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RESEARCH IN

GEOMETRICAL OPTICS

BY A. LÉVISTAL

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Translated by D. H. Delphenich

FIRST ARTICLE

ON THE GENERAL PROPERTIES OF OPTICAL SYSTEMS OF RECTILINEAR RAYS

INTRODUCTION.

1. The first research relating to the general properties of systems that are composed of light rays goes back to the year 1682, where caustic curves were treated, as well. It is due to Tschirnhausen ([1]), who proposed to find the equation of the caustic of reflection in the particular case where the incident rays are parallel and contained in the same plane, while the reflection takes place on the circumference of a circle. This problem, which seems quite simple today, was quite difficult back then, and the solution that was given by Tschirnhausen was falsely attributed to Cassini, Mariotte, and de la Hire, who were named curators by the Academy of Sciences. Moreover, the attention of the geometers was directed to that genre of questions, but they did not pause to appreciate all of their importance in optics. Bernoulli, Carré, l'Hôpital, and several other mathematicians pointed out some general method for obtaining the equation of plane caustics that were due to reflection and refraction, regardless of the nature of the reflecting or refracting curve. In a more recent epoch, the theory of planar caustics was perfected, notably, by Quetelet, who showed that in a great number of cases these caustics, although they might

([1]) TSCHIRNHAUSEN, *Inventa nova, etc., Acta eruditorum*, year 1682, pp. 364.

be quite complicated in their own right, are only the developments of other curves that are really quite simple, and which are generally epicycloids ([1]).

2. Malus was the first to consider systems of rays in space. It was in a paper on optics that was published in 1810 ([2]) that he proved, by means of a very laborious analysis, the theorem that often bears his name. That theorem can be stated in the following manner: *When all of the rays that comprise a luminous pencil are normal to the same surface, they will preserve that property after an arbitrary number of reflections by arbitrary surfaces, and an arbitrary number of refractions during their passage through bounded media that exhibit arbitrary refringent powers.* An error in calculation led Malus to refuse to give that proposition all of the generality that it deserved, and to restrict it to the case of a single reflection or refraction. In 1822, Charles Dupin recognized that Malus's theorem extended to an arbitrary number of reflections and refractions, and substituted a very simple geometric proof in the case of reflection for the analytic considerations that Malus had used ([3]). Timmermans ([4]) and, after him, Gergonne ([5]), treated the case of refraction in an analogous manner.

The importance of Malus's theorem, which is, moreover, applicable to only isotropic media, and its relation to the theory of caustics, are easy to conceive. Indeed, if a system of light rays can be considered to be composed of lines that are normal to the same surface then it will result immediately that the locus of the intersections of those rays – i.e., the caustic surface – is nothing but the surface with two sheets that is the locus of centers of curvature of that orthogonal trajectory of rays; the theory of caustics is thus found to be very closely linked with that of the curvature of surfaces.

3. The geometers that we just spoke of were occupied exclusively with isotropic media. As we will see, the systems of rays that propagate in media of this type enjoy the property of always being normal to the same surface. In birefringent or anisotropic media, this is not generally true, which is a fact that leads one to qualify the systems that propagate in such media as being *irregular*. The study of these irregular systems was begun for the first time by Hamilton in his treatise that was entitled *Theory of systems of rays*, and especially in a supplement that was published in 1830 ([6]). In these important papers, Hamilton took the principle of least action as his starting point, which, when applied to optical phenomena, will lead to the condition:

$$\delta \int v\, ds = 0,$$

([1]) QUETELET, "Mémoire sur une nouvelle manèiere de considérer les caustiques, soit par réflexion, soit par réfraction," Nouveaux Mémoires de l'Académie de Bruxelles, t. III, pp. 15.

([2]) MALUS, "Mémoire sur l'Optique," Journal de l'École Polytechnique, Letter XIV, pp. 1.

([3]) CH. DUPIN, *Développments de Géométrie*, 4th memoir: "Sur les routes suivies par la lumiere dans les phénomènes de la réflexion et de la réfraction," pp. 187.

([4]) TIMMERMANS, *Correspondance mathématique et physique*, t. I, pp. 336.

([5]) GERGONNE, "Démonstration purement géométrique du principe fondamental de la théorie des caustiques," Annales de Mathématiques pures et appliquées, t. XVI, pp. 307.

([6]) HAMILTON, "Theory of systems of rays," Transactions of the Irish Academy, v. XV, pp. 69 and v. XVI, pp. 1 and 94.

where ds is the element of the trajectory that is followed by the light ray, and v is the velocity of light that is calculated under the hypothesis of emission. This makes the solution of all of the questions that relate to reflection and refraction – whether ordinary or extraordinary – depend upon the existence of a *characteristic function* for every optical system of rays. That function is generally defined in the following manner:

$$V = \int v\, ds = f(x, y, z, x', y', z', \psi).$$

x, y, z are the coordinates of the final point of a luminous trajectory, x', y', z' are those of the initial point, and ψ is a constant that depends upon the color. What distinguishes Hamilton's research on the subject of that integral is that it regards the variable coordinates as depending upon the extremities of the ray and the color, while in the statement of the principle of least action these extreme coordinates and color are regarded as constants, and only the intermediate points where reflection or refraction takes place are assumed to vary. In the report that Hamilton made on this work to the British Association ([1]), he remarked that under the hypothesis of undulations, the characteristic function V will represent the *time of propagation of light from one point to another*. However, he confined himself to indicating that viewpoint without otherwise taking advantage of it, which is easily explained by the lack of favor that the wave theory of light enjoyed in England in that era. Moreover, Hamilton was occupied almost exclusively with regular systems of rays, and spoke only incidentally of the modifications that would be introduced into these systems by passing through birefringent media.

4. Hamilton's work did not attract the attention that it deserved at first, and it was only after a long interval that we see Kummer return to the same subject in 1859, in a very remarkable paper that bore the title of "General theory of rectilinear ray systems ([2])." The celebrated German geometer proposed *to study, in full generality, the properties of a system of lines that fills up all of space or a portion of space, in such a manner that one ray or a well-defined number of rays will pass through any given point.* Taking a purely geometric viewpoint, at no point did he concern himself with the changes in the directions of rays that were produced by reflection or refraction, nor the relations between these directions and the system of light waves. Moreover, he rose to a degree of generality that was, without a doubt, of considerable interest to geometry, but useless to optics. Indeed, the rays that move through an arbitrary, homogeneous medium, and regardless of how they were introduced, will enjoy the property – and one will find the proof of this later on – that their direction has a relationship with the tangent plane to the wave surface that is well-defined and constant for the same medium. It results from this that the optically-realizable systems of rays in a given medium are not arbitrary, but there constitution is intimately connected with that of the medium, and that it presents a certain

([1]) *Report of the first and the second meetings of British Association for the advancement of science.*, pp. 545.
([2]) KUMMER, "Allgemeine Theorie der gradlinigen Strahlensysteme," Journal de Crelle, t. LVII, pp. 189.

number of peculiarities that would not belong to a system of lines that was chosen completely arbitrarily.

The consequences that Kummer's theory implied when one restricted oneself to optically-possible rays were finally developed by Meibauer ([1]). That author was concerned almost exclusively with caustic surfaces. It was in his paper that the relation that exists in an arbitrary homogeneous medium between the direction of a ray and that of the tangent plane to the wave at the point where it is met by that ray was proved for the first time, and stated neatly.

5. The rapid historical outline that you just read sufficiently indicates the type of questions that I propose to treat. My goal is to both simplify and generalize the proof of the principles of geometrical optics. The method to which I have taken recourse consists essentially in introducing the consideration of light waves into the solution of problems that spring from that branch of applied mathematics, and also that of the time that is employed by light in order to propagate from one point to another. That manner of proceeding will offer the advantage of permitting one to take for one's starting point, not the laws of reflection and refraction in isotropic media, but the general principle that was stated for the first time by Huyghens under the name of *the principle of enveloping waves*, and which is applicable to any type of homogeneous medium. With the introduction of time, the theorems of geometrical optics will thus take on a form that permits one embrace the properties of luminous trajectories in any homogeneous medium – whether isotropic or anisotropic – in a single statement.

The present paper has the objective of studying the general properties of optical systems of rectilinear rays; i.e., the systems that propagate in an arbitrary *homogeneous* medium ([2]). The principles that are presented in it will ultimately receive their application in other papers that are dedicated to aplanatic surfaces and caustic surfaces. Moreover, the results that one gains for homogeneous media can be extended to heterogeneous media, which can always be considered to be composed by the juxtaposition of an infinitude of infinitely-thin homogeneous layers.

I. – **Definitions and notations.**

6. Let there be a center of disturbance in an arbitrary homogeneous medium, from which a homogeneous wave emanates. The locus of points to which the vibratory motion is communicated after a given time interval will constitute a certain surface. By following each of the directions that start from the luminous point, the velocity of propagation of that motion will be constant, and the surfaces that correspond to different time points will all be similar and similarly-placed with respect to the luminous point.

([1]) MEIBAUER, *Theorie der gradlinigen Strahlensysteme des Lichts*, Berlin, 1864.

([2]) The principle results that are contained in this paper have been recorded in a note that was presented to the Academy of Sciences on 10 September 1866. *Comptes rendus des séances de l'Academie des Sciences,* t. LXIII, pp. 458.

Since the medium is homogeneous, the form of these surfaces and their orientation in the medium will be independent of the point from which the light emanates. Each of them will obviously have its center at the luminous point to which it refers; moreover, since the velocity of light cannot become infinite along any direction, they will be closed surfaces. We call them *characteristic wave surfaces of the medium*. Indeed, it is clear that a homogeneous medium is optically defined, at least, for a given color, when one knows that locus of points to which a vibratory motion that corresponds to that color and emanates from an arbitrary point of the medium will be communicated after a given length of time. Knowing any of the characteristic wave surfaces of a homogeneous medium – for example, the one that has its center at the origin and which corresponds to a unit of time – it is easy to find all of the others; in order for one of those surfaces to be well-defined, it will then suffice to give its center and the time interval to which it corresponds.

In order for two homogeneous media to be identical from the optical viewpoint (while considering only light of one well-defined color), it is necessary and sufficient that the characteristic wave surfaces of these two media that correspond to the same time – for example, a unit of time – be identical. Meanwhile, we remark that if two homogeneous media are optically identical then light can nonetheless experience a change of direction upon passing from one to the other; that is what always happens whenever these media are placed in such a fashion that the homologous radius vectors of the characteristic wave surfaces are not parallel, but that is true when one superposes two layers of the same crystal that was sliced along different planes. In order for two homogeneous media to be considered as composing a continuous whole, it is therefore necessary and sufficient that:

1. The characteristic wave surfaces that correspond to the unit time be identical.
2. The homologous radius vectors of these surfaces be parallel.

For isotropic, homogeneous media the characteristic wave surfaces will always be spherical, so the first condition will be sufficient.

The homogeneous media whose characteristic wave surfaces that correspond to a unit of time are similar, without being identical, constitute a group and enjoy a certain number of common properties. For example, one has the group of isotropic, homogeneous media whose characteristic wave surfaces that correspond to a unit of time are all spheres, but of different radii.

7. The characteristic wave surfaces of a homogeneous medium have one or more sheets, according to whether or not the light motion in each direction can propagate in that medium with a unique velocity or with several velocities, respectively. The media whose characteristic wave surfaces have only one sheet are called *unifringent;* the ones for which these surfaces have two sheets are called *birefringent*. From another viewpoint, one distinguishes the *isotropic* media; i.e., the ones in which light propagates with the same velocity in all directions, and the *anisotropic* media, in which the velocity of propagation of light will vary with the direction. Calculation and experiment have consistently shown that if one accounts for only the transverse vibrations, which seem to be the only ones that are suitable for producing the luminous effect, then any anisotropic, homogeneous medium will necessarily be birefringent. In the unifringent or isotropic, homogeneous media, the characteristic wave surfaces will obviously be spheres. As for the birefringent or anisotropic media, physical optics tells us that they divide into two classes: For the one, the characteristic wave surfaces are composed of one spherical sheet and another sheet that takes the form of an ellipsoid of revolution, such that these two sheets will envelop each other and touch at the extremities of the axis of the ellipsoidal sheet: These are the *uniaxial* media. For the others, the characteristic wave surfaces will be surfaces of fourth degree that are indecomposable into second-degree surfaces: these are the *biaxial* media. We remark that at present the uniaxial media can be considered to be isotropic relative to the rays that correspond to the spherical sheet and which one calls *ordinary rays*. All of the theorems that were proved for isotropic media will thus be applicable to uniaxial birefringent media when one confines oneself to ordinary rays in the latter media.

