2\alpha_0 \cos \beta}, \tag{14}$$

which will assume the following simple form when r_1 and r_2 are replaced with their values $r_1 = m - d$ and $r_1 = m + d$:

$$(15)\qquad R = \frac{\rho_1\rho_2 \sin\beta}{m\sin\beta + \sqrt{d^2-\delta^2}\cos\beta - d\sin(2\alpha_0+\beta)}.$$

If one now considers R to be a function of only α_0 and β to be a given constant quantity then R will attain its largest value for $\sin(2\alpha_0 + \beta) = +1$, so $\alpha_0 = \tfrac{1}{4}\pi - \tfrac{1}{2}\beta$, and its smallest value for $\sin(2\alpha_0 + \beta) = -1$, so $\alpha_0 = \tfrac{3}{4}\pi - \tfrac{1}{2}\beta$, and if the largest value of R is denoted by R_1, while the smallest value is denoted by R_2 then one will have:

$$(16)\qquad \left\{\begin{aligned} R_1 &= \frac{\rho_1\rho_2 \sin\beta}{m\sin\beta + \sqrt{d^2-\delta^2}\cos\beta - d}, \\ R_2 &= \frac{\rho_1\rho_2 \sin\beta}{m\sin\beta + \sqrt{d^2-\delta^2}\cos\beta + d}. \end{aligned}\right.$$

It further follow from this that:

$$(17)\qquad \left\{\begin{aligned} \frac{1}{R}-\frac{1}{R_1} &= \frac{(1-\sin(2\alpha_0+\beta))\,d}{\rho_1\rho_2\sin\beta} = \frac{2\sin^2(\alpha_0+\tfrac{1}{2}\beta-\tfrac{1}{4}\pi))\,d}{\rho_1\rho_2\sin\beta}, \\ \frac{1}{R_2}-\frac{1}{R} &= \frac{(1+\sin(2\alpha_0+\beta))\,d}{\rho_1\rho_2\sin\beta} = \frac{2\cos^2(\alpha_0+\tfrac{1}{2}\beta-\tfrac{1}{4}\pi))\,d}{\rho_1\rho_2\sin\beta}, \end{aligned}\right.$$

and from these two equations, one will get:

$$(18)\qquad \frac{1}{R} = \frac{\cos^2(\alpha_0+\tfrac{1}{2}\beta-\tfrac{1}{4}\pi)}{R_1} + \frac{\sin^2(\alpha_0+\tfrac{1}{2}\beta-\tfrac{1}{4}\pi)}{R_2}.$$

Thus: If one takes any ray that is infinitely-close to a ray that starts from an arbitrary point of it at *the* length at which it makes a constant rotation angle with it then that length will be one of the largest and smallest lengths amongst the infinitely-close rays, and will be determined from the angle that the direction of its starting point makes with the direction of the starting point of the largest ray by precisely the same equations that determine the radius of curvature of a normal section of a surface in terms of the largest and smallest radii of curvature and the angle that this normal section defines with the principal plane by way of the well-known *Euler* equation. *Euler's* theorem itself is included as a special case of this general theorem, as will be shown below. In the special case where the constant rotation angle β equals a right-angle, one will have:

$$(19)\qquad \frac{1}{R} = \frac{\cos^2\alpha_0}{R_1} + \frac{\sin^2\alpha_0}{R_2}.$$

Hamilton first established the special property of the general ray system that is expressed by this equation in the cited Supplement, and in fact, by considering the projection of a ray that was infinitely close to a given ray onto a plane that was drawn through the first ray and the starting point of the infinitely-close ray. However, nowhere did *Hamilton* apply the concept of the rotation of rays relative to each other and the rotation angle, which is extraordinarily fruitful for understanding the properties of ray systems.

§ 8.

Infinitely thin ray bundles and principal rays.

