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Foundations of an invariant theory of contact transformations

By SOPHUS LIE in Christiania

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Without a doubt, Jacobi's epoch-making papers on first-order, partial differential equations [*]) will always assume a distinguished place in science. Still, to some mathematicians they seemed to be over-rated, or, in any event, not correctly assessed. Namely, they propagated the impression that by means of Jacobi's investigations the theory of first-order, partial differential equations had been brought to a conclusion. Such an opinion must, however, be regarded as incorrect, now that recent papers have improved upon it, and, in turn, given new methods of integration, and have initiated likewise fruitful directions of investigation.

In the present treatise, I will give a systematic representation of a new theory that I have communicated to the Academy in Christiania in 1872 and 1873. I will make some remarks in advance that refer to how my other investigations on partial differential equations relate to the simultaneous papers of Mayer. Of the many important papers of this author, I have discussed only the ones that are most closely connected with my own work.

Résumé of some older investigations.

Starting from a *geometric* investigation of the relationship between Plücker's line geometry and general curvature theory, I was gradually drawn over to the realm of partial differential equations. Thus, it was readily conspicuous to me that the mathematics that Monge had employed to such great effect had abandoned the simultaneous use of synthetic and analytic methods. It seems obvious to me (cf., the paper "Ueber Complexe, etc." in these Ann., Bd. V) that such a *mixed* method would lead to new results more easily than pure analysis, which had been applied almost exclusively to the examination of partial differential equations since the time of Monge. I nurture the hope that the discoveries that I made might serve to reinforce such a viewpoint.

In 1871, I posed the problem for myself of working through the Jacobi integration method conceptually, and in particular, the Poisson-Jacobi theorem. It became clear to

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[*]) The first method of integration for first-order, partial differential equations goes back to Pfaff (1814). Somewhat later (1819), Cauchy gave a method that was formulated in newer ways by Jacobi (1837) and, in latter times, Mayer (Math. Ann., Bd. IV). This method must be called the Cauchy method and not the Jacobi-Hamilton. Finally, Jacobi 1837-1840 found a new method that was then first published in 1862. I will call this latter method the Jacobi method, and not the new Jacobi method, as is customary.

me almost immediately that it was possible to give a *new* method of integration [*]) that required fewer integrations than that of Jacobi, and therefore did not use the Poisson-Jacobi method, at all. This new method, which, like all of the other ones, began with the search for an integral of the known simultaneous system, represented, in a sense, an intermediate step between the Cauchy and Jacobi methods. It was based upon my extensions of Cauchy's method.

At the same time, Mayer [**]) gave a *fundamental theorem* – I shall call it *Mayer's theorem* – that allowed him to improve upon the Jacobi method of integrating first-order, partial differential equations essentially, as well as Clebsch's treatment of the Pfaff problem. By it, he achieved, in particular, and admittedly, by a completely different route from my own, the same reduction in the number of operations that were necessary for the integration of a partial differential equation of first order.

Thereafter, I developed (Göttinger Nachrichten, 1872, no. 25) a general approach to the concept of a complete solution, by which I consequently worked through the Pfaff formulation of the integration problem and further exploited it. By that means, I eliminated, *inter alia*, certain shortcomings that were inherent to the integration methods up to that point in time.

I found the theory that was just cited by *purely synthetic* considerations *when I consequently generalized Monge's concept of "characteristic,"* and used only simple arguments from the modern theory of manifolds, moreover. My old synthetic representation of this theory, which I had developed only in the most general terms, was, in an obvious way, satisfactory to only those readers that were quite familiar with manifold considerations. Unfortunately, I have not found the time to present everything thoroughly. I am therefore greatly indebted to Mayer, who, in many elegant treatises [***]) has given a clear, analytical formulation and foundation of these investigations, to the extent that they are employed in the sequel. I refer the reader to the cited papers of Mayer.

On the contents of this treatise.

In the investigation of partial differential equations, those properties that remain unchanged under arbitrary contact transformations – i.e., analytic transformations – deserve special attention [****]). Such a study is important because, *inter alia*, just such properties come under consideration in the ordinary methods of integration. In particular, for first-order equations, to which this treatise is dedicated, such investigations take on a very simple and beautiful form. It is then possible to resolve several fundamental problems of the type spoken of. In this way, one achieves, *inter alia*, the foundation of a rational treatment of those first-order, partial differential equations, by whose integration, one *has already made some steps forward*. It always lets one decide how one must

[*]) Abhandlungen der Akademie zu Christiania, 3 and 10, May 1872; Göttinger Nachrichten 1872, no. 16.

[**]) Math. Annalen, Bd. V, pp. 448; Göttinger Nachrichten 1872, no. 15.

[***]) Göttinger Nachrichten 1872, no. 24, 1873, no. 11; Math. Ann., Bd VI, pp. 162, 192.

[****]) Klein has directed my attention to the fact that in many mathematical directions of inquiry, one is dealing with the determination of properties that remain invariant under some group of transformations.

proceed in order to resolve the matters that still remain to be integrated by the simplest means. In order to clarify this with a good example, I consider the equation:

$$F(x_1, \dots, x_9, p_1, \dots, p_9) = a,$$

to whose integration, Hamilton and Jacobi were reduced in the three-body problem. As is well-known, one knows eight integrals of the associated simultaneous system. My general theories now allow one to exhibit an equation:

$$f(x_1, \dots, x_9, p_1, \dots, p_9) = 0,$$

whose integration implies that of $F = 0$. Thus, this known result is reduced to its intrinsic basis [*]). Apart from that, I give less weight to the new integration method of my treatise than I do to the *deeper insight into the essence of first-order, partial differential equations* that it gives one. Hopefully, my future research will remedy this situation.

Among the new theories in this paper, I also emphasize the following ones: Let F, Φ_1, and Φ_2 be functions of $x_1, \dots, x_n$, $p_1, \dots, p_n$ for which one has:

$$(F, \Phi_1) = 0, \qquad (F, \Phi_2) = 0.$$

The Poisson-Jacobi theorem then says that one also has:

$$(F, (\Phi_1, \Phi_2)) = 0.$$

If one knows two solutions Φ_1, Φ_2 of the equation:

$$(F, \Phi) = 0$$

then there is an operation that allows one to find several such solutions, in general.

I know prove that any operation that serves to make new solution known essentially coincides with the stated one; thus, it will only be assumed that the type of operation in question should be independent of the form of the function F.

I found the following theory by the application of a *combined synthetic-analytic* method. If the editing had been less tiresome for me, I would have sought to develop everything simultaneously in a synthetic and analytic way, following the model of Monge. Since I have very little confidence in my editing talents, and am, moreover, concerned with new investigations, I have then chosen to present results in the *ordinary* analytical form. As a result, the first section especially has lost its simplicity. Fortunately, I can refer the reader to a beautiful and *exceptionally simple* analytic foundation that Mayer has given on just the results of the first section [**]).

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[*]) The previously-cited papers of Mayer and myself have essentially reduced the three-body problem, insofar as they simplified the integration of the equation $f = 0$.

[**]) Göttinger Nachrichten, 1874, no. 13: "Ueber die Lie'schen Berührungstransformationen." Cf., also the following note of Mayer.

PART ONE

Theory of contact transformations.

I will seek to develop the theory of contact transformations, which defines the foundation for my work on partial differential equations up to now, as well as my future work, analytically and for n variables. As I already said, the cited paper of Mayer gives an elegant development of this theory that, *inter alia*, is to be preferred over my own since it is direct, while my treatment rests upon the Clebsch theory of the Pfaff problem. (Cf., § 8).

§ 1.

Definition of the concept of contact transformation.

1. The origin of the theory of contact transformations goes back to Euler; later on, Jacobi, in particular, presented the theory that appears here in connection with his work on the perturbation theory of developments. If I am not incorrect, I am then the first one to explain the general meaning of that theory and emphasize its importance. I also believe that I am the first to set down the true essence *) of these matters in a precise and rigorous way; the term "contact transformation" originated with me **).

Before I define the concept of contact transformation, I find it convenient to present some simple geometric considerations that lead to this notion in a natural way. Indeed, I shall refer them only to a space of three dimensions, but they can still be extended to arbitrary manifolds.

If the Cartesian point-space, in the ordinary sense of the word, were subjected to a point transformation then surfaces would go to surfaces and surfaces that contact each other would to other such surfaces. Admittedly, there are exceptional cases that transform in other ways, but they appear only in a limited number. However, besides the point transformations, there are still other transformations that possess a useful character. For instance, a dualistic transformation, *in general*, also takes surfaces to surfaces and contacting surfaces to other such surfaces. Thus, it should be remarked that there are unboundedly many surfaces – namely, the developable surfaces – that do not transform into surfaces under a dualistic transformation, but into curves. In particular, all planes go to the points of space.

It can be proved that, in addition to point transformations, there is an extended category of transformations that generally take surfaces to surfaces and surfaces that

*) I recall, in particular, that I have shown that any contact transformation, in the Plücker sense, is based in a change of space element or the introduction of a new coordinate system. This remark is fundamental for a synthetic treatment of the theory of first-order, partial differential equations.

**) After I published my first papers on contact transformations, Darboux wrote to me at that point in time that he, too, had been concerned with this theory. I must apologize that I could have derived no benefit from his investigations, which were still unpublished at that point in time. Du Bois-Reymond has concerned himself with the contact transformations of a triple-extended space. The results that his work on partial differential equations contained are still not complete.

contact each other to other such surfaces. For such a transformation that is not a point transformation, there are unboundedly many surfaces that transform into curves. In particular, there are ∞^3 *surfaces that go to the points of space.*

This is, however, not a definition of the contact transformations of space; we have given only the essential properties of them.

2. In previous treatises, I gave perhaps the following definition: If the independent variables $x_1, .., x_n$, the function z of these variables, and the partial derivatives of z with respect to $x_1, .., x_n$, which might be called $p_1, .., p_n$, are coupled with a corresponding system of variables $z', x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ in such a way that any quantity in either of the two sequences:

$$z, \;\; x_1, \ldots, \;\; x_n, \;\; p_1, \ldots, \;\; p_n,$$
$$z', \; x'_1, \ldots, \; x'_n, \;\; p'_1, \ldots, \;\; p'_n,$$

can be expressed in terms of quantities in the other sequence, so I called the transformation in question a *contact transformation*. However, this definition is not sufficiently clear, and perhaps also not entirely correct, insofar as it implicitly rests upon assumptions that do not always apply. Thus, I shall replace this definition with the following one, which, in my opinion, completely captures the essence of things:

Definition. *If* $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$ *are functions of* $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ *for which one has:*

$$dZ - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k) \tag{1}$$

identically, then the equations:

$$z' = Z, \qquad x'_i = X_i, \qquad p'_i = P_i \tag{2}$$

define a transformation that shall be called a contact transformation.

The fact that equations (2) always define a transformation rests upon the fact that equation (1) necessarily implies that $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$ are mutually independent functions.

Terminology. If F and Φ are functions of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ then I will write, as usual, $[F, \Phi]$, instead of:

$$\sum_{k=1}^{k=n} \left\{ \frac{\partial F}{\partial p_k}\left(\frac{\partial \Phi}{\partial x_k} + p_k \frac{\partial \Phi}{\partial z}\right) - \frac{\partial \Phi}{\partial p_k}\left(\frac{\partial F}{\partial x_k} + p_k \frac{\partial F}{\partial z}\right) \right\},$$

and likewise, when F and Φ do not include z, (F, Φ), instead of:

$$\sum_{k=1}^{k=n}\left(\frac{\partial F}{\partial p_k}\frac{\partial \Phi}{\partial x_k}-\frac{\partial \Phi}{\partial p_k}\frac{\partial F}{\partial x_k}\right).$$

If I wish to stress here that F and Φ are considered to be functions of $x_1, \ldots, p_n$, and not perhaps $x_1', \ldots, p_n'$, then I will write $(F, \Phi)_{xp}$.

One knows that the Pfaffian expression $\sum_{k=1}^{k=1} X_k dx_k$ can generally be reduced to a form with n terms:

$$\sum_{k=1}^{k=1} X_k dx_k = F_1\, df_1 + \ldots, + F_n\, dx_n\,.$$

Here, according to Clebsch (Crelle's Journal, Bd. 61, pp. 153), the functions f are an arbitrary system of solutions of $\frac{n(n+1)}{1\cdot 2}$ equations, which I will denote by the symbols:

$$((f_i)) = 0, \qquad ((f_i, f_k)) = 0.$$

§ 2.

Determination of all contact transformations.

In these paragraphs, I will give two very different ways of determining all contact transformations. Thus, I shall base this on the known, established theory of Pfaffian problems, which, in my opinion, should be placed in the foreground of all investigations of first-order, partial differential equations, rather than the one that came about since the time of Cauchy and Jacobi. In particular, I cannot sufficiently stress that the Pfaff conception of the problem of integrating an equation:

$$F(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$$

gives this theory a generality that is completely lacking in the ordinary theory. Admittedly, no one seems to have commented on this fundamental asset of the Pfaffian way of looking at things.

3. Let:

$$X_1\, dx_1 + \ldots + X_{2n+1}\, dx_{2n+1}$$

be a given Pfaffian expression whose canonical form [*]) includes $n + 1$ terms. If:

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[*]) If a Pfaffian expression $X_1\, dx_1 + \ldots + X_m\, dx_m$ can be reduced to a p-term form $F_1\, df_1 + \ldots + F_m\, df_m$, and not to a form with less than p terms then I call $F_1\, df_1 + \ldots + F_p\, df_p$ a *canonical form* – or *normal form* – for the given expression.

$$\alpha\,(df_{n+1} + F_1\,df_1 + \ldots + F_n\,df_n)$$

is such a given form then, as is known, it is possible to find arbitrarily many canonical forms:

$$\beta\,(d\varphi_{n+1} + \Phi_1\,d\varphi_1 + \ldots + \Phi_n\,d\varphi_n).$$

In fact, in order to satisfy the equation:

$$df_{n+1} + \sum_{k=1}^{k=n} F_k\,df_k = \rho\,(d\varphi_{n+1} + \sum_{k=1}^{k=n} \Phi_k\,d\varphi_k) \tag{3}$$

one chooses $q + 1$ equations between the f and φ arbitrarily:

$$\Pi_0 = 0, \qquad \Pi_1 = 0, \ \ldots, \qquad \Pi_q = 0,$$

and sets:

$$F_i = \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial f_i} : \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial f_{n+1}},$$

$$\Phi_i = \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial \varphi_i} : \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial \varphi_{n+1}},$$

$$(i = 1, 2, \ldots, n).$$

If one eliminates $\lambda_1, \ldots, \lambda_q$ from the $2n + q + 1$ equations and then solves them for f_i and F_i then one finds values for these quantities that fulfill (3) identically.

4. By my definition, the problem of determining all contact transformations turns into the problem of determining all the quantities $z', x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ in the most general way as functions of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ that make the equation:

$$dz' - \sum_{k=1}^{k=n} p'_k\,dx'_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k\,dx_k)$$

true identically. How, since $z, x_1, \ldots, p_n$ are mutually independent quantities, we are allowed to consider $dz - \sum_{k=1}^{k=n} p_k\,dx_k$ as the canonical form of a $(2n + 1)$-term Pfaffian problem, and one then immediately obtains the following theorem from the known results of Pfaff theory that were just given:

Theorem. 1. *Every contact transformation can be obtained in the following way: One takes $q + 1$ equations between the $z, x_1, \ldots, x_n, x'_1, \ldots, x'_n$:*

$$\Pi_0 = 0, \qquad \Pi_1 = 0, \ \ldots, \qquad \Pi_q = 0,$$

and sets:

$$-p_i = \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial x_i} : \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial z},$$

$$-p_i' = \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial x_i'} : \frac{\partial(\Pi_0 + \lambda_1\Pi_1 + \cdots + \lambda_q\Pi_q)}{\partial z'},$$

$$(i = 1, \ldots, n).$$

If one eliminates $\lambda_1, \ldots, \lambda_q$ *from the* $2n + q + 1$ *equations then the remaining* $2n + 1$ *equations always determine a contact transformation between the two systems of variables* $z, x_1, \ldots, p_n$ *and* $z', x_1', \ldots, p_n'$.

Jacobi likewise considered all of the transformations and indeed asserted that they are the most general transformations of a partial differential equation of first order. We shall not go into this assertion here, whose validity is likewise not obvious, *a priori*. Furthermore, Jacobi gave no explicit definition of the concept of the most general transformation of a partial differential equation of first order *).

5. The determination of all contact transformations that was just given is, *inter alia*, not satisfactory, since it introduces a classification of contact transformations in terms of the value of the number q, even if that was only implicit. However, such a classification in no way corresponds to the nature of things, insofar as it rests upon a random choice, in a sense. I shall thus give a new general method for the determination of contact transformations. If one were to apply it to a special case then it would be clearly necessary to perform, not just differentiations and eliminations, as in the usual method, but also certain integrations, moreover. However, this is completely valid whenever one is only dealing with the establishment of the concept.

It is known from the theory of the Pfaff problems that one can reduce a $(2n + 1)$-term expression $\sum_k X_k dx_k$ to an $(n + 1)$-term expression in the following way: One takes an arbitrary function φ of $x_1, \ldots, x_{2n+1}$, removes the quantities x_{2n+1} and dx_{2n+1} by means of the equations:

$$\varphi = a, \qquad \sum_{k=1}^{k=2n+1} \frac{\partial\varphi}{\partial x_k} dx_k = 0,$$

and thus obtains a $2n$-term expression:

*) I shall take this occasion to address two questions, the second of which, in particular, seems important:

1. Are there transformations that are not contact transformations and for which contact of higher order is an invariant relation? – This question seems to be answered in the negative.

2. Do partial differential equations of higher order admit transformations that are not contact transformations? – This question must indeed be answered in the affirmative. If this were the case then this would open up an important domain of research.

$$X_1^a dx_1 + \ldots + X_{2n}^a dx_{2n},$$

whose coefficients include a, in addition to $x_1, \ldots, x_{2n}$. One brings it into the form:

$$\sum_{k=1}^{k=n} X_k^a dx_k^a = \sum_{k=1}^{k=n} \Phi_k^a d\varphi_k^a$$

in the same way, and then replaces the quantities a with φ in $\varphi_1^a, \ldots, \varphi_n^a$, such that these functions go to functions of $x_1, \ldots, x_{2n+1}$ that will be denoted by $\varphi_1, \ldots, \varphi_n$. In this way, the original $(2n + 1)$-term expression can take on the form:

$$\Phi\, d\varphi + \Phi_1\, d\varphi_1 + \ldots + \Phi_n\, d\varphi_n,$$

where $\Phi, \Phi_1, \ldots, \Phi_n$ are determined in the ordinary way. I add to this the fact that $\varphi_1^a, \ldots, \varphi_n^a$ are defined by the Clebsch equations:

$$((\varphi_i^a)) = 0, \qquad ((\varphi_i^a, \varphi_k^a)) = 0, \qquad (i = 1, \ldots, n), \quad (k = 1, \ldots, n).$$

6. By my definition, the problem of determining all contact transformations comes down to the problem of bringing the Pfaffian expression $dz - \sum_{k=1}^{k=n} p_k dx_k$, which already possesses the canonical form, into a new canonical form in the most general way. To that end, from the foregoing, one can proceed in the following way:

One chooses a function Z of $z, x_1, \ldots, p_n$ arbitrarily, and solves the equation:

$$Z = a$$

for p_n, which might take the form:

$$p_n = f(z, x_1, \ldots, p_{n-1}, a).$$

In so doing, one brings the expression:

$$dz - p_1\, dx_1 - \ldots - p_{n-1}\, dx_{n-1} - f\, dx_n$$

into the form:

$$Y_1^a dX_1^a + \ldots + Y_n^a dX_n^a,$$

where X_i^a are determined by the equations:

$$((X_i^a)) = 0, \qquad ((X_i^a, X_k^a)) = 0, \qquad (i = 1, \ldots, n), \quad (k = 1, \ldots, n).$$

In our case, as one easily discovers, these equations assume the form:

$$[p_n - f,\ X_i^a] = 0, \qquad [X_i^a, X_k^a] = 0, \qquad (i = 1, \ldots, n)\ (k = 1, \ldots, n).$$

However, if follows from this that the quantities X_i – that is, those functions of z, x_1, …, p_n that emerge when the quantity a is replaced with Z in X_i^a – are defined by the system of equations [*]):

$$(A) \qquad [Z, X_i] = 0, \qquad [X_i, X_k] = 0 \qquad (i = 1, \ldots, n)\ (k = 1, \ldots, n)$$

If one knows functions Z, X_1, …, X_n that fulfill these relations then it is possible to satisfy the equation:

$$dZ - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k)$$

identically. The quantities ρ, P_1, …, P_n will be determined by n + 1 of the 2n + 1 equations:

$$\frac{\partial Z}{\partial z} - \sum_k P_k \frac{\partial X_k}{\partial x_k} = \rho, \quad \frac{\partial Z}{\partial x_i} - \sum_k P_k \frac{\partial X_k}{\partial x_i} = -\rho\, p_i, \quad \frac{\partial Z}{\partial p_i} - \sum_k P_k \frac{\partial X_k}{\partial p_i} = 0, \quad (i = 1, \ldots, n).$$

It must certainly be remarked that the functions Z, X_1, …, X_n are subject to no other restriction than the demand that equation (A) must be fulfilled. The result obtained can be summarized as follows:

Theorem I. *If one knows* n + 1 *mutually independent functions* Z, X_1, …, X_n *of* z, x_1, …, p_n *that fulfill the equations:*

$$dZ - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k)$$

identically then the relations:

$$z' = Z, \qquad x_i' = X_i, \qquad p_i' = P_i$$

define a contact transformation. The condition equations that we just gave are not, by themselves, sufficient, but they are necessary.

The fact that it is at all possible to find n + 1 functions H_0, H_1, …, H_n of z, x_1, …, x_n, p_1, …, p_n that pair-wise satisfy the conditions $[H_i, H_k] = 0$ rests upon the following theorem:

[*]) By the way, one can interpret this to mean a new formal treatment of the Pfaff problem. It is completely symmetric, which the Clebsch treatment is not.

Theorem 2. *If $H_0, H_1, \ldots, H_q$ are functions of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ and all $[H_i, H_k]$ vanish then the linear equations:*

$$[H_0, H] = 0, \quad \ldots, [H_q, H] = 0$$

define a complete system *).

Namely, if one sets $[H_k, H] = A_k(H)$ and then forms the expressions $A_i(A_k(H)) - A_k(A_i(H))$, then one sees that they can be expressed linearly in terms of the $A_i(H)$.

§ 3.

Contact transformations that take functions of $x_1', \ldots, p_n'$ to functions of $x_1, \ldots, p_n$.

