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30 July. Session of the physical-mathematical class.

Kummer presented three models of the general, infinitely-thin rectilinear ray bundles that are produced by filaments, and gave the following communication:

As I have established in a treatise that appeared in volume 57 of Borchardt's mathematical journal on 17 October, the general, infinitely-thin ray bundles are bounded by rectilinear surfaces whose generating lines will always go through two straight lines that are perpendicular to the axis of the ray bundle, and likewise though an infinitely-small, closed curve that surrounds the axis. In the present model, this small, closed curve will be chosen to be a circle whose plane is perpendicular to the axis, and whose center will lie on the axis. The bounding surface of the ray bundle will then be a fourth-degree, rectilinear surface whose sections that are perpendicular to the axis will be ellipses everywhere, of which, two of them will degenerate into straight lines for the ray bundles that are represented by the first and second models. The two light rays that are perpendicular to the axis, and which correspond to the two rectilinear sections of the ray bundle, and with them, likewise the two planes that are drawn through the axis and any of the straight light rays – which I call the *focal planes* of the ray bundle – will define a right angle in the first model, and an acute angle in the second one, but in the third one, they will be imaginary and define an imaginary angle, such that the ray bundle and its surrounding surface will remain real. The three types of ray bundle that are represented by this model, along with their limiting cases – namely, the conics and cylinders – are, as I have proved in the cited treatise, the only mathematically-possible ones. Since then, I have now also examined the question of whether, and under what circumstances, these can and must actually occur in nature as optical ray bundles, and in this regard, I have found a very general and simple theorem that gives the complete answer to this question, and indeed, not only for simply-refracting media – whose wave surface is a sphere – but also for uniaxial crystals – whose wave surface us a sphere and an ellipsoid of rotation – and for biaxial crystals – which are associated with the Fresnel wave surface – and even for all possible transparent media or crystals that might be associated with any other wave surface of light. This theorem is the following one:

Theorem: *Any infinitely-thin, optical ray bundle, in whose interior a homogeneous, transparent medium has the property that its two focal planes will cut out two curves that will intersect in conjugate directions from the wave surface of light that is associated with that medium, and whose center is assumed to lie on the axis of the ray bundle; any ray bundle that has this property will also be optically-representable.*

Amongst the conjugate directions on the wave surface, one will find the directions of two conjugate diameters of the infinitely-small Dupin conic section that is associated with the point of the wave surface in question – namely, the *indicatrix* – if one understands that this conic section will be an ellipse or a hyperbola, according to whether the surface is convex-convex or convex-concave at this location.

For any chosen direction in the crystal, and for any point of intersection of the radius vector that is parallel to it with the wave surface, one can choose the position of one focal plane to be arbitrary, and the position of the other focal plane will then be determined completely by the theorem that we gave. There will always be a well-defined position of the first focal plane, for which, the second focal plane will make a right angle with it, such that the ray bundles of the first kind – whose focal planes are perpendicular to each other – can then exist for all arbitrary directions of their axes in any crystal, but, in general, only for a certain position of the focal planes.

It is only when the wave surface is convex-convex at the point of the wave surface at which the radius vector meets it – so the indicatrix will be an ellipse – and one rotates the first focal plane around the radius vector as axis, starting from the position in which that focal plane is perpendicular to the second one, that the angle between the two focal planes will become smaller and attain a well-defined minimum, for which the two focal planes will lie in such a way that the angle between them will bisect the mutually-perpendicular focal planes. If one lets α denote the angle around which the first focal plane will rotated from the given initial position and denotes the angle between the two associated focal planes by γ then one will find the smallest value of γ when $\gamma = 2\alpha$.

Secondly, if the wave surface at the endpoint of the radius vector in question is convex-concave – so the indicatrix will be a hyperbola – and one rotates the first focal plane from the position in which the second one is perpendicular to it then the angle γ between the two focal planes will become smaller, and it will attain the value zero at a well-defined position, and if one continues from that position then the angle γ will again increase to 90°, and will then assume the value zero for a second time. The two positions of the focal planes for which $\gamma = 0$ will correspond to the directions of infinitely-small radius of curvature on the wave surface or – what amounts to the same thing – the asymptotes of the hyperbolic indicatrix. Since the hyperbola possesses imaginary conjugate diameters, in addition to its real conjugate diameters, it will then follow that for those directions in which the radius vector meets a convex-concave part of the wave surface, there will also exist infinitely-thin ray bundles of the third kind that have imaginary focal planes.