In the two equations (6), § 7:

$$(1)\qquad \begin{cases} dq\cos\alpha = \left(1-\dfrac{Rr_1}{\rho_1\rho_2}\right)dq_0\cos\alpha_0 - \dfrac{R\sqrt{d^2-\delta^2}}{\rho_1\rho_2}dq_0\sin\alpha_0, \\ dq\sin\alpha = \dfrac{R\sqrt{d^2-\delta^2}}{\rho_1\rho_2}dq_0\cos\alpha_0 - \left(1-\dfrac{Rr_1}{\rho_1\rho_2}\right)dq_0\sin\alpha_0 \end{cases}$$

dq and α can be regarded as the polar coordinates of the boundary curve of that cross-section of an infinitely-thin ray bundle that belongs to the abscissa R, and dq_0 and α_0, as the polar coordinates of the curve around the cross-section that is found at the starting point. These equations can, in turn, be employed not only to compare the cross-sections of an infinitely-thin ray bundle in terms of surface area, which will already be accomplished completely by the density measure, but also to determine how the form of any cross-section will depend upon that of a given one. If one then goes from the polar coordinates of the two cross-sections to rectangular coordinates whose axes lie in the two principal planes of the ray, relative to which all other rays of the ray bundle will be regarded as infinitely-close rays, then one will have to set:

$$(2)\qquad \begin{cases} dq\cos\alpha = x, \qquad dq\sin\alpha = y, \\ dq_0\cos\alpha_0 = x_0, \quad dq_0\sin\alpha_0 = y_0, \end{cases}$$

where x, y and x_0, y_0 are the infinitely-small coordinates of the two cross-sections. Equations (1) will then give:

$$(3)\qquad \begin{cases} x = \left(1-\dfrac{Rr_1}{\rho_1\rho_2}\right)x_0 - \dfrac{R\sqrt{d^2-\delta^2}}{\rho_1\rho_2}y_0, \\ y = \dfrac{R\sqrt{d^2-\delta^2}}{\rho_1\rho_2}x_0 + \left(1-\dfrac{Rr_2}{\rho_1\rho_2}\right)y_0, \end{cases}$$

and when, conversely, x_0 and y_0 are expressed in terms of x and y, one will get:

$$\text{(4)} \qquad \begin{cases} (\rho_1 - R)(\rho_2 - R)\, x_0 = (\rho_1\rho_2 - Rr_2)\, x + R\sqrt{d^2 - \delta^2}\, y, \\ (\rho_1 - R)(\rho_2 - R)\, y_0 = -R\sqrt{d^2 - \delta^2}\, x + (\rho_1\rho_2 - Rr_1)\, y. \end{cases}$$

The boundary curves of the cross-section of an infinitely-thin ray bundle are thus not only all curves of the same degree, but they are also related to each by the collineation that is expressed by these equations.

Special attention is warranted for the cross-sections at the two focal points of infinitely-thin ray bundles, for which, as was already shown above, the density measure will be infinitely large, so the surface area will be infinitely small of a higher order. If one takes $R = \rho_1$ or $R = \rho_2$ then, from the two equations (4) and, in turn, also from equations (3), the one will become identical with the other one, and they will give:

$$\text{(5)} \qquad \begin{cases} y = \sqrt{\dfrac{d - \delta}{d + \delta}}\, x, \qquad \text{for } R = \rho_1, \\ y = \sqrt{\dfrac{d + \delta}{d - \delta}}\, x, \qquad \text{for } R = \rho_2, \end{cases}$$

which are the equations of straight lines, and in fact, infinitely-small straight lines, because y and x can have only infinitely-small values.

The cross-sections of an infinitely-thin ray bundle at the two focal points are then infinitely-small straight lines; i.e., of the two dimensions of the cross-section, which are generally infinitely-small quantities of first order, one of them will become an infinitely-small quantity of higher order at the two focal points.

From this, it also follows that *the bounding surface of any infinitely-thin ray bundle with real focal points can be constructed by moving a straight line that always goes through an infinitely-small plane curve and two straight lines that can be perpendicular to a perpendicular that is erected to the plane of the small curve in its interior.*