8. We let:

$$f(x, y, z) = 0$$

represent the equation of the characteristic wave surface of a homogeneous medium that has its center at the coordinate origin and corresponds to a unit of time. That of the characteristic wave surface that has its center at the origin and corresponds to time T will be, in turn:

$$f\left(\frac{x}{T}, \frac{y}{T}, \frac{z}{T}\right) = 0.$$

When the latter equation is assumed to have been solved for T, we will write:

$$\varphi(x, y, z) = T.$$

The equation of the characteristic wave surface that has its center at a point whose coordinates are (a, b, c) and corresponds to the time T will be:

$$f\left(\frac{x-a}{T}, \frac{y-b}{T}, \frac{z-c}{T}\right) = 0,$$

or

$$\varphi(x - a, y - b, z - c) = T.$$

If there is reason to consider two different media then we will give the symbol of the function f or φ the index 1 when the medium is the one in which the incident and reflected rays propagate, and the index 2 for the media in which the refracted rays move. When one is dealing with a birefringent medium, the indices o or e, when added to the symbol of the function, will serve to distinguish the two sheets of the characteristic wave surfaces.

When we speak of reflection and refraction, we always suppose implicitly that the separation surface where the change of direction of rays takes place is continuous; i.e., that one can draw only one tangent plane at each point of that surface. That restriction is essential, because most of the theorems that we will arrive at will cease to be true when the separation surface is composed of several portions of surfaces that intersect at angles that are different from zero; for example, when the surface is polyhedral. The case where the separation surface presents one or more projecting (*saillants*) points will likewise be reserved.

9. To abbreviate, we denote the ordinary rays with the symbol (*o*) and the extraordinary rays by the symbol (*e*). When the light passes from a birefringent medium into another, likewise birefringent, medium, there will generally be four types of refracted rays: We let (*o*, *o*) denote the ones that one obtains by taking the ordinary sheets of the characteristic wave surfaces in the two media, while the other three types of refracted rays will be denoted in an analogous manner by the symbols (*e*, *e*), (*o*, *e*), (*e*, *o*), where the first letter will always refer to the first medium – i.e., the one in which the incident rays move. Moreover, one knows that the construction that gives some of these refracted rays can become impossible, and their number can reduce to 3, 2, 1, and even zero. When the light that emanates from a point that is situated in a birefringent medium is subject to a reflection, there is likewise reason to distinguish four types of reflected rays, which we denote by the same notations as in the case of refraction. However, here, one must remark that the reflected rays (*o*, *o*) and (*e*, *e*) will always exist, whereas the construction that gives the reflected rays (*o*, *e*) or (*e*, *o*) can become impossible. Moreover, there is an essential distinction to be made between the reflected rays (*o*, *o*) and (*e*, *e*), on the one hand, and the reflected rays (*o*, *e*) and (*e*, *o*), on the other. For the rays of the first group, the same sheet of the characteristic wave surface will correspond to the incident and reflected rays; for those of the second group, the two sheets will be different. As is easy to imagine, it will then result that the latter type of reflection will offer more of an analogy with refraction than it does with reflection, properly speaking. We say that there is *homologous reflection* when the incident and reflected rays correspond to the same sheet of the characteristic wave surface of the medium, and that there is *antilogous reflection* in the contrary case. The rays (*o*, *o*) and (*e*, *e*) will therefore be the rays that are subject to a homologous reflection, while the rays (*o*, *e*) and (*e*, *o*) will experience an antilogous reflection. In unifringent media, reflection is necessarily homologous.

10. If the directions of the incident rays that fall on a reflecting or refracting surface in a homogeneous medium agree at the same point that is situated *on one side* of that surface then that point – whether or not it is the one that provides light – will bear the name of *real, luminous point*. If the point of agreement of the incident is situated *on the other side* of the reflecting or refracting surface then we will say, to abbreviate, that these rays emanate from a *virtual, luminous point*. When all of the reflected or refracted rays that propagate in a homogeneous medium are directed in such a fashion that these rays or their prolongations converge to the same point, we will call that point a *total focus*. If, amongst the reflected or refracted rays that move in a homogeneous medium, there is an infinitude of them that form a conical surface, and which, by themselves or their prolongations, agree at a point then that point will be called a *partial focus*. In the case of reflection, a total or partial focus will be called *real* if it is in front to the reflecting surface and *virtual* if it is behind it; in the case of refraction, the opposite will be true. If there exists a sequence of partial foci that form a continuous line then that line will receive the name of *focal line*.

11. We conclude these preliminary considerations with some important remarks that relate to waves.

When the rays that move in a homogeneous medium do not issue from a point that is situated in that medium, or when, although they emanate from a point of the medium, they are subject to one or more reflections, the locus of the points that are attained at the same instant by the vibratory motion that propagates along these rays will again bear the name of *wave surface*. However – and this is the point on which one cannot insist enough – these reflected or refracted waves will not, as we will see later on, generally have the form of characteristic wave surfaces in the medium in which they propagate, and will no longer necessarily constitute a system of similar, concentric surfaces. For example, in an isotropic medium, these waves will be spherical in only three special cases.

Let a system of rays originate from a luminous point that is situated in a homogeneous medium, and currently propagates, either in that medium or in another homogeneous medium, where these rays can have been subjected to an arbitrary number of reflections or refractions, but have traversed only homogeneous media. The wave that corresponds to these rays can have several sheets, even when the medium in which it presently moves is isotropic; this will be the case, in general, if these rays have previously traversed birefringent media. *A fortiori*, the wave will have several sheets when the rays presently propagate in a birefringent medium. From that, it is essential to give more precision to the theorems that we shall state, to group the rays that, having originally issued from the same luminous point, propagate in an arbitrary, homogeneous medium in systems such that *the rays of the same system will correspond to waves that are each composed of one unique and continuous sheet.* We call the rays that belong to such a system *rays of the same kind*.

In order for rays that originally issued from the same point, having traversed only homogeneous media and presently propagating in such a medium, to be of the same kind, from the definition that we just gave, it is obviously necessary and sufficient that:

1. If the medium where the luminous point is found is birefringent then these rays should all be originally *of the same nature;* i.e., all ordinary or all extraordinary.

2. They have been subjected to the same reflections and refractions by the same surfaces in the same order, and consequently, have traversed the same homogeneous media.

3. In each of these reflections or refractions, these rays will either all preserve their nature – i.e., their ordinary or extraordinary quality – or they will all change their nature together.

One should not confuse the term "rays of the same kind" with the term "rays of the same nature." Some rays can be of the same nature – i.e., all ordinary or all extraordinary – relative to the medium in which they move, without also being of the same kind. Rays of the same kind are necessarily of the same nature.

12. Finally, consider a system of rays of the same kind that propagates in an arbitrary, homogeneous medium. Let *S* denote a wave that is composed of one unique and continuous sheet that corresponds to these rays at a certain instant, and let *R* denote an arbitrary ray of the system. The ray *R* can meet the wave *S* at more than one point; however, *among the intersection points of the ray R with the wave S, there will always be one and only one of them where the vibratory motion takes place on the ray R at the instant considered,* and it is always that point that we will be referring to when we speak of the point where the ray *R* meets the wave *S*. That remark must not be lost from sight, because what we will have to say about that point will never be applicable to the other points of intersection of the ray *R* with the wave *S* when there exist more than one of them. Moreover, one sees that whenever a wave is met by a ray of the system to which it corresponds at more than one point, all of these points, except for the ones where the vibratory motion takes place on the ray *R* at the instant considered, will necessarily be points of intersection of the ray *R* with the other rays of the system.

II. – **Huyghens's principle, or enveloping waves. – General construction of reflected or refracted waves.**

13. The only notions that geometrical optics borrows from the theory of wave mechanics are:

1. The knowledge of the form of the characteristic wave surfaces of different homogeneous media.

2. The known theorem that goes by the name of *Huyghens's principle* or *the principle of enveloping waves* ([1]).

It is, above all, important to clarify the significance and scope of that fundamental principle, since we will, so to speak, develop only some of its consequences in all of what follows.

We first place ourselves in the simplest case, where the light that emanates from a point that is situated in an indefinite homogeneous medium propagates in that medium without being subject to reflection or refraction. Let *O* be the luminous point, and let *S* denote the position that is occupied by the wave that emanates from the luminous point after the time $T + t$. The surfaces *S* and S' will be the characteristic wave surfaces of the medium; they will therefore be similar and similarly-placed with respect to the luminous point *O*. It follows from this that the wave S' can be regarded as the envelope of the characteristic wave surfaces of the medium that are described when the various points of the wave *S* are taken to be the centers and corresponding to the time *t*; these characteristic wave surfaces will be what *Huyghens* called *elementary waves*. One sees, moreover, that in order to get the direction of the ray that passes through an arbitrary point *A* of the wave S', it will suffice to join that point to the center of the elementary wave that touches the wave S' at *A*. Of course, if the medium is birefringent then one must take the elementary wave to be one or the other of the sheets of the characteristic wave surface of the medium, according to whether one is dealing with the propagation of ordinary or

([1]) HUYGHENS, *Traité de la lumiére,* Leyden, 1690.

extraordinary waves. The results that we just stated can be expressed in another form by saying that in order to study the propagation of the wave after the time *T*, one can suppose that the luminous point *O* has been suppressed, on the condition that one considers each of the points of the wave *S* to be a center of disturbance, but while regarding each of the elementary waves that emanates from the various points of the surface *S* as being active only at the point where it touches the common envelope. As long as one confines oneself to the simple case of the propagation of light in the same homogeneous medium, Huyghens's principle is only the expression of an identity, and there is no need for a special proof in order to see that, at a given instant, the perceptible motion on each of the elementary waves cannot be at the point where it touches the enveloping wave. However, Huyghens did not stop there: By a sort of intuition, he arrived at the generalization of the principle of enveloping waves and applied it to the construction of reflected and refracted waves. The principle, when conceived in its full scope, can be stated as follows: *As the various points of a surface are successively or simultaneously arrived at by the vibratory motion that emanates from a luminous point, the wave, at any instant, will always be the envelope of the elementary waves that emanate from the different points of the surface, when it is considered in the position that it occupies at that moment; moreover, the ray that passes through an arbitrary point of the wave will also pass through the center of the elementary wave that touches that point of the enveloping wave.* The surface where one finds it can be either a wave surface or a reflecting or refracting surface, and in turn, the elementary waves can correspond to equal or unequal times. Taken with that general meaning, Huyghens's principle is far from obvious in itself, since it pertains to the case in which the wave propagates in the same homogeneous medium without reflecting or refracting. It does not suffice to remark, as Huyghens's did – and after, him Young and many other authors – that the elementary waves shrink more and more in measure as one approaches the enveloping wave. As long as one does not introduce the notion of interference, one can indeed prove that on each elementary wave at a finite distance from the common envelope the vibratory motion will be very small with respect to that on the envelope, but it is not infinitely small ([1]). It was only in the work of Fresnel that it was proved rigorously that the elementary waves will be destroyed by interference at all of the points that do not belong to the common envelope, and that the use of Huyghens's principle is found to be justified in every case.

15. We therefore take that principle to be our point of departure, and before passing to the general construction of reflected or refracted waves, we shall apply it to the solution of a problem that will constantly present itself to us in the sequel. Let a wave *S* be given that corresponds to a system of rays of the same type and propagates in a homogeneous medium. As we have already remarked, that wave will not generally have the form of the characteristic waves surfaces of the medium in which it propagates, because the rays cannot issue from a point that is situated in that medium, and even when they do emanate from a point that is situated in that medium, they can be subjected to one or more reflections. When the position of the wave *S* is known at a certain instant, it will amount to finding the position of that same wave after a positive or negative time *T* – i.e.,

([1]) That is what Verdet proved using analysis in the first lessons in the course in higher optics that he taught at the Sorbonne in 1865.

an epoch that is later or previous by T to the instant considered – when one supposes that during the time T, the wave is not subject to either reflection or refraction. To that effect, from the principle of enveloping waves, it will suffice to describe each of the points of the first wave as being the center of a characteristic wave surface of the medium that corresponds to a time T. If the medium is birefringent then one can take the ordinary or extraordinary sheet of the characteristic wave surface, according to whether the rays of the system themselves are ordinary or extraordinary, resp.. The envelope of the portions of the characteristic wave surface thus described, which are found in front of the first wave when T is positive, and behind that wave when T is negative, will be the desired wave. It can happen that the wave that is obtained by means of that construction is, in whole or in part, outside the limits of the medium; that is what we express by saying that the wave is, in whole or in part, *virtual*. The consideration of virtual positions of the wave – i.e., the positions that it occupies in the epochs before or after the one in which that wave passes through its real positions, if the medium in which it moves is continued beyond the limits that separate the contiguous media – will be of great help to us in the proof of several theorems.

We further remark that, from the construction that we just pointed out, whenever a wave that propagates in a homogeneous medium does not coincide with one of the characteristic wave surfaces of that medium, the various positions that it will successively occupy will no longer constitute a system of similar and concentric surfaces.

15. It is easy to translate the preceding construction into analytical language. Indeed, let:

(1) $$\psi(x', y', z') = 0$$

denote the equation of the wave considered in the position that it occupies at a certain moment that is taken to be the origin, and let:

$$f_1(x, y, z) = 0$$

be the equation of the sheet with the same nature as the rays of the characteristic wave surface of the medium that is described by having its origin as the center and that corresponds to a unit of time. The equation of the characteristic wave surface of the medium, when it is described by taking an arbitrary point (x', y', z') of the original wave to be the center and corresponding to the time T, will be:

(2) $$f_1\left(\frac{x-x'}{T}, \frac{y-y'}{T}, \frac{z-z'}{T}\right) = 0,$$

if one considers only the sheets of that surface that have the same nature as the rays.