I will now show the existence of a very important category of contact transformations. The characteristic property of them consists of the idea that they take functions of $x_1', \ldots, p_n'$ to functions of $x_1, \ldots, p_n$. Thus, if:

$$z' = Z, \; x_i' = X_i, = P_i$$

are the equations of such a transformation then the quantities X_i and P_i do not include z at all, but only $x_1, \ldots, p_n$. In the first two sections, I gave two methods of finding arbitrarily many such transformations. In the last section, I showed that both methods are general, in the sense that the one of them, as well as the other one, gives a transformation of type discussed.

In the foregoing paragraphs, we saw that the determination of all contact transformations follows immediately from the theory of the determinate cases of the Pfaff problem. It lets us easily show that the reasoning of these paragraphs have a close connection with a new theory of the indeterminate case that goes back to Clebsch (Borchardt’s Journal, Bd. 61).

7. If we choose $q + 1$ arbitrary equations between $z', x_1', \ldots, x_n', a, x_1, \ldots, x_n$ that include z and z' only in the combination $z' - Az$, where A is a constant, put them into the form:

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*) If q linear partial differential equations in n variables:

$$A_i(H) = 0, \ldots, A_q(H) = 0,$$

which are independent of each other, have such a reciprocal relationship that each $A_i(A_k(H)) - A_k(A_i(H))$ can be expressed linearly in terms of the $A_i(H)$ then, as Clebsch has proved (Borchardt’s Journal, Bd. 65), they have $n - q$ different common solutions. With Clebsch, I call such a system a *complete system*. Clebsch based the integration of linear partial differential equations with common solutions on the consideration of such systems. The corresponding theory for arbitrary – i.e., not just linear – equations was first given by Mayer. Thus, it must be remarked that Mayer, as well as Clebsch, took his starting point from an idea that goes back to Bour. Mayer has remarked on that subject that Bour’s formulation of this theory was not rigorous.

$$z' - Az = \Pi(x'_1, \cdots, x'_n, x_1, \cdots, x_n), \ \Pi_1(x'_1, \cdots, x_n) = 0, \ \ldots, \ \Pi_q(x'_1, \cdots, x_n) = 0,$$

and search for the contact transformations that, from theorem 1, belong to these equations then we obtain the formula:

$$\begin{aligned} p'_i &= \frac{\partial \Pi}{\partial x'_i} + \lambda_1 \frac{\partial \Pi_1}{\partial x'_i} + \cdots + \lambda_q \frac{\partial \Pi_q}{\partial x'_i}, \\ -Ap_i &= \frac{\partial \Pi}{\partial x_i} + \lambda_1 \frac{\partial \Pi_1}{\partial x_i} + \cdots + \lambda_q \frac{\partial \Pi_q}{\partial x_i}. \end{aligned}$$

These $2n$ equations, combined with the q equations $\Pi_1 = 0, \ldots, \Pi_q = 0$, express the x'_i and p'_i in terms of only the $x_1, \ldots, x_n, p_1, \ldots, p_n$, and when one substitutes the values thus found into the equation $z' - Az = \Pi$ that would make it take the form:

$$z' = Az + F(x_1, \ldots, x_n, p_1, \ldots, p_n).$$

Thus:

Theorem 3. *Equations between* $z', x'_1, \ldots, x'_n, a, x_1, \ldots, x_n$ *that include* z *and* z' *only in the combination* $z' - Az$ *define a contact transformation that is expressed by equations of the following form:*

$$z' = Az + F, \qquad x'_i = X_i ; \qquad p'_i = P_i .$$

Here, A *is a constant,* F, X_i*, and* P_i *are functions of only* $x_1, \ldots, x_n, p_1, \ldots, p_n$*. I refer to such a transformation briefly as a transformation between* x, p.

8. The method that we just developed for finding contact transformations between x, p has the inconvenient aspect that it introduces a classification that does not correspond to the nature of things, namely, in terms of the value of the number q. The following method is free of this drawback; nonetheless, both methods have a self-sufficient justification. I shall next present a lemma.

Theorem 4. *If* $X_1, \ldots, X_q$ *are functions of* $x_1, \ldots, p_n$ *that pair-wise satisfy the conditions* $(X_i, X_k) = 0$ *then among the solutions* F *of the complete system* (theorem 2):

$$[X_1, F] = 0, \ldots, [X_q, F] = 0$$

there is one of them that possesses the form $Az + \Pi$*. Here,* A *is a constant and* Π *is a function of only* $x_1, \ldots, p_n$.

Our theorem comes about from the fact that the equations:

$$[X_1, F] = 0, \ldots, [X_q, F] = 0, \qquad \frac{\partial F}{\partial z} = A$$

possess common solutions. In order to verify this, we look for a function Φ of z, x_1, …, p_n, and F such that any solution of the equation Φ = const. gives a function F of the desired property. It shows that Φ must fulfill the following relations:

$$[X_1, \Phi] = 0, \ldots, [X_q, \Phi] = 0, \quad \frac{\partial\Phi}{\partial z} + A\frac{\partial\Phi}{\partial F} = 0;$$

however, as one easily verifies, these define a complete system. Our theorem is thus proved.

Here, the following theorem, which I will need later, might find a place:

Theorem 5. *If one knows a solution F of the complete system:*

$$[X_1, F] = 0, \ldots, [X_q, F] = 0$$

that possesses the form $z + \Pi(x_1, \ldots, p_n)$ *then* $Az + A\Pi + \Omega(X_1, \ldots, X_n)$ *is a general solution of it, and indeed, it is the most general one that is linear in z.* (Ω *denotes an arbitrary function of the argument in question.*)

Namely, let:

$$F_1 = A_1 z + \Pi_1, \qquad F_2 = A_2 z + \Pi_2$$

be two solutions of the stated form. Therefore, $A_2F - A_1F$, or, what amounts to the same thing, $A_2\Pi_1 - A_1\Pi_2$, in which z does not enter anywhere, also satisfies our complete system. However, as is known, it follows from this that:

$$A_2\Pi_1 - A_1\Pi_2 = W(X_1, \ldots, X_n),$$

an equation that proves our theorem.

Theorem 6. *If* $X_1, \ldots, X_n$ *are functions of* $x_1, \ldots, p_n$ *that satisfy the conditions* $(X_i, X_k) = 0$ *pair-wise then (theorem 4) there are functions of the form* $Az + \Pi(x_1, \ldots, p_n)$ *that fulfill all equations:*

$$[X_i, Az + \Pi] = 0, \qquad (i = 1, \ldots, n),$$

and thus (theorem I) it is possible to satisfy the equation:

$$d(Az + \Pi) - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k).$$

Thus, all of the P_i*, as we likewise prove, are functions of* $x_1, \ldots, p_n$. *Thus, the contact transformation:*

$$z' = Az + \Pi; \quad x'_i = X_i, \qquad p'_i = P_i$$

possesses the property that it transforms functions of $x'_1, \ldots, p'_n$ *into functions of* $x_1, \ldots, p_n$.

Therefore, if the equation:

$$d(Az + \Pi) - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k)$$

is to be satisfied identically then, since z does enter into $\Pi, X_1, \ldots, X_n$, one must have:

$$A = \rho$$

and:

$$\frac{\partial \Pi}{\partial p_i} - \sum_k P_k \frac{\partial X_k}{\partial p_i} = 0,$$
$$\frac{\partial \Pi}{\partial x_i} - \sum_k P_k \frac{\partial X_k}{\partial x_i} = -A\,p_i\,,$$

equations that show that all P_i depend upon only $x_1, \ldots, p_n$.

9. If one eliminates $p_1, \ldots, p_n$ from the equations:

(a) $$z' = Az + \Pi, \qquad x'_i = X_i,$$

in which:

$$(X_i, X_k) = 0, \qquad [Az + \Pi, X_i] = 0,$$

then one finds a number of equations of the form:

(b) $$z' - Az = \Omega(x'_1, \ldots, x'_n, x_1, \ldots, x_n), \qquad \Omega_1(x'_1, \ldots, x'_n) = 0, \; \ldots, \Omega_q(x'_1, \ldots, x'_n) = 0.$$

Thus, the latter method gives only such transformations that also can be obtained by the previously-developed method. Now, since, conversely, a system of equations of the form (b), as we showed earlier, always leads to transformation equations of the form (a), it is clear that both of our methods overlap; their difference is only formal. We will show that that they give us *every* contact transformation between x, p.

One obtains all contact transformations between x, p when one determines the quantities $X_1, \ldots, X_n, P_1, \ldots, P_n$ in a general way as functions of $x_1, \ldots, p_n$ such that the equation:

(e) $$dZ - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k)$$

is true identically. By development, it will assume the form:

$$U\,dz + \sum_i V_i dx_i + \sum_i W_i dp_i = \rho\,(dz - \sum_i p_i dx_i),$$

where:

$$U = \frac{\partial Z}{\partial z}, \qquad V_i = \frac{\partial Z}{\partial x_i} - P_1 \frac{\partial X_1}{\partial x_i} - \cdots - P_n \frac{\partial X_n}{\partial x_i}$$

$$W_i = \frac{\partial Z}{\partial p_i} - P_1 \frac{\partial X_1}{\partial p_i} - \cdots - P_n \frac{\partial X_n}{\partial p_i}.$$

We thus obtain $2n + 1$ relations:

$$\frac{\partial Z}{\partial z} = \rho, \qquad \frac{\partial Z}{\partial p_i} - P_1 \frac{\partial X_1}{\partial p_i} - \cdots - P_n \frac{\partial X_n}{\partial p_i} = 0 \qquad (i = 1, \ldots, n)$$

$$\frac{\partial Z}{\partial x_i} - P_1 \frac{\partial X_1}{\partial x_i} - \cdots - P_n \frac{\partial X_n}{\partial x_i} = -\rho\, p_i = -p_i \frac{\partial Z}{\partial z} \qquad (i = 1, \ldots, n),$$

which shows that the differential quotients depend upon only $x_1, \ldots, p_n$. Thus, Z has the form:

$$Z = Z_1(z, x_1, \ldots, x_n) + Z_2(x_1, \ldots, p_n),$$

where Z_1 must satisfy the following relations:

$$\frac{\partial Z_1}{\partial z} = \rho, \qquad \frac{\partial Z}{\partial x_i} + \frac{\partial Z}{\partial x_i} - \sum_{k=1}^{k=n} P_k \frac{\partial X_k}{\partial x_i} = -p_i \frac{\partial Z_1}{\partial z}.$$

By differentiation with respect to z it follows from the last one that:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial Z_1}{\partial z}\right) = 0, \qquad \frac{\partial}{\partial z}\left(\frac{\partial Z_1}{\partial z}\right) = 0, \qquad (i = 1, \ldots, n).$$

Thus, $\frac{\partial Z_1}{\partial z}$ is equal to a constant A – i.e., Z_1 is linear relative to z. With that, we have proved that Z possesses the form:

$$Z = Az + \Pi(x_1, \ldots, p_n).$$

However, in the second paragraph we saw that the expressions $[Z, X_i]$, $[X_i, X_k]$ must necessarily vanish when the condition equation (c) is valid, and as a result, we can assert that the two methods that were given in this paragraph give every contact transformation between x, p.

I summarize the results of this paragraph in the following way:

Theorem I. *There is an extended category of contact transformations that possess the characteristic property that functions of* $x_1', \ldots, p_n'$ *go to functions of* $x_1, \ldots, p_n$. *All such transformations possess the form:*

$$z' = Az + \Pi(x_1, \ldots, p_n), \qquad x_i' = X_i, \qquad p_i' = P_i,$$

where A denotes a constant. Relations between z', x_1', …, x_n', z, x_1, …, x_n *that include the quantities* z' *and* z *only in the combination* $z' - Az$ *always determine such a function. On the other hand, if* X_1, …, X_n *are functions of* x_1, …, p_n *such that all of the* (X_i, X_k) *are equal to zero then there always exists a function* $Az + \Pi(x_1, \ldots, p_n)$ *such that all expressions* $[Az + \Pi, X_i]$ *vanish, and the equations:*

$$z' = Az + \Pi, \qquad x_i' = X_i$$

then, in turn, define a contact transformation of the stated type.

§ 4.

Presentation of some characteristic relations.

If the equations:

$$z' = Z, \qquad x_i' = X_i, \qquad p_i' = P_i$$

define a contact transformation between x, p then the functions X_i and P_i satisfy certain relations that will now be developed.

10. I first address the following problem: I assume that X_1, …, X_n, P_1, …, P_n are *given* functions of x_1, …, p_n and that it is possible to find a function $Az + \Pi(x_1, \ldots, p_n)$ such that the equations:

$$z' = Az + \Pi, \qquad x_i' = X_i, \qquad p_i' = P_i$$

define a contact transformation. This will achieve the determination of the quantities A and Π in the *most general* way. It will show that A is defined completely by the X_i and P_i and Π, up to an arbitrary constant.

Since ρ must be equal A, the identity equation:

$$\text{(d)} \qquad d(Az + \Pi) - \sum_{k=1}^{k=n} P_k dX_k = \rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k)$$

reduces to:

$$d\Pi - \sum_k P_k dX_k = -A \sum_k p_k dx_k .$$

This equation can be solved into the following $2n$:

$$\frac{\partial \Pi}{\partial x_i} - \sum_k P_k \frac{\partial X_k}{\partial x_i} = -A\, p_i\,; \qquad \frac{\partial \Pi}{\partial p_i} - \sum_k P_k \frac{\partial X_k}{\partial p_i} = 0.$$

If one now differentiates with respect to p_i and x_i and sets the two expressions for $\frac{\partial^2 \Pi}{\partial x_i \, \partial p_i}$ equal to each other then one finds:

$$A = \sum_{k=1}^{k=n} \left(\frac{\partial X_k}{\partial x_i} \frac{\partial P_k}{\partial p_i} - \frac{\partial X_k}{\partial p_i} \frac{\partial P_k}{\partial x_i} \right) = W_i ,$$

in which i refers to any one of the numbers 1, …, n.

Above, we determined $\frac{\partial \Pi}{\partial x_i}$ and $\frac{\partial \Pi}{\partial p_i}$ as functions of $x_1, \ldots, p_n$:

$$\frac{\partial \Pi}{\partial x_i} = M_i , \qquad \frac{\partial \Pi}{\partial p_i} = N_i ,$$

by integration:

$$\Pi = \int (M_1 \, dx_1 + \ldots + M_n \, dx_n + N_1 \, dp_1 + \ldots + N_n \, dp_n) + \text{const.}$$

The constant introduced is arbitrary, since Π only appears as the differential in (d). Therefore:

Theorem 7. *If* $X_1, \ldots, X_n, P_1, \ldots, P_n$ *are* given *functions of* $x_1, \ldots, p_n$ *such that the equation:*

$$dZ - \sum_{k=1}^{k=n} P_k dX_k = \rho \, (dz - \sum_{k=1}^{k=n} p_k dx_k)$$

is satisfied then this can happen in essentially only one way. Z has (§ 3) *the form* $Az + \Pi(x_1, \ldots, p_n)$; *A is a completely determined constant, and* Π *includes an arbitrary additive constant.*

Corollary. *If* $X_1, \ldots, P_n$ *are given functions of* $x_1, \ldots, p_n$ *, and the two systems of equations:*

$$z' = Z_1, \qquad x_i' = X_i, \qquad p_i' = P_i,$$

and

$$z'' = Z_2, \qquad x_i'' = X_i, \qquad p_i'' = P_i$$

determine two contact transformations between x, p then $Z_1 - Z_2$ *is a constant.*

Incidentally, it is simple to prove this corollary directly.

11. The characteristic relations that we mentioned rest upon the fact that, in a sense that we will likewise define (Theorems 8 and 11), the expression $(\omega_1, \omega_2)_{x,p}$ remains *invariant* under contact transformations.

Theorem 8. *Let* ω_1' *and* ω_2' *be functions of* z', x_1', …, p_n' *that go to functions of* z, x_1, …, p_n, *which might be called* ω_1 *and* ω_2, *resp., under contact transformations. If the expression* $[\omega_1', \omega_2']_{x',p'}$ *vanishes then this is also the case with* $[\omega_1, \omega_2]_{x,p}$.

If $[\omega_1', \omega_2']_{x',p'}$ vanishes then (theorem 2) it is possible to determine further functions ω_3', …, ω_{n+1}' of z', x_1', …, p_n' such that all of the $[\omega_i', \omega_k']_{x',p'}$ are equal to zero. Then, however, (Theorem 1) one has an identity of the form:

$$dz' - \sum_{k=1}^{k=n} p_k' dx_k' = \sum_{k=1}^{k=n+1} \Omega_k' d\omega_k' \; .$$

If we now express z', x_1', …, p_n' in terms of z, x_1, …, p_n then the left-hand side of our equation goes to $\rho\,(dz - \sum p_k dx_k)$, and the transformed equation then possesses the form:

$$\rho\,(dz - \sum_{k=1}^{k=n} p_k dx_k) = \sum_{k=1}^{k=n+1} \Omega_k d\omega_k \,,$$

if ω_1, …, ω_{n+1} denote the functions of z, x_1, …, p_n that ω_1', …, ω_{n+1}' go to. However, this new equation shows (Theorem I) that all of the $[\omega_i, \omega_k]_{x,p} = 0$. Thus, one also has, in particular, $[\omega_1, \omega_2]_{x,p} = 0$.

Theorem 9. *If the equations:*

$$z' = Z, \qquad x_i' = X_i, \qquad p_i' = P_i$$

define a contact transformation between x, p then all of the expressions (P_i, P_k) *vanish, and when i = k, so do all* (X_i, P_k).

We then know that the expressions $(x_i', p_k')_{x',p'}$, $(p_i', p_k')_{x',p'}$ are all equal to zero, so, from the foregoing theorem, $(X_i, P_k)_{x,p}$, $(P_i, P_k)_{x,p}$ do, as well.

Theorem 10. *If the equations:*

$$z' = Az + \Pi, \qquad x_i' = X_i, \qquad p_i' = P_i$$

define a contact transformation between x, p then all of the expressions (X_1, P_1), …, (X_n, P_n) *equal the constants A.*

Namely, let F' and Φ' be two functions of x_1', …, p_n' and let F, Φ be the corresponding functions of x_1, …, p_n . We know that the expressions $(F', \Phi')_{x',p'}$ and $(F, \Phi)_{x,p}$ vanish simultaneously. If we now consider that:

$$(x_i', x_k')_{x,p} = (x_i', p_k') = (p_i', p_k') = 0$$

then we find that:

$$(F, \Phi)_{x,p} = \sum_{k=1}^{k=n} \left(\frac{\partial F}{\partial x_k'} \frac{\partial \Phi}{\partial p_k'} - \frac{\partial F}{\partial p_k'} \frac{\partial \Phi}{\partial x_k'} \right) \cdot (x_k', p_k')_{x,p},$$

or, when we recall that F and Φ, when regarded as functions of x_1', …, p_n', are denoted by F' and Φ':

$$(F, \Phi)_{x,p} = \sum_{k=1}^{k=n} \left(\frac{\partial F'}{\partial x_k'} \frac{\partial \Phi'}{\partial p_k'} - \frac{\partial F'}{\partial p_k'} \frac{\partial \Phi'}{\partial x_k'} \right) \cdot (x_k', p_k')_{x,p};$$

one further has:

$$(F', \Phi')_{x',p'} = \sum_{k=1}^{k=n} \left(\frac{\partial F'}{\partial x_k'} \frac{\partial \Phi'}{\partial p_k'} - \frac{\partial F'}{\partial p_k'} \frac{\partial \Phi'}{\partial x_k'} \right).$$

In order for these expressions to vanish identically, one must necessarily have:

$$(x_1', p_1')_{x,p} = (x_2', p_2')_{x,p} = \ldots = (x_n', p_n')_{x,p} = \frac{1}{n} \sum_{i=1}^{i=n} (x_i', p_i')_{x,p} .$$

However, we previously found:

$$\sum_{k=1}^{k=n} \left(\frac{\partial x_k'}{\partial x_k} \frac{\partial p_k'}{\partial p_k'} - \frac{\partial x_k'}{\partial p_k'} \frac{\partial p_k'}{\partial x_k'} \right) = W_i = A,$$

so this means that:

$$\frac{1}{n} \sum_{i=1}^{i=n} W_i = A.$$

Now, one easily verifies that:

$$\frac{1}{n} \sum_{i=1}^{i=n} W_i = \frac{1}{n} \sum_{i=1}^{i=n} (x_i', p_{i)_{x,p}}' ,$$

so one has:

$$\frac{1}{n} \sum_{i=1}^{i=n} (x_i', p_{i)_{x,p}}' = A,$$

and, as a result, all of the expressions $(x_i', p_i')_{x,p}$ equal zero.

Theorem 11. *If F' and Φ' are functions of x_1', …, p_n' that go to the functions F and Φ of x_1, …, p_n, resp., then, at the same time, $(F', \Phi')_{x',p'}$ goes to* $\frac{1}{A}(F, \Phi)_{x,p}$.

We then see that:

$$(F, \Phi)_{x,p} = \sum_{i=1}^{i=n} \left(\frac{\partial F'}{\partial x_i'} \frac{\partial \Phi'}{\partial p_i'} - \frac{\partial F'}{\partial p_i'} \frac{\partial \Phi'}{\partial x_i'} \right) (x_i', p_i')_{x,p} ,$$

and furthermore, that:

$$(x_i', p_i')_{x,p} = A.$$

One thus has:

$$(F, \Phi)_{x,p} = A\,(F', \Phi')_{x',p'} .$$

With that, our assertion is proved.

12. I would now like to show that the relations that we found are not only necessary, but also sufficient.

Theorem 12. *If* $X_1, \ldots, X_n, P_1, \ldots, P_n$ *are functions of* $x_1, \ldots, p_n$ *that satisfy the relations:*

$$(X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0, \qquad (X_i, P_i) = A = \text{const.}$$

then there is always one and essentially only one *contact transformation of the form:*

$$z' = F, \qquad x_i' = X_i, \qquad p_i' = P_i .$$

Proof. We take a function $Az + \Pi$ that satisfies all relations:

$$[X_i, Az + \Psi] = 0,$$

and thus determine, as before, functions $\Pi_1, \ldots, \Pi_m$ such that the equation:

$$d(Az + \Psi) - \sum \Pi_k\, dX_k = A\,(dz - \sum p_k\, dx_k)$$

is verified identically; Thus, $\Pi_1, \ldots, \Pi_m$ will generally become other functions besides $P_1, \ldots, P_m$. From the previous theorems, one now has:

$$(X_i, \Pi_k) = 0, \quad (X_k, \Pi_k) = A,$$

and from our assumptions one has:

$$(X_i, P_k) = 0, \quad (X_k, P_k) = A,$$

so one obtains:

$$(X_i, \Pi_k - P_k) = 0, \qquad (X_k, \Pi_k - P_k) = 0,$$

from which:

$$\Pi_k - P_k = W_k\,(X_1, \ldots, X_k)$$

and

$$P_k = \Pi_k - W_k .$$

Now, one has:

$$(P_i, P_k) = 0,$$

or

$$(\Pi_i - W_i, \Pi_k - W_k) = 0,$$

from which, upon consideration of the known relations:

$$\frac{\partial W_k}{\partial X_i} = \frac{\partial W_i}{\partial X_k},$$

one has:

$$W_i = \frac{\partial F(X_1, \cdots, X_n)}{\partial X_i}$$

and it follows that:

$$P_i = \Pi_i - \frac{\partial F}{\partial X_i}$$

If we now write the equation:

$$d(Az + \Psi) - \sum \Pi_k\, dX_k = A\,(dz - \sum p_k\, dx_k)$$

in the equivalent form:

$$d(Az + \Psi - F) - \sum \left(\Pi_k - \frac{\partial F}{\partial X_k}\right) dX_k = A\,(dz - \sum p_k\, dx_k)$$

then we discover a function Z that satisfies the equation:

$$dZ - \sum P_k\, dX_k = A\,(dz - \sum p_k\, dx_k).$$

From Theorem 7:

$$Az + \Psi - F + \text{const.}$$

is the most general function that satisfies this requirement.