In order to determine the two cross-sections at the focal points, which are infinitely-small lines, and their lengths in relation to the dimensions of the cross-section that is given at the starting point of the ray bundle, it is preferable to revert to the polar coordinates dq, α and dq_0, α_0 . If one sets $R = \rho_1$ in equations (1) and observes that $\rho_1 - r_1 = d + \delta$, $\rho_2 - r_2 = - d + \delta$ then one will get:

$$\text{(6)} \qquad \begin{cases} \rho_1\, dq \cos\alpha = (d + \delta)\, dq_0 \cos\alpha_0 - \sqrt{d^2 - \delta^2}\, dq_0 \sin\alpha_0, \\ \rho_2\, dq \cos\alpha = \sqrt{d^2 - \delta^2}\, dq_0 \cos\alpha_0 - (d - \delta)\, dq_0 \sin\alpha_0 \end{cases}$$

for the cross-section at the first focal point. By introducing the angle ω_1 that the first focal plane makes with the first principal plane, for which, as was shown above in (18), § 4:

$$\sin\omega_1 = \sqrt{\frac{d - \delta}{2d}}\,, \qquad \cos\omega_1 = \sqrt{\frac{d + \delta}{2d}}\,,$$

these equations can be represented in the following form:

$$(7)\qquad \begin{cases} \rho_2 dq \cos\alpha = 2d \cos\omega_1 \cos(\alpha_0 + \omega_1)\, dq_0, \\ \rho_2 dq \sin\alpha = 2d \sin\omega_1 \cos(\alpha_0 + \omega_1)\, dq_0, \end{cases}$$

and one will get by dividing them:

$$(8)\qquad \tan\alpha = \tan\omega_1, \qquad \alpha = \omega_1 .$$

One can conclude the fact that the angle α has a constant value from this, but in order to conclude the fact that the cross-section whose polar coordinates are dq and α must be part of a straight line in which the pole lies, one must give the direction of that straight line, along with the constant value $\alpha = \omega_1$, since it makes the angle ω_1 with the first principal plane.

In the same way, for $R = \rho_2$ – i.e., for the cross-section at the second focal point – one will get:

$$(9)\qquad \begin{cases} \rho_1\, dq \cos\alpha = 2d \cos\omega_2 \cos(\alpha_0 + \omega_2)\, dq_0, \\ \rho_1\, dq \sin\alpha = 2d \sin\omega_2 \cos(\alpha_0 + \omega_2)\, dq_0, \end{cases}$$

$$(10)\qquad \tan\alpha = \tan\omega_2, \qquad \alpha = \omega_2 .$$

One has the following theorem:

The two infinitely-small straight lines that define the cross-section of an infinitely-thin ray bundle at the focal points will lie in the their two focal planes.

For the cross-section at the first focal point, where $\alpha = \omega_1$, one will have, from equation (7):

$$(11)\qquad dq = dq_0 \cos(\alpha_0 + \omega_1).$$

Now if, as we have assumed here, the bounding curve of the one cross-section at the starting point of the ray bundle is completely determined and given then one will have its radius vector dq_0 given as a function of the angle α_0, and then dq will also be determined as a function of α_0 by the equation (11). However, since the curve whose radius vector is dq is a straight line, and the pole lies on that straight line, its length will necessarily be equal to the difference between the two extreme values that this radius vector dq can have as a function of α_0, or because one of these two extreme values is necessarily positive, the other one must be negative, so the desired length of that line will be equal to the sum of the absolute values of that maximum and minimum. One likewise obtains the length of the cross-section at the two focal points when one adds the largest positive and negative values that dq can assume as a function of α_0 from the equation:

$$(12)\qquad dq = \frac{2d}{\rho_1}\, dq_0 \cos(\alpha_0 + \omega_1),$$

while ignoring the sign.

In the simplest case, where the cross-section at the starting point is assumed to be an infinitely-small circle, so dq_0 will be constant, as the radius of that circle, one will have the two extreme values of dq for the cross-section at the first focal point when $\alpha_0 + \omega_1 = 0$ and $\alpha_0 + \omega_1 = \pi$, so they will be equal to $\frac{2d}{\rho_2}dq_0$ and $-\frac{2d}{\rho_2}dq_0$; when they are added, ignoring the signs, that will give $\frac{4d}{\rho_2}dq_0$ as the length of the rectilinear cross-section at the first focal point. One will likewise find that length of the cross-section at the second focal point is equal to $\frac{4d}{\rho_2}dq_0$. The lengths of these two cross-sections at the focal points will then behave like their distances from the circular cross-section at the starting point of the ray bundle.