The wave after the time T will be the envelope of the surfaces that are represented by (2). In order to obtain that envelope, it is necessary to eliminate one of the three variable parameters x', y', z' from equations (1) and (2) – for example, z' – which will give an equation of the form:

(3) $$\Phi(x, y, z, x', y', z', T) = 0,$$

which contains only the two arbitrary parameters x' and y'. If one eliminates x' and y' from equation (3) and the two equations that one obtains by differentiating them with respect to the two variables x' and y', which are:

$$\frac{d\Phi}{dx'} = 0, \qquad \frac{d\Phi}{dy'} = 0,$$

then one will arrive at an equation of the form:

$$F(x, y, z, T) = 0,$$

which will represent the wave in the position that it occupies at the time T.

16. We can now begin the general problem of the construction of reflected or refracted waves. Let two homogeneous media be separated by a continuous surface. Suppose that a system of rays of the same kind propagates in one of these media that originally issue from a luminous point that is situated in that medium or in any other homogeneous medium, but having had to cross only homogeneous media. That system of rays will correspond to a system of waves that we call *incident waves*, and since the rays will be of the same type, each incident wave will be composed of a unique and continuous sheet. When the incident rays encounter the separation surface of the two media, the light will divide – at least, in general – into two parts: One of them will be the path that turns back into the first medium and is called *reflecting*, while the other one will penetrate into the second medium and is called *refracted*.

Having said that, consider the incident wave in a certain position S where it cuts the separation surface of the two media, and propose to find the position of the reflected or refracted wave; we further make use of the principle of enveloping waves. In the first place, suppose that one is dealing with a refracted wave. When the various points of the curve along which the wave S cuts the refringent surface are taken to be centers, one will describe characteristic wave surfaces of the second medium that correspond to the time T. One will then seek the intersection of the incident wave at a time τ after the wave S with the refringent surface, and when the various points of that curve are taken to be centers, one will describe characteristic wave surfaces in the second medium that correspond to the time $T - \tau$. One gives τ all of the positive values that are between zero and T and all of the negative values that are between zero and a certain limiting value for which the incident wave becomes tangent to the refringent surface and ceases to intersect it. The envelope of the portions that are situated in the second medium of the characteristic wave surfaces thus described at the various points of the refringent surface, when taken to be centers, will be the refracted wave after the time T. In other words, when each point of the refringent surface is taken to be a center, one will describe a characteristic wave surface in the second medium that corresponds to a time $T - \tau$, where τ is the interval that elapses between the moment at which the incident wave passes through the position S that was taken to be the origin and the one at which it passes through the point considered, and that time τ will be taken to be positive or negative, according to whether the first of the two moments is earlier than or later than the second one. The envelope of the portions of all these characteristic wave surfaces that are situated in the second medium will be the desired refracted wave. A construction that is entirely similar to the preceding one will give the position of the reflected wave after a time T. However, in that case, the characteristic wave surfaces that one must describe at the various points of the reflecting surface, when they are taken to be centers, will be those of the first medium, and it is the envelope of the portions of the characteristic wave surfaces that are situated in the first medium that will be the desired reflected wave.

In order to get the directions of the reflected or refracted rays that provide a given incident ray, one must look for the points where the characteristic wave surface, which is described at the point A where the incident ray meets the separation surface, which is taken to be a center, touches the common envelope – i.e., reflected or refracted – and join these points to the point A. In the case of reflection, one will always obtain a unique reflected ray by that construction if the first medium is unirefringent, but one can find two of them if the medium is birefringent. In the case of the refraction, that construction will never give a refracted ray if the second medium is refringent, but can give two if it is birefringent.

If the second medium is birefringent in the case of refraction, or if the first medium is birefringent in the case of reflection, then the reflected or refracted wave will generally have two sheets, which signifies that a system of incident rays of the same type will then correspond to two systems of reflected or refracted rays that must be regarded as having different types.

It might happen that when a certain point of the separation surface is chosen to be a center the characteristic wave surface that was described in the preceding construction does not intersect any of the ones that are described when the other points of that surface are chosen to be centers, and consequently does not touch the common envelope. When that is the case, there will be no possibility of reflection or refraction. If the second medium is unirefringent and the characteristic wave surface that is described when a certain point A is chosen to be a center does not touch the common envelope then that will indicate that there is no refraction at A and that, in turn, the incident ray that reaches A will be subject to a total reflection at that point. If the second medium is birefringent, in that one of the sheets of the characteristic wave surface that is described when A is taken to be a center does not touch the common envelope, then one must conclude that there is only one refracted ray. However, when any of the sheets of that surface touch that common envelope there will be total reflection. Now, examine the case of reflection. If the first medium is unirefringent then each of the characteristic wave surfaces will necessarily touch the common envelope, and in turn, each incident ray will correspond to a unique reflected ray. If the first medium is birefringent then that sheet of each of the characteristic waves surfaces that has the same nature as the incident ray will necessarily touch the common envelope, and in turn, each incident ray will always correspond to at least one reflected ray, which will be either the ray (o, o) or the ray (e, e), according to whether the incident ray is ordinary or extraordinary, resp. However, the other reflected ray, which will either the ray (o, e) or the ray (e, o), can be absent, and will indeed be absent when the sheet of the characteristic wave surface that is described when the point of incidence is taken to be its center, which will have a different nature from that of the rays, does not touch the common envelope.

17. The general construction that we just presented permits us to find the equation of the reflected or refracted wave when it is considered in the position that it occupies after a given time, while the equation that represents the incident wave is known at a certain moment.

Indeed, let the position of the incident wave be given at a certain moment that we take to be the time origin. The equation of that incident wave in the position that it occupies after a time t – whether positive or negative – when measured by starting from that instant, can be obtained in the way that we indicated above (15), and that equation will be of the form:

$$F(x, y, z, t) = 0.$$

Moreover, let:

$$f_1(x, y, z) = 0$$

and

$$f_2(x, y, z) = 0$$

be the equations of the characteristic wave surfaces of the first and second medium, when they are described by taking the origin to be the center and corresponding to a unit of time, and finally let:

$$\varphi(\xi, \eta, \zeta) = 0$$

be the equation of the reflecting or refracting surface.

We propose to find the equation of the refracted wave, when it is considered in the position that it occupies after a time T, after starting from the instant that is taken to be the origin. Let ξ, η, ζ be the coordinates of a point of the refringent surface that is attained by the incident wave after a positive or negative time that denote by θ. We will have:

(1) $$F(\xi, \eta, \zeta, \theta) = 0,$$

and

(2) $$\varphi(\xi, \eta, \zeta) = 0.$$

From the preceding construction, one must describe the characteristic wave surface in the second medium that corresponds to the time $T - \theta$ when the point ξ, η, ζ is taken to be the center, a surface whose equation is:

(3) $$f_2\left(\frac{x-\xi}{T-\theta}, \frac{y-\eta}{T-\theta}, \frac{z-\zeta}{T-\theta}\right) = 0.$$

The desired wave is the envelope of the surfaces that are represented by equation (3). In order to find that envelope, to begin with, one must eliminate θ and one of the three variables ξ, η, ζ – for example, ζ – from equations (1), (2), and (3), which will lead to one equation that only refers to the two variable parameters ξ and η, an equation that will be of the form:

$$\Phi(x, y, z, T, \xi, \eta) = 0.$$

If one eliminates the two variables ξ and η from that equation and the ones that one obtains by differentiating it with respect to the variable parameters ξ and η, equations that will be:

$$\frac{d\Phi}{d\xi} = 0,$$

and

$$\frac{d\Phi}{d\eta} = 0,$$

resp., then one will definitively arrive at an equation of the form:

$$\mathcal{F}(x, y, z, T) = 0,$$

which will represent the desired wave.

The path that one must follow in order to find the equation of the reflected wave is entirely similar to the one that we just pointed out, with the one difference that in equation (3), the function f_2 must be replaced with the function f_1, which represents the characteristic wave surface in the first medium.

In the particular case where the directions of the incident rays agree at the same luminous point – whether real or virtual – the equation of the incident wave must be, upon denoting the coordinates of the luminous point by (a, b, c) and measuring time by starting from the instant where the light begins at that point:

$$f_1\left(\frac{x-a}{t}, \frac{y-b}{t}, \frac{z-c}{t}\right) = 0,$$

and equation (1) will take the form:

$$f_1\left(\frac{\xi-a}{\theta}, \frac{\eta-b}{\theta}, \frac{\zeta-c}{\theta}\right) = 0.$$

18. In order to clarify the preceding by way of an example, we shall perform the calculation of the refracted wave in the simple case where the two media are isotropic, the separation surface is planar, and the rays emanate from a point that is situated in the first medium. We take the refringent surface to be the xy-plane, and the perpendicular that is based at the luminous point on that plane to be the z-axis. We let c represent the distance from the luminous point to the refringent plane, and we let v and v' be the velocities of light in the first and second media, resp. We take the time origin to be the moment at which the light starts from the luminous point, and we propose to calculate the equation of the refracted wave in the position that it will occupy when the time T is equal to zero. That position will obviously be virtual, but we choose it anyway, because that will be when the refracted wave has the simplest form. In the particular case that we have chosen, equations (1), (2), and (3) will become:

$$\xi^2 + \eta^2 + (\xi - c)^2 = v^2\,\theta^2, \qquad \zeta = 0, \quad (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = v'^2\,\theta^2.$$

Upon eliminating θ and ζ from these three equations, one will get:

$$(A) \qquad v'^2\,(\xi^2 + \eta^2 + c^2) = v^2\,[(x - \xi)^2 + (y - \eta)^2 + z^2].$$

When the latter equation is differentiated with respect to the two variable parameters ξ and η that it contains, that will give:

$$v'^2\xi + v^2\,(x - \xi) = 0, \qquad v'^2\eta + v^2\,(y - \eta) = 0,$$

resp., so:

$$\xi = \frac{v^2 x}{v^2 - v'^2}, \qquad \eta = \frac{v^2 y}{v^2 - v'^2},$$

$$x - \xi = -\frac{v'^2 x}{v^2 - v'^2}, \qquad y - \eta = -\frac{v'^2 y}{v^2 - v'^2}.$$

Upon substituting these values into equation (A), one will obtain the equation:

$$x^2 + y^2 + \frac{(v'^2 - v^2)}{v'^2}\,z^2 = \frac{(v'^2 - v^2)}{v^2}\,c^2$$

for the desired refracted wave, which, upon setting:

$$\frac{v}{v'} = n,$$

will become:

$$x^2 + y^2 + (1 - n^2)\,z^2 = \frac{(1 - n^2)}{n^2}\,c^2.$$

One immediately sees that the surfaces that this equation can represent when one gives various values to n will all be ellipsoids or hyperboloids of revolution around the z-axis that have one of their foci at the luminous point and their center at the origin – i.e., at the foot of the perpendicular that is based at the luminous point on the refringent plane. The refracted wave that corresponds to a time that is equal to zero will be an ellipsoid of revolution if n is smaller than unity; i.e., if the second medium is less refringent than the first one. It will be a hyperboloid of revolution when the second medium is more refringent than the first one. If the position of the luminous point with respect to the refringent plane changes – i.e., if one makes c vary – then the ratio of the axes of the meridian curve will remain constant, because that ratio will be equal to $\sqrt{n^2 - 1}$ or $\sqrt{1 - n^2}$, according to whether n is greater than or less than unity, resp. It will result from this that if the meridian curve is a hyperbola then it will keep the same asymptotes when one makes c vary.

From Malus's theorem, since the rays are all normal to the wave in an isotropic medium, one will see that when two isotropic media are separated by a planar surface, and the incident rays emanate from a luminous point that is situated in the first medium, the refracted rays will be normal to a second-degree surface of revolution. In order to have the refracted wave that corresponds to an arbitrary time T, it will suffice to give each of these normals to that surface a length that is equal to $v'T$. The surface that passes through all of the points thus obtained will be the desired wave.