Since the variables $x_1, \ldots, p_n$ are independent of each other under the contact transformation between x, p, it is, in general, more convenient to write down the equations:

$$x_i' = X_i, \qquad p_i' = P_i.$$

I finally summarize the results of this paragraph:

Theorem III. *If the* 2*n equations:*

$$x_i' = X_i, \qquad p_i' = P_i$$

define a contact transformation between x, p then one finds the following relations:

$$(X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0, \qquad (X_i, P_i) = A = \text{const.}$$

On the other hand, if these relations are valid then the first set of equations always determines a contact transformation.

This theorem may be generalized in the following way, moreover:

Theorem 13. *The following characteristic relations are true between the* $2n + 1$ *functions Z, X, P that determine a contact transformation:*

$$[Z, X_i] = [X_i, X_k] = [X_i, P_k] = [P_i, P_k] = 0 = [Z, P_i] - P_i\,[X_i, P_i],$$
$$[X_1, P_1] = [X_2, P_2] = \ldots = [X_n, P_n].$$

This theorem, which I will not need here and will therefore not prove here, plays an important role in the theory of Pfaffian problems.

§ 5.

Homogeneous contact transformations.

There is an important class of contact transformations between *x*, *p* that possess the characteristic property that they take functions of x_1', …, p_n' that are homogeneous in the differential quotients to other such functions. I will determine all functions of this type, which I call *homogeneous contact transformations.* Corresponding to them, I will refer to functions of x_1, …, p_n that are homogeneous in p_1, …, p_n briefly as *homogeneous* functions.

The importance of this new theory lies in the fact that it overlaps with the *general* theory of contact transformations from a certain standpoint.

13. *Theorem 14. If* X_1, …, X_n *are homogeneous functions of degree zero that pair-wise give* $(X_i, X_k) = 0$ *then it is possible to satisfy the equation:*

$$dZ - \sum P_k\,dx_k = A\,(dz - \sum p_k\,dx_k)$$

in such a way that all P_i *become homogeneous functions of degree one. The contact transformation:*

$$x_i' = X_i\,, \qquad p_i' = P_i$$

is then homogeneous; i.e., they transform homogeneous functions into homogeneous functions of the same degree.

Proof: The fact that all X_i are homogeneous of degree zero is expressed by:

$$p_1 \frac{\partial X_i}{\partial p_1} + \ldots + p_n \frac{\partial X_i}{\partial p_n} = 0,$$

or, what amounts to the same thing, by:

$$[z, X_i] = 0.$$

Now, since all (X_i, X_k) are likewise zero, one can satisfy the equation:

$$dz - \sum P_k \, dx_k = dz - \sum p_k \, dx_k .$$

The quantities P_i then satisfy the relations:

$$P_1 \frac{\partial X_1}{\partial x_i} + \ldots + P_n \frac{\partial X_n}{\partial x_i} = p_i,$$
$$P_1 \frac{\partial X_1}{\partial p_i} + \ldots + P_n \frac{\partial X_n}{\partial p_i} = 0,$$

and are, as a result, homogeneous functions of the degree one.

If we apply the transformation:

$$x_i' = X_i, \qquad p_i' = P_i$$

to any homogeneous functions of degree s, such as:

$$p_n' \cdot H\left(x_1', \cdots, x_n', \frac{p_1'}{p_n'}, \cdots, \frac{p_{n-1}'}{p_n'} \right),$$

then they are converted into:

$$P_n' \cdot H\left(X_1, \cdots, X_n, \frac{P_1}{P_n}, \cdots, \frac{P_{n-1}}{P_n} \right),$$

which is again a homogeneous function of degree s.

14. I will now determine all homogeneous contact transformations of degree zerol that pair-wise give $(X_i, X_k) = 0$. One then has the relations:

$$[z, X_1] = 0, \ldots, [z, X_n] = 0,$$

and thus (Theorem 5) $Az + \Pi(X_1, \ldots, X_n)$, where A denotes a constant and Π, an arbitrary function, is the most general function that is linear in z that fulfills the relations:

$$[X_1, F] = 0, \ldots, [X_n, F] = 0.$$

One is now, in turn, dealing with the problem of determining Π in the most general way such that the relations:

$$z' = Az + \Pi, \qquad x_i' = X_i, \qquad p_i' = P_i$$

define a homogeneous contact transformation. The identity equation:

$$d(Az + \Pi) - \sum P_k\, dX_k = A\,(dz - \sum p_k\, dx_k)$$

gives:

$$\frac{\partial \Pi}{\partial x_i} = \sum P_k \frac{\partial X_k}{\partial x_i} - A\, p_k\,, \qquad \frac{\partial \Pi}{\partial p_i} = \sum P_k \frac{\partial X_k}{\partial p_i}\,.$$

These equations show that the quantities $\partial\Pi / \partial x_i$ and $\partial\Pi / \partial p_i$ must be of degree zero and one, respectively, if they are not perhaps equal to zero. Now, Π is, however, of degree zero and thus $\partial\Pi / \partial x_i$ and $\partial\Pi / \partial p_i$ must be of degree zero and – 1, respectively, if they are non-zero. These considerations show that $\partial\Pi / \partial x_i$, as well as $\partial\Pi / \partial p_i$, must vanish, so Π is a constant – say, B – and $Az + B$ is the most general form of the desired function.

15. If one eliminates the quantities $p_1, \ldots, p_n$ from the equations:

$$z' = Az + B, \qquad x_i' = X_i$$

then one obtains relations of the form:

$$z' = Az + B, \qquad \Omega_1(x_1', \ldots, x_n', x_1, \ldots, x_n) = 0, \ldots, \Omega_q(x_1', \ldots, x_n) = 0.$$

Conversely, one may show that relations of this form always determine a homogeneous contact transformation.

From our general theory, one has that in order to find this contact transformation one must append the following relations to the foregoing ones:

$$p_i' = \frac{\partial(\lambda_1\Omega_1 + \cdots + \lambda_q\Omega_q)}{\partial x_i'}\,,$$

$$p_i = -\frac{1}{A}\frac{\partial(\lambda_1\Omega_1 + \cdots + \lambda_q\Omega_q)}{\partial x_i}\,.$$

However, the form of these equations shows that x_i' and p_i' will be homogeneous functions of degrees zero and one, resp., of $p_1, \ldots, p_n$. Thus:

Theorem IV. *If* $X_1, \ldots, X_n$ *are homogeneous functions of degree zero that pair-wise give* $(X_i, X_k) = 0$ *then the equations:*

$$z' = Az + B, \qquad x_i' = X_i,$$

always determine a homogeneous contact transformation. Such a transformation can also be obtained when one takes $q + 1$ *equations of the form:*

$$z' = Az + B, \qquad \Omega_k(x_1', \ldots, x_n', x_1, \ldots, x_n) = 0 \qquad (k = 1, \ldots, q)$$

and looks for the corresponding contact transformation. Finally, it is self-explanatory (Theorem III) that when $X_1, \ldots, X_n, P_1, \ldots, P_n$ *are homogeneous functions of degrees zero and one, respectively, that fulfill the relations:*

$$(X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0; \qquad (X_i, P_i) = A,$$

the equations:

$$x_i' = X_i, \qquad p_i' = P_i$$

always determine a homogeneous contact transformation.

§ 6.

Infinitesimal homogeneous contact transformations.

16. I say that a homogeneous contact transformation:

$$x_i' = X_i, \qquad p_i' = P_i\,, \qquad (X_i, P_i) = 1$$

is *infinitesimal* if it can assume the form:

$$x_i' = x_i + \varepsilon M_i, \qquad p_i' = p_i + \varepsilon \Pi_i,$$

where ε is an infinitesimal quantity, M_i and Π_i are homogeneous functions of degree zero and one, respectively. I will show that there is always a homogeneous function of degree one whose partial derivatives with respect to p_i and x_i are just M_i and $-\Pi_i$. This remark, which will not be used further in this treatise, possesses a fundamental importance: For me, it was the starting point of some recent investigations of transformation groups.

If one substitutes $x_i + \varepsilon M_i$ and $p_i + \varepsilon \Pi_i$, in place of X_i and P_i, resp., in the relations:

$$(X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0, \qquad (X_i, P_i) = 1$$

then one finds by developing them and dropping quantities that are infinitesimal of second order that:

$$\frac{\partial M_i}{\partial p_k} = \frac{\partial M_k}{\partial p_i}, \qquad \frac{\partial M_i}{\partial x_k} = -\frac{\partial \Pi_k}{\partial p_i}, \qquad \frac{\partial \Pi_i}{\partial x_k} = \frac{\partial \Pi_k}{\partial x_i},$$

where i and k may assume all possible values, and, in particular, the same value. These equations show that there is a function Φ of $x_1, \dots, p_n$ for which:

$$M_i = \frac{\partial \Phi}{\partial p_i}, \qquad \Pi_i = -\frac{\partial \Phi}{\partial x_i}.$$

Here, Φ is subject to only the restriction that $\partial\Phi/\partial p_i$ and $\partial\Phi/\partial x_i$ should be homogeneous of degree zero and one, resp. One must then have:

$$\sum_{k=1}^{k=n} p_k \frac{\partial}{\partial p_k}\left(\frac{\partial \Phi}{\partial p_i}\right) = 0, \qquad \sum_{k=1}^{k=n} p_k \frac{\partial}{\partial p_k}\left(\frac{\partial \Phi}{\partial x_i}\right) = \frac{\partial \Phi}{\partial x_i},$$

from which:

$$\frac{\partial}{\partial p_i}\sum_k\left(p_k \frac{\partial \Phi}{\partial p_k}\right) = \frac{\partial \Phi}{\partial p_i}, \qquad \frac{\partial}{\partial x_i}\sum_k\left(p_k \frac{\partial \Phi}{\partial p_k}\right) = \frac{\partial \Phi}{\partial x_i},$$

and by integrating and omitting some inessential constants, it follows that:

$$\sum_k p_k \frac{\partial \Phi}{\partial p_k} = \Phi,$$

i.e., Φ itself must be a homogeneous function of degree one. It is also clear that $\partial\Phi/\partial p_i$ and $\partial\Phi/\partial x_i$ should be homogeneous of degree zero and first, resp., when Φ is homogeneous of degree one.

For the sake of brevity, if we now set δx_i and δp_i, instead of $x_i' - x_i$ and $p_i' - p_i$, resp., and denote any auxiliary variable by t then we can summarize the aforementioned as follows:

Theorem V. *Any infinitesimal homogeneous contact transformation possesses the form:*

$$\frac{\delta x_i}{\frac{\partial H}{\partial p_i}} = \frac{\delta p_i}{-\frac{\partial H}{\partial x_i}} = \delta t;$$

here, H denotes any homogeneous function of degree one [*]).

---

[*]) From this theorem it follows, *inter alia*, as one easily recognizes, that the determination of all infinitesimal contact transformation that take an equation:

$$f(z, x_1, \dots, x_n, p_1, \dots, p_n) = \text{const.}$$

to itself overlaps with the integration of this equation. For equations of higher order these two problems are, in general, different, and for that reason, they both have their independent justification.

§ 7.

On an improvement of the Jacobi-Mayer method of integration.

The Jacobi method of integration, as well as the Jacobi-Weiler and the Jacobi-Mayer methods, rest upon the fact that when n functions $F_1, .., F_n$ of $x_1, \ldots, x_n, p_1, \ldots, p_n$ pairwise give:

$$(F_i, F_k) = 0,$$

and it is therefore possible to solve the equations:

$$F_1 = a_1, .., F_n = a_n$$

for the differential quotients, it is possible to integrate each of these partial differential equations. This requirement – viz., that our equations must be soluble for the p – implies, as is known, certain difficulties that Jacobi has indeed reduced, but still not completely. One must therefore consider it to be an essential improvement of the methods in question that one can drop the stated requirement completely, as shall now be shown. I first consider equations in which the unknown function enters explicitly, and then ones in which this is not the case.

17. I base the discussion on the Clebsch theory of the Pfaff problem. Let $\sum_{k=1}^{k=2n} X_k dx_k$ be a given Pfaff expression that can be brought into the n-term form:

$$\sum_{k=1}^{k=2n} X_k dx_k = \sum_{k=1}^{k=n} F_k df_k \,.$$

From Clebsch, the quantity f will be determined from the simultaneous system:

$$((f_i)) = 0, \qquad ((f_i, f_k)) = 0.$$

If n functions of f are found that satisfy them then it is possible to exhibit all $2n - 1$ solutions of the equation:

$$((f)) = 0$$

by means of executable operations; i.e., to integrate this equation. This known theorem shall now be utilized.

Let φ be a function of $z, x_1, \ldots, x_n, p_1, \ldots, p_{n-1}$ and let:

$$dz - p_1\, dx_1 - \ldots - p_{n-1}\, dx_{n-1} - \varphi\, dx_n$$

be the Pfaff expression, which can be reduced to an n-term form $K_1\, dH_1 + \ldots + K_n\, dH_n$. The simultaneous system that was given above then assumes the form:

$$[p_n - \varphi, H_i] = 0, \qquad [H_i, H_k] = 0,$$

and, as a result, we obtain the following theorem:

Theorem 15. *If* $\varphi, H_1, \dots, H_n$ *are given functions of* $z, x_1, \dots, x_n, p_1, \dots, p_{n-1}$ *that pair-wise satisfy the equations:*

$$[p_n - \varphi, H_i] = 0, \qquad [H_i, H_k] = 0$$

then it is always possible to exhibit all $2n - 1$ solutions H *of the equation* $[p_n - \varphi, H] = 0$.

If one considers that the integration of the equation:

$$p_n - \varphi = 0$$

by the Cauchy method comes down to the determination of all solutions H to the equation $[p_n - \varphi, H] = 0$ then one can state the following theorem:

Theorem 16. *The integration of the partial differential equation:*

$$p_n - \varphi(z, x_1, \dots, x_n, p_1, \dots, p_{n-1}) = 0$$

can always be achieved when one has found n mutually independent functions $H_1, \dots, H_n$ *of* $z, x_1, \dots, x_n, p_1, \dots, p_{n-1}$ *that satisfy all equations:*

$$[p_n - \varphi, H_i] = 0, \qquad [H_i, H_k] = 0.$$

Therefore, it is entirely irrelevant whether one eliminates all p from the equations:

$$H_1 = a_1, \dots, H_n = a_n$$

or not. It is, in turn, conceivable that some of these functions H_i do not include these differential quotients at all.

This theorem may also be reproduced in the following way:

Theorem 17. *If* $H_0, H_1, \dots, H_n$ *are given functions of* $z, x_1, \dots, x_n, p_1, \dots, p_n$ *that pair-wise satisfy the equations* $[H_i, H_\lambda]$ *then each of the equations* $H_i = a_i$ *can be integrated.*

Thus, we can formulate the Jacobi-Mayer integration method in the following way:

Should the equation:

$$H_0(z, x_1, \dots, x_n, p_1, \dots, p_n) = a_0$$

have been integrated, then one first looks for a solution H_1 of:

$$[H_0, H] = 0$$

that is different from H_0. This requires a $2n - 1$ operation [*]). One then seeks a solution H_2 of the complete system:

$$[H_0, H] = 0, \quad [H_1, H] = 0$$

that is different from H_0 and H_1 . By means of Mayer's theorem, this happens by means of a $2n - 3$ operation. By means of a $2n - 5$ operation, one then finds a solution of the complete system:

$$[H_0, H] = 0, \quad [H_1, H] = 0, \quad [H_2, H] = 0$$

that is different from H_0, H_1, and H_2 , etc. By means of a 1 operation, one ultimately finds a solution H_n of the complete system:

$$[H_0, H] = 0, \quad [H_1, H] = 0, \ldots, [H_{n-1}, H] = 0$$

that is different from $H_0, H_1, \ldots, H_{n-1}$. Thus, from the aforementioned developments, the integration process can be considered to be concluded.

The foregoing theorem likewise includes the complete solution of the important problem:

From the complete solution of a given partial differential equation of first order:

$$H_0(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = a_0$$

find the complete solution of any other partial differential equation:

$$H_0'(z', x_1', \cdots, x_n', p_1', \cdots, p_n') = a_0$$

that arises from the given one by means of any contact transformation,

a problem that Jacobi [**]) was already involved with and was first rigorously solved by Mayer [***]), if only for special types of contact transformations.

Namely, if:

$$z = Z(x_1, \ldots, x_n, a_1, \ldots, a_n)$$

is a complete solution of the given equation $H_0 = a_0$ then one must have that the $n + 1$ equations:

$$z = Z, \; p_1 = \frac{\partial Z}{\partial x_1}, \ldots, p_n = \frac{\partial Z}{\partial x_n}$$

allow us to determine the $n + 1$ constants $a_0, a_1, \ldots, a_n$, and the value of a_0 that is obtained from them must be a given function H_0 . If one further lets:

[*]) By the term "an m operation," I understand this to mean the discovery of an integral of a system of m ordinary differential equations.

[**]) Nova Methodus, § 61 and Vorles. über Dynamik, pp. 469.

[***]) Göttinger Nachrichten, 1872, no. 21.

$$a_1 = H_1,\ \ldots,\ a_n = H_n$$

denote the value of the n remaining constants then H_0, H_1, …, H_n are mutually independent functions that have the mutual relationship:

$$[H_i, H_k] = 0.$$

Now, if H_1, …, H_n go to H_1', …, H_n' by the application of any contact transformation that transforms H_0 into H_0' then H_0', H_1', …, H_n' are also mutually independent functions that, from Theorem 8, pair-wise satisfy the equations:

$$[H_i', H_k'] = 0.$$

From Theorem 17, one can thus obtain a complete solution to the transformed equation $H_0' = a_0$ by just algebraic operations.

Since the complete solution to the partial differential equation $H_0 = a_0$ requires no further sort of integrations, as long as one has found n functions H_1, …, H_n that are independent of each other, as well as H_0, and satisfy the conditions:

$$[H_i, H_k] = 0,$$

one is then close to the concept of extending the complete solution that one immediately calls the n equations that are defined by these functions:

$$H_1 = a_1,\ \ldots,\ H_n = a_n$$

a *complete solution* of the given equation $H_0 = a_0$.

By establishing this extended definition of the complete system, one can immediately say:

A contact transformation that takes the given partial differential equation:

$$H_0(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = a_0$$

to the equation:

$$H_0'(z', x_1', \cdots, x_n', p_1', \cdots, p_n') = a_0$$

also simultaneously takes any complete solution of the former equation to a complete solution of the latter one.

18. In order to be able to extend this theory to partial differential equations that do not include the unknown function itself, we present some lemmas.

Theorem 18. *Let V be a function of x_1, …, x_m, y_1, …, y_q that is defined by q linear partial differential equations:*

$$\sum_{k=1}^{k=m} X_{ik} \frac{\partial V}{\partial x_k} + \sum_{k=1}^{k=q} Y_{ik} \frac{\partial V}{\partial y_k} = W_i(x_1, \ldots, x_m, y_1, \ldots, y_q) \qquad (i = 1, \ldots, q).$$

If these equations possess a common solution of the form:

$$V = F + \Phi(x_1, \ldots, x_m)$$

and if Φ *denotes an arbitrary function then all* X_{ik} *are equal to zero.*

Then, by assumption, the given equations shall be satisfied simultaneously by $V = F$ and by $V = F + \Phi$. Therefore, one must have:

$$\sum_{k=1}^{k=m} X_{ik} \frac{\partial \Phi}{\partial x_k} = 0.$$

However, this equation, as the assumption $\Phi = x_k$ shows immediately, can be true for an arbitrary function Φ only when each $X_{ik} = 0$.

Theorem 19. *Let V be a function of* $x_1, \ldots, x_n$ *that is defined by q linear partial differential equations:*

$$\sum_{k=1}^{k=n} X_{ik} \frac{\partial F}{\partial x_k} = W_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, q).$$

If these equations possess a common solution of the form:

$$V = F + \Phi(\xi_1, \ldots, \xi_n),$$

and Φ *refers to an arbitrary function of the quantities* ξ*, which shall be known, then the determination of V requires only one quadrature.*

Namely, if one chooses q functions $y_1, \ldots, y_q$ of $x_1, \ldots, x_n$ such that no relation exists between $x_1, \ldots, x_{n-q}, y_1, \ldots, y_q$, which is always possible, and then introduces these quantities as independent variables in our partial differential equations then, from the foregoing theorem, these equations take on the form:

$$\frac{\partial V}{\partial y_i} = \Omega(\xi_1, \ldots, \xi_{n-q}, y_1, \ldots, y_q), \qquad (i = 1, \ldots, q);$$

One then finds V by quadrature.

Theorem 20. *If* $X_1, \ldots, X_n$ *are given functions of* $x_1, \ldots, x_n, p_1, \ldots, p_n$ *that pair-wise yield* $(X_i, X_k) = 0$ *then it is always possible to find a function F of* $z, x_1, \ldots, p_n$ *that fulfills all n equations* $[X_i, F] = 0$ *by mere quadrature.*

Previously (Theorem 6), we saw that, in fact, the equations:

$$[X_1, Az + \Pi] = 0, \ldots, [X_n, Az + \Pi] = 0,$$

in which Π denotes an unknown function of $x_1, \ldots, p_n$, possess a common solution of the form:

$$\Pi + \Phi(X_1, \ldots, X_n);$$

here, Φ is an arbitrary function of $X_1, \ldots, X_n$, and thus, from the foregoing theorem, the determination of Π is achieved by only one quadrature.

Moreover, we can also formulate the Jacobi-Mayer method for the case in which the equation in question does not include the unknown function z.