If one investigates the condition for the length of a cross-section to be zero at the focal points of the ray bundle – i.e., infinitely-small of higher order than the former – then one will recognize immediately from equations (11) and (12) that this case will occur when $d = 0$, and that it can occur only when that condition is fulfilled. The condition $d = 0$ also necessarily implies that $\delta = 0$, since δ, when it is real, is nowhere greater than d, so one must have $r_2 = r_1$ and $\rho_2 = \rho_1$; i.e., the two boundary points of the shortest distance, and the two focal points must coincide for those ray bundles that have the one midpoint. If one, with *Hamilton*, calls those rays whose infinitely-close rays all go through a single point *principal rays* then it will follow that the principal rays can exist, and also exist in reality, where the two boundary surfaces, and with them, likewise the two focal surfaces, have common points, which can be either contact points or intersection points or points on the intersecting lines.

The two principal planes will be undetermined for the principal rays, because the shortest distance to the infinitely-close rays will always be zero for them, and in turn, cannot determine a direction.

In the completely special ray systems, whose rays all go through a single point, all rays will be principal rays; it is also easy to see that this is the only degenerate case. However, there are infinitely many ray systems that have continuous sequences of principal rays that collectively define a surface, such as, e.g., the system of common tangents to two confocal second-degree surfaces, in which all tangent are the intersection curves of these two confocal surfaces of principal rays. Similarly, there are infinitely many ray systems that have isolated principal rays, but as a rule, principal rays do not exist in general systems, because the values of the two independent variable u and v for which a ray becomes a principal ray will be determined by three equations. Namely, since the directions of the two principal planes will be undetermined for a principal ray, the quadratic equation (4), § 2, whose roots determine the directions of the principal planes, must be fulfilled identically, and one must then simultaneously have:

$$(13) \qquad \mathrm{gF} - \tfrac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{G} = 0, \quad \mathrm{eG} - \mathrm{gE} = 0, \quad \tfrac{1}{2}(\mathrm{f} + \mathrm{f}')\,\mathrm{E} - \mathrm{e\,F} = 0.$$

These three equations will, in general, reduce to two, because, except for the case of F = 0, one of them is a necessary consequence of the other two; however, a third condition equation will arise, because the ray must have a *real* focal point – namely, $\delta = 0$ – which will yield:

(14) $$\mathrm{f} = \mathrm{f}'.$$

If the two focal points of a ray coalesce with the midpoint, but not, at the same time, the two limit points of shortest distance, as well, then the infinitely-thin ray bundle that neighbors it will have *only one* rectilinear cross-section at that midpoint, which will likewise contain the two focal points, and that cross-section will lie in the plane into which the two focal planes will coalesce in this case, since, from the equation $\sin \gamma = \delta / d$, the angle γ, along with δ – viz., one-half the distance between the two focal points – will be equal to zero. Since the condition for the two focal points to coincide gives only one equation between the two independent variables u and v, it will then follow that the ray system, as a rule, will not contain individual rays of this kind, but a continuous sequence of them that define rectilinear surfaces, and that the focal surfaces, as a rule, will intersect in well-define curves, since all tangents to the intersection curve of the two focal surfaces will be rays whose focal points coincide. However, there is also an entire category of ray systems for which all of the rays have that property, because their two focal surfaces cover each other in such a way that they combine into a single surface.

§ 9.

Comparison between the general theory of rays systems and the special theory of the curvature of surfaces and their systems of normals.

In the special case for which two ray systems whose general theory was developed in the foregoing are denoted by f and f′, and the expressions that are composed of the partial differential quotients of x, y, z and ξ, η, ζ are equal to each other, those ray systems will become special systems whose rays are all normals to one and the same surface. Namely, if there is a surface for which any ray is a normal, and one lets x', y', z' denote the coordinates of that point at which the ray of the system that is determined by x, y, z, ξ, η, ζ is normal to it, and calls the distance from these points to the stating point x, y, z, of the ray r then one will have:

(1) $$x' = x + r\,\xi, \quad y' = y + r\,\eta, \quad z' = z + r\,\zeta,$$

and because this ray must be perpendicular to the surface, one must have:

(2) $$\xi\,dx' + \eta\,dy' + \zeta\,dz' = 0.$$

When x', y', z' are replaced with their values, this condition will give:

(3) $$\xi\,dx + \eta\,dy + \zeta\,dz + dr\,(\xi^2 + \eta^2 + \zeta^2) + r\,(\xi\,d\xi + \eta\,d\eta + \zeta\,d\zeta) = 0,$$

and as a result [*sic*]:

(3) $$\xi\,dx + \eta\,dy + \zeta\,dz = -\,dr,$$

or:

(4) $$(\xi\, a + \eta\, b + \zeta\, c)\, du + (\xi\, a' + \eta\, b' + \zeta\, c')\, dv = -\, dr.$$

The expression on the left-hand side of this equation must then be a complete differential of a function – r of the two independent variables u and v. One must therefore have:

(5) $$\frac{\partial(\xi a + \eta b + \zeta c)}{\partial v} = \frac{\partial(\xi a' + \eta b' + \zeta c')}{\partial u},$$

and from this, by performing the partial differentiations, since:

$$\frac{\partial a}{\partial v} = \frac{\partial a'}{\partial u}, \qquad \frac{\partial b}{\partial v} = \frac{\partial b'}{\partial u}, \qquad \frac{\partial c}{\partial v} = \frac{\partial c'}{\partial u},$$

one will get the following condition:

$$a\mathrm{a}' + b\mathrm{b}' + c\mathrm{c}' = a'\mathrm{a} + b'\mathrm{b} + c'\mathrm{c},$$

so

(6) $$\mathrm{f} = \mathrm{f}',$$

which must be fulfilled identically in order for the ray system to be a system of normals to a surface. The fact that this condition is also sufficient will emerge from the fact that when it is fulfilled, the quantity r can be determined from equation (40 as a function of u and v, and the fact that for such a value of r, equations (1) will represent a surface whose normals are rays of the system. Since an arbitrary constant can be added to the value of r that is determined from the differential equation (4), one will then have, not just one surface that satisfies this condition, but an entire family of them, which are known by the name of *parallel surfaces*.

For f = f′, the quadratic equation (5), § 4, whose roots are τ_1 and τ_2, will be identical with the quadratic equation (9), § 4, whose roots are t_1 and t_2, and likewise the quadratic equation (9), § 4, whose roots are ρ_1 and ρ_2 , will be identical with the quadratic equation (16), § 2, whose roots are r_1 and r_2 . It will follow from this that:

In those systems whose rays are normals to a surface, the two focal planes of any line will coincide with the principal planes, and the two focal points will coincide with the two limit points of shortest distance.

If one chooses one of the surfaces in this case for which all of rays of the systems are normals to be the surface from which all rays can be regarded as starting then the abscissas of the focal points ρ_1 and ρ_2 – or what amounts to the same thing here, the abscissas of the limit points r_1 and r_2 – will be the two principal radii of curvature of this surface, and the focal surfaces of the system, which will coincide with the limit surfaces of shortest distance, will be the surfaces that were treated by *Monge* on which the centers of all principal curvature circles will lie. The theory of the curvature of surfaces can then be regarded as a special case of the general theory of ray systems, and it is not without interest to discuss the connection between the general theorems that were developed in the foregoing and the known theorems on the curvature of surfaces somewhat more closely.

If one next examines whether the general of ray systems might perhaps yield new theorems for the theory of curvature and the normals to surfaces then, as one might expect, one will derive no great profit from that. In this regard, the theorem that is expressed through equation (16), § 3, viz.:

$$r = r_1 \cos^2 \omega + r_2 \sin^2 \omega,$$

can be cited as such a thing that, since it likewise expresses a general property of the normals to a surface, will serve as a distinguished one in this special case. Furthermore, from the property of infinitely-thin ray bundles that was proved in § 8 that their cross-sections at the two focal points are not infinitely-small surfaces, but infinitely-small lines that lie in the two focal planes, one can obtain the following not-uninteresting – and I believe, still not well-known – theorem for the normals to surfaces:

The two principal normal planes at a point of a surface will be intersected by all of the normals that are infinitely-close to that point in such a way that the distances from the intersection points to the given point of the surface will be equal to the larger of the radii of curvature in the one principal normal plane and the smaller of them in the other one.