19. The well-known construction that Huyghens described in order to find the direction of the reflected or refracted ray when one knows the direction of the incident ray and also that of the plane that is tangent to the reflecting or refracting surface at the point of incidence is easily deduced from the general construction of the reflected or refracted wave. In order to do that, it will suffice to imagine that the incident ray is part of a system of parallel rays of the same type, that the separation surface is planar, and that it agrees with the tangent to the reflecting or refringent surface at the point of incidence, which is a hypothesis that changes nothing regarding the direction of the reflected or refracted wave. Let OA be the incident ray, P, the incident plane wave that passes through the point of incidence A, and let P' be the position of the reflected or refracted plane wave after a time that is equal to unity, when measured from the moment when the incident wave occupies the position P. The wave P' is tangent to that sheet of the characteristic wave surface that is described by making the point A its center and corresponding to a unit of time, and which has the same nature as the rays that refer to the wave P', where that characteristic wave surface is that of the first medium when one is dealing with a reflection and that of the second one when one is dealing with a refraction. One will get the desired reflected or refracted ray by joining the point of contact to the point A. Moreover, the plane P' will intersect the planar separation surface of the two media along the same straight line as the incident wave, when considered in the position that it occupies after a unit of time. In order to find that position of the incident wave, it will obviously suffice to describe the sheet of the characteristic wave surface in the first medium that has the point A for its center, corresponds to a unit of time, and has the same nature as the incident rays, and to draw a tangent plane that is parallel to the plane P on the portion of that sheet that is found in the second medium. One can further – and because it amounts to exactly the same thing – draw a tangent plane to that sheet at the point where it is met by the prolonged incident ray OA, which is a plane that is necessarily parallel to the plane P.

We thus arrive at the following construction, which is nothing but that of Huyghens: When the point of incidence is taken to be a center, one describes the characteristic wave surfaces Σ and Σ' of the first and second media, resp., that correspond to a unit of time. At the point where the prolonged incident ray meets the surface Σ, if the first medium is unirefringent, or the point at which it meets the sheet of the surface Σ that has the same nature as it, when the first medium is birefringent, the incident ray will be only ordinary or only extraordinary, and one draws a tangent plane to that surface or that sheet, resp. If the first medium is birefringent, and the incident ray is both ordinary and extraordinary then one will draw tangent planes to the two sheets of the surface Σ at the points where that prolonged ray meets those sheets. In order to get the refracted rays, through the line or lines of intersection of that tangent plane or planes with the tangent to the separation surface at the point of incidence, one draws as many tangent planes as possible to the portion of the surface Σ' that is found in the second medium, and one joins the points of contact to the point of incidence. In order to get the reflected rays, one draws as many tangent planes as possible to the portion of the surface Σ that is found in the first medium that go through the same lines of intersection, and one joins the points of contact to the point of incidence.

We shall not stop to discuss the known consequences that result from that construction for birefringent media with two axes where the characteristic wave surface presents singular points and singular tangent planes. The particular phenomena that will then be produced for certain directions of the incident rays have been studied from the theoretical viewpoint by Hamilton and experimentally by Lloyd, and constitute what is convenient to call *internal or external conical refraction*.

III. – **Relations between the direction of the ray and that of the tangent plane to the wave. – Generalization of Malus's theorem.**

20. Let a system of waves be given in an arbitrary homogeneous medium that corresponds to a system of rays that originally issue from the same point with the same type. Take one of these waves, which we denote by Σ, to be the starting point, and let S, S', S'', … be the positions that are successively occupied by the wave as it propagates through the medium without reflecting or refracting. Consider a ray of the system that meets the wave Σ at the point O, and the waves S, S', S'', …, at the points A, A', A'', …, resp.

From the construction that was described above (14), which was based upon the principle of enveloping waves, the waves S, S', S'', …, at the points A, A', A'', …, resp., will be tangent to the sheets of the characteristic wave surfaces of the medium that have the point O for their center and correspond to different times, while having the same nature as the rays. Since these sheets are similar surfaces that are similarly-placed with respect to the point O, and the points A, A', A'', … are found on the same line, the tangent planes to the sheets at the points A, A', A'', …, and in turn, also the tangent planes to the waves S, S', S'', … at the same points, will be parallel to each other. We then have the following proposition:

THEOREM I. – *When a system of rays that originally issue from the same point and all have the same type propagates in an arbitrary homogeneous medium, the tangent planes that are drawn to the waves that correspond to these rays at the points where these waves are met by the same ray will be parallel to each other.*

We remark that this theorem is true, no matter what number of reflections and refractions the rays have been subjected to.

21. It results from the preceding that when a system of rays that originally issued from the same point and all have the same type propagates through an arbitrary homogeneous medium, the direction of one of these rays is determined when one knows that of the tangent plane that is drawn on one of the waves that corresponds to the system of ray at the point where it is met by that ray, and that furthermore, if the medium is birefringent then the nature of the rays will be given. Indeed, suppose one wishes to find the direction of the ray that meets the wave S at a certain point A. When an arbitrary point is taken to be the center, one describes the sheet of the characteristic wave surface of the medium that has the same nature as the rays and corresponds to an arbitrary time, and one draws a tangent plane to that sheet that is parallel to the tangent to the wave S at A. The radius vector that that joins the point of contact to the center of the surface will be parallel to the ray that passes through the point A.

Conversely, if one is given the direction of one of the rays of the system and its nature then it will be easy to find the common direction of the tangent planes to the waves at the points where they are met by that ray. In order to do that, it will suffice to describe the sheet of a characteristic wave surface of the medium that corresponds to an arbitrary time, and has the same nature as the rays from an arbitrary point that is taken to be the center, and to draw a radius vector on that surface that is parallel to the given ray. The tangent plane to the sheet thus-described at the point where is met by the radius vector will have the desired direction.

These two constructions, which are reciprocal to each other, can be combined into the following statement:

THEOREM II. – *When a system of rays that originally issued from the same point and all have the same type propagates in an arbitrary, homogeneous medium, there will exist a relationship between the direction of the tangent plane to the wave that corresponds to these rays and the direction of the ray that passes through the point of contact that is constant in the same homogeneous medium for rays of the same nature, and that relationship will be the same as the one that exists between the direction of the tangent plane to the sheet of one of the characteristic wave surfaces of the medium that has the same nature as the rays and the direction of the radius vector on that surface that passes through the point of contact.*

We express the relation that exists between the direction of the ray and that of the tangent plane to the wave at the point where it is met by that ray by saying that these two directions are *conjugate* to each other. If the medium is birefringent then we will say that the two directions are *ordinarily* or *extraordinarily conjugate* to each other when the ray is ordinary or extraordinary, resp. As a special case, it can happen that in a birefringent medium the directions of the ordinarily or extraordinarily conjugate planes coincide with the direction of the same line. In order for this to come about, it is necessary and sufficient that upon describing the two sheets of a characteristic wave surface of the medium when an arbitrary point is chosen to be its center, and upon drawing a radius vector that is parallel to that line, the tangent planes to these two sheets at the points where they are met by that radius vector will be mutually parallel. We express the relation that exists between the direction of a plane and a line that are conjugate to each other, both ordinarily and extraordinarily, by saying that these directions are *doubly conjugate*. Therefore, in a birefringent medium with one axis, the direction of a line that is parallel or perpendicular to the axis of the medium and that of a plane that is perpendicular to that line will be doubly conjugate.

22. The preceding theorem, of which Malus's theorem is only a special case, is the key to most of the questions of geometrical optics. It plays an important role in the theory of caustic surfaces and in that of aplanatic surfaces. We shall develop some immediate consequences of it.

In isotropic or uniaxial homogeneous media, the characteristic wave surfaces present neither singular points nor singular tangent planes – i.e., these surfaces are tangent to just one plane at each of their points, and each of the tangent planes to these surfaces touch at only one point – each given direction for the ray is conjugate to a unique direction for the tangent plane to the wave, when the nature of the ray has been assigned. Conversely, in these media, each direction of the tangent to the wave is conjugate to a unique direction for a ray of a given nature. The same thing will be true in birefringent media with two axes, except for two exceptions, which are due to the fact that in such media every characteristic wave surface will present four singular points and four singular planes. The four singular planes are the ones where the surface is tangent to a cone, instead of a plane; they are distributed pair-wise symmetrically on two lines that pass through the center. We call the directions of these two lines *singular directions* of the medium. The singular tangent planes are the ones that touch the surface along a curve, instead of a unique point. We shall examine the peculiarities that result for the propagation of light in birefringent media with two axes from the existence of these singular points and singular tangent planes.

23. In a biaxial medium, let there be a ray that belongs to a system of rays that issue from the same point and have the type. If that ray propagates parallel to one of the singular directions of the medium then Theorem II will be found to break down, and it is no longer sufficient to determine the direction of the tangent plane to the wave at the point where it is met by that ray. All that one can assert is that it this tangent plane will be parallel to one of the tangent planes to the cone that touches the characteristic wave surface of the medium at that one of its singular points that is situated on a radius vector that is parallel to the ray in question, and that this tangent plane will remain parallel to itself while the wave displaces. In general, the wave does not present a singular point at the point where it is met by a ray that is parallel to one of the singular directions of the medium, because if the wave at that point touches a characteristic wave surface of the medium at one of its singular points then it does not result that it must be tangent to the cone that touches the characteristic wave surface at that singular point. It will suffice that the tangent plane to the wave at that point also be tangent to the cone. The direction of the tangent plane to the wave at the point where it is met by a ray that is parallel to one of the singular directions of the medium can generally be determined in a complete manner only when one knows the changes in direction and nature that the ray was subjected to before it propagated along the line that it presently traverses.

Meanwhile, there exists a completely special case where one can affirm that the wave presents a singular point at the point where it is met by a ray that is parallel to one of the singular directions of the medium. That is the one where the ray comes from an infinitude of rays that combine into just one after reflecting or refracting. The tangent cone to the wave at the point where it is met by that ray then has its generators parallel to those of the tangent cone to the characteristic wave surfaces of the medium at that one of its singular points that is situated on the radius vector that is parallel to the ray in question. When the wave propagates in a medium, its singular point will displace along a line that will be parallel to one of the singular directions of the medium.

24. In birefringent media with two axes, every singular tangent plane to the characteristic wave surface will be conjugate to an infinite of radius vectors that form a conical surface and will point to the different points of the curve on which that plane will touch the surface.

From that, it will be easy to see that if, among the rays that issue from the same point and have the same type and propagate in such a medium, there is one of them that is parallel to one of the radius vectors of the characteristic wave surface of the medium that is conjugate to a singular tangent plane then there will necessarily be an infinitude of other rays in the system that are each parallel to one of the radius vectors of the characteristic wave surface that are conjugate to the same singular tangent plane. This is evident when the rays emanate directly from a luminous point that is situated in the medium. If they were subjected to a certain number of reflections and refractions before taking their present directions then one can remark that during the last reflection or refraction that made these rays take their present directions, an incident ray that had given rise to a reflected or refracted ray that was parallel to one of the radius vectors that we just spoke of, the tangent plane to the characteristic wave surface of the medium that is described by making the point of incidence be the center at the point it is met by that reflected or refracted ray will touch that surface along a curve. Therefore, from Huyghens's construction, all of the lines that join the various point of the portion of that curve that is situated in the medium at the point of incidence will be either reflected or refracted rays that come from the same incident ray. These rays will define a conical surface, and each wave will be tangent to the same plane along the curve that is defined where it cuts that conical surface, which will be parallel to one of the singular tangent planes to the characteristic wave surface of the medium.

We thus arrive at the following theorem:

THEOREM III. – *When a system of rays that issues from the same point and has the same type propagates in a biaxial, homogeneous medium, if one draws a tangent plane to any of the waves that corresponds to that system that is parallel to one of the singular tangent planes to the characteristic wave surface of the medium then that plane will touch the wave along a curve. The rays that pass through the various points of that curve will define a conical surface whose summit will be found on the surface where these rays are subjected to the reflection or refraction that has given them their present directions, or at the luminous point, if the rays emanate directly from a point that is situated in the medium. Each of these rays will be parallel to one of the radius vectors that are conjugate in the characteristic wave surface of the medium to the singular tangent plane to that surface that is parallel to the singular tangent plane to the wave. Finally, while* *the wave propagates in the medium, its singular tangent plane will displace while remaining parallel to itself.*

It follows form this that whenever the tangent plane to the wave in a biaxial medium is parallel to one of the singular tangent planes to the characteristic wave surface of the medium, the direction of the ray that passes through the point of contact will be indeterminate, in the sense that this ray can be parallel to any of the radius vectors that are conjugate to that singular tangent plane.

25. In a birefringent medium with one axis, the two sheets of the characteristic wave surface of the medium are tangent to each other at two points that are situated on a line that passes through the common center of these two sheets, and whose direction is the one that one calls the *axis* of the medium. Imagine a system of rays in such a medium that have the same type before the reflection or refraction that gives them their present direction; these rays will correspond to a system of waves with two sheets. If one finds that some of these rays are parallel to the axis of the medium then each of these rays that is parallel to the axis will obviously meet the two sheets of each wave at the same point, because the ordinary and extraordinary velocities of light will be equal to each other along the direction of the axis. Moreover, at this common point, the two sheets of the wave must be tangent to each other, because at that point the tangent plane to each of these sheets must be parallel to the tangent plane to the sheets that corresponds to the characteristic wave surface of the medium at the point where it is met by the axis, which is a plane that is perpendicular to the axis. Thus:

THEOREM IV. – *When a system of rays that issue from the same point propagates in a birefringent medium with one axis, and those rays all had the same type before they were subjected to the reflections or refractions that gave them their present directions, the two sheets of each of the waves that correspond to that system of rays will be tangent to each other at the point where they are met by any ray that is parallel to the axis of the medium, and the common tangent plane to the two sheets of the wave at that point will be perpendicular to the axis.*

26. In isotropic media, the radius vectors of the characteristic surface are all normal to that surface, which is spherical. In birefringent media with one axis, the same is true if one confines oneself to the consideration of the ordinary sheet of the characteristic wave surface. That remark will permit is to deduce the following theorem from Theorem IV, which is nothing but that of Malus:

THEOREM V. – *When a system of rays that issue from the same point and all have the same type propagate in an isotropic, homogeneous medium, or when a system of ordinary rays that issue from the same point and all have the same type propagate in a uniaxial, homogeneous medium, these rays will always be normal to the wave that they correspond to, when it is considered at any of the positions that it successively occupies.*

Malus's theorem is found to be generalized in that statement, in the sense that before the rays penetrated into the medium in which they presently move, they can have traversed arbitrary homogeneous media that are anisotropic, as well as isotropic, and can have been subjected to any number of reflections in these media without this theorem ceasing to be applicable.