Should the equation:

$$X_1(x_1, \ldots, x_n, p_1, \ldots, p_n) = a_1$$

be integrated, then one would first seek a solution X_2 of the equation:

$$(X_1, X) = 0$$

that is different from X_1 by means of a $2n - 2$ operation, and then, by means of a $2n - 4$ operation, a solution X_3 of the complete system:

$$(X_1, X) = 0, \qquad (X_2, X) = 0,$$

that is different from X_1 and X_2, etc. Ultimately, one would find a solution X_n of the complete system:

$$(X_1, X) = 0, \ldots, (X_{n-1}, X) = 0$$

by means of a 2 operation. If this has happened then one would determine a function $Az + \Pi(x_1, \ldots, x_n, p_1, \ldots, p_n)$ that satisfies all of the equations:

$$[X_1, Az + \Pi] = 0, \ldots, [X_n, Az + \Pi] = 0.$$

The integration process of the foregoing number is thus concluded.

§ 8.

Response to a remark of Mayer.

19. Since 1872, Mayer and I entered into a lively state of communication that was inspiring to me in several directions. In particular, it was at Mayer's suggestion in 1873 that I sought to find an algebraic representation of the foregoing theory, which I had, for the most part, found by considering manifolds. Thus, I was prepared to find my analytic form incomplete. In fact, Mayer immediately made me aware of some inaccuracies that I had perpetrated in that treatise. At the same time, he made the essential objection *) that I had employed the Clebsch theory of the Pfaff problem:

$$\sum_{k=1}^{k=2n} X_k dx_k = F_1\, df_1 + \ldots + F_n\, df_n$$

to a greater extent that Clebsch had proved. Namely, it was only under the assumption that the determinant R that is constructed from the elements:

$$a_{ik} = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i}$$

does not vanish that Clebsch had proved that the f are an arbitrary system of solutions to the simultaneous equations:

$$((f_i)) = 0, \qquad ((f_i, f_k)) = 0.$$

My response to him is that these equations, when multiplied by the determinant itself:

$$R\,((f_i)) = 0, \qquad R\,((f_i, f_k)) = 0,$$

define the quantities f under all circumstances.

Namely, let:

$$\sum_{k=1}^{k=2n} X_k dx_k = F_1\, df_1 + \ldots + F_n\, df_n$$

and

$$\sum_{k=1}^{k=2n} Y_k dy_k = \Phi_1\, d\varphi_1 + \ldots + \Phi_n\, d\varphi_n$$

be two Pfaff expressions in the variables x and y, resp., whose canonical forms include n terms. Each of the sequences:

$$f, \ldots, f_n, \frac{F_1}{F_n}, \ldots, \frac{F_{n-1}}{F_n},$$

and

$$\varphi, \ldots, \varphi_n, \frac{\Phi_1}{\Phi_n}, \ldots, \frac{\Phi_{n-1}}{\Phi_n}$$

consist of functions, between which no functional relation exists. Thus, one can choose two functions $F(x_1, \ldots, x_{2n})$ and $\Phi(y_1, \ldots, y_{2n})$ such that the $2n$ equations:

*) Cf., his note in the Göttinger Nachr., 1874, no. 13: "Ueber die Lie'schen Berührungstransformationen."

$$f_i = \varphi_i, \qquad \frac{F_k}{F_n} = \frac{\Phi_k}{\Phi_n}, \qquad F = \Phi$$

determine a transformation between the two systems of variables x and y. Such a transformation, however, takes the one Pfaff expression to the other one, but multiplied by a certain quantity. Thus:

If:

$$\sum_{k=1}^{k=2n} X_k dx_k \qquad \textit{and} \qquad \sum_{k=1}^{k=2n} Y_k dy_k$$

are two Pfaff expressions whose canonical forms include n terms then the one expression can take on those properties of the other one that remain unperturbed under a change of variables, when multiplied by a suitable quantity.

20. This important remark (which may be extended immediately to arbitrary Pfaff problems) also addresses the stated difficulty quite easily.

The vanishing or non-vanishing of the determinant R is, in fact, a property that remains undisturbed when new variables are introduced. Indeed, for Clebsch, the fact that R is equal to zero meant that an equation of the form:

$$\sum_{k=1}^{k=2n} X_k dx_k = d\pi_1 + \Pi_2\, d\pi_2 + \ldots + \Pi_n\, d\pi_n$$

is possible. As a result, the vanishing or non-vanishing of the determinant can be arranged by multiplication by a suitable quantity.

Now, let $\sum X_k\, dx_k$ be an expression whose determinant is non-vanishing. We choose a quantity ρ such that the determinant of the expression $\sum \rho\, X_k\, dx_k$ is non-vanishing. Now, if:

(a) $((f_i)) = 0, \qquad ((f_i, f_k)) = 0$

and

(b) $((f_i))_\rho = 0, \qquad ((f_i, f_k))_\rho = 0$

are two simultaneous system that correspond to these two expressions then it is clear that they can differ from each other only by a factor. The quantities $f_1, \ldots, f_n$, which must satisfy the one system must, from the nature of things, also satisfy the second one. Therefore, our equations (a) and (b) can take on a common form that remains valid when the determinant vanishes. For Clebsch, our equations had the form:

$$\frac{1}{R}\sum_i \sum_k X_i \frac{\partial f}{\partial x_k} R_{ik} = 0, \qquad \frac{1}{R}\sum_i \sum_k \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_k} R_{ik} = 0,$$

which become illusory when $R = 0$. By contrast, the equivalent equations:

$$\sum_i \sum_k X_i \frac{\partial f}{\partial x_k} R_{ik} = 0, \qquad \sum_i \sum_k \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_k} R_{ik} = 0,$$

are never illusory, because, in fact, the sub-determinants R_{ik} may not all vanish. These equations are the ones that that define f under all situations, and in this form I have also employed the Clebsch equations in the foregoing.

It should only be added here that the aforementioned multiplier should be regarded as an *integrating factor.*

PART TWO

Theory of groups.

In this section, I will consider a sequence of functions $F_1, \ldots, F_r$ of $x_1, \ldots, x_n, p_1, \ldots, p_n$, and determine all of the relations that exist between them that remain unperturbed under arbitrary contact transformations between x, p:

$$x_i' = X_i \,, \qquad p_i' = P_i \,.$$

In order to be able to give the results the simplest possible form, I will assume that the constants (X_i, P_i) are equal to 1. This is, however, only a formal restriction. In connection with the results obtained, I will develop a rational method that teaches one how to exploit the circumstances that occur during the integration of a first-order, partial differential equation as best as possible.

§ 9.

Group. System in involution. Statement of two problems.

21. The theory that follows has its origin in the explicit introduction of two concepts, the first of which essentially goes back to Jacobi.

Definition. I say that r mutually independent functions $u_1, \ldots, u_r$ of $x_1, \ldots, x_n, p_1, \ldots, p_n$ define a *one-parameter group* when any $(u_i\,, u_k)$ can be expressed as a function of the u. I say that any function of the quantities u *belongs* to the group.

If the functions $u_1, u_2, \ldots, u_\rho$ of an ρ-parameter group belong to a group with more terms $u_1, \ldots, u_\rho, u_{\rho+1}, \ldots, u_r$ then I say that the latter group contains the former one, or that the former is a *subgroup* of the latter.

Theorem 21. If q relations exist between $u_1, \ldots, u_r$ and therefore any $(u_i\,, u_k)$ is expressible in terms of the u such that one has:

$$(u_i\,, u_k) = f_{ik}(u_1, \ldots, u_r)$$

then there is an $(r - q)$-parameter group that all u belong to.

Then, from our assumption, it is possible to find quantities among the $r - q$ quantities u – say, $u_1, \ldots, u_{r-q}$ – that can be expressed in terms of the remaining ones. If one substitutes the values of $u_{r-q+1}, \ldots, u_r$ thus found into:

$$(u_i\,, u_k) = f_{ik}(u_1, \ldots, u_r)$$

then this expression assumes the form:

$$(u_i\,,\,u_k) = \varphi_{ik}(u_1,\,\ldots,\,u_{r-q})$$

and as a result the $u_1, \ldots, u_{r-q}$ define a group that $u_{r-q+1}, \ldots, u_r$ also belong to.

Theorem 22. If $v_1, \ldots, v_r$ belong to the group $u_1, \ldots, u_r$, such that one has:

$$v_i = V_i(u_1,\,\ldots,\,u_r),$$

and if $V_1, \ldots, V_r$ define mutually independent functions of the u then $v_1, \ldots, v_r$ also define an r-parameter group, which regards as *another form* of the given one.

Then, by our assumption, the $v_1, \ldots, v_r$ can also be regarded as mutually independent functions of the $x_1, \ldots, p_n$. Furthermore, one has:

$$(v_i\,,\,v_k) = \sum_{m=1}^{r}\sum_{n=1}^{r}\frac{\partial V_i}{\partial u_m}\frac{\partial V_k}{\partial u_n}(u_m\,,\,u_n),$$

From which, it follows that $(v_i\,,\,v_k)$ is a function of the quantities u, and thus it is likewise a function of the quantities v.

Definition. If $u_1, \ldots, u_r$ define a group and all $(u_i\,,\,u_k)$ vanish then the group shall be called an r-parameter *system in involution.*

I call two groups $u_1, \ldots, u_r$ and $w_1, \ldots, w_r$ *involutory* groups when any $(u_i\,,\,w_k) = 0$.
In Jacobi's theory, systems in involution $u_1, \ldots, u_r$ that are subject to the bothersome restriction that the equations:

$$u_1 = a_1\,,\,\ldots,\,u_r = a_r$$

should be solved for r of the quantities p play a fundamental role. The introduction of the general concept of system in involution belongs to me.

Theorem 23. A contact transformation between $x_1, \ldots, p_n,\ x_1', \ldots,\ p_n'$:

$$x_i' = X_i, \qquad p_n' = P_i \qquad (X_i, P_i) = 1$$

takes the functions of an r-parameter group $u_1', \ldots, u_r'$ to the functions of a new r-parameter group $u_1, \ldots, u_r$. Thus, any $(u_i\,,\,u_k)$ can be expressed in terms of $u_1, \ldots, u_r$ in the same way as the corresponding (u_i', u_k') in terms $u_1', \ldots, u_r'$.

Namely, we have seen (Theorem 11) that:

$$(u_i',\,u_k')_{x'p'} = (u_i\,,\,u_k)_{xp}\,.$$

Now, we assume that:

$$(u_i',\,u_k')_{x'p'} = f_{ik}\,(u_1',\,\ldots,\,u_r'),$$

so we find that:

$$(u_i, u_k)_{xp} = f_{ik}(u'_1, \ldots, u'_r),$$

or, when we recall that $u'_1, \ldots, u'_r$ are regarded as functions of $u_1, \ldots, p_n$ that are denoted by $u_1, \ldots, u_r$:

$$(u_i, u_k) = f_{ik}(u_1, \ldots, u_r),$$

with which, our theorem is proved.

Corollary. A contact transformation takes a system in involution to another system in involution.

22. I can now formulate the two main problems of this section.

Problem 1. Let one be given two r-parameter groups $v'_1, \ldots, v'_r$ and $v_1, \ldots, v_r$. One must decide whether there is a contact transformation that transforms each v'_i into a function of $v_1, \ldots, v_r$, or, as I will say, for the sake of brevity, that transforms the one group into the other one.

We will see that any r-parameter group can be characterized by a certain positive whole number that is less than r. Should an r-parameter group be capable of being transformed into another one, then it would be necessary and sufficient that this number would be the same for both groups. This important theorem can also be expressed as follows: An r-term group possesses only one property that is independent of its form and remains invariant under contact transformations. This property can be expressed in terms of a positive whole number that is less than r.

Problem II. Let one be given two systems of any r functions:

$$F'_1, \ldots, F'_r \qquad \text{and} \qquad F_1, \ldots, F_r$$

of $x'_1, \ldots, p'_n$ and $x_1, \ldots, p_n$, resp. One must decide whether there is a contact transformation:

$$x'_i = X_i, \qquad p'_i = P_i \qquad (X_i\, P_i) = 1$$

that transforms any F'_k into the corresponding F_k.

The solution to this problem, which we will give in § 16, is also very simple.

§ 10.

Reciprocal groups.

The analytical starting point *) for my investigations into groups was the following theorem:

Theorem 24. If $u_1, \ldots, u_r$ is a group and V is an unknown function of $x_1, \ldots, x_n, p_1, \ldots, p_n$, then the r linear equations:

$$(u_1, V) = 0, \quad \ldots, \quad (u_r, V) = 0$$

define a complete system.

Proof. It is clear, to begin with, that these equation are mutually independent, since otherwise a sequence of functional determinants would vanish, and as a result, there would exist relations between $u_1, \ldots, u_r$, which are regarded as functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$. However, this contradicts our assumptions.

If we now write $A_i(V)$, instead of (u_i, V), then we find by carrying out the calculations that **):

$$A_i(A_k(V)) - A_k(A_i(V)) = ((u_i, u_k), V).$$

However, one has (no. 21):

$$(u_i, u_k) = f_{ik}(u_1, \ldots, u_r),$$

so one gets:

$$((u_i, u_k), V) = \frac{\partial f_{ik}}{\partial u_1}(u_1, V) + \ldots + \frac{\partial f_{ik}}{\partial u_r}(u_r, V),$$

i.e.:

$$A_i(A_k(V)) - A_k(A_i(V)) = \frac{\partial f_{ik}}{\partial u_1}(A_1, V) + \ldots + \frac{\partial f_{ik}}{\partial u_r}(A_r, V),$$

with which, our theorem is proved.

The complete system:

$$(u_1, V) = 0, \ldots, (u_r, V) = 0,$$

has $2n - r$ solutions $v_1, v_2, \ldots, v_{2n-r}$, between which no functional relation exists, and any other solution can be represented as a function of these quantities. Now, the Poisson-Jacobi theorem says that any (v_i, v_k) is such a common solution. As a result, (v_i, v_k) is a function of the v:

*) It was by synthetic speculations about the Poisson-Jacobi theorem and the intrinsic essence of things that led me to this theorem. I remark that it is the manifolds that are generated by characteristic strips of two or more equations that are to be examined.

**) The fact that the two equations $(u_1, V) = 0$, $(u_2, V) = 0$ imply that $((u_1, u_2), V) =$ is a well-known proof of the Poisson-Jacobi theorem.

$$(v_i , v_k) = \varphi_{ik} (v_1 , v_2 , \ldots, v_{2n-r}),$$

i.e., $v_1 , v_2 , \ldots, v_{2n-r}$ define a new group.

Therefore, the equations:

$$(v_1 , U) = 0, \ldots, (v_{2n-r} , U) = 0,$$

define a complete system with $2n - (2n - r) = r$ solutions. Obviously, $u_1, \ldots, u_r$ satisfy this system, whose solutions therefore all belong to the original group. Thus:

Theorem VI. *Any group $u_1, \ldots, u_r$ determines a second group with $2n - r$ terms that has a completely reciprocal relationship to the first one. Any group consists of all functions that are in involution with the functions of the second group. Two such groups shall be called reciprocal groups. I also frequently call the one group the* polar group *of the other one* *).

If $u_1, \ldots, u_r$ and $v_1, \ldots, v_{2n-r}$ are two reciprocal groups that are taken to $u_1', \ldots, u_r'$ and $v_1', \ldots, v_{2n-r}'$, resp., by a contact transformation then these two new groups are also reciprocal groups. Then, since each (u_i , u_k) vanishes, this is also the case (Theorem 8) for any expression (u_i', v_k').

§ 11.

The distinguished functions of a group.

24. A new fundamental concept will be introduced in this paragraph.

Definition. *Functions U that belong to a group $u_1, \ldots, u_r$ and satisfy all the relations:*

$$(u_1 , U) = \ldots = (u_r , U) = 0$$

shall be called distinguished *functions.*

It is clear that the number of mutually independent distinguished functions of a group is independent of the form of the group. It is also clear that a group with m distinguished functions will go to a a group with m distinguished function under any contact transformation.

Theorem 25. If m relations exist between the functions of two reciprocal groups then there are m functions that simultaneously belong to both groups.

Proof. I assume that $u_1, \ldots, u_r$ and $v_1, \ldots, v_{2n-r}$ are two reciprocal groups, between whose functions m relations exist. We now recall that:

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*) A general theory of reciprocity is based upon this theorem. Any theorem about groups corresponds to a reciprocal theorem. On the other hand, the two groups are possibly reciprocally inter-related by pair-wise relations. These suggestions shall not be developed further, here.

$$(u_i\,,\,u_k) = f_{ik}\,(u_1,\,\ldots,\,u_r), \qquad (v_i\,,\,v_k) = \varphi_{ik}(v_1,\,\ldots,\,v_{2n-r}), \qquad (u_i\,,\,v_k) = 0,$$

and further consider Theorem 21, then this yields that $u_1, \ldots, u_r, v_1, \ldots, v_{2n-r}$ belongs to a certain $(2n - m)$-parameter group:

$$W_1, \ldots, W_{2n-m}$$

that can assume the form:

$$u_1, \ldots, u_r, v_1, \ldots, v_{2n-r-m}\,,$$

as well as the form:

$$v_1, \ldots, v_{2n-r}\,,\, u_1, \ldots, u_{r-m}\,.$$

From this, it follows that the m solutions $F_1, \ldots, F_m$ of the complete system:

$$(W_1, F) = 0, \quad \ldots, \quad (W_{2n-m}\,, F) = 0$$

satisfy, on the one hand, the equations:

$$(u_1\,, F) = 0, \quad \ldots, \quad (u_r\,, F) = 0,$$

and thus belong to the group $v_1, \ldots, v_{2n-r}$, and, on the other hand, fulfill the equations:

$$(v_1\; F) = 0, \quad \ldots, \quad (v_{2n-r}\; F) = 0,$$

and thus likewise belong to the group $u_1, \ldots, u_r$. There are then actually m functions that belong to both groups.

Theorem 26. If a function F simultaneously belongs to two reciprocal groups then it is a distinguished function in both groups.

As an element of the group $v_1, \ldots, v_{2n-r}$, F then satisfies the equations:

$$(u_1, F) = 0, \quad \ldots, (u_r, F) = 0.$$

Now, F is a function of the quantities u, and any such function that fulfills the equations that we just presented is a distinguished function of the group $u_1, \ldots, u_r$. In a corresponding way, one sees that F is a distinguished function of the group v.

Theorem 27. Any distinguished function of a group belongs to the reciprocal group.

Then, if U is a distinguished function of the group $u_1, \ldots, u_r$ then the relations:

$$(u_1, U) = 0, \quad \ldots, \quad (u_r, U) = 0$$

are true; However, these are just the equations that must be true if U is to belong to the reciprocal group.

Theorem 28. Any distinguished function of a group is a distinguished function in the reciprocal group.

This theorem follows as a corollary of the previous two.

Theorem 29. If a group $u_1, \ldots, u_r$ contains m functions $U_1, \ldots, U_r$ then m relations exist between the functions of this group and those of the reciprocal group $v_1, \ldots, v_{2n-r}$.

$U_1, \ldots, U_r$ then belong to the two groups. If one then expresses them, in one case, as functions of u and in the other case, functions of v, and sets these expressions pair-wise equal to each other then one finds the relations that we spoke of:

$$F_1 (u_1, \ldots, u_r) = \Phi_1 (v_1, \ldots, v_{2n-r}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$F_m (u_1, \ldots, u_r) = \Phi_m (v_1, \ldots, v_{2n-r}).$$

I summarize the results of this paragraph in the following way:

Theorem VII. *Two reciprocal groups contain the same distinguished functions, and exactly as many relations exist between the functions of two reciprocal groups as the number of distinguished functions that the groups contain. These relations always have the form:*

$$F_i(u_1, \ldots, u_r) = \Phi_i(v_1, \ldots, v_{2n-r}) \qquad (i = 1, \ldots, m).$$

They simply express the idea that the m distinguished functions belong to the one group, as well as the other.

25. I will show that the number of distinguished functions can be determined in such a way that one constructs a certain number of determinants and then examines whether they and their sub-determinants of 1st, 2nd, … order vanish. If our group contains m distinguished functions then the determination requires an m, $m - 1$, …, 3, 2, 1 operation, respectively.

Let $u_1, \ldots, u_r$ be a group and let U be function of $u_1, \ldots, u_r$. Should it be a distinguished function then it would be necessary and sufficient that the equations:

$$(u_1, U) = 0, \qquad \ldots, (u_r , U) = 0,$$

exist, or, when developed:

$$A_1(U) = (u_1, u_1)\frac{\partial U}{\partial u_1} + (u_1, u_2)\frac{\partial U}{\partial u_2} + \ldots + (u_1, u_r)\frac{\partial U}{\partial u_r} = 0,$$

$$A_2(U) = (u_2, u_1)\frac{\partial U}{\partial u_1} + (u_2, u_2)\frac{\partial U}{\partial u_2} + \ldots + (u_2, u_r)\frac{\partial U}{\partial u_r} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$A_r(U) = (u_r, u_1)\frac{\partial U}{\partial u_1} + (u_r, u_2)\frac{\partial U}{\partial u_2} + \ldots + (u_r, u_r)\frac{\partial U}{\partial u_r} = 0.$$

If one sets (u_i, u_k) everywhere here, instead of the corresponding function $f_{ik}(u_1, \ldots, u_r)$, then one obtains r linear, partial differential equations with r independent variables for the determination of U. Therefore, should the group contain m distinguished functions then our r equations would have to be capable of being replaced with $r - m$ of them that define a complete system – say:

$$A_1(U) = 0, \ldots, A_{r-m}(U) = 0.$$

In order for this to be true, it is obviously requisite that $A_{r-m+1}(U), \ldots, A_r(U)$ could be expressed linearly in terms of $A_1(U), \ldots, A_{r-m}(U)$. Conversely, it is clear that our $r - m$ equations define a complete system when this requirement is fulfilled. The expression $A_i(A_k(U)) - A_k(A_i(U))$ is then expressed linearly in terms of $A_1(U), \ldots, A_r(U)$ as:

$$A_i(A_k(U)) - A_k(A_i(U)) = \lambda_1 A_1(U) + \ldots + \lambda_r A_r(U).$$

However, if one replaces $A_{r-m+1}(U), \ldots, A_r(U)$ with their expressions in terms of $A_1(U), \ldots, A_{r-m}(U)$ in this then one obtains relations of the form:

$$A_i(A_k(U)) - A_k(A_i(U)) = \rho_1 A_1(U) + \ldots + \rho_{r-m} A_{r-m}(U),$$

with which our assertion is proved.

This then shows that one must construct the determinant:

$$D = \begin{vmatrix} (u_1,u_1) & (u_1,u_2) & \cdots & (u_1,u_r) \\ (u_2,u_1) & (u_2,u_2) & \cdots & (u_2,u_r) \\ \cdots & \cdots & \cdots & \cdots \\ (u_r,u_1) & \cdots & \cdots & (u_r,u_r) \end{vmatrix}.$$

If this is non-zero then the expressions $A_1(U), \ldots, A_r(U)$ are independent of each other, and our group then has no distinguished functions. By contrast, if this determinant and its 1st, 2nd, …$(m - 1)$th-order sub-determinants vanish, while the mth-order sub-determinants do not vanish simultaneously, then among the $A_i(U)$ there are m of them that can be expressed in terms of the remaining ones, and thus the group contains m distinguished functions.