If one goes through the known theorems on the curvature and the normals to surface then one will find them again in a general form, and with a general interpretation, in the general theory of ray systems.

If one next considers the two principal normal intersections for a point of a surface that yields the larger and the smaller of the radii of curvature then in the general theory one will get, on the one hand, the two principal planes, and on the other hand, the focal planes that correspond to these planes. The properties of the normals to surfaces that are connected with the principal normals will arrange themselves in the general theory in such a way that one part of them will contain the principal planes and the other part will contain the focal planes. The principal planes preserve the properties that they are always real and perpendicular to each other, while the focal planes preserve the property that both of the intersecting rays that are infinitely close to the given ray will lie in them. Likewise, in the general theory, the centers of principal curvature of the surface will split into the limit points of shortest distance and the focal points, and correspondingly, the surfaces in which the centers of principal curvature lie will also split into the limit surfaces of shortest distance and focal surfaces. Here, the limit surfaces will retain only the property that was already expressed in their definition that they bound the space, inside of which, all of the shortest distances to any two infinitely-close rays will lie, but the focal surfaces will retain the property that they will be tangent to all rays of the system. The two beautiful properties of surfaces of centers of principal curvature that were found by *Monge*, namely, that first of all their outlines always intersect perpendicularly, from which, one might also consider points of space, and secondly, that the curves of regression of all developable surfaces into which the normals can be combined are the shortest lines on the surface of centers of principal curvature, will be lost for the limit surfaces of shortest distance, as well as the focal surfaces of the most general ray system, and will become special properties that belong to the system of normals to a surface.

The two families of curvature lines of surfaces, insofar as they have the property that the normals that belong to them will define developable surfaces, will appear in the general ray systems as the two families of developable surfaces that were denoted by Ω_1 and Ω_2 in § 5. However, on the other hand, the rectilinear surfaces O_1 and O_2 can also be considered to be the curvature lines of the surfaces, because the curvature lines will sweep out the surface in the special case in which all rays are normals to a surface when they coincide with them.

The umbilic points of surfaces, for which the two principal curvature centers will coalesce, such that all infinitely-close normals will go through the same point of coalescence, and at which the principal normal planes will lose their well-defined directions, are found in the general theory as the principal rays whose infinitely-close rays all go through a point, and their principal planes, as focal planes, are undefined, as well.

Euler's theorem, which teaches us how the radii of curvature of an arbitrary normal section is determined from the two principal radii of curvature and the angle that its plane makes with one of the principal planes, is included as a special case in the general equation (18), § 7, which goes to *Euler's* equation for $\beta = \pi / 2$, $r_1 = \rho_1$, $r_2 = \rho_2$. The general method in § 7 also allows one to understand the radius of curvature of the normal section of a surface from a new, not-uninteresting, viewpoint, in that it shows that the rotation angle of the radius of curvature of a normal section with a normal that starts at an infinitely-close point on the plane of that section, when computed along the entire length of the radius of curvature, will be equal to a right angle, or:

If one draws the normals to a surface at two infinitely-close points, and gives them the well-defined length for which their rotation angle is equal to a right angle then they will represent the curvature radius of the surface at these two infinitely-close points for the normal section that goes through them.

The *Gaussian* curvature measure of the surface is found in the general ray system as the general concept of the density measure, and the expression for it as the reciprocal values of the products of two principal radii of curvature corresponds completely to the expression for the density measure that was given in § 6, from which, that would be equal to the reciprocal value of the products of the distances from the two focal points of the ray to the point in question. For the ray systems that are normals to a surface, and in turn, also normals to an entire family of its parallel surfaces, the density measure will be completely identical with the curvature measure, since at any point of space the density measure of the rays will be equal to the curvature measure of the parallel surface that goes through that point. This also shows how the concepts that *Gauss* introduced into science rigorously carry with them the character of true generality, by which they can extend their influence far across the domain in which they originally came about.

Berlin, in October 1859.