Theorem V leads us to the following corollary: The waves that correspond to a system of rays that issue from the same point and all have the same type, and which propagate in an isotropic medium will define a system of surfaces such that any line that is normal to one of them will be, at the same time, normal to all of the other ones. That is why one agrees to call them *parallel surfaces.* The same thing will be true for ordinary waves in a birefringent medium with one axis. On the contrary, if one considers a system of rays that issue from the same point and all have the same type – viz., extraordinary, in a birefringent medium with one axis, ordinary or extraordinary in a birefringent medium with two axes – then one will see that these rays, except for some special directions, will no longer be normal to the waves, and that consequently they will no longer constitute parallel surfaces, at least in the sense that one attaches to that expression most frequently. These waves will even have their tangent planes parallel at some points that are situated on the same line, but the lines that pass through the points of contact of the parallel tangent planes will not generally be normal to the waves.

The converse to Theorem V is true. Indeed, if a system of rays that issue from the same point and have the same type, *but are not all mutually parallel*, propagate in a homogeneous medium in such a way that they are a;; normal to the waves that they correspond to then one must conclude that the ray vectors of the characteristic wave surface of the medium – at least, on the sheet of that surface that corresponds to the rays – will all be normal to that surface or to that sheet, which in turn, cannot be spherical. Therefore:

THEOREM VI. – *When a system of rays that issue from the same point and all have the same type, but are not mutually parallel, propagate in a homogeneous medium in such a way that they are all normal to the waves that correspond to them, that medium will be isotropic or birefringent with one axis, and in the latter case, the rays will necessarily be ordinary.*

If the rays of the system are all parallel to each other, so they are perpendicular to the waves that they correspond to, then it will not be permissible for us to conclude that the medium is isotropic, or even uniaxial. That would result only when the radius vector of the sheet of the characteristic wave surface of the medium that has the same nature as the rays, which is drawn parallel to the common direction of the rays, is normal to that sheet, which can be true for certain directions of the radius vector, even in biaxial media.

27. From Theorem II and the various propositions that we deduced from it as corollaries, we can infer this conclusion, which summarizes all of the preceding developments: If the waves in a homogeneous medium that correspond to a system of rays that issue from the same point and all have the same type are not necessarily present when these rays were subjected to a certain number of reflections and refractions before taking their present directions, then the form of these characteristic wave surfaces of the medium – at least, the nature of these characteristic wave surfaces – will, so to speak, imprint a special stamp on all of the waves that can propagate in the medium, especially as far as the relations between the directions of the rays and those of the tangent planes to the waves are concerned, and certain peculiarities of these characteristic wave surfaces will be found to be reproduced on all of the waves that can traverse that medium.

28. We shall now exhibit several consequences of Theorem II that relate to the case where the rays that propagate in a homogeneous medium are mutually parallel. We first remark that by virtue of the previously-indicated construction (14), a plane wave in an arbitrary, homogeneous medium, no matter what its direction, will always remain planar while it displaces parallel to itself. Having said that, we can state the following two propositions:

THEOREM VII. – *If the rays that issue from the same point with the same type that propagate in an arbitrary homogeneous medium are all parallel to each other then the wave that corresponds to these rays will be planar at each of the positions that it successively occupies and will propagate while remaining parallel to itself.*

Indeed, from Theorem II, since the rays are parallel, the tangent plane to the wave must have the same direction at all of the points of that wave, which can be true only as long as the wave is planar. The direction of the plane wave is always conjugate to the common direction of the parallel rays, which are ordinary or extraordinary, according to whether these rays are themselves ordinary or extraordinary, respectively. As a result, if the medium is isotropic or uniaxial then since the rays are ordinary, the plane of the wave will always be perpendicular to the parallel rays.

However, from the fact that the plane wave is perpendicular to the parallel rays, one can conclude only that the radius vector of the characteristic wave surface of the medium that is parallel to these rays will be normal to that of the sheet of that surface that has the same nature as the rays. As a result, if the medium is isotropic or uniaxial then if the rays are ordinary then those rays can have an arbitrary direction. However, if the medium is uniaxial and the rays are extraordinary then they will necessarily be parallel or perpendicular to the axis of the medium, and if the medium is biaxial then they will necessarily be parallel to one of the three symmetry axes of the medium.

The truth of Theorem VII will suffer a very special exception, but it is one that we cannot neglect to point out: That exception will present itself in the case where the medium is biaxial and the rays are parallel to one of the singular directions of the medium. In that case, as we saw (23), the direction of the tangent plane to the wave can vary, that of the ray can remain constant, and in turn, although the rays might be parallel to each other, the wave that they correspond to will no longer be necessarily planar.

THEOREM VIII. – *If the wave in an arbitrary homogeneous medium that corresponds to a system of rays that issue from the same point and all have the same type is planar at one of the positions that it occupies successively then all of these rays will be parallel to each other.*

Indeed, since the direction of the tangent plane to the wave is constant at the various points of that wave, the same must be true for the direction of the rays that pass through these points.

Like the preceding one, this theorem is subject to an exception. If a plane wave that propagate in a biaxial medium is parallel to one of the singular planes of that characteristic wave surface of the medium then that wave will remain parallel to itself while it displaces, but the rays that correspond to it will no longer be parallel to each other. Indeed, these rays will then define conical surfaces whose summits are found on the surface where these rays were subject to their last reflection or refraction (24). The generators of all these cones will all be parallel. Consequently, an infinitude of systems of parallel rays will propagate in the medium, where each system will have a different direction, and all of these systems of rays will correspond to a unique system of plane waves that are mutually parallel.

Instead of supposing that all of the rays of a system are mutually parallel, one can imagine that only a subset of these rays, which defines a continuous surface, are parallel. One will then arrive at the following theorems:

THEOREM XI. – *If, among the rays that issue from the same point and all have the same type, and propagate in an arbitrary homogeneous medium, one finds an infinitude of them that define a continuous surface and are mutually parallel then each of the waves that correspond to the system of rays will be tangent to the same plane along the curve along which it cuts the cylindrical surface that is defined by the parallel rays (except for the case where the medium is biaxial and the rays are parallel to one of the singular directions of the medium).*

THEOREM X. – *If a wave in an arbitrary homogeneous medium that corresponds to a system of rays that issue from the same point and all have the same type is tangent to the same plane along a continuous curve then the rays that pass through the various point of the line of contact will be mutually parallel and will define a continuous, cylindrical surface, at least when the medium is biaxial and that plane is not parallel to one of the singular tangent planes of the characteristic wave surface of the medium.*

IV. – Research on the reflecting or refringent surface when one is given the incident wave and the reflected or refracted wave. – Reciprocal construction to that of Huyghens.

29. The general construction of the reflected or refracted (16) immediately shows that the reflecting or refracting surface can be considered to be the locus of the intersections of the incident waves with the reflected or refracted waves that correspond to the same time. That remark will permit us to solve the following problem:

Being given a real position S of the incident wave and a likewise real position S′ of the reflected or refracted wave – or one of the sheets of that wave, if there are two of them – and knowing, moreover, the time T that it takes for the light to propagate from the wave S to the wave S′, find the reflecting or refracting surface.

The problem comes down to uniquely determining the incident waves and the reflected or refracted wave that correspond to the same time. Let there be an incident wave that is τ later than the wave S, and the reflected or refracted wave that corresponds at the same time is $T - \tau$ prior to the wave S' if τ is between zero and T and $\tau - T$ later than the wave S' if τ greater than T. Now, let there be a incident wave that is τ later than the wave S, so the reflected or refracted wave that corresponds to it at the same time will be $T + \tau$ prior to the wave S'. By definition, the reflecting or refracting surface will therefore be the locus of the intersections of the incident waves that are separated from the wave S by an interval of time τ, with the reflected or refracted waves being separated from the wave S' by a time interval $T - \tau$, if we agree that the incident wave must be regarded as later than or prior to the wave S according to whether τ is positive or negative, resp., and that the reflected or refracted wave must considered to be prior to or later than the wave S' according to whether $T - \tau$ is positive or negative, resp. Moreover, τ must take on all of the values for which there is an intersection. Solution will be impossible when, for any value of τ there is no intersection between the incident wave and the reflected or refracted wave that corresponds to it at the same time. Moreover, it is obvious that for any waves S and S' there will always exist certain values of T for which solution will be possible, because if one imagines an incident wave that is τ later than the wave S then one can always find a value for T such that this incident wave should be cut by the reflected wave that is $T - \tau$ prior to the wave S'. If one gives all values to T for which the problem admits a solution then one will obtain the system of reflecting or refracting surfaces that can transform the incident wave S into the reflected or refracted wave S'.

As a special case, one can suppose that the waves S and S' each reduce to a point – i.e., one looks for the reflecting or refracting surfaces that make the rays that emanate from a given point converge to a likewise given point. The surfaces will receive the qualification *aplanatic*, and their complete theory will require some very extensive developments that will be the object of a special chapter.

30. Now, return to the general case, which we propose to treat by calculation. Being given the equation of the incident wave S, and knowing, moreover, the nature of the rays, and in turn, the equation of the sheet of the characteristic wave surface for the first medium that has the same nature as these rays, one can, upon following the path that was indicated in no. 15, find the equation of the incident wave τ later than the wave S, which is an equation that will be of the form:

$$F(x, y, z) = 0.$$

Likewise, being given the equation of the reflected to refracted wave S', and knowing, moreover, the nature of the reflected or refracted rays, and in turn, the equation of the sheet of the characteristic wave surface of the medium in which they propagate that has the same nature as these rays, one can, by following an analogous path, find the equation of the reflected or refracted wave t later than the wave S', which is an equation that will be of the form:

$$\mathcal{F}(x, y, z, \tau) = 0.$$

From the construction that was described in the preceding paragraph, one will obtain the equation of the reflecting or refracting surface that is capable of transforming the incident wave S into the reflected or refracted wave S' by eliminating τ from the two equations:

$$F(x, y, z, \tau) = 0.$$

and

$$\mathcal{F}(x, y, z, \tau - T) = 0,$$

which will lead to an equation of the form:

$$\Phi(x, y, z, T) = 0,$$

in which T can take on all possible values.

31. Let S be a wave that corresponds to a system of rays that issue from the same point and all have the same type. Suppose that these rays, after having been subjected to an arbitrary number of reflections and refractions upon traversing an arbitrary number of homogeneous media – whether isotopic or anisotropic – still have the same type, and then let S' be one of the waves that correspond to them. From what we just saw, it is always possible to find a reflecting or refracting surface that is capable of transforming the wave S into the wave S'. If we remark that, furthermore, if the nature of the rays in the same medium is given then the same system of waves will necessarily correspond to the same system of rays then we will be led to the following theorem, which was proved by Gergonne ([1]) for the special case of isotropic media, and which was then found to extend to any type of homogeneous media.

THEOREM XI. – *When rays that issue from the same point and all have the same type are subjected to an arbitrary number of reflections and refractions upon traversing an arbitrary number of homogeneous media, whether isotropic or anisotropic, the effect of these reflections and refractions can always be replaced with either a single reflection or a single refraction.*

32. The Huyghens construction will further permit us, when we are given the direction of an incident ray and that of a reflected or refracted ray that results from the incident ray, to determine the direction of the tangent plane to the reflecting or refringent surface at the point of incidence, upon supposing that the nature of each of the rays is known, if the medium in which it propagates is birefringent. To that effect, when the point of incidence is taken to be the center, one will describe the characteristic wave surfaces Σ and Σ' in the first and second medium at that point that correspond to a unit of

([1]) GERGONNE, Annales, t. XIV, pp. 129.

time. At the point where the incident ray meets that sheet of the surface Σ that has the same nature as it, one draws a tangent plane P to that sheet. At the point where the reflected ray meets that sheet of the surface Σ that has the same nature as it, or even at the point where the refracted ray meets that sheet of the surface Σ' that has the same nature as it, one draws a tangent plane P' to that sheet. Finally, one passes a plane through the line of intersection of the planes P and P' and the point of incidence, which is the desired plane. If the two tangent planes P and P' are parallel to each other then the required plane will be parallel to those two planes. The line of intersection of the planes P and P' can never pass through the point of incidence, so the construction that we just described will always give one and only one plane, except in the special case where one of the two media is biaxial and the incident ray or the reflected ray, if it is the first medium, and the refracted case, if it is the second one, meets the characteristic wave surface of the medium at one of its singular points. Meanwhile, the solution will be possible for any given directions of the two rays only when one is dealing with a homologous reflection. If there is refraction or antilogous reflection then the solution can become impossible for certain direction of the given rays. That is what will happen in the case of refraction if the plane that is determined by the construction that we just described is found within the angle that is defined by the prolonged incident ray and the refracted ray, and for antilogous reflection if that plane is found within the angle that is defined by the incident ray and the reflected ray.