One can remark that D is a skew determinant. Thus, if r denotes an arbitrary *odd* number then D is equal to zero, and in any case the group contains *one* distinguished function.

If one finds that our r-parameter group $u_1, \ldots, u_r$ contains m distinguished functions then, as is always possible, one takes $r - m$ of the expressions $A_k(U)$ – say, $A_1(U), \ldots, A_{r-m}(U)$ – that are mutually independent. The equations:

$$A_1(U) = 0, \ldots, A_{r-m}(U) = 0,$$

in turn, define a complete system, whose m solutions are precisely the distinguished functions of the group. As Mayer and I have remarked in our previous papers, their determination requires only:

$$m, m-1, \ldots, 3, 2, 1$$

operations, respectively. If one applies Mayer's theorem then one will very frequently be able to determine the distinguished functions through even simpler operations.

It is reasonable that when m distinguished functions are already known, the determination of the remaining ones, in turn, requires only $m - \mu$, $m - \mu - 1, \ldots, 3, 2, 1$ operations.

Theorem VII. *Should one wish to decide how many distinguished functions a group* $u_1, \ldots, u_r$ *contains, one would have to construct the determinant with r rows and columns whose elements are the quantities* (u_i, u_k), *when expressed as functions of* $u_1, \ldots, u_r$. *If these determinants should vanish, along with their sub-determinants of 1st, 2nd, …, up to* $(m-1)^{\text{th}}$*-order, then the group would have m distinguished functions. One finds them when one chooses r – m of the r expressions:*

$$A_i(U) = (u_i, u_1)\frac{\partial U}{\partial u_1} + (u_i, u_2)\frac{\partial U}{\partial u_2} + \ldots + (u_i, u_r)\frac{\partial U}{\partial u_r},$$

say, $A_1(U), \ldots, A_{r-m}(U)$, *that are mutually independent. The equations:*

$$A_1(U) = 0, \ldots, A_{r-m}(U) = 0,$$

in turn, define a complete system whose m solutions are precisely the distinguished functions of the group. One thus finds them by means of m, m – 1, …, 3, 2, 1 *operations.*

§ 12.

Canonical form of a group.

In these paragraphs, we will first prove some lemmas and show that any group can be brought into a remarkable form that I call its *canonical* form.

26. Theorem 30. If an r-parameter group contains more than $r - 2$ distinguished functions then it is a system in involution, and thus possesses r distinguished functions.

Proof. Assume that the group $u_1, \ldots, u_r$ possesses $r - 1$ distinguished functions $U_1, \ldots, U_{r-1}$. We will see that they must include yet another such function in a noteworthy way. We then bring the group into the equivalent form $U_1, \ldots, U_{r-1}, V$, so since U_1 is a distinguished function, one must have:

$$(U_1, V) = 0.$$

(U_2, V) likewise vanishes, because U_2 is a distinguished function. In this way, we recognize the existence of the relations:

$$(U_1, V) = 0, \quad \ldots, \quad (U_{r-1}, V) = 0,$$

which show that V is also a distinguished function. Our group thus actually possesses r distinguished functions.

Theorem 31. If u_1 is not a distinguished function of a group $u_1, \ldots, u_r$ then there are always functions $F(u_1, \ldots, u_r)$ that fulfill the equation $(u_1, F) = 1$.

Then, from our assumption, there are, in any case, some of the expressions (u_1, u_2), (u_1, u_3), …, (u_1, u_r) that do not vanish identically. Therefore, if F denotes an undetermined function of $u_1, \ldots, u_r$ then one has:

$$(u_1, u_2)\frac{\partial F}{\partial u_2} + (u_1, u_3)\frac{\partial F}{\partial u_3} + \ldots + (u_1, u_r)\frac{\partial F}{\partial u_r},$$

or, when one introduces the corresponding function $f_{1k}(u_1, \ldots, u)$, instead of $(u_1\ u_k)$:

$$f_{12}\frac{\partial F}{\partial u_2} + f_{13}\frac{\partial F}{\partial u_3} + \ldots + f_{1r}\frac{\partial F}{\partial u_r}$$

is non-zero. Thus:

$$f_{12}\frac{\partial F}{\partial u_2} + f_{13}\frac{\partial F}{\partial u_3} + \ldots + f_{1r}\frac{\partial F}{\partial u_r} = 1$$

is a linear, partial differential equation whose solutions F satisfy the conditions:

$$(u_1, F) = 1.$$

Theorem 32. If the group $u_1, \ldots, u_r$ contains a sub-group $u_1, \ldots, u_\rho$ then the polar group of the former is contained in the polar group of the latter.

The parameters of the polar group of $u_1, \ldots, u_r$ are then defined by:

$$(u_1, v) = 0, \ldots, (u_\rho, v) = 0, \ldots, (u_r, v) = 0,$$

and the parameters of the polar group of $u_1, \ldots, u_\rho$ satisfy the equations:

$$(u_1, V) = 0, \ldots, (u_\rho, V) = 0.$$

We see that the solutions of the former system also satisfy the latter system, while the converse is not true. The theorem is thus proved.

Theorem 33. If the expression (u_1, u_2) equals 1 then any group $u_1, u_2, \ldots, u_r$ can be brought into the form $u_1, u_2, u'_1, \ldots, u'_{r-2}$, where all (u_1, u'_k) and (u_2, u'_k) are equal to zero, while all (u'_i, u'_k) can be expressed as functions of $u'_1, \ldots, u'_{r-2}$.

Namely, if:

$$v_1, \ldots, v_{2n-r},$$

is the polar group of $u_1, \ldots, u_r$ then by our assumption:

$$u_1, u_2, v_1, \ldots, v_{2n-r}$$

is also a group whose $(r-2)$-parameter polar group:

$$u'_1, \ldots, u'_{r-2}$$

is contained in $u_1, \ldots, u_r$ (Theorem 32), and is in involution with the group u_1, u_2. It is clear that no relation can exist between $u_1, u_2, u'_1, \ldots, u'_{r-2}$. Such a parameter could then be brought into the form:

$$u_1 = \Psi(u_2, u'_1, \ldots, u'_{r-2})$$

and one would thus have:

$$(u_1, u_2) = (u'_1, u_2)\frac{\partial\Psi}{\partial u'_1} + \ldots + (u'_{r-2}, u_2)\frac{\partial\Psi}{\partial u'_{r-2}},$$

an equation in which the right-hand side would vanish, while the left-hand side would equal 1. However, this is absurd. Thus, $u_1, u_2, u'_1, \ldots, u'_{r-2}$ is one form of our group that possesses the desired property.

Theorem 34. Any r-parameter group that is not a system in involution can be decomposed into a two-parameter group and an $(r-2)$-parameter group that is in involution with it.

By our assumption, the given group $u_1, \ldots, u_r$ contains functions that are not in involution with all of the remaining functions of the group. We take one of them – say, u_1 – and determine (Theorem 21) a second function u_2 of the group that gives:

$$(u_1, u_2) = 1.$$

If we consider the previous theorem then we recognize the validity of our present theorem.

27. A general and exceptionally important reduction of any group to a *canonical* form flows from the foregoing theorems.

Theorem 35. Any group can take on the form $X_1, \ldots, X_r, P_1, \ldots, P_\mu$, where the expressions (X_i, X_k), (X_i, P_k), (P_i, P_k) are equal to zero and all (X_i, P_i) are equal to 1. This form is what I call a *canonical* form.

Namely, if our r-parameter group is a system in involution then it already has the canonical form, and indeed, $\nu = r$, $\mu = 0$.

By contrast, if $u_1, \ldots, u_r$ is not a system in involution then one decomposes it (Theorem 34) into a two-parameter group and an $(r - 2)$-parameter group:

$$(A) \qquad X_1, P_1, u'_1, \ldots, u'_{r-2}$$

that are both in involution. If the $(r - 2)$-parameter group is a system in involution then (A) is the canonical form for the original group, where $\nu = r - 1$, $\mu = 1$.

If $u'_1, \ldots, u'_{r-2}$ is not a system in involution then one decomposes this $(r - 2)$-parameter group into a two-parameter group and an $(r - 4)$-parameter group:

$$X_2, P_2, u''_1, \ldots, u''_{r-4}.$$

With that, the original group assumes the form:

$$X_1, P_1, X_2, P_2, u''_1, \ldots, u''_{r-4}$$

that is the desired canonical form if the $(r - 4)$-parameter group is a system in involution. One proceeds in this way, until one ultimately comes – say, after q decompositions – to an $(r - 2q)$-parameter group $u_1^{(q)}, \ldots, u_{r-2q}^{(q)}$ that is a system in involution. Consequently:

$$X_1, P_1, X_2, P_2, \ldots X_q, P_q, u_1^{(q)}, \ldots, u_{r-2q}^{(q)}$$

is the canonical form of the r-parameter group. Here, one has $\nu = r - q$, $\mu = q$.

Theorem 36. In a canonical group $X_1, \ldots, X_{q+m}, P_1, \ldots, P_q$, the $X_{q+1}, \ldots, X_{q+m}$ are the only distinguished functions.

Namely, if Π belongs to the given canonical group then one will have:

$$(X_i, \Pi) = \frac{\partial \Pi}{\partial P_i}, \qquad (P_i, \Pi) = -\frac{\partial \Pi}{\partial X_i}.$$

Should Π then be a distinguished function then one would have to have:

$$\frac{\partial \Pi}{\partial P_i} = 0, \qquad \frac{\partial \Pi}{\partial X_i} = 0$$

for $i = 1, \dots, q$; i.e., Π is merely a function of $X_{q+1}, \dots, X_{q+m}$.

Theorem 37. If $X_1, \dots, X_{q+m}, P_1, \dots, P_q$ satisfy the relations:

$$(X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0, \qquad\qquad (X_i, P_i) = 1,$$

and there is thus no relation between $X_{q+1}, \dots, X_{q+m}$ then our $2q + m$ functions define a $(2q + m)$-parameter group.

Our theorem emerges from the fact that, under the assumptions that we made, no relation exists between our $2q + m$ functions. If that were the case then, in any event, one of them would contain one of the $2q$ quantities $X_1, \dots, X_q, P_1, \dots, P_q$ – say, X_1 – and could thus take on the form:

$$X_1 = W(X_2, \dots, X_{q+m}, P_1, \dots, P_q).$$

However, it would follow from this that:

$$(X_1, P_1) = (W, P_1),$$

which is contradictory, in that the left-hand side equals 1 and the right-hand side equals 0.

Theorem 38. The difference between the number of parameters in a group and the number of its distinguished functions is an even number.

Any group can then take on the form:

$$X_1, \dots, X_{q+m}, P_1, \dots, P_q,$$

where $X_{q+1}, \dots, X_{q+m}$ are the distinguished functions; the stated difference is then equal to $2q$.

Corollary 1. A $2q$-parameter group contains either $2q$ or $2q - 2$ or $2q - 4$, …, or 2 or no distinguished functions.

Corollary 2. A $(2q + 1)$-parameter group contains either $2q + 1$ or $2q - 1$, …, or 3 or 1 distinguished functions. Such a group then always contains at least *one* distinguished function.

We finally summarize our results.

Theorem IX. *Any group can take on the form:*

$$X_1, \dots, X_{q+m}, P_1, \dots, P_q,$$

where the following relations exist:

$$(X_i\,, X_k) = (X_i\,, P_k) = (P_i\,, P_k) = 0, \qquad (X_i\,, P_i) = 1.$$

Here, X_{q+1}, …, X_{q+m} are the distinguished functions of the group. The difference between the number of parameters and the number of distinguished functions is always an even number.

§ 13.

Determination of the invariant properties of a group.

28. We next show that one can always find canonical groups that contain a given canonical group. Therefore, we shall deal with the first of the two problems that we posed in the beginning of this section.

Theorem 39. If X_1, …, X_{q+m}, P_1, …, P_q is a canonical group then there are always functions P_{q+1} that fulfill the equations:

$$(X_i\,, P_{q+1}) = (P_i\,, P_{q+1}) = 0, \qquad (X_{q+1}\,, P_{q+1}) = 0.$$

Consequently, X_1, …, X_{q+m}, P_1, …, P_{q+1} is a new canonical group that contains the given one.

In fact:

(*A*) $$X_1, \ldots, X_q, X_{q+2}, \ldots, X_{q+m}, P_1, \ldots, P_q$$

is obviously a group whose polar group contains X_{q+1}, and perhaps possesses the form:

(*B*) $$X_{q+1}, U_1, U_2, \ldots$$

Now, X_{q+1} does not belong to the group (*A*), and is therefore (Theorem 27) not a distinguished function of (*B*), so the latter group contains (Theorem 31) functions P_{q+1} that yield:

$$(X_{q+1}\,, P_{q+1}) = 1.$$

However, because it belongs to the group (*B*), any such function P_{q+1} is in involution with all functions of the group (*A*), and thus possesses all of the desired properties.

Theorem 40. If X_1, …, X_{q+m}, P_1, …, P_q is a canonical group then there are always further functions P_{q+1}, P_{q+2}, …, P_{q+m} such that:

$$X_1, \ldots, X_{q+m}, P_1, \ldots, P_{q+m}$$

define a new canonical group that contains the given one.

This theorem is obtained immediately by an *m*-fold application of the foregoing one.

Theorem 41. If $X_1, \ldots, X_q, P_1, \ldots, P_q$ is a canonical group and $q < n$ then there are always functions X_{q+1} that are in involution with the functions of our group. Consequently:

$$X_1, \ldots, X_{q+1}, P_1, \ldots, P_q$$

is a new canonical group that contains the given one.

Any function that belongs to the polar group of the given group then possesses the properties that we required of the desired function X_{q+1}.

Theorem 42. If $X_1, \ldots, X_{q+m}, P_1, \ldots, P_q$ is a canonical group then there are always further functions $X_{q+m+1}, \ldots, X_n, P_{q+1}, \ldots, P_n$ such that:

$$X_1, \ldots, X_n, P_1, \ldots, P_n$$

is also a canonical group.

From theorem 40, there is then a canonical group:

$$X_1, \ldots, X_{q+m}, P_1, \ldots, P_{q+m}$$

that contains the given one. Thereafter, by means of theorem 41, one finds a canonical group:

$$X_1, \ldots, X_{q+m+1}, P_1, \ldots, P_{q+m}$$

and thus (Theorem 39), a canonical group:

$$X_1, \ldots, X_{q+m+1}, P_1, \ldots, P_{q+m+1},$$

etc.

29. In Part I (Theorem III), we saw that equations of the form:

$$x_i' = X_i, \qquad p_i' = P_i,$$

in which X_i and P_i denote functions of $x_1, \ldots, p_n$ that fulfill the conditions:

$$(X_i, X_k) = (X_i, P_k) = (P_i, P_k) = 0, \qquad (X_i, P_i) = 1,$$

always determine a contact transformation. By the use of this theorem, we can now prove the following theorem, and in so doing, resolve problem I:

Theorem X. *If two r-parameter group possess just as many distinguished functions then there is always a contact transformation that takes the one group to the other. On the other hand, this condition is not just sufficient, but also necessary.*

Let $u_1, \ldots, u_r$ be functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$ and let $w_1, \ldots, w_r$ be functions of $y_1, \ldots, y_n, \pi_1, \ldots, \pi_n$. If $u_1, \ldots, u_r$, as well as $w_1, \ldots, w_r$, then define a group, and both groups possess the same number of distinguished functions then the two groups can assume the canonical forms:

$$X_1, \ldots, X_\mu, P_1, \ldots, P_\nu \text{ and } Y_1, \ldots, Y_\mu, \Pi_1, \ldots, \Pi_\nu ,$$

respectively. Moreover, from Theorem 42, there are always functions X, P of $x_1, \ldots, p_n$ and Y, Π of $y_1, \ldots, \pi_n$ such that:

$$X_1, \ldots, X_n, P_1, \ldots, P_n \text{ and } Y_1, \ldots, Y_n, \Pi_1, \ldots, \Pi_n ,$$

are also, in turn, canonical groups. Therefore:

$$x_i' = X_i, \qquad p_i' = P_i,$$

as well as:

$$x_i' = Y_i, \qquad p_i' = \Pi_i,$$

is a contact transformation. However, it also follows from this that the $2n$ equations:

$$X_i = Y_i, \qquad P_i = \Pi_i$$

define a contact transformation, and one sees that this transformation takes the one group to the other one.

With that, the first part of our theorem is proved. The last part of it follows immediately from the fact that for any contact transformation the number of terms and the number of distinguished functions of a group remain unchanged (§ 9 and § 11).

Corollary. *The only properties of a group that are independent of the form of the group and remain unchanged by a contact transformation are the number of parameters and the number of distinguished functions.*

§ 14.

Invariant relations between a group and a subgroup of it.

I shall now address the following problem:

Problem. *Let two r-parameter groups be given, each of which contains a ρ-parameter subgroup. Decide whether there is a contact transformation that takes the one r-parameter group and its subgroup into the second r-parameter group and its subgroup, respectively.*

30. First, some lemmas.

Theorem 43. Let $u_1, \dots, u_r$ be a group that is contained in a larger group $u_1, \dots, u_\rho, \dots, u_r$. Furthermore, let U be a function of the latter group. If our groups contain no common distinguished functions then the equations:

$$(u_1, U) = 0, \dots, (u_r, U) = 0$$

define a complete system whose $r - \rho$ solutions $w_1, \dots, w_{r-\rho}$ define a new group. In particular, if $u_1, \dots, u_\rho$ contains no distinguished function then:

$$u_1, \dots, u_\rho, w_1, \dots, w_{r-\rho},$$

is a form of the group $u_1, \dots, u_r$, which is thereby decomposed into two involutory groups $u_1, \dots, u_\rho$ and $w_1, \dots, w_{r-\rho}$.

In fact, let $v_1, \dots, v_{2n-r}$ be the polar group of $u_1, \dots, u_r$. Previously (Theorem VII), we saw that any relation between the u and v possesses the form:

$$F(u_1, \dots, u_r) = \Phi(v_1, \dots, v_{2n-r}),$$

where F is a distinguished function of the group $u_1, \dots, u_r$. From our assumption, this group contains no distinguished functions of the form $F(u_1, \dots, u_\rho)$. There then exists no functional relation between $u_1, \dots, u_\rho, v_1, \dots, v_{2n-r}$. As a result, these quantities define a group, and the equations:

$$(A) \qquad (u_1, W) = 0, \quad \dots, (u_r, W) = 0, \qquad (v_1, W) = 0, \dots, (v_{2n-r}, W) = 0$$

define a complete system whose $r - \rho$ solutions $w_1, \dots, w_{r-\rho}$, as solutions of:

$$(v_1, W) = 0, \quad \dots, \quad (v_{2n-r}, W) = 0,$$

belong to the group $u_1, \dots, u_r$.

The fact that $w_1, \dots, w_{r-\rho}$ define a group follows from the fact that from the Poisson-Jacobi theorem any (w_i, w_k) is a solution of the system (A).

In particular, if $u_1, \dots, u_\rho$ contains no distinguished functions then there exists no relation between $u_1, \dots, u_\rho$ and $w_1, \dots, w_{r-\rho}$, since (Theorem VII) one would then have the form:

$$F(u_1, \dots, u_\rho) = \Phi(w_1, \dots, w_{r-\rho}),$$

where F would be a distinguished function of the group $u_1, \dots, u_r$. Thus:

$$u_1, \dots, u_\rho, w_1, \dots, w_{r-\rho}$$

is a form of the group $u_1, \dots, u_r$ that is then decomposed into two involutory groups.

Theorem 44. If $X_1, \dots, X_\alpha, P_1, \dots, P_\alpha$ is a canonical group that is contained in a group G then G can assume the canonical form $X_1, \dots, X_\beta, P_1, \dots, P_\gamma$.

We then decompose G into the two involutory groups:

$$X_1, \dots, X_\alpha, P_1, \dots, P_\alpha \qquad \text{and} \qquad w_1, \dots, w_\rho$$

by using the foregoing theorems, and then bring $w_1, \dots, w_\rho$ into a canonical form:

$$X_{\alpha+1}, \dots, X_\beta, P_{\alpha+1}, \dots, P_\gamma$$

then

$$X_1, \dots, X_\beta, P_1, \dots, P_\gamma$$

is obviously the desired canonical form of G.

Theorem 45. If a group G contains a system in involution $X_1, \dots, X_\rho$, and if no function of the X is a distinguished function in G then this group can assume the canonical form:

$$X_1, \dots, X_\rho, \dots, X_\alpha, P_1, \dots, , P_\rho, \dots, P_\beta .$$

Namely, let $X_1, \dots, X_\rho, u_1, \dots, u_{r-\rho}$ be a form for G, and let $v_1, \dots, v_{2n-r}$ be its polar group. From our assumption, there exists no relation between the X and the v. Therefore:

$$X_2, \dots, X_\rho, v_1, \dots, v_{2n-r}$$

define a group whose polar group (Theorem 32) is contained in G and contains X_1 . This polar group thus possesses the form:

$$X_1, \dots, X_\rho, w_1, \dots, w_{r-2\rho+1} \qquad (G').$$

X_1 is not (Theorem 27) a distinguished function in G' so that group contains (Theorem 31) a function P_1 that yields:

$$(X_1, P_1) = 1.$$

With that, the group G, which is contained in G', is brought into the form:

$$X_1, P_1, X_2, \dots X_\rho, \varphi_1, \dots$$

It can therefore (Theorem 43) be decomposed into two involutory groups, one of which is X_1, P_1, while the other one contains $X_2, \dots, X_\rho$, and possesses the form:

$$X_2, \dots X_\rho, u_1', \dots, u_{r-\rho-1}' .$$

This group contains no distinguished function of the form $F(X_2, \dots, X_\rho)$, so it can likewise be decomposed into two involutory groups:

$$X_2, P_2 \qquad \text{and} \qquad X_3, \dots, X_\rho, u_1'', \dots, u_{r-\rho-2}'' .$$

In order to go further in this way, we finally bring G into the desired form.