If the transparent medium is simply-refracting – so its wave surface is the outer surface of a sphere – then all of its indicatrices will be circles, and it follows that all of the conjugate directions can only be mutually-perpendicular, and since the radius vectors will be everywhere-perpendicular to the wave surface here, it will then also follow that the focal planes must be everywhere-perpendicular to the ray bundle. In a simply-refracting medium, one will then find no other optical ray bundles besides ones of the first kind, whose focal planes will be mutually perpendicular.

When the transparent medium is an optically-uniaxial crystal whose irregular rays form an ellipsoid of rotation on the wave surface then the indicatrices will only be ellipses. The direction in which the first focal plane must lie (which is then perpendicular to the second one) is the only one in which the optical axis can lie, here. If the rotational semi-axis of the wave ellipsoid is equal to c, the semi-axis that is perpendicular to it is equal to a, and furthermore, ω is the angle that the axis of the ray bundle makes with the optical axis, and:

$$\rho = \frac{ac}{\sqrt{a^2 \cos^2 \omega + c^2 \sin^2 \omega}}$$

is the radius vector that corresponds to this direction then the smallest angle γ between the two focal planes of the ray bundle that lies in this direction will be given by the formula:

$$\tan \frac{\gamma}{2} = \frac{c}{\rho} \quad \text{or} \quad \tan \frac{\gamma}{2} = \frac{\rho}{c},$$

according to whether $c < a$ or $c > a$, resp.; i.e., according to whether the uniaxial crystal is a negative or positive one, resp. For $\omega = 90^{\circ}$ – i.e., for the position that is perpendicular to the optical axis, one gets the ray bundle with the smallest angle between the focal planes that can exist in such a crystal, at all, namely:

$$\tan \frac{\gamma}{2} = \frac{c}{a} \quad \text{or} \quad \tan \frac{\gamma}{2} = \frac{a}{c},$$

according to whether $c < a$ or $c > a$, resp.

For the double path, for which:

$$\frac{1}{a} = 1.483, \qquad \frac{1}{c} = 1.654,$$

one will then obtain:

$$\gamma = 83^{\circ}\ 45'\ 50''.$$

Inside of the double path, one will then find no other ray bundles, as such, for which the angle between the two focal planes lies between 90° and $83^{\circ}\ 45'\ 50''$. For the ray bundles that are perpendicular to two parallel, double paths on the natural surfaces of the rhombohedron, and define an angle of $44^{\circ}\ 36'\ 30''$ with the optical axis, one will find that the smallest angle between the focal planes will be $\gamma = 87^{\circ}\ 5'$.

In optically-biaxial crystals – which are associated with the Fresnel wave surface – one finds, not only the ray bundles of the first and second kind – and indeed for all angles between the two focal planes from a right angle down to zero – but also the ray bundles of the third kind that have imaginary focal planes. In fact, the Fresnel wave surface has four places on its exterior sheet at which it is convex-concave, which are places that will be bounded by the four, well-known circles at which one finds singular tangential planes contacting the surface. The ray bundles of the third kind and the ones for which the angle between the focal planes drops down to zero are found in only those directions of the crystal whose corresponding radius vectors meet the wave surface inside these circles. For any of the directions that are included in these limiting directions, there is a well-defined minimum of the angle γ between the two focal planes, which will get larger as the radius vector gets further away from the aforementioned four circles. The value of the angle γ as a function of the direction of the axis of the ray bundle and the direction of the first focal plane, as well as the value of the minimum of γ for any given direction of the axis of the ray bundle, can be given with no particular difficulty, but since the expressions are nonetheless somewhat complicated, I would like to pass over them here.

If one allows a ray bundle of the first, second, or third kind that exists inside of a crystal to go from it into a simply-refracting medium – e.g., air – then it will always be converted into a ray bundle of the first kind with mutually-perpendicular focal planes, and for that reason, one can conversely, generate any ray bundle that is possible in a crystal in such a way that one lets a suitable ray bundle of the first kind fall upon the crystal.

One can generate a ray bundle of the first kind that has arbitrary given distances to the two mutually-perpendicular rectilinear sections in the simplest way by means of a convex spherical lens, through which one lets the light that emanates from a point go, and which must go through a narrow opening, in addition, in order for the ray bundle to be sufficiently thin. If one arranges the lens in such a way that its axis lies in the direction of the incident light itself then one will obtain only a conical ray bundle in which the two rectilinear sections combine into a single point – viz., the *focal point*. However, if one rotates the lens in such a way that its axis defines an acute angle with the direction of the incident light then the two rectilinear sections will drift apart from each other, and their separation will get larger as this angle gets smaller; likewise, the two rectilinear sections will increase in length proportionately. A white paper that is held perpendicular to the axis of the ray bundle at various distances will illustrate its different sections, among which, the two rectilinear, mutually-perpendicular ones will emerge as entirely apparent.