33. We shall now propose to seek the conditions that must be satisfied in order for a ray to reflect back onto itself or to refract with no deviation.

To begin with, consider the case of a homologous reflection. If a ray is then reflected in such a fashion that that it returns to itself then the construction of the preceding paragraph will show immediately that the tangent plane to the reflecting surface at the point of incidence will be parallel to the tangent plane that is drawn to that sheet of the characteristic wave surface of the medium that is described by choosing the point of incidence to be its center and corresponding to a unit time, and which has the same nature as the incident ray at the point where that sheet is met by the prolonged incident ray. The Huyghens construction will show, moreover, that this condition is sufficient. Therefore:

THEOREM XII. – *In order for a ray that propagates in an arbitrary, homogeneous medium to reflect back onto itself (the reflection being homologous), it is necessary and sufficient that the tangent plane to the reflecting surface at the point of incidence have a direction that is conjugate to that of the incident ray, which will be ordinary or extraordinary, according to the nature of that ray.*

If the medium is isotropic, of if the medium is uniaxial and the ray is ordinary, then in order for a ray to return to itself as a result of a homologous reflection, from the preceding theorem, it will be necessary and sufficient that the ray be normal to the reflecting surface.

Upon passing through to the case of refraction, if we are given the direction of the incident ray and its nature, as well as the nature of the refracted ray, then we can propose to determine the direction that the tangent plane to the refringent surface at the point of incidence must have in order for the incident ray and the refracted ray to be on the prolongation of each other. The Huyghens construction will again provide us with the solution to the problem. When an arbitrary point O is taken to be the center, one will describe the sheets of the characteristic wave surface of the medium that corresponds to unit time, which will have the same nature as the incident ray and the refracted ray, respectively. One then draws a plane through the point O that is parallel to the given direction of the incident ray. That line will meet the sheets Σ and Σ' at two points that are situated on the same side of the point O, where one will draw two tangent planes P and P' to those sheets. Finally, one passes a plane through the line of intersection of the two planes P and P' and the point O whose direction is the desired direction. One sees that this problem will always involve one and only one solution.

When the sheets Σ and Σ' are similar and similarly-placed, the planes P and P' will be parallel to each other, and in turn, in order for the incident ray and the refracted ray to be a straight line, it is necessary that the tangent plane to the refringent surface at the point of incidence be parallel to the planes P and P'. That condition will obviously be sufficient, so:

THEOREM XIII. – *When the sheets of the characteristic wave surfaces of two homogeneous media that correspond to the incident ray and the refracted ray, respectively, are similar and similarly-placed, in order for the incident ray and the refracted ray to be on the prolongation of each other, it is necessary and sufficient that the tangent plane to the refringent surface at the point of incidence have the direction that would be conjugate to that of the incident ray in either of the media, which will be ordinary or extraordinary according to the nature of that ray.*

One can remark that in this case, the incident ray and the refracted ray will always have the same nature. One further sees that if the sheets of the characteristic wave surfaces of the two media that correspond to the incident and refracted rays, respectively, are spherical, then in order for the incident ray and the refracted rays to be prolongations of each other, it will be necessary and sufficient that the incident ray be normal to the refringent surface.

An inverse problem to the preceding one is the one that consists of being given the direction of the tangent plane to the refringent surface at the point of incidence and the nature of the incident and refracted rays and looking for the direction that the incident ray must have in order to penetrate the second medium without breaking. In order to answer that question, one further describes the sheets Σ and Σ' when an arbitrary point O is taken to be the center. One draws a plane through the point O that is parallel to the given one, and one looks for a line on that plane such that the tangent planes that are drawn through that line to subsets of the sheets Σ and Σ' that are found on the same side of that plane will touch these surfaces at two points, which will be on the same straight line as the point O. The line that passes through the point O and the two contact points will have the desired direction.

As for the case where one ray must be reflected back onto itself, if the reflection is antilogous then it will be treated by following exactly the same path that we did for refraction. The surfaces Σ and Σ' will then be the two sheets of the characteristic wave surface of the first medium. Since these two sheets can never be similar, there is no reason to state a proposition that is analogous to Theorem XIII.

V. – On total and partial foci and focal lines. – Tautochronism of luminous trajectories that end at the same focus.

34. Suppose that in an arbitrary, homogeneous medium – whether isotropic or anisotropic – all of the reflected or refracted rays of a certain type are directed in such a fashion that they coincide at a total focus *O*, which is either real or virtual. Let *S* be any of the real or virtual waves that correspond to that system of rays. Take an arbitrary point *A* on that wave. If we describe the sheet of a characteristic wave surface in the medium when the point *O* is taken to be its center and that has the same nature as the rays at the time that corresponds to the time that sheet that passes through *A* then, by virtue of Theorem II, that sheet will be tangent to the wave *S* at *A*. Since this argument is applicable to all of the points of the wave *S*, that wave must be tangent at each of its points to the sheet that has the same nature as the rays of a certain characteristic wave surface and is described with *O* for its center. If that characteristic wave surface is not the same for all points of the wave *S* then that wave will envelop sheets of the same nature as the characteristic wave surfaces that are described with the same point *O* as their centers and corresponding to different times, which is impossible, since the sheets cannot intersect. At all of its points, the wave *S* will therefore be tangent to the sheet that has the nature as the rays of the same characteristic wave surface that is described with *O* as its center, and in turn, will coincide with that sheet. Conversely, if any of the waves, whether real or virtual, that correspond to the system of rays coincides with the sheet that has the same nature as the rays of a characteristic wave surface of the medium that is described with *O* for its center then the directions of all the rays of the system will agree at the point *O*. Indeed, if that condition is satisfied then the conjugate directions, which will be ordinary or extraordinary, according to the nature of the rays, to the tangent planes that are drawn to that wave at its various points will all pass through the point *O*. Now, from Theorem II, these directions will be precisely those of the rays that pass through the various points of the wave; these rays will all therefore agree at *O*.

We thus arrive at the fundamental theorem that we will state as:

THEOREM XIV. – *In order for the reflected or refracted rays that issue from the same point and have a certain type and that propagate in an arbitrary, homogeneous medium to have a total focus O, whether real or virtual, it is necessary and sufficient that the wave that corresponds to that system of rays, when considered in any of its real or virtual positions, coincide with the sheet that has the same nature as the rays of a characteristic wave surface of the medium that has the point O for its center. It will then follow that:*

1. *The wave will reduce to a point at the moment when it passes through the total focus.*

2. *If all of the reflected or refracted rays in an arbitrary, homogeneous medium that issue from the same point and all have a certain type converge at the same real focus*

then all of these rays will take the same time to propagate from the luminous point to the focus and consequently will arrive there with no difference in phase.

This theorem is true for any sort of reflections and refraction that the rays might have be subject to before taking their present directions, but on the condition that these rays originally emanate from the same point and that they remain of the same type.

The tautochronism of luminous trajectories that end at the same focus constitutes a proposition of paramount importance. It is the principal basis for the theory of aplanatic surfaces, and one must necessarily take recourse to it in order to justify the use of lenses in the observation of interference phenomena and diffraction ([1]).

It results from the preceding theorem that in order for the reflected or refracted waves that propagate in an arbitrary, homogenous medium to take on the form that is presented in that medium by that sheet of the characteristic wave surface that has the same nature as these waves, it will be necessary and sufficient that all of the reflected or refracted rays be directed in such a fashion that they will agree at the same point.

We finally remark that if one describes the sheet that has the same nature as the rays of a characteristic wave surface that has its center at the total focus O, and which corresponds to an arbitrary time T, then that sheet will coincide with both of the two waves, namely, with the one that is T prior to the moment when the wave reduces to the total focus and with the one that is T later than that moment. If the reflected or refracted rays do not fill up all of space then these two waves will coincide with two subsets of the characteristic wave surface that will be bounded by two curves that are symmetric with respect to the focus, and can impinge upon each other. However, if the reflected or refracted rays fill up all of space before they arrive at the focus, which can happen when the rays that emanate from a luminous point that is situated in a homogeneous medium that is bounded by a closed surface converge to a unique focus after reflection, then the two waves will each coincide with the sheet of the characteristic wave surface over the entire extent of that sheet, and in turn, will coincide with each other.

35. It can happen that amongst the reflected or refracted rays of a certain type that propagate in an arbitrary, homogeneous medium there is only one subset of them that defines a conical surface that converges to the same real or virtual focus. That focus will then bear the name of *partial focus.* By employing exactly the same line of reasoning that we did in the previous paragraph, one will arrive at the following proposition:

THEOREM XV. – *In order for the reflected or refracted rays that issue from the same point and all have a certain type and propagate in an arbitrary, homogeneous medium to contain an infinitude of them that define a continuous surface that converges to a partial focus O, which is real or virtual, it is necessary and sufficient that the wave that corresponds to that system of rays, when considered in any of its real or virtual positions, be tangent along the curve along which it is intersected by the surface that the rays defines to that sheet that has the same nature as the rays of a characteristic wave*

([1]) This theorem has been known for some time for isotropic media. Huyghens (*Traité de la Lumière*, chap. VI) gave a proof by appealing to the laws of reflection and refraction in these media. It remains to be seen whether it likewise applies to birefringent media.

surface in the medium that is described by taking O to be its center. Therefore, it will follow that if an infinitude of reflected or refracted rays in an arbitrary, homogeneous medium issue from the same point with the same type and define a continuous surface that converges to the same real partial focus then those rays will take the same time to propagate from the luminous point to that focus, and will consequently arrive with no difference in phase.

As a special case, it can happen that the conical surface that is defined by the rays whose directions converge to the partial focus *O* reduces to a planar surface. The preceding theorem will then persist without modification.

36. When a system of rays issues from the same with the same type and propagates in an arbitrary, homogeneous medium, four distinct cases can present themselves:

1. There is neither a total focus nor a partial one.
2. There are a finite number of mutually isolated partial foci.

We will then say *aplanatic lines*, which will lie either on the reflecting or refracting surface or on the wave that correspond to the system of rays, in order to refer to the intersections of those surfaces with the cones that are defined by the rays that converge to the same partial focus, where these conical surfaces can reduce to planes in some special cases.

3. There is an infinitude of partial foci that define a continuous line that we call the *focal line.* There then exists a continuous system of aplanatic lines on the reflecting or refracting surface and on each of the waves that correspond to the system of reflecting or refracting rays, and the system of rays will decompose into an infinitude of conical surfaces whose summits will define a continuous line.
4. There is a total focus, and then the reflecting or refringent surface will be called *aplanatic.*

The last case was just studied (34). We shall now stop for some time in order to examine the consequences that result from the existence of a focal line. That focal line can be either entirely real, entirely virtual, or partially real and partially virtual. However, there is, in addition, an essential distinction to be made, according to whether the rays take the same time or unequal times in order to go from the reflected or refracted wave considered to any of the real or virtual positions that it successively occupies at the various points of the focal line (upon supposing that the medium in which the prolonged rays move is prolonged to its limits in such a fashion as to include both the wave and the focal line, if that is necessary in order to evaluate those times). If these times are equal then we will say that the focal line is *isochronous;* in the contrary case, we will call it *anisochronous.* One sees that when a real focal line is isochronous, all of the reflected or refracted rays will take the same time to propagate from the luminous point to the various points of that focal line, while, in general, it is only the rays that end on the same point of the focal line will take the same time in order to arrive at that partial focus.

37. Let *L* be a real or virtual, isochronous, focal line in an arbitrary, homogeneous, medium. From Theorem XV, if one considers the wave *S* that corresponds to the system of reflected or refracted rays in any of its real or virtual positions then that wave must be tangent along a curve to the sheet that has the same nature as the rays of the characteristic wave surfaces that are described when each of the points of the line *L* are taken to be centers. In order for the focal line to isochronous, it is obviously necessary that these characteristic wave surfaces that are described when the various points of the line *L* are taken to be centers, and each of which is tangent to the wave *S* along a curve, must correspond to the same time. Conversely, if that condition is satisfied then the rays must end on the various points of the line *L* and take equal times to go from the wave *S* to the line *L*, which will be, in turn, isochronous. Therefore:

THEOREM XVI. – *In order for a line L to be an isochronous focal line for the reflected or refracted rays that issue from the same point with the same type and propagate in an arbitrary medium, it is necessary and sufficient that the wave that corresponds to these rays be, in any of its real or virtual positions, the envelope of the sheets that have the same nature as the rays of the characteristic wave surfaces of the medium that are described when the various points of the line L are taken to be centers and corresponding to the same time. Therefore, it follows that the wave must reduce to an isochronous focal line in some position, which will be either real or virtual.*

For example, if the medium is isotropic then the wave will be the envelope of a sphere whose center will traverse the isochronous focal line and whose radius will remain constant.