Theorem 46. Now, suppose we are given a group G with r parameters $u_1, \ldots, u_r$ and a subgroup of it $u_1, \ldots, u_\rho$ that has ϖ distinguished functions $X_1, \ldots, X_\varpi$ in common with G. If:

$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_{\varpi+\alpha}, P_{\varpi+1}, \ldots, P_{\varpi+\alpha}, X_{\varpi+\alpha+1}, \ldots, X_{\varpi+\alpha+\beta}$$

is a canonical form for the subgroup then G can always assume the canonical form:

$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_{\varpi+\alpha+\beta}, \ldots, X_\gamma, P_{\varpi+1}, \ldots, P_{\varpi+\alpha+\beta}, P_\delta .$$

Then, from our assumption:

$$X_{\varpi+1}, \ldots, X_{\varpi+\alpha}, P_{\varpi+1}, \ldots, P_{\varpi+\alpha} \qquad (G')$$

define a group G' that is contained in G. Therefore, from Theorem 43, G can be decomposed into two involutory groups G' and G'', the latter of which obviously contains the functions $X_1, \ldots, X_\varpi, X_{\varpi+\alpha+1}, \ldots, X_{\varpi+\alpha+\beta}$, and thus possesses the form:

$$X_1, \ldots, X_\varpi, X_{\varpi+\alpha+1}, \ldots, X_{\varpi+\alpha+\beta}, U_1, U_2, \ldots \quad (G'').$$

Now, G'' contains the system in involution $X_{\varpi+\alpha+1}, \ldots, X_{\varpi+\alpha+\beta}$, which contains no distinguished function of G''. If we then apply the previous theorem then we see that G'' can assume the form:

$$X_1, \ldots, X_\varpi, X_{\varpi+\alpha+1}, \ldots, X_{\varpi+\alpha+\beta}, \ldots, X_\gamma, P_{\varpi+\alpha+1}, \ldots, P_{\varpi+\alpha+\beta}, P_\delta .$$

With that, the group G, which consists of the functions of the two groups G' and G'', is brought into the desired form.

Corollary. If a group has ϖ distinguished functions with a common subgroup then these two groups can assume the canonical forms:

$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_\beta, \ldots, X_\gamma, P_{\varpi+1}, \ldots, P_\beta, \ldots, P_\delta ,$$
$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_{\varpi+\alpha}, \ldots, \quad P_{\varpi+1}, \ldots, P_{\varpi+\alpha}, \ldots, X_{\varpi+\alpha+1}, \ldots, X_\beta .$$

31. We can now resolve the problem that was posed at the beginning of this paragraph.

Theorem XI. *Let two groups G and G' be given with the same number of parameters and distinguished functions. Any of the two groups further contains a subgroup g (g', respectively) with the same number of parameters and distinguished functions. Finally, G, as well as G', might have ϖ distinguished functions in common with the subgroup in question. Consequently, there is a contact transformation that simultaneously takes G*

and g to G′ and g′, respectively. Conversely, such a transformation is possible if all of the stated conditions are fulfilled.

Namely, from the previous corollary, if we bring g and G into the simultaneous canonical forms:

$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_{\varpi+\alpha}, \ldots, \quad P_{\varpi+1}, \ldots, P_{\varpi+\alpha}, \ldots, X_{\varpi+\alpha+1}, \ldots, X_\beta,$$
$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_\beta, \ldots, X_\gamma, P_{\varpi+1}, \ldots, P_\beta, \ldots, P_\delta,$$

resp., then it is possible to bring g' and G' into the simultaneous canonical forms:

$$X_1', \ldots, X_\varpi', X_{\varpi+1}', \ldots, X_{\varpi+\alpha}', \ldots, \quad P_{\varpi+1}', \ldots, P_{\varpi+\alpha}', \ldots, X_{\varpi+\alpha+1}', \ldots, X_\beta',$$
$$X_1', \ldots, X_\varpi', X_{\varpi+1}', \ldots, X_\beta', \ldots, X_\gamma', P_{\varpi+1}', \ldots, P_\beta', \ldots, P_\delta',$$

resp. Thus, (see the proof of Theorem X) G can be transformed into G' in such a way that X_i and P_i go to the corresponding X_i' and P_i', resp. Therefore, it is obvious that g will simultaneously go to g'. Thus, the requirements presented are sufficient; the fact that they are necessary comes from the fact that they relate to relations that remain invariant under contact transformations.

Corollary. All invariant relations between a group and a subgroup will be determined by the number of common distinguished functions, coupled with the number of parameters and distinguished functions in both groups. From the foregoing, the latter numbers define the individual invariants of each of the two groups.

This suggests the question of how one must proceed if one would like to investigate how many common distinguished functions a group $u_1, \ldots, u_\rho, \ldots, u_r$ and a subgroup $u_1, \ldots, u_r$ of it contain.

If one denotes a function of $u_1, \ldots, u_r$ by F then it is clear that the stated functions are defined by the simultaneous equations:

$$(u_1\, F) = 0, \ldots, (u_r\, F) = 0, \qquad \frac{\partial F}{\partial u_{\rho+1}} = 0, \ldots, \frac{\partial F}{\partial u_r} = 0.$$

One thus examines how many common solutions that these equations have in the usual way. If there are ϖ of them then their determination requires $\varpi, \varpi - 1, \ldots, 3, 2, 1$ operations. Therefore:

Theorem 47. If one knows a group and a subgroup of it then one can, with no integration, decide how many distinguished functions that these two groups have in common. If there are ϖ of them then one finds them by means of $\varpi, \varpi - 1, \ldots, 3, 2, 1$ operations.

Finally, I also need the following theorem:

Theorem 48. Let $u_1, \ldots, u_\rho, \ldots, u_r$ be a group, and let $u_1, \ldots, u_\rho$ be a subgroup of it that has ϖ distinguished functions in common with it. If one then lets F denote a function of $u_1, \ldots, u_r$ then the equations:

$$(u_1, F) = 0, \ldots, (u_\rho, F) = 0$$

have $r - \rho + \varpi$ common solutions, and can therefore be replaced with $\rho - \varpi$ equations that define an involutory system.

Our groups can then be brought into the simultaneous canonical forms:

$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_{\varpi+\alpha+\beta}, \ldots, X_\gamma, P_{\varpi+1}, \ldots, P_{\varpi+\alpha+\beta}, \ldots, P_\delta,$$
$$X_1, \ldots, X_\varpi, X_{\varpi+1}, \ldots, X_{\varpi+\alpha}, \ldots, \quad P_{\varpi+1}, \ldots, P_{\varpi+\alpha}, \ldots, X_{\varpi+\alpha+1}, \ldots, X_{\varpi+\alpha+\beta},$$

resp. As a result:

$$X_1, \ldots, X_\varpi, X_{\varpi+\alpha+1}, \ldots, X_\gamma, P_{\varpi+\alpha+\beta+1}, \ldots, P_\delta$$

are those functions of the larger group that are in involution with all functions of the subgroup. If one now makes a simple count then one recognizes the validity of our theorem.

§ 15.

Determination of the system of involution that is included in a group.

32. Theorem 49. $(m + q)$-parameter systems in involution can be selected from a group with m distinguished functions and $m + 2q$ terms.

Such a group then possesses the canonical form:

$$X_1, \ldots, X_{q+m}, P_1, \ldots, P_q,$$

and here $X_1, \ldots, X_{q+m}$ define a system in involution with $q + m$ parameters.

Theorem 50. A system in involution that is contained in a $(2q + m)$-parameter group with m distinguished functions can consist of at most $q + m$ functions.

Then, let $\Phi_1, \ldots, \Phi_\nu$ be a system in involution that contains the group $X_1, \ldots, X_{q+m}, P_1, \ldots, P_q$. One determines further functions X and P such that:

$$X_1, \ldots, X_n, P_1, \ldots, P_n$$

is a canonical group, between whose functions it is known that no relation can exist. There is now a function of the system in involution:

$$X_{q+m+1}, \ldots, X_n$$

that is in involution with all functions of the original group, in particular, with $\Phi_1, \ldots, \Phi_\nu$, as well. Therefore:

$$X_{q+m+1}, \ldots, X_n, \Phi_1, \ldots, \Phi_\nu$$

is a system in involution with $\nu + n - q - m$ mutually independent functions. However, it is known that a system in involution contains at most n parameters, so one must have:

$$\nu + n - q - m < n,$$

that is:

$$\nu < q + m,$$

and that was precisely our assertion.

We now show how one must proceed in general in order to select systems in involution with as many parameters as possible from a given group.

If $u_1, \ldots, u_{2q+1}$ is a given group with m distinguished functions $U_1, \ldots, U_m$ then one first finds the latter by integrating the system:

$$(u_1, U) = 0, \ldots, (u_{2q+1}, U) = 0,$$

which requires $m, m - 1, \ldots, 3, 2, 1$ operations. We know that $U_1, \ldots, U_m$ belong to a $(q + m)$-parameter system in involution of our group.

Thereupon, one takes an arbitrary, but not distinguished, function of the group – e.g., u_1 – and determines a further function $F(u_1, u_2, \ldots)$ from the equation:

$$(u_1, F) = \sum_{k=1}^{2q+m} (u_1, u_k) \frac{\partial F}{\partial u_k} = 0.$$

This partial differential equation, in which one replaces (u_1, u_k) with the corresponding function of the u everywhere, possesses $m + 1$ known solutions, namely, $U_1, \ldots, U_m, u_1$; one thus finds a further solution $F = w_2$ by means of $2q - 2$ operations.

One then defines:

$$(u_1, F) = 0, \qquad (w_2, F) = 0$$

with the two functions u_1 and w_2, replaces (u_1, u_k) and (w_2, u_k) with the functions in question of u, and thus obtains a complete system that consists of two equations in $2q + m$ with $m + 2$ known solutions, namely, $U_1, \ldots, U_m, u_1, w_2$. One then finds a further common solution w_3 by $2q - 4$ operations.

When one goes further in this way, one recognizes that the determination of a $(q + m)$-parameter system in involution in a $(2q + m)$-parameter group with m distinguished functions generally requires:

$$m, m - 1, \ldots, 3, 2, 1, 2q - 2, 2q - 4, \ldots, 4, 2$$

operations.

33. This method can be replaced with another one that requires simpler integrations, as long as the given group contains a known subgroup.

Therefore, let a group G with a known subgroup g be given. One seeks a system in involution in G that has as many parameters as possible. To this end, one first determines the ϖ common distinguished functions $U_1, \dots, U_\varpi$ of our two groups. This requires (Theorem 47):

$$\varpi, \varpi - 1, \dots, 3, 2, 1$$

operations. One then seeks the remaining $m' - \varpi$ distinguished functions that g contains by means of:

$$m' - \varpi, m' - \varpi - 1, \dots, 3, 2, 1$$

operations (§ 11, conclusion).

After one has found all of the distinguished functions of the group g in this way, one then determines a system in involution:

$$U_1, \dots, U_\varpi, u_1, \dots, u_\rho$$

that is contained in g and has the largest possible number of parameters using the method that was previously described.

Of the m distinguished functions of the group G, one now already knows ϖ of them, namely, $U_1, \dots, U_\varpi$. One then finds the $m - \varpi$ remaining ones $U_{\varpi+1}, \dots, U_m$ by means of:

$$m - \varpi, m - \varpi - 1, \dots, 3, 2, 1$$

operations. One then knows all of the distinguished functions:

$$U_1, \dots, U_m$$

of the group G, and, in addition, a system in involution:

$$u_1, \dots, u_\rho$$

that is contained in G whose functions are independent of the U. One now proceeds as in the general case.

Still greater simplifications emerge when, e.g., the subgroup g itself contains a known subgroup. Without going into all of the cases the can arise, I only emphasize that in each case my general theory allows one to give the number and order of the necessary integrations *a priori*.

Theorem XII. *A group with m distinguished functions and $2q + m$ parameters contains systems in involution with $q + m$ parameters. The determination of such systems generally requires:*

$$m, m - 1, \dots, 3, 2, 1, 2q - 2, 2q - 4, \dots, 4, 2$$

operations. If one already knows such subgroups then simplifications in the integration arise that can always be given a priori. Our group contains no system in involution with more than $q + m$ parameters.

34. In this section, I prove that there is a maximum number for the distinguished functions in a group with more than n parameters. An important theorem follows from this about groups that contain the largest possible number of distinguished functions.

A group with m distinguished functions and $2q + m$ parameters contains $(q + m)$-parameter systems in involution, so one must have:

$$q + m \leq n.$$

If we call the number of parameters r then this condition assumes the form:

$$\frac{r+m}{2} \leq n.$$

Finally, if we call the number of parameters $n + k$ then we obtain the third form:

$$m \leq n - k,$$

which says that the when the number of parameters is larger than n, the number of distinguished functions has a maximum value.

Theorem XIII. *If a given group $u_1, \ldots, u_{n+k}$ possesses the largest possible number of distinguished functions $\Phi_1, \ldots, \Phi_{n-k}$ then the integration of the system in involution:*

$$\Phi_1 = a_1, \ \ldots, \Phi_{n-k} = a_{n-k}$$

requires only permissible operations.

My extension of Cauchy's method then says that the integration of a system in involution:

$$\Phi_1 = a_1, \ \ldots, \Phi_{n-k} = a_{n-k}$$

can be accomplished when all solutions of the complete system:

$$(\Phi_1 , F) = 0, \ldots, (\Phi_{n-k} , F) = 0$$

are found. However, such solutions are just $u_1, \ldots, u_{n+k}$, and indeed there are no others. My theorem is thus proved.

§ 16.

Resolution of the main problem.

We first solve a special case of the second main problem, and in so doing, show that the general problem can revert to this special case.

35. We assume that $F_1, \dots, F_r$ and $F_1', \dots, F_r'$ are two r-parameter groups. We will deduce whether there is a contact transformation that takes any F_i into the corresponding F_i'. If such a transformation exists then (Theorem 11) it would take the equation:

$$(F_i\,,F_k)_{xp} = \Omega_{ik}(F_1, \dots, F_r)$$

into

$$(F_i', F_k')_{x'p'} = \Omega_{ik}(F_1', \dots, F_r').$$

Should the stated transformation then be possible then any $(F_i'\,F_k')_{x'p'}$ would have to be expressible in terms of $F_1', \dots, F_r'$ in the same way that the corresponding $(F_i\,F_k)_{xp}$ are expressible in terms of the $F_1, \dots, F_r$. Conversely, it can be shown that this necessary condition is also sufficient.

In fact, let $F_1, \dots, F_r$ and $F_1', \dots, F_r'$ be two such r-parameter groups such that:

$$(A) \qquad (F_i\,,F_k) = \Omega_{ik}(F_1, \dots, F_r), \qquad (F_i', F_k') = \Omega_{ik}'(F_1', \dots, F_r'),$$

and let $X_1, \dots, X_\alpha, X_1, \dots, X_\beta$, where:

$$X_i = \Phi_i(F_1, \dots, F_r), \qquad P_i = \Psi_i(F_1, \dots, F_r)$$

be a canonical form for the former group.

I define the functions:

$$X_i' = \Phi_i(F_1', \dots, F_r'), \qquad P_i' = \Psi_i(F_1', \dots, F_r')$$

and the expressions:

$$(X_i', X_k'), \qquad (X_i', P_k'), \qquad (P_i', P_k'),$$

which, due to (A), are the same functions of $F_1', \dots, F_r'$ that:

$$(X_i\,,X_k), \qquad (X_i\,,P_k), \qquad (P_i\,,P_k)$$

are of $F_1, \dots, F_r$. However, from our assumption, the relations:

$$(X_i\,,X_k) = (X_i\,,P_k) = (P_i\,,P_k) = 0, \qquad (X_i\,,P_i) = 1,$$

are true, so one also finds that the corresponding equations:

$$(X_i', X_k') = (X_i', P_k') = (P_i', P_k') = 0, \qquad (X_i', P_i') = 1$$

are true. Now, $X_1', \ldots, X_\alpha', P_1', \ldots, P_\beta'$ are obviously mutually independent functions, so:

$$X_1', \ldots, X_\alpha', P_1', \ldots, P_\beta'$$

is a canonical form for the group $F_1', \ldots, F_r'$. As a result (Theorem X), there is a contact transformation that transforms any X_i and P_i into the corresponding X_i' and P_i'. Thus, as one immediately sees, any F_i goes to the corresponding F_i'. Thus:

Theorem 51. *Let $F_1, \ldots, F_r$ and $F_1', \ldots, F_r'$ be two r-parameter groups. Should a contact transformation be given that takes any F_i to the corresponding F_i' then it would be necessary and sufficient that any (F_i, F_k) be expressible in terms of $F_1, \ldots, F_r$ as the corresponding (F_i', F_k') are in terms of $F_1', \ldots, F_r'$.*

36. We can now address the general problem.

Therefore, let two systems of functions $F_1, \ldots, F_r$ and $F_1', \ldots, F_r'$ be given. One must decide whether there is a contact transformation that takes every F_i to the corresponding F_i'.

It is first of all clear that we can assume that all of the F_i (and likewise, all of the F_i') are mutually independent; if, e.g., only $F_1, \ldots, F_\alpha$ were mutually independent, and by contrast:

$$F_{\alpha+k} = W_k(F_1, \ldots, F_\alpha) \qquad (k = 1, \ldots, r - \alpha)$$

then obviously $F_1', \ldots, F_\alpha'$ would also be mutually independent, while the remaining $F_{\alpha+k}'$ would be expressible in terms of the $F_1', \ldots, F_\alpha'$:

$$F_{\alpha+k}' = W_k(F_1', \ldots, F_\alpha').$$

However, if this is the case then it is also clear that a contact transformation that transforms $F_1, \ldots, F_\alpha$ into $F_1', \ldots, F_\alpha'$, resp., simultaneously takes $F_{\alpha+1}, \ldots, F_r$ to $F_{\alpha+1}', \ldots, F_r'$, resp.

Thus, let $F_1, \ldots, F_r$, and likewise $F_1', \ldots, F_r'$, be mutually independent. If the desired contact transformation exists then it would take every (F_i, F_k) to the corresponding (F_i', F_k'). I now define new functions by setting:

$$(F_{a^{(1)}}, F_{b^{(1)}}) = F_{r+1}, \quad (F_{a^{(2)}}, F_{b^{(2)}}) = F_{r+2}, \quad \ldots, \quad (F_{a^{(\rho)}}, F_{b^{(\rho)}}) = F_{r+\rho},$$

where the numbers $a^{(k)}$ and $b^{(k)}$ are subject to the restriction that one must have:

$$a^{(k)} < r + \rho, \qquad b^{(k)} < r + \rho,$$

and that F_{r+k} cannot be expressible in terms of $F_1, \ldots, F_r, \ldots, F_{r+k-1}$. I proceed as far as possible in this way; i.e., until I have found the groups:

$$F_1, \ldots, F_r, \ldots, F_{r+\rho}$$

that are determined by $F_1, \ldots, F_r$ and which contain at most $2n$ parameters. If I now set:

$$(F'_{a^{(k)}}, F'_{b^{(k)}}) = F'_{r+k}$$

in a corresponding way then the desired contact transformation must transform every F_{r+k} into the corresponding F'_{r+k}. Therefore:

$$F'_1, \ldots, F'_r, \ldots, F'_{r+\rho}$$

must also define a group. Furthermore, from the previous theorems, any (F'_i, F'_k) can be expressed in terms of $F'_1, \ldots, F'_{r+\rho}$ in the same way as the corresponding (F_i, F_k) is in terms of $F_1, \ldots, F_r$. On the other hand, from the above, this necessary requirement is also sufficient. Therefore:

Theorem XIV. *Let two systems of functions:*

$$F_1, \ldots, F_\alpha \text{ and } F'_1, \ldots, F'_\alpha$$

of x, p and x′, p′, resp., be given. If one wishes to decide whether there is a contact transformation that transforms any F_i into the corresponding F'_i then one should proceed in the following way: Among the F, one takes r mutually independent ones – say, $F_1, \ldots, F_r$ – in terms of which the remaining ones can be expressed:

$$F_{r+k} = W_k(F_1, \ldots, F_r) \qquad (k = 1, \ldots, \alpha - r).$$

A first condition is then that $F'_1, \ldots, F'_r$ should be independent functions, in terms of which, the F'_{r+k} could be expressed in a corresponding way:

$$F'_{r+k} = W_k(F'_1, \ldots, F'_r).$$

If this is the case then one defines the group that is determined by $F_1, \ldots, F_r$ when one sets:

$$(F_{a^{(1)}}, F_{b^{(1)}}) = F_{r+1}, \quad \ldots, \quad (F_{a^{(k)}}, F_{b^{(k)}}) = F_{r+k},$$

and thus chooses the numbers $a^{(k)}$, $b^{(k)}$ in such that one always has:

$$a^{(k)} < r + k, \qquad b^{(k)} < r + k,$$

and that one F_{r+k} is expressible in terms of $F_1, \ldots, F_{r+k-1}$. Let:

$$F_1, \ldots, F_{r+\rho}$$

be one of the groups that are obtained in this way. If one then sets, in a corresponding way:

$$(F'_{a^{(1)}}, F'_{b^{(1)}}) = F'_{r+1}, \quad \ldots, (F'_{a^{(k)}}, F'_{b^{(k)}}) = F'_{r+\rho}$$

then the functions:

$$F'_1, \ldots, F'_{r+\rho}$$

must also define a group with $r + \rho$ parameters, and in addition, any (F'_i, F'_k) of this group must be expressible in terms of $F'_1, \ldots, F'_{r+\rho}$ in the same way as the corresponding (F_i, F_k) are expressible in terms of $F_1, \ldots, F_{r+\rho}$. If all of the conditions are verified then the desired transformation is possible.

This theorem determines all of the relations that exist between the given functions $F_1, \ldots, F_\alpha$ that remain unchanged under contact transformations. As one sees, all such relations can be expressed by means of the differential symbols (Φ, Π), when coupled with finite functional relations.

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§ 17.

Integration methods that are based on the previous developments.

37. I assume that a system in involution:

$$F_1 = C_1, \ldots, F_q = C_q$$

is supposed to be integrated and that one already knows a sequence of functions $\Phi_1, \ldots, \Phi_r$ that all satisfy the equations:

$$(F_i, \Phi) = 0.$$

If one can find no further solutions by means of the Poisson-Jacobi theorem then $F_1, \ldots, F_q, \Phi_1, \ldots, \Phi_r$ define a group in which $F_1, \ldots, F_q$ are distinguished functions. If there are, in addition, μ such functions:

$$F_{q+1}, \ldots, F_{q+\mu},$$

then (§ 11, conclusion) one determines them by means of:

$$\mu, \mu - 1, \ldots, 3, 2, 1$$

operations. Consequently:

$$F_1 = C_1, \ldots, F_{q+\mu} = C_{q+\mu}$$

is a new system in involution with $r - \mu$ known solutions $\Phi_1, \ldots, \Phi_{r-\mu}$ of the $q + m$ equations $(F_i\ \Phi) = 0$, and the integration of the given system in involution is converted into that of the new system.

One then remarks that $r - \mu$ must be an even number. $r - \mu$ is then the difference between the number of parameters $r + q$ and the number of distinguished functions $q + \mu$, and, from a previous theorem (Theorem IX), it is then an even number.

38. We are thus led to the especially important problem of integrating a system in involution:

$$F_1 = C_1, \ldots, F_{q+\mu} = C_{q+\mu}$$

in the simplest possible way when one knows $2q$ solutions $\Phi_1, \ldots, \Phi_{2q}$ of the system $(F_i\ \Phi) = 0$ that, together with the F, define a group whose only distinguished functions are the F.