When there exists an isochronous focal line, every wave that corresponds to the system of reflected or refracted rays will present a continuous system of aplanatic lines. In order to find the aplanatic line on one of these waves that corresponds to a given point of the focal line, it will obviously suffice to describe a characteristic wave surface with that point for its center whose sheet that has the same nature as the rays is tangent to the wave. The contact between the two surfaces is defined along a line that will be the desired aplanatic line. It is likewise easy to find the aplanatic line on the reflecting or refracting surface that will correspond to a given point *A* of the isochronous focal line. Indeed, since the wave will reduce to that line in one of its positions, one will obtain the rays that converge to *A* by drawing the lines whose directions are ordinary or extraordinary conjugates according to the nature of the rays to those of the tangent planes to the focal line at *A* – i.e., to all of the planes that pass through the tangent line at *A* to that line. These rays that define a conical surface can reduce to a planar surface in special cases, whose intersection with the reflecting or refringent surface will be the required aplanatic line.

If the medium is isotropic, or if the medium is uniaxial and the rays are ordinary then the rays that converge to the same point *A* of the isochronous focal line must be perpendicular to the tangent plane to that focal line at *A*, and in turn, to the tangent that is drawn to that line at the point *A*. These rays will therefore all be contained in the same normal plane to the focal line at *A*. This will give the following theorem:

THEOREM XVII. – *In an isotropic, homogeneous medium or a uniaxial, homogeneous medium when the rays are ordinary, if rays that issue from the same point and all have the same type give rise to an isochronous focal line then those of the rays whose directions converge to the same real or virtual point of that line will be contained in the same normal plane to the focal line, and in turn, the aplanatic lines, as long as the reflecting or refringent surface and the waves that correspond to the system of rays are planar.*

38. When there exists an anisochronous focal line, the wave, when considered in any of its real or virtual positions, will again be the envelope of the sheets that have the same nature as the rays of the characteristic wave surfaces of the medium that are described by taking the various points of that line to be their centers. However, these characteristic wave surfaces will no longer correspond to equal times. The aplanatic lines in the waves that correspond to the reflected or refracted waves will be obtained in this case by the same construction as in the case of an isochronous focal line (37). That construction will be further applicable when there exist a finite number of isolated partial foci.

When the focal line is anisochronous, the wave does not reduce to that line in any of its positions, whether real or virtual. It will then successively pass through the various points of that line. Consider the wave in the real or virtual position where it passes through an arbitrary point A of the anisochronous focal line. From Theorem XV, all of the rays that converge at A will arrive at that point at the same time. The wave at the point A must then be simultaneously tangent to all of the planes that are conjugate to the rays that arrive at A, and will be ordinary or extraordinary according to the nature of the rays. The envelope of these planes is, in general, a conical surface, but that surface can reduce to a straight line. That is what will happen when, for example, the medium is isotropic, and the rays that end at the point A are contained in the same plane.

It will result from what we just said that the wave will present what one calls a *singular point* at A. There is an exception to this rule only in a very special case: It is the one where the medium is biaxial and each of the rays that converge to A is parallel to one of the radius vectors that are conjugate to the singular tangent planes in the characteristic wave surface of the medium. All of what we just said will obviously apply to the case where the wave passes through an isolated partial focus. Conversely, if the wave presents a singular point in any of its real or virtual positions then one can assert that the point will either belong to an anisochronous focal line or constitute an isolated partial focus, because at that point the wave will be tangent to an infinitude of planes that have a conical surface for their envelope, or in some special cases, a straight line, and these planes will be conjugate to an infinitude of rays that will define a conical surface that can reduce to a planar surface. Here again, the general rule will present a particular exception: Indeed, if the medium is biaxial and the conical surface that is tangent to the wave at the singular point has all of its generators parallel to those of the cone that is tangent to the characteristic wave surface of the medium at one of its singular points then all of the tangent planes to the wave at the singular point will be conjugate to a unique direction, which will be one of the singular directions of the medium. Only one ray will pass through A, but, from a remark that was made previously (23), that ray will be composed of the superposition of an infinitude of reflected or refracted rays that come from a cone of incident rays.

The preceding remarks can be summarized into the following theorem:

THEOREM XVIII. – *Whenever the wave in an arbitrary homogeneous medium that corresponds to a system of rays that issue from the same point and all have the same type passes through a real or virtual partial focus without belonging to an isochronous focal line, it will present a singular point at that focus, at least when the medium is biaxial and the tangent plane to the wave that is drawn through that focus is not parallel to one of the singular tangent planes to the characteristic wave surface of the medium. Conversely, whenever the wave presents a singular point in any of its real or virtual positions, that point will be a partial focus that does not belong to an isochronous focal line, at least when the medium is biaxial and the tangent cone to the wave at that singular point has no generators that are parallel to those of the tangent cone to the characteristic wave surface of the medium at one of its singular points.*

A construction results from this theorem that will permit one to find the aplanatic line on the reflecting or refracting surface that corresponds to a point of an anisochronous focal line or an isolated focal point. Indeed, except for the exception that was pointed out above, the wave will be tangent to a cone when it passes through such a point. If one draws lines through the singular point of the wave whose directions are conjugate, ordinary or extraordinary, according to the nature of the rays, to those of the tangent planes to that cone then these lines will define a conical surface whose intersection with the reflecting or refracting surface will be the desired aplanatic line.

If the medium is isotropic then the cone that is defined by the rays that end at the partial focus will have generators that are perpendicular to those of the cone that is tangent to the wave at that partial focus, and then the first cone will reduce to a planar surface when the second one reduces to a straight line.

39. The remarks that we just made about the relationship between the existence of foci or focal lines and the peculiarities that are presented by waves can be summarized as follows:

1. If the wave that corresponds to a system of reflected or refracted rays that issue from the same point and all have the same type propagates in an arbitrary, homogeneous medium and does not reduce to a point or a line at any of the real or virtual positions that it successively occupies, and it presents no singular point at any of its positions then there will be neither a total nor a partial focus (at least, when the medium is biaxial and an infinitude of incident rays that define a continuous surface gives rise to an infinitude of conical sheaves of reflected or refracted rays that can create partial foci when they cross each other without the wave presenting any singular point).

2. If the wave, while never reducing to either a point or a line, presents one or more singular points in one or more of its isolated positions then these singular points will also be isolated partial foci. (Nonetheless, if the medium is biaxial then the singular points of the wave, where the tangent cone to the wave has generators that are parallel to those of one of the tangent

cones to the characteristic wave surface of the medium at its singular points, will be the reproduction of the singular points of that characteristic wave surface, and will not be responsible for the existence of partial foci.)

3. If the wave, while never reducing to either a point or a line, presents a singular point in a continuous series of positions then the line that is described by that point will be an anisochronous focal line (except for the exception that was pointed out in the preceding case).

4. If the wave reduces to a line in certain positions that are either real or virtual then that line will be an isochronous focal line.

5. If the wave reduces to a point in a certain position that is either real or virtual then that point will be a total focus.

VI. – Fundamental property of luminous trajectories: the time that it takes to traverse is, in general, a minimum or a maximum.

40. While we do not propose to present the theory of aplanatic surfaces here – especially, in order to arrive at the simple and general proof of a fundamental property of luminous trajectories – it will be, nonetheless, indispensible for us to go further than we have done in the study of those surfaces up to now.

Suppose that the reflected or refracted rays of a certain type that correspond to the incident rays that emanate from a real, luminous point O converge to the same real focus O'. Let T denote the time that it takes for a ray of the same nature as the incident rays to propagate in the first medium from the point O to an arbitrary point M, and let T' denote the time that it takes for a ray of the same nature as the reflected or refracted rays to propagate from the point O' to the same point M, where one is dealing with refraction in the second medium and reflection in the first medium. It results immediately from Theorem XIV that for all points of the reflecting or refringent surface, the sum $T + T'$ will be constant, and that conversely, if that sum is constant on a reflecting or refringent surface then that surface will be aplanatic for the positions O and O' of the luminous point and focus, respectively (which are both assumed to be real), and if the nature of the incident rays is assigned, as well as that of the reflecting or refracting rays, then there will exist an infinitude of aplanatic surfaces that constitutes a certain system. The sum $T + T'$ will have a constant value on each of these surfaces of the system. However, that sum will vary when one passes from one of these surfaces to another one. It then follows that the aplanatic surfaces that define a subset of the same system can never intersect, and that they will completely envelop each other.

To begin with, consider the case of reflection. The aplanatic surfaces that correspond to the given positions of the luminous point O and focus O', which are both assumed to be real, will then be closed, and each of them will include the points O and O' in its interior. Moreover, it is obvious that the sum $T + T'$, which is constant on each of these surfaces, must increase in measure when one considers a surface that is more and more separated from the points O and O', or in other words, when one passes from an aplanatic surface to another one that includes the first one in its interior. It will result from that, if one denotes the particular value of $T + T'$ on one of the aplanatic surfaces by C – a surface that we shall call S – that for any point that is external to the surface S, the sum T

+ T' will have a value that is greater than C, and for any point that is internal to that surface, that sum will have a value that is less than C. One will further see that if, at an arbitrary point of the surface S, one draws a tangent plane to that surface then the sum $T + T'$ will have a value that is greater than the one that it takes at the point of contact at every point of that plane other than that point of contact; i.e., a value that is greater than C. It will then have a minimum value on the plane at the point of contact.

Now, pass to the case of refraction. The aplanatic surfaces will then necessarily pass between the points O and O'. First, suppose that in the direction OO', the velocity of a ray in the first medium that has the same nature as the incident rays is greater than the velocity of a ray in the second medium that has the same nature as the refracted rays. The sum $T + T'$ will then increase in measure until the aplanatic surface meets the line OO' at a point that is closer to the point O. Moreover, in that case, the aplanatic surfaces – at least, in the part of their extent that actually contributes to refraction – will have their convexity turned towards the point O. Having said that, we let C denote the value of the sum $T + T'$ on one of these aplanatic surfaces – a surface that we shall call S – and let M be a point that is subject to only the condition that it be found on the subset of any of the aplanatic surfaces of the system that actually contribute to refraction. One must conclude from the preceding that the sum $T + T'$ will have a value that is greater than C at the point M if that point is found outside of the surface – i.e., on the same side of that surface as the point O – and a value that is less than C if that point is situated inside the surface S – i.e., on the same side as the point O'.

On the contrary, if when one follows the direction OO', the velocity in the first medium of a ray that has the same nature as the incident rays is smaller than the velocity in the second medium of a ray that has the same nature as the refracted rays then the sum $T + T'$ will increase in measure until the aplanatic surface meets the line OO' at a point that is closer to the point O'. However, since the aplanatic surface – at least, the part of its extent that actually contributes to the refraction – has its convexity turned towards the point O', one will see that, as in the preceding case, the sum $T + T'$ will have a value that is greater than C at the point M if that point is found outside of the surface S – i.e., on the same side of that surface as the point O' – and a value that is less than C if that point is situated inside that surface – i.e., on the same side as the point O.

In either case, if one draws a tangent plane to one of the aplanatic surfaces at a point that is situated on the part of that surface that actually contributes to the refraction then since all of the points of that plane other than the point of contact will be external to the aplanatic surface, the sum $T + T'$ will have a minimum value at the point of contact. Since aplanatic surfaces – either by reflection or refraction – will always be convex, the proposition that we just stated, just like the analogous proposition that relates to reflection, will suffer no exception.

41. Now, suppose that light propagates from a point O to another point O' while being subject to a reflection or refraction, where these two points are situated in one arbitrary, homogeneous medium if one dealing with reflection, or two arbitrary, homogeneous media if one is dealing with refraction, and the reflecting or refringent surface is likewise arbitrary. Call the incident ray R, the reflected or refracted ray R', the point of incidence I, and the reflecting or refracting surface S. Let T denote the time that a ray in the first medium that has the same nature as the ray R will take in order to propagate from the point O to an arbitrary point M, and let T' be the time that it takes, in the first or second medium, according to whether one is dealing with a reflection or refraction, respectively, for a ray that has the same nature as the ray R' to propagate from the point O' to the same point M. If we consider the point O to be a real, luminous point and the point O' to be a real focus, while the incident rays have the same nature as the ray R and the reflected or refracted rays have the same nature as the ray R', then, from what we just saw, there will be an infinitude of aplanatic surfaces for these positions of the luminous point and the focus. Among these aplanatic surfaces, there will necessarily be one of them that passes through the point of incidence I. That aplanatic surface, which we shall denote by Σ, will be tangent to the reflecting or refringent surface S, because at the point I the surface Σ must, like the surface S, reflect or refract a ray that comes from O towards the point O'. The sum $T + T'$ will be constant on the aplanatic surface Σ (40). Therefore, since the surface S is tangent to the surface Σ at I, if one passes from the point I on that reflecting or refracting surface to an infinitely-close point then the variation of the sum $T + T'$ will be infinitely small of order higher than one. In other words, if one considers the sum $T + T'$ on the surface S to be a function of the coordinates of the points on that surface then the differential of that function will be zero at the point I.

Conversely, if the differential of the function $T + T'$ is zero at the point I on the surface S then that function will be constant on the surface S to an infinitely-small extent when one starts with the point I. As a result, the surface S will be tangent to one of the aplanatic surfaces at I for the positions O and O' of the luminous point and the focus. Therefore, the point I will be such that a ray that emanates from the point O must pass through the point O' after reflecting or refracting.

We thus arrive at the following proposition:

THEOREM XIX. – *If one is given two points O and O′ and a reflecting or refringent surface S, while the nature of the incident rays is assigned, as well as that of the reflected or refracted rays, then in order for a trajectory that starts at the point O and ends at the point O′, while touching the surface S, to be actually followed by light, it is necessary and sufficient that upon passing from that trajectory to an arbitrary infinitely-close trajectory, the variation of the time that is taken by the light to propagate from O to the point O′ be infinitely small of order higher than one.*

42. In the theory of waves, the preceding theorem is the analogue of the one that takes the form of the principle of least action in the theory of emission.

One can further express this by saying: In order for a trajectory that starts at the point O and ends at the point O' to be actually followed by light, it is necessary and sufficient that the differential of the sum $T + T'$ be zero at the point I where that trajectory touches the reflecting or refracting surface, where that sum is considered to be a function of the coordinates of the points of the surface S.

If one is given the coordinates of the points O and O', the equation of the reflecting or refracting surface, the nature of the incident rays and that of the reflected or refracted rays then this theorem will permit one to calculate the coordinates of the points where a ray that starts at O must touch the surface S in order to pass through the point O' after reflecting or refracting.

Indeed, let (a, b, c) denote the coordinates of the point O and let (a', b', c') denote those of the point O'. Let:

(1) $$\varphi(\xi, \eta, \zeta) = 0$$

be the equation of the reflecting or refringent surface S, let:

$$f_1(x, y, z) = 0$$

be the equation of the sheet that has the same nature as the incident ray of the characteristic wave surface in the first medium that is described by taking the origin to be its center and that corresponds to a unit of time, and let:

$$f_2(x, y, z) = 0$$

be the equation of the sheet that has the same nature as the reflected or refracted ray of the characteristic wave surface of the first or second medium, according to whether one is dealing with reflection or refraction, respectively, that is described by taking the origin to be its center and that corresponds to a unit of time. The time T that is taken by light to propagate from the point O to an arbitrary point of the surface S whose coordinates are (ξ, η, ζ) will be given by the equation:

(2) $$f_1\left(\frac{\xi - a}{T}, \frac{\eta - b}{T}, \frac{\zeta - c}{T}\right) = 0.$$

Similarly, the time T' that is taken by light to propagate from the point O' to the same point of the surface S will be given by the equation:

(3) $$f_2\left(\frac{\xi - a'}{T'}, \frac{\eta - b'}{T'}, \frac{\zeta - c'}{T'}\right) = 0,$$

where the two functions f_1 and f_2 will be identical when one is dealing with a homologous reflection.

If one deduces values for T and T' from equations (2) and (3), resp., and adds them then one will obtain an equation of the form:

$$T + T' = F(\xi, \eta, \zeta, a, b, c, a', b', c') = 0.$$

If one eliminates one of the three variables ξ, η, ζ – for example, ζ – from that equation and equation (1) then one will arrive at an equation of the form:

$$T + T' = \mathcal{F}(\xi, \eta, a, b, c, a', b', c') = 0.$$

From Theorem XIX, one must have:

$$d(T + T') = 0$$

for the desired points of the surface *S*, an equation that will decompose into two other ones, namely:

$$\frac{d\mathcal{F}}{d\xi} = 0$$

and

$$\frac{d\mathcal{F}}{d\eta} = 0,$$

since there are two independent variables. These two equations, when combined with that of the surface, will determine the coordinates of the desired points.

43. Based upon the considerations that already helped us prove Theorem XIX, it is easy to see that the time that is employed by light in order to propagate from one point to another, while experiencing a reflection or refraction, is generally a maximum or a minimum.

Indeed, at the point of incidence *I*, the reflecting or refringent surface *S* will be tangent to an aplanatic surface Σ on which the sum $T + T'$ is constant (41). If the contact between the two surfaces *S* and Σ at the point *I* is simply of first order, which is true in general, or if that contact is of odd order, then the surface *S* will be internal or external to the surface Σ in the neighborhood of the point *I*, because they cannot intersect in the case of two surfaces. If the surface *S* is external to the aplanatic surface Σ around the point *I* then, from what we saw previously (40), the sum $T + T'$ will have a value that is greater than it is on the surface Σ for the points that are external to the surface Σ, while that sum will have a minimum value on the surface *S* at *I*. On the contrary, if the surface *S* is external to the aplanatic surface Σ everywhere around the point *I* then the sum $T + T'$ will have a maximum value on the surface *S* at the point *I*. When the two surfaces *S* and Σ have a contact of even order at the point *I*, which is true only in exceptional cases, these two surfaces must intersect everywhere that they are tangent, and then the value of the sum $T + T'$ on the surface *S* will be neither a maximum nor a minimum at *I*, although the variation of the sum $T + T'$ must again be infinitely-small of order higher than one when one passes from the point *I* to an infinitely-close point on the surface *S*, and in this case, that variation must even be of at least third order. We can thus state the following theorem:

THEOREM XX. – *When light propagates from a point O to a point O′ while experiencing reflection or refraction, the time that is employed by the light in order to traverse its trajectory will always be a minimum or a maximum, at least when the point of incidence of the reflecting or refringent surface does not have a contact point of even order with that of the aplanatic surfaces that relate to the positions O and O′ of the luminous point and focus, respectively.*

The time will be a minimum when the reflecting or refrignent surface is external to the aplanatic surface everywhere around the point of incidence, and a maximum when the reflecting or refringent surface is internal to the aplanatic surface everywhere around the point of incidence.

One often believes that the time that is employed by light in order to propagate from one point to another while reflecting or refracting is always a minimum. The preceding theorem shows that this is not true, and indicates the circumference under which one will know whether that time is a maximum or a minimum. Moreover, it lets one know the exceptional cases in which that time is neither a minimum nor a maximum. It should be remarked that these exceptional cases will never present themselves when the reflecting or refringent surface is convex; i.e., situated entirely on the same side of any of the planes that are tangent to it.

The error that we just pointed out stems from the fact that one ordinarily considers only planar reflecting or refringent surfaces. When the reflecting or refringent surface is planar, one will find that the time is indeed a minimum everywhere around a point of incidence that is external to the aplanatic surface to which it is tangent at that point.

44. Up to now, we have considered only the case of a single reflection or refraction. However, Theorem XIX has a more extensive significance and will permit one to solve the general problem, which consists of *determining the trajectory that a light ray must follow in order to propagate from one given point to another given point, while experiencing an arbitrary number of reflections and refractions on given surfaces and traversing an arbitrary number of homogeneous media that are likewise given, while the nature of the ray on each part of its trajectory is assigned, moreover.*

Let a, b, c be the coordinates of the point of departure and let a', b', c' denote those of the point of arrival. Let:

$$(A) \qquad \begin{cases} \varphi_1(\xi_1, \eta_1, \zeta_1) = 0, \\ \varphi_2(\xi_2, \eta_2, \zeta_2) = 0, \\ \dots\dots\dots\dots\dots\dots, \\ \varphi_n(\xi_n, \eta_n, \zeta_n) = 0 \end{cases}$$

be the equation of the reflecting or refringent surfaces, which we assume to n in number, when they are written in the order in which the trajectory meets these surfaces. Call the part of the trajectory that is found between the point of departure and the first of these surfaces R_1, the part that is between the first and second one, R_2, …, and the part that is between the last surface and the point of arrival R_{n+1} .

Let ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; …; ξ_n, η_n, ζ_n be the coordinates of the points where the trajectory meets the first, second, …, and n^{th} surface, respectively. These will be the coordinates that one must determine. Furthermore, let:

$$(B)\qquad \begin{cases} f_1(x,y,z)=0, \\ f_2(x,y,z)=0, \\ \dots\dots\dots\dots\dots\dots, \\ f_n(x,y,z)=0, \\ f_{n+1}(x,y,z)=0 \end{cases}$$

be the equations of the characteristic wave surfaces of the media in which the rays R_1, R_2, …, R_n, R_{n+1} move, respectively, which will be surfaces that are described by taking the origin to be their centers and corresponding to a unit of time, and each of these equations will be considered to represent only that sheet of the characteristic wave surface that has the same nature as the corresponding portion of the trajectory.

Finally, let T_1 denote the time that it takes for light to propagate from the point a, b, c to the point ξ_1, η_1, ζ_1, T_2, the time that it takes to propagate from the point ξ_1, η_1, ζ_1 to the point ξ_2, η_2, ζ_2, …, and T_{n+1}, the time that it takes to propagate from the point ξ_n, η_n, ζ_n to the point a', b', c'.

From equations (B), one will get:

$$(C)\qquad \begin{cases} f_1\left(\dfrac{\xi_1-a}{T_1},\dfrac{\eta_1-b}{T_1},\dfrac{\zeta_1-c}{T_1}\right)=0, \\ f_2\left(\dfrac{\xi_2-\xi_1}{T_2},\dfrac{\eta_2-\eta_1}{T_2},\dfrac{\zeta_2-\zeta_1}{T_2}\right)=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ f_{n+1}\left(\dfrac{a'-\xi_n}{T_{n+1}},\dfrac{b'-\eta_n}{T_{n+1}},\dfrac{c'-\zeta_n}{T_{n+1}}\right)=0. \end{cases}$$

If one eliminates the variables ζ_1, ζ_2, …, ζ_{n+1} from each of equations (C) then one can deduce the values of T_1, T_2, …, T_{n+1} from the resulting equations, and when one finally adds these values to each other pair-wise, one will get n equations of the form:

$$(D)\qquad \begin{cases} T_1+T_2=\mathcal{F}_1(a,b,c,\xi_1,\eta_1,\xi_2,\eta_2), \\ T_2+T_3=\mathcal{F}_2(\xi_1,\eta_1,\xi_2,\eta_2,\xi_3,\eta_3), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ T_n+T_{n+1}=\mathcal{F}_n(\xi_{n-1},\eta_{n-1},\xi_n,\eta_n,a',b',c'). \end{cases}$$

From Theorem XIX, if one considers ξ_1 and η_1 to be the only variables in the first of these equations, ξ_2 and η_2 to be the only variables in the second of these equations, …, and ξ_n and η_n to be the only variables in the last one then one must have:

$$d(T_1+T_2)=0,$$
$$d(T_2+T_3)=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$d(T_n + T_{n+1}) = 0,$$

and in turn:

$$\frac{d\mathcal{F}_1}{d\xi_1} = 0, \qquad \frac{d\mathcal{F}_1}{d\eta_1} = 0,$$
$$\frac{d\mathcal{F}_2}{d\xi_2} = 0, \qquad \frac{d\mathcal{F}_2}{d\eta_2} = 0,$$
$$\ldots\ldots\ldots, \qquad \ldots\ldots\ldots,$$
$$\frac{d\mathcal{F}_n}{d\xi_n} = 0, \qquad \frac{d\mathcal{F}_n}{d\eta_n} = 0.$$

These equations, which are $2n$ in number, when combined with the n equations (A), will determine the $3n$ coordinates ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; …; ξ_n, η_n, ζ_n, and in turn, the desired trajectory.

In general, there will be a unique or finite number of solutions when a solution is possible, which is not always true, as is easy to assure oneself. However, it can happen that there will be an infinitude of solutions that correspond to a curve on each of the reflecting or refringent surfaces. The point of arrival will then be a partial focus for a certain group of rays that define a continuous surface. Finally, in some very special cases it can happen that the values of the unknowns are completely indeterminate, which indicates that all of the rays that emanate from the point of departure must agree at the point of arrival after reflecting or refracting from the given surfaces, or in other words, that the reflecting or refringent surface will constitute an aplanatic system with respect to the two given points.

45. We further remark that when light can propagate from one point to the other, while experiencing an arbitrary number of reflections and refractions by given surfaces, *for several trajectories, which are finite in number, the times that it takes to traverse these trajectories will not necessarily be equal.* However, when light can follow an infinitude of trajectories in order to go from the point of departure to the point of arrival, which will then be a total or partial focus, the times that the light takes to traverse these trajectories will necessarily be equal (34 and 35). In other words, two trajectories that start from the same point in order to arrive at the same point will not necessarily traverse them in equal times when there exists a continuous sequence of trajectories that are likewise traversed by light, and to which, these two trajectories will belong.