To that end, one exhibits the complete system:

$$(F_1, F) = 0, \ldots, (F_m, F) = 0, \quad (\Phi_1, F) = 0, \ldots, (\Phi_{2q}, F) = 0,$$

among whose $2n - 2q - m$ solutions m are already known, namely, $F_1, \ldots, F_m$. One determines a further solution F_{m+1} by means of a:

$$2n - 2q - 2m$$

operation. On this, it must be remarked that F_{m+1} cannot belong to the group $F_1, \ldots, F_m, \Phi_1, \ldots, \Phi_{2q}$. $F_1, \ldots, F_m$ are then the only functions of this group that likewise belong to the polar group and, from our procedure, F_{m+1} is not a function of $F_1, \ldots, F_m$.

With that, our problem is reduced to that of the integration of the system in involution:

$$F_1 = C_1, \ldots, F_{m+1} = C_{m+1}$$

with $2q$ solutions $\Phi_1, \ldots, \Phi_{2q}$ of the corresponding complete system $(F_i, \Phi) = 0$. Here, we go further in the same manner. We then pose the complete system:

$$(F_1, F) = 0, \ldots, (F_{m+1}, F) = 0, \quad (\Phi_1, F) = 0, \ldots, (\Phi_{2q}, F) = 0,$$

among whose $2n - 2q - m - 1$ solutions $m + 1$ of them are known, namely, $F_1, \ldots, F_{m+1}$. We determine a further solution F_{m+2} by means of a:

$$2n - 2q - 2m - 2$$

operation, and remark, as before, that F_{m+2} cannot belong to the group $F_1, \ldots, F_{m+1}$.

We then treat the system in involution:

$$F_1 = C_1, \dots, F_{m+2} = C_{m+2}$$

with the unknown solutions $\Phi_1, \dots, \Phi_{2q}$ to the equations $(F_i, \Phi) = 0$, and find a function F_{m+3} by means of a:

$$2n - 2q - 2m - 4$$

operation, and then a function F_{m+4} by means of a:

$$2n - 2q - 2m - 6$$

operation, etc., until finally we get a function F_{n-q} by means of a:

$$2$$

operation.

With that, the integration of the original system in involution is converted into that of the system:

$$F_1 = C_1, \dots, F_{n-q} = C_{n-q}$$

with $2q$ known solutions:

$$\Phi_1, \dots, \Phi_{2q}$$

of the $n - q$ equations $(F_i, \Phi) = 0$. However, the integration of this system will be accomplished (Theorem XIII) by my extension of Cauchy's method with nothing further. Therefore:

Theorem 52. The integration of a system in involution:

$$F_1 = C_1, \dots, F_m = C_m$$

with $2q$ known solutions $\Phi_1, \dots, \Phi_{2q}$ of the m equations:

$$(F_1\,\Phi) = 0, \dots, (F_m\,\Phi) = 0$$

requires a:

$$2n - 2q - 2m,\ \ 2n - 2q - 2m - 2, \dots, 6, 4, 2$$

operation, while a:

$$2n - 2q - 2m,\ \ 2n - 2q - 2m - 1, 2n - 2q - 2m - 2, \dots, 3, 2, 1$$

operation would be required for the direct application of the extended Cauchy method. In this, it is assumed that the application of the *Poisson-Jacobi* theorem gives no further solutions Φ, that is, that $F_1, \dots, F_m, \Phi_1, \dots, \Phi_{2q}$ do not define a group, and that the F are the only distinguished functions of this group.

If we then combine the content of the foregoing paragraph then we obtain the following theorem, which gives the most important simplification of the integrations that one can deduce from the foregoing in a schematic way.

Theorem XV. *Should one wish to integrate a system in involution:*

$$F_1 = C_1, \ldots, F_q = C_q ,$$

and one then knows $2n + m$ *solutions* $\Phi_1, \ldots, \Phi_{2n+2}$ *of the* q *equations* $(F_i , \Phi) = 0$ *that define a group, along with* $F_1, \ldots, F_q$*, that contains m distinguished functions, in addition to F, then the execution of our integration procedure would require an:*

$$m, m - 1, m - 2, \ldots, 3, 2, 1,$$
$$2n - 2q - 2\nu - 2m, 2n - 2q - 2\nu - 2m - 2, \ldots, 6, 4, 2.$$

operation. The direct application of the extended Cauchy method requires a:

$$2n - 2q - 2n - m, 2n - 2q - 2n - m - 1, \ldots, 3, 2, 1$$

operation. In general, the Jacobi method would make much less use of the functions Φ.

Moreover, one easily recognizes that still greater simplifications can often be achieved, namely, when one already knows subgroups.

39. In order to compare the accomplishments of this theory with that of the extended Cauchy method, I shall go back to the previously-found (no. 34) relation between the number r of parameters and the number m of distinguished functions of a group:

$$\frac{r+m}{2} \leq n.$$

In the present case, since the group:

$$F_1, \ldots, F_q, \Phi_1, \ldots, \Phi_{2r+m}$$

contains $2\nu + m + q$ parameters and $q + m$ distinguished functions, this equation assumes the following form:

$$\frac{2\nu+2q+2m}{2} \leq n$$

or

$$2n - 2\nu - 2q - 2m \geq 0.$$

We first consider the case:

$$2n - 2\nu - 2q - 2m > 0,$$

and then the case:

$$2n - 2\nu - 2q - 2m = 0.$$

A. If

$$2n - 2\nu - 2q - 2m > 0$$

then one easily convinces oneself that the new method requires simpler integrations that the previous method. Then, in this case, one has:

$$2n - 2\nu - 2q - m > m,$$

and therefore the numbers:

$$m, m - 1, \ldots, 3, 2, 1,$$
$$2n - 2\nu - 2q - 2m, 2n - 2\nu - 2q - 2m - 2, \ldots, 4, 2$$

are smaller than the numbers:

$$2n - 2\nu - 2q - 2m, 2n - 2\nu - 2q - 2m - 1, \ldots, 3, 2, 1.$$

B. By contrast, in the case:

$$2n - 2\nu - 2q - 2m = 0,$$

the two methods require just as many operations. Namely, in this case, the new method requires an:

$$m, m - 1, \ldots, 3, 2, 1$$

operation, while the old one requires a:

$$2n - 2q - 2\nu - m, 2n - 2q - 2\nu - 2m - 1, \ldots, 3, 2, 1$$

operation, which comes to precisely the same thing.

Finally, we would like to consider the case $q = 1$ somewhat closer. One must integrate an equation:

$$F(x_1, \ldots, x_n, p_1, \ldots, p_n) = \text{const.},$$

and one knows $2\nu + m$ solutions $\Phi_1, \ldots, \Phi_{2\nu+m}$ of the equation $(F, \Phi) = 0$, from which no new solution can be found by applying the Poisson-Jacobi theorem. We assume that the group:

$$F, \Phi_1, \ldots, \Phi_{2\nu+m}$$

contains m distinguished functions, in addition to F.

If the number of known solutions is:

$$2\nu + m < n - 1$$

here, and thus also:

$$m < n - 1$$

then one will have:

$$2\nu + 2m < 2n - 2,$$

so

$$2n - 2\nu - 2m - 2 > 0.$$

From our reasoning above, our method thus requires simpler integrations than the Cauchy method, in this case.

Now, let:

$$2\nu + m = n - 1.$$

If $\nu = 0$ then $m = n - 1$, and the equations:

$$F = C, \qquad \Phi_1 = C_1, \qquad \ldots, \qquad \Phi_{n-1} = C_{n-1}$$

define a system in involution whose integration using my improvement of the Jacobi method requires only a quadrature in all situations.

By contract, if:

$$2\nu + m = n - 1$$

and

$$\nu > 0$$

then one has:

$$m \leq n - 3,$$

and then:

$$2\nu + 2m \leq 2n - 4,$$

or

$$2n - 2\nu - 2m - 2 > 0.$$

Therefore, the new theory, in turn, requires simpler operations than Cauchy's in this case.

Finally, if $2\nu + m$ is equal to 2 then one can either choose one of the two known solutions, and then apply Jacobi's method, or also employ both of them and follow the theory above. Both methods require just as many integrations. This situation, i.e., that one can derive the same benefits from one known solution as from two of them, in no way represents a defect of the method. It can be proved that it lies in the nature of things. If $2\nu + m$ is greater than 2 then I do not need to compare my new method with Jacobi's. In that case, the latter is in the background of Cauchy's.

We now consider the case:

$$2\nu + m \geq n.$$

From my previous argument, the unfavorable case in which my method offers no simplification shall emerge when:

$$2n - 2n - 2m - 2 = 0.$$

This condition enters in when the group:

$$F, \Phi_1, \ldots, \Phi_{2\nu+m}$$

contains the largest possible number of distinguished functions, and otherwise never. Thus:

Theore, XVI. *Should one wish to integrate an equation:*

$$F(x_1, \ldots, x_n, p_1, \ldots, p_n) = \text{const.},$$

and one knows more than two solutions $\Phi_1, \dots \Phi_r$ *of the equation* $(F, \Phi) = 0$ *then my new theory always simplifies the background integration difficulties, assuming only that:*

$$r \geq n,$$

along with the demand that the group $F, \Phi_1, \dots \Phi_r$ *should contain the largest possible number of distinguished functions, in which case, my method demands just as many integrations as the older theory.*

§ 18.

Schematically executed examples.

40. In order to make the meaning of the foregoing theories emerge clearly, I will treat some examples schematically.

A. Let:

$$p_{10} - f(x_1, \dots, x_{10}, p_1, \dots, p_9) = 0$$

be given, with seven known solutions $\varphi_1, \dots, \varphi_7$ of the equation $(p_{10} - f, \varphi) = 0$, which, together with $p_{10} - f$, define a group. Here, four different cases are imaginable that require a different treatment.

1) Our group contains only one distinguished function besides $p_{10} - f$. In that case, the background integration process requires a:

$$1, 10, 8, 6, 4, 2$$

operation.

2) Our group contains three distinguished functions besides $p_{10} - f$. In that case, a:

$$3, 2, 1, 8, 6, 4, 2$$

operation is necessary.

3) If the group contains five distinguished functions then besides $p_{10} - f$ then a:

$$5, 4, 3, 2, 1, 6, 4, 2$$

operation is necessary.

4) Finally, if the group is a system in involution then only a:

$$4, 2$$

operation is necessary.

Previously, one wished only to treat the latter case in such a simple way, and indeed this only when the system in involution in question fulfilled the known condition (§ 7). *The remaining cases were not known.* One always required an:

$$11, 10, 9, 8, \ldots, 3, 2, 1$$

operation, or with the use of Jacobi's theory of multipliers, an:

$$11, 10, 9, \ldots, 4, 3, 2$$

operation.

I summarize this example by means of the following table:

1 distinguished function	1, 10, 8, 6, 4, 2
3 distinguished functions	3, 2, 1, 8, 6, 4, 2
5 distinguished functions	5, 4, 3, 2, 1, 6, 4, 2
7 distinguished functions	4, 2
Except in the last case, with the use of the theory of multipliers, one previously needed the following operations	11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1

B. Let:

$$p_{10} - f = 0$$

be given, with 8 known solutions $\varphi_1, \ldots, \varphi_8$ to the equation $(p_{10} - f, \varphi) = 0$. They, along with $p_{10} - f$ define a group that contains 8 or 6 or 4 or 2 or no distinguished functions in addition to $p_{10} - f$. The following table gives the necessary operations in these cases.

no distinguished functions	10, 8, 6, 4, 2
2 distinguished functions	2, 1, 8, 6, 4, 2
4 distinguished functions	4, 3, 2, 1, 6, 4, 2
6 distinguished functions	6, 5, 4, 3, 2, 1, 4, 2
8 distinguished functions	3
Except in the last case, with the use of the theory of multipliers, one previously needed the following operations	10, 9, 8, 7, 6, 5, 4, 3, 2

C. Let:

$$p_{10} - f = 0$$

be given, with 12 known solutions $\varphi_1, \ldots, \varphi_{12}$ to the equation $(p_{10} - f, \varphi) = 0$, from which, no further ones could be derived by using the Poisson-Jacobi theorem. The following table explains the possible cases, as compared to the older method.

no distinguished function	6, 4, 2
2 distinguished functions	2, 1, 4, 2
4 distinguished functions	4, 3, 2, 1, 2
6 distinguished functions	6, 5, 4, 3, 2, 1
Except in the last case, with the use of the theory of multipliers, one previously needed the operations	6, 5, 4, 3, 2

Except in the last case, my theory thus gives a reduction in the number of integrations.

§ 19.

Suggestions for some further simplifications of the integration.

41. The great importance of the integration theory that was developed lies especially in the fact that in the treatment of a first-order, partial differential equations by the methods that Mayer and I gave in the Spring of 1872, one often finds oneself in the following position:

A system in involution:

$$F_1, = C_1, \qquad \ldots, \qquad F_m = C_m$$

is to be integrated, and one already knows a *sequence* of functions $\Phi_1, \ldots, \Phi_r$ that satisfy all of the equations $(F_i , \Phi) = 0$.

It therefore seems natural to pose the question: How must one proceed in order to reduce the background integrations as much as possible in regard to their number and order?

Those simplifications that are thus always achieved are given in the last paragraphs. All that remains to be shown is how one can exploit the situations that can arise in the further treatment of the problem to the best advantage.

Let a system in involution be given:

$$F_1, = C_1, \qquad \ldots, \qquad F_m = C_m$$

with r solutions $\Phi_1, \ldots, \Phi_r$ that satisfy the m equations $(F_i, \Phi) = 0$, from which no further solution can be calculated by the use of the Poisson-Jacobi theorem. (In this, we can assume that the group $F_1, \ldots, F_m, \Phi_1, \ldots, \Phi_r$ contains no distinguished functions other than F. In the contrary case, one can determine them and then add those functions to F.)

From our general theory, we pose the complete system:

$$(F_1, F) = 0, \quad \ldots, \quad (F_m, F) = 0, \quad (\Phi_1, F) = 0, \quad \ldots, (\Phi_r, F) = 0$$

and seek common solution of it that is different from $F_1, \ldots, F_m$ by the use of Mayer's theorem. If one succeeds in determining such a solution then, as is known, one very often simultaneously finds more of them – perhaps ρ of them:

$$\Pi_1, \dots, \Pi_\rho .$$

It is now conceivable that the application of the Poisson-Jacobi theorem gives still further solutions Π [*]). In any case, one can always calculate the group that is determined by our functions:

$$F_1, \dots, F_m, \Phi_1, \dots, \Phi_r, \Pi_1, \dots, \Pi_\varpi .$$

The original problem is thus reduced to the integration of the system in involution:

$$F_1 = C_1, \dots, F_m = C_m ,$$

with the known solutions $\Phi_1, \dots, \Phi_r, \Pi_1, \dots, \Pi_\varpi$.

Before one goes any further here, one must, as usual, investigate whether the group $F_1, \dots, F_m, \Phi_1, \dots, \Phi_r, \Pi_1, \dots, \Pi_\varpi$ contains still more distinguished functions besides the F. If such functions exist then one determines them, and thereby our problem again assumes the original form:

A system in involution:

$$F_1 = C_1, \dots, F_{m+q} = C_{m+q} ,$$

with k known solutions $\Omega_1, \dots, \Omega_k$ of the $m+q$ equations $(F_i , \Omega) = 0$ is to be integrated, where the F are the only distinguished functions of the group in question. Here, one proceeds in the same way.

Here, the remark can find its place that the foregoing theory can take on another form, in part, namely, by applying a theorem that has a close connection with my new method of integration:

Theorem XVI. Let a system in involution be given:

$$F_1 = C_1, \dots, F_m = C_m ,$$

in the variables $x_1, \dots, x_n, p_1, \dots, p_n$, and let $\Phi_1, \dots, \Phi_q$ be known solutions of the m equations $(F_i , \Phi) = 0$. One can, in turn, reduce the system in involution to a single equation of the form:

$$f(x_1, \dots, x_{n-m} , p_1, \dots, p_{n-m}) = \text{const.},$$

in such a way that the integration of this one equation amounts to that of the system in involution, and q solutions $\varphi_1, \dots, \varphi_q$ of $(f, \varphi) = 0$ can likewise be given.

[*]) I have convinced myself that this case can actually arise by an example.

§ 20.

Treatment of the three-body problem using my general method.

42. It is known that Hamilton and Jacobi have shown that any problem in celestial mechanics can be expressed by a certain first-order, partial differential equation:

$$H(x_1, \ldots, x_n, p_1, \ldots, p_n) = a.$$

The known integrals of the simultaneous differential equations that the problem in question immediately defines give just as many solutions of the linear equation:

$$(H, F) = 0.$$

My general theory now teaches us how one must employ the known solutions in any individual case in order to reduce the background integrations as much as possible in regard to their number and order. As an example, I shall choose the three-body problem, and thus first assume that one body is fixed. I will then give a direct treatment of the general case.

If three material bodies, one of which is fixed, move by means of their mutual attraction then the three-surface theorem is valid. I denote the partial differential equation that expresses the problem by:

$$H(x_1, \ldots, x_6, p_1, \ldots, p_6) = a,$$

and the three solutions of the equation $(H, F) = 0$ that correspond to the surface theorem by:

$$F_1, F_2, F_3 .$$

As is known, the relations:

$$(F_1 , F_2) = F_3, \quad (F_2 , F_3) = F_1, \quad (F_3 , F_1) = F_2$$

exist between them, and therefore F_1, F_2, F_3 define a three-parameter group that is not a system in involution, and therefore contains *one* distinguished function Φ. The same thing will be determined by any two of the equations:

$$(F_1 , \Phi) = 0 = F_3 \frac{\partial \Phi}{\partial F_2} - F_2 \frac{\partial \Phi}{\partial F_3},$$

$$(F_2 , \Phi) = 0 = - F_3 \frac{\partial \Phi}{\partial F_1} + F_1 \frac{\partial \Phi}{\partial F_3},$$

$$(F_3 , \Phi) = 0 = F_2 \frac{\partial \Phi}{\partial F_1} - F_1 \frac{\partial \Phi}{\partial F_2}.$$

Here, if one integrates by the usual rules then one finds that:

$$\Phi = F_1^2 + F_2^2 + F_3^2 .$$

It is clear that the four-parameter group:

$$H, F_1, F_2, F_3$$

contains two distinguished functions:

$$H \text{ and } F_1^2 + F_2^2 + F_3^2 .$$

Thus, any system in involution that contains this group consists of at most three parameters. One such system is:

$$H = a, \qquad F_1 = b, \qquad F_1^2 + F_2^2 + F_3^2 = c.$$

The original problem is then reduced to the integration of this system. However, my new integration method teaches us that it is always possible to exhibit an equation of the form:

$$f(x_1, \ldots, x_4, p_1, \ldots, p_4) = 0$$

that is equivalent to the system in involution above. From Mayer's and my older theory, the solution of the original problem thus requires only a:

$$6, 4, 2$$

operation.

43. Now, let:

$$H(x_1, \ldots, x_9, p_1, \ldots, p_9) = a$$

be the partial differential equation that is equivalent to the general three-body problem. Let:

$$\varphi_1, \varphi_2, \varphi_3$$

be the three solutions of the equation $(H, \varphi) = 0$ that correspond to the three center-of-mass integrals, and furthermore let:

$$\varphi_4, \varphi_5, \varphi_6$$

be three solutions that correspond to the surface theorems, and finally, let:

$$\varphi_7, \varphi_8, \varphi_9$$

be the solutions that arise from the center-of-mass integrals by eliminating time. One must then remark that one relation:

$$\varphi_1\varphi_7 + \varphi_2\varphi_8 + \varphi_3\varphi_9 = 0$$

exists between the 9 functions. The functions φ_1, …, φ_8 define an eight-parameter group. We will find that it contains two distinguished functions, and that as a result, it is possible to select five-parameter systems in involution from our group.

From our general theory, we must exhibit the determinant with 8 rows and columns that is defined by all (φ_i, φ_k):

$$\Delta = \begin{vmatrix} (\varphi_1,\varphi_1) & \cdots & (\varphi_1,\varphi_8) \\ \cdots & \cdots & \cdots \\ (\varphi_8,\varphi_1) & \cdots & (\varphi_8,\varphi_8) \end{vmatrix}.$$

One finds that:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & 0 & \varphi_3 & -\varphi_2 & 0 & M\varphi_3 \\ 0 & 0 & 0 & -\varphi_3 & 0 & \varphi_1 & -M\varphi_3 & 0 \\ 0 & 0 & 0 & \varphi_2 & -\varphi_1 & 0 & M\varphi_2 & -M\varphi_1 \\ 0 & \varphi_3 & -\varphi_2 & 0 & \varphi_6 & -\varphi_5 & 0 & \varphi_9 \\ -\varphi_3 & 0 & \varphi_1 & -\varphi_6 & 0 & \varphi_1 & -\varphi_9 & 0 \\ \varphi_2 & -\varphi_1 & 0 & \varphi_5 & -\varphi_4 & 0 & \varphi_8 & -\varphi_7 \\ 0 & M\varphi_3 & -M\varphi_2 & 0 & \varphi_4 & -\varphi_8 & 0 & M\varphi_9 \\ -M\varphi_3 & 0 & M\varphi_1 & -\varphi_9 & 0 & \varphi_7 & -M\varphi_9 & 0 \end{vmatrix},$$

where M is a constant, and φ_9 is determined by the identity:

$$\varphi_1\varphi_7 + \varphi_2\varphi_8 + \varphi_3\varphi_9 = 0.$$

The calculation of the determinant shows that it is equal to zero. Thus, our group contains, in any case, one, and as a result, at least two, distinguished functions. If it had more than two such functions then there number would be four or larger. However, all sub-determinants of second and third order would then vanish, and one verifies, with no difficulty, that there are sub-determinants of third (and also second) order that are non-zero. Therefore, our group has two distinguished functions and thus contains systems in involution with five parameters, and none with more than five parameters. If such a system is found then its parameters, together with H, define a 6-parameter system in involution whose integration by my method can be reduced to that of a single equation:

$$f(x_1, \ldots, x_4, p_1, \ldots, p_4) = 0.$$

In order to select a five-parameter system in involution by the simplest possible operations (§ 15, 33) from the eight-parameter group, we remark that φ_1, φ_2, φ_3, φ_4, φ_5, φ_6 define a six-parameter subgroup that contains the system in involution φ_1, φ_2, φ_3. If we examine the determinant of the six-parameter group then we find that this group also possesses two distinguished functions. Thus, it contains systems in involution with four parameters and none with more than four. We seek such a system that possesses the form:

$$\varphi_1, \varphi_2, \varphi_3, \Phi(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6).$$

The function Φ will be determined by two of the equations:

$$(\varphi_1, \Phi) = 0, \quad (\varphi_2, \Phi) = 0, \quad (\varphi_3, \Phi) = 0,$$

which, by developing and substituting the values of (φ_i, φ_k), assume the form:

$$\varphi_3 \frac{\partial \Phi}{\partial \varphi_5} - \varphi_2 \frac{\partial \Phi}{\partial \varphi_6} = 0,$$

$$-\varphi_3 \frac{\partial \Phi}{\partial \varphi_4} + \varphi_1 \frac{\partial \Phi}{\partial \varphi_6} = 0.$$

If one integrates these by the usual rules then one finds that:

$$\Phi = \varphi_1\varphi_4 + \varphi_2\varphi_5 + \varphi_3\varphi_6 .$$

We thus know one four-parameter system in involution:

$$(A) \qquad \varphi_1, \varphi_2, \varphi_3, \varphi_1\varphi_4 + \varphi_2\varphi_5 + \varphi_3\varphi_6$$

of the eight-parameter group. In order to now find five-parameter systems in involution, we need to determine only one function Π of the group that is in involution with the functions (*A*). When one follows the usual rules, one finds Π to be the function:

$$(M\varphi_4 - \varphi_7)^2 + (M\varphi_5 - \varphi_8)^2 + (M\varphi_6 - \varphi_9)^2.$$

With that, the desired system in involution is found [*]). After that, an elimination will give an equation of the form:

$$f(x_1, \dots, x_4, p_1, \dots, p_4) = 0,$$

to which the integration of the three-body problem will reduce. This known result is thus reduced by its intrinsic nature.

Before Mayer and I published our integration method in the year 1872, the solution of the problem by the Jacobi-Weiler method required a:

6, 4, 4, 2, 2

operation. Our work showed that only a:

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[*]) Clebsch reduced the integration to the system in involution that was presented here in his lectures on the three-body problem.

6, 4, 2

operation was required.

It is self-explanatory that the reasoning of this paragraph extend to the general problem of n bodies with no further assumptions *).

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*) I will develop the mechanics of an n-fold extended manifold with constant scalar curvature on another occasion. The integrals of the equations of motion, which have their basis in the free mobility of the space in question, can be presented by means of a general principle that I will give at another time. This treatise will then show how one is to employ these integrals best. It is not known to me whether the theory that I just suggested has already been presented.

Part three

Theory of homogeneous groups.

In this Part, I will consider a number of homogeneous functions $H_1, \ldots, H_r$ of $x_1, .., x_n$, $p_1, \ldots, p_n$ and determine all of the relations that exist between them that remain unchanged under homogeneous contact transformations. At the same time, the corresponding problem for arbitrary functions of $z, x_1, .., x_n, p_1, \ldots, p_n$ that are subject to arbitrary contact transformations will be solved.

§ 21.

Homogeneous groups.

44. First, we shall introduce a new concept. It rests upon the following theorem:

Theorem 53. If H_α and H_β are homogeneous functions of degree α and β, resp. then (H_α, H_β) is homogeneous of degree $(\alpha + \beta + 1)$.

One then has:

$$(H_\alpha, H_\beta) = \sum_{i=1}^{n} \left(\frac{\partial H_\alpha}{\partial x_i} \frac{\partial H_\beta}{\partial p_i} - \frac{\partial H_\alpha}{\partial p_i} \frac{\partial H_\beta}{\partial x_i} \right).$$

Now, $\partial H_\alpha / \partial x_i$ and $\partial H_\beta / \partial x_i$ are homogeneous of degree α and β, resp. Furthermore, $\partial H_\alpha / \partial p_i$ and $\partial H_\beta / \partial p_i$ are homogeneous of degree $\alpha - 1$ and $\beta - 1$, resp. Therefore, $\frac{\partial H_\alpha}{\partial x_i} \cdot \frac{\partial H_\beta}{\partial p_i}$, as well as $\frac{\partial H_\alpha}{\partial p_i} \cdot \frac{\partial H_\beta}{\partial x_i}$, are of degree $\alpha + \beta + 1$. Thus, (H_α, H_β) is also of degree $\alpha + \beta + 1$.

Corollary. If two or more homogeneous functions $H_1, \ldots, H_\rho$ generate an r-parameter group then in any form in which they are featured they consist of r homogeneous parameters. As far as that is concerned, one must remark that functions that belong to such a group are not homogeneous, in general.

Definition. *An r-parameter group is called* homogeneous *when it contains r homogeneous, mutually independent functions.*

Theorem 54. If $H_1, \ldots, H_r$ are homogeneous functions that define a group, and F denotes an arbitrary function of this group then $\sum_{k=1}^{n} p_k \frac{\partial F}{\partial p_k}$ also belongs to our group.

When one first sums over k, and in so doing recalls that all H_i are homogeneous – perhaps, of degree s – the equation:

$$\sum_{k=1}^{n} p_k \frac{\partial F}{\partial p_k} = \sum_{k=1}^{n}\sum_{i=1}^{r} p_k \frac{\partial F}{\partial H_i}\frac{\partial H_i}{\partial p_k}$$

goes to:

$$\sum_{k=1}^{n} p_k \frac{\partial F}{\partial p_k} = \sum_{i=1}^{r} s_i \frac{\partial F}{\partial H_i}\cdot H_i \,.$$

However, the right-hand side is a function of $H_1, \ldots, H_r$, here.

Theorem 55. If $K_1, \ldots, K_r$ define a group that possesses the property that $\sum_{k=1}^{n} p_k \frac{\partial K}{\partial p_k}$ can be expressed in terms of K then the group is homogeneous.

Namely, if some of the expressions:

$$\sum_{k=1}^{n} p_k \frac{\partial K}{\partial p_k} = \Omega_i(K_1, \ldots, K_r)$$

are non-zero then the equation:

$$\sum_{k=1}^{n} p_k \frac{\partial \Phi}{\partial p_k} = \Phi,$$

or the corresponding one:

$$\sum_{i=1}^{r} \Omega_i \frac{\partial \Phi}{\partial K_i} = \Phi,$$

is a linear, partial differential equation with r mutually independent solutions:

$$\Phi_1, \ldots, \Phi_r$$

that are homogeneous of degree 1 and belong to our group. Finally, if all Ω_i are equal to zero then this means that all K are of degree zero; the group is also homogeneous in this case.

Theorem 56. If all functions of a homogeneous group have degree zero then the group is a system in involution.

In fact, let $N_1, \ldots, N_r$ be functions of degree zero that define an r-parameter group. If the expression (N_i, N_k) is non-zero then (Theorem 53) it must be of degree – 1. Now, it is, however, possible to express a function of degree – 1 in terms of quantities of $N_1, \ldots, N_r$ of degree zero. Thus, all (N_i, N_k) must be zero, and the group is a system in involution.

Theorem 57. If a homogeneous group contains functions that are not all of degree zero then the group can take the form $N_1, \ldots, N_{r-1}, H$. Here, all N functions are of degree zero and H is a function of degree one.

Namely, if $H_1, \ldots, H_r$ are homogeneous functions of our group then it is always possible, when one replaces each H with a certain power of it, to given the group a form:

$$N_1, \ldots, N_\rho, H_{\rho+1}, \ldots, H_r$$

that includes only parameters of degree zero and one. Now, if one sets:

$$\frac{H_{\rho+1}}{H_r} = N_{\rho+1}, \ \ldots, \ \frac{H_{r-1}}{H_r} = N_{r-1}$$

then

$$N_1, \ldots, N_{r-1}, H$$

is a form for our group that fulfills the stated requirements.

§ 22.

The polar group and the distinguished functions of a homogeneous group are homogeneous.

45. The theory of this Part rests upon a theorem that we will now prove. First, we give a lemma.

Theorem 58. *The equations:*

$$(H, K) = 0, \qquad \sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} = s\,H,$$

in which s denotes a constant, imply the following one:

$$\left(H, \sum_{k=1}^{n} p_k \frac{\partial K}{\partial p_k}\right) = 0.$$

If we set:

$$A(H) = (H, K), \qquad B(H) = \sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} - s\,H$$

then, since $A(0) = B(0) = 0$, any common solution H of our two equations is also a solution of the equation:

$$A(B(H)) - B(A(H)) = 0.$$

However, by performing the calculations one finds that:

$$A(B(H)) - B(A(H)) = (H, K) - \left(H, \sum_{k=1}^{n} p_k \frac{\partial K}{\partial p_k} \right).$$

Thus, our two equations do, in fact, imply the third one:

$$\left(H, \sum_{k=1}^{n} p_k \frac{\partial K}{\partial p_k} \right) = 0.$$

Theorem XVIII. *The polar group of a homogeneous group is homogeneous.*

Proof. Let $H_1, \ldots, H_r$ be homogeneous functions that define an r-parameter group, and let $K_1, \ldots, K_{2n-r}$ be the polar group. The equations:

$$(H_i , K_\rho) = 0, \quad \sum_{k=1}^{n} p_k \frac{\partial H_i}{\partial p_k} = s_i \, H_i ,$$

are, in turn, valid, which, from the foregoing theorem, implies the following ones:

$$\left(H_i, \sum_{k=1}^{n} p_k \frac{\partial K_\rho}{\partial p_k} \right) = 0 \qquad (i = 1, \ldots, r).$$

Therefore, the group $K_1, \ldots, K_{2n-r}$ possesses the property that the expression $\sum_{k=1}^{n} p_k \frac{\partial K_\rho}{\partial p_k}$ is a function of the quantities $K_1, \ldots, K_{2n-r}$, in any case. Thus (Theorem 55), $K_1, \ldots, K_{2n-r}$ define a homogeneous group.

Theorem 59. Let $H_1, \ldots, H_r$ be a homogeneous group. The equations:

$$(H_1 , \Phi) = 0, \ldots, (H_r , \Phi) = 0, \qquad \sum_{k=1}^{n} p_k \frac{\partial \Phi}{\partial p_k} = 0,$$

in turn, define a complete system if the last equation is not an accidental algebraic consequence of the remaining ones.

The polar group of $H_1, \ldots, H_r$ is, in fact, homogeneous, and thus possesses (Theorem 57) either the form $N_1, \ldots, N_{2n-r}$, H or the form $N_1, \ldots, N_{2n-r}$. In the first case, there are $2n - r - 1$ of the $2n - r$ solutions of the complete system:

$$(A) \qquad (H_1, \Phi) = 0, \ldots, (H_r, \Phi) = 0,$$

namely, $N_1, \ldots, N_{2n-r}$, that likewise satisfy the equation:

$$(B) \qquad \sum_{k=1}^{n} p_k \frac{\partial \Phi}{\partial p_k} = 0.$$

Thus:

$$(H_1, \Phi) = 0, \ldots, (H_r, \Phi) = 0, \quad \sum_{k=1}^{n} p_k \frac{\partial \Phi}{\partial p_k} = 0,$$

in turn, define a complete system. In the second case, all solutions of equations (A) are likewise solutions of (B), an equation that is then a consequence, and indeed and algebraic consequence, of (A).

It must be remarked that in this last case one must have $r \geq n$. The polar group, since it consists of functions of degree zero, is then (Theorem 56) a system in involution, and can thus contain at most n parameters.

With the use of the last theorem, I will give an integration method for the equation:

$$N_1\left(x_1, \cdots, x_n, \frac{p_1}{p_n}, \cdots, \frac{p_{n-1}}{p_n}\right) = \text{const.}$$

that agrees with Mayer's and my previous theories in regard to the number and order of necessary integrations.

I pose the complete system:

$$(N_1 , F) = 0, \quad \sum_{k=1}^{n} p_k \frac{\partial F}{\partial p_k} = 0,$$

and determine a solution N_2 of it by means of a $2n - 3$ operation. N_1, N_2 is, in turn, a system in involution. I pose the complete system:

$$(N_1, F) = 0, \qquad (N_2, F) = 0, \qquad \sum_{k=1}^{n} p_k \frac{\partial F}{\partial p_k} = 0,$$

and determine a solution N_3 of it that is different from N_1 and N_2 by means of a $2n - 5$ operation. In this way, one finally finds a system in involution:

$$N_1 = a_1, \qquad N_2 = a_2, \qquad \ldots, \qquad N_n = a_n .$$

One eliminates the differential quotients $p_1, \ldots, p_n$ from these equations, which is always possible, since the p appear only as ratios, then one obtains an equation in $x_1, \ldots, x_n$, or more in some situations, that represents the one complete solution of the given equation.

46. We now turn to the distinguished functions of homogeneous groups.

Theorem XIX. *The distinguished functions of a homogeneous group define a homogeneous group.*

Namely, let $H_1, \ldots, H_r$ be a homogeneous group, and let $K_1, \ldots, K_{2n-r}$ be its polar group. If these two groups possess m common distinguished functions then (Theorem VII) it is always possible to choose $r - m$ parameters in the first group – say, $H_1, \ldots, H_{r-m}$ – such that no relation exists between the $2n - m$ quantities:

$$H_1, \ldots, H_{r-m}, K_1, \ldots, K_{2n-r}.$$

These functions, in turn, define a group, and indeed a homogeneous group, whose polar group, which must likewise be homogeneous, consists of the distinguished functions of the original group (Theorem 25, proof). Our theorem is thus proved.

Theorem 60. *If some of the m distinguished functions of a homogeneous group have non-zero degrees then one can determine all of the distinguished functions by means of an:*

$$m-1,\ m-2,\ \ldots,\ 3,\ 2,\ 1,\ 1$$

operation. My older method required an m, $m - 1, \ldots, 3, 2, 1$ operation.

Proof. We restrict ourselves to the case in which our group already possesses the form $N_1, \ldots, N_{r-1}$. If we denote a function of $N_1, \ldots, N_{r-1}$ by N then the equations:

$$(N_1, \mathsf{N}) = 0, \quad \ldots, \quad (N_{r-1}, \mathsf{N}) = 0, \quad (H, \mathsf{N}) = 0,$$

or, when they are developed:

$$(\alpha) \qquad \sum_{k=1}^{r-1} (N_1, N_k) \frac{\partial \mathsf{N}}{\partial N_k} = 0, \quad \ldots, \quad \sum_{k=1}^{r-1} (H, N_k) \frac{\partial \mathsf{N}}{\partial N_k} = 0,$$

determine the distinguished functions of degree zero. Now, the (N_i, N_k) are functions of degree -1, and (H, N_k) is a function of degree zero in $N_1, \ldots, N_{r-1}, H$. Therefore, this expression must have the form:

$$(N_i, N_k) = \frac{f_{ik}(N_1, \ldots, N_{r-1})}{H},$$

$$(H, N_k) = \varphi_{ik}(N_1, \ldots, N_{r-1}).$$

By substituting these values, one converts equations (α), when one multiplies the first $r - 1$ of them by H, into r equations that only contain the $r - 1$ independent variables $N_1, \ldots, N_{r-1}$; the variable H has vanished completely. One now establishes how many common solutions our r linear equations possess in the usual way. If they have m of them – i.e., if all distinguished functions are of degree zero – then, as usual, their determination requires:

$$m-1,\ m-2,\ \ldots,\ 3,\ 2,\ 1$$

operations. After one has determined $m - 1$ distinguished functions of degree zero in this way, one finds a further distinguished function that is not of degree zero by one more operation.

§ 23.

Canonical forms for homogeneous groups.

We shall first prove some lemmas. We then present two canonical forms; any homogeneous group can assume the one or the other of these two forms. For the sake of convenience, I will always need that symbol P in the sequel in order to denote a homogeneous function of degree one.

47. Theorem 61. Among the functions F of a homogeneous group $N_1, \ldots, N_{r-1}, P$ that satisfy the equation:

$$(N_1, F) = 1,$$

there is one that has degree one, and thus possesses the form $P \cdot \mathsf{N}\,(N_1, \ldots, N_{r-1})$. Obviously, N_1 cannot be a distinguished function.

By performing the calculation, we then find that:

$$(N_1, P\mathsf{N}) = (N_1, P)\,\mathsf{N} + (N_1, \mathsf{N})\,P,$$

or

$$(N_1, P\mathsf{N}) = (N_1, P)\,\mathsf{N} + \sum_{k=1}^{r-1} (N_1, N_k) P \frac{\partial \mathsf{N}}{\partial N_k}.$$

Here, (N_1, P) has degree zero, while (N_1, N_k) has degree -1, and thus $(N_1, N_k)\,P$ also has degree zero. Therefore, these expressions, which are known functions of $N_1, \ldots, N_{r-1}, P$, must then have the form:

$$(N_1, P) = \varphi(N_1, \ldots, N_{r-1}),$$
$$(N_1, N_k)\,P = f_k(N_1, \ldots, N_{r-1}).$$

By substituting these values, however, the equation $(N_1, P\mathsf{N}) = 1$ is converted into the following one:

$$\varphi \cdot \mathsf{N} + \sum_{k=1}^{r-1} f_k \frac{\partial \mathsf{N}}{\partial N_k} = 1,$$

which no longer contains P, at all, and is a linear, partial differential equation with the independent variables $N_1, \ldots, N_{r-1}$. If N is an arbitrary solution of it then $P\,\mathsf{N}$ is a function of degree one of our group that fulfills the requirements of our theorem.

Theorem 62. A homogeneous group of the form $N_1, \ldots, N_{r-1}, P$ contains functions $\mathsf{N}(N_1, \ldots, N_{r-1})$ of degree zero that satisfy the equation:

$$(P, \mathrm{N}) = 1,$$

when one naturally assumes that P is not a distinguished function.

One then has:

$$(P, \mathrm{N}) = \sum_{k=1}^{r-1} (P, N_k) \frac{\partial \mathrm{N}}{\partial N_k},$$

and (P, N_k), as functions of degree zero, must be expressible in terms of only $N_1, \dots, N_{r-1}$. Thus, if:

$$(P, N_k) = f_k(N_1, \dots, N_{r-1})$$

then

$$\sum_{k=1}^{r-1} f_k \frac{\partial \mathrm{N}}{\partial N_k} = 1$$

is a linear, partial differential equation on N and the independent variables $N_1, \dots, N_{r-1}$ whose solutions belong to our group, and have the desired relationship with the given function P.

Theorem 63. If a homogeneous group $N_1, \dots, N_{r-1}, P$ contains a two-parameter subgroup N_1, P then the $(r - 2)$-parameter subgroup that is in involution with the two-parameter one is also homogeneous (Theorem 34).

Namely, if $H_1, \dots, H_{2n-r}$ is the polar group of $N_1, \dots, N_{r-1}, P$ then it is known that:

$$H_1, \dots, H_{2n-r}, N_1, P$$

is a homogeneous group whose homogeneous polar group is just the $(r - 2)$-parameter subgroup that we spoke of. Our theorem is thereby proved.

48. Theorem XX. *A homogeneous group can always assume the form:*

$$X_1, P_1, \dots, X_q, P_q, U_1, \dots, U_m .$$

Here, X_i and P_i are functions of degree zero and one, resp., that have the known reciprocal relationships. $U_1, \dots, U_m$ are the distinguished functions of the group that define a homogeneous group, in their own right.

Proof. If the given homogeneous group $H_1, \dots, H_r$ is a system in involution then it already has the desired form. If that is not the case then one takes X_1 to be a function of degree zero and then determines a function of degree one P_1 of the group from the first theorem of this paragraph using the equation:

$$(X_1, P_1) = 1.$$

One then determines (prev. theorem) the homogeneous (r – 2)-parameter subgroup:

$$H_1^{(1)}, \ldots, H_{r-2}^{(1)}$$

that is in involution with X_1, P_1. With that, the original group assumes the form:

$$X_1, P_1, H_1^{(1)}, \ldots, H_{r-2}^{(1)}.$$

If $H_1^{(1)}, \ldots, H_{r-2}^{(1)}$ is a system in involution then the original group is already in the desired form. If that is not the case then we decompose $H_1^{(1)}, \ldots, H_{r-2}^{(1)}$ into two homogeneous groups X_2, P_2 and $H_1^{(2)}, \ldots, H_{r-4}^{(2)}$ that are in involution, from which the original group assumes the form:

$$X_1, P_1, X_2, P_2, H_1^{(2)}, \ldots, H_{r-4}^{(2)}.$$

If $H_1^{(2)}, \ldots, H_{r-4}^{(2)}$ is a system in involution then the desired form has been found. In the contrary case, we perform another decomposition, etc.

If finitely many decompositions – say, q – are possible then our group has assumed the desired form:

$$X_1, P_1, \ldots, X_q, P_q, H_1^{(q)}, \ldots, H_{r-2q}^{(q)}.$$

Here, two further cases are now conceivable. Either all of the distinguished functions are of degree zero, or there are some distinguished functions whose degree is non-zero. Thus:

Corollary 1. *If all distinguished functions of a homogeneous group are of degree zero then:*

$$X_1, P_1, \ldots, X_q, P_q, X_{q+1}, \ldots, X_{q+m}$$

is the canonical form of the group. Here X_i and P_i denote functions of degree zero and one, resp., that have the known mutual relationships.

Corollary 2. *If a homogeneous group contains distinguished functions that are not of degree zero then:*

$$X_1, P_1, \ldots, X_q, P_q, X_{q+1}, \ldots, X_{q+m-1}, P_{q+m},$$

or, what amounts to the same thing:

$$X_1, P_1, \ldots, X_q, P_q, P_{q+1}, \ldots, P_{q+m},$$

is the canonical form of the group.

The arguments at the end of the foregoing paragraph show how one decides whether a given homogeneous group belongs to the one or the other of the two stated categories.

§ 24.

Invariant properties of a homogeneous group.

I will now prove that the only properties of a homogeneous group that are independent of the form of it, and thus remain unchanged under arbitrary homogeneous contact transformations (which obviously always take the given group to a new *homogeneous* group) can be expressed by means of three positive whole numbers:

1) The number of parameters.
2) The number of distinguished functions.
3) The number of distinguished functions of degree zero.

For this investigation, I embark upon a path that is very similar to the one that I followed in § 13, to which I shall refer.

49. First, I consider groups whose distinguished functions are all of degree zero.

Theorem 64. If $X_1, \ldots, X_{q+m}, P_1, \ldots, P_q$ define a homogeneous group then there are always functions P_{q+1} such that $X_1, \ldots, X_{q+m}, P_1, \ldots, P_{q+1}$ defines a new canonical homogeneous group that envelops the given one.

The polar group of $X_1, \ldots, X_q, X_{q+2}, \ldots, X_{q+m}, P_1, \ldots, P_{q+1}$ is then homogeneous and contains X_{q+1}, which is not a distinguished function. From the second theorem of the foregoing paragraph, our polar group thus contains functions of degree one – say, P_{q+1} – that yield:

$$(X_{q+1}, P_{q+1}) = 1,$$

and thus satisfy all of our requirements.

Theorem 65. If $X_1, \ldots, X_{q+m}, P_1, \ldots, P_q$ define a homogeneous group then there are always functions $P_{q+1}, \ldots, P_{q+m}$ such that $X_1, \ldots, X_{q+m}, P_1, \ldots, P_{q+m}$ is a canonical homogeneous group that envelops the original one.

This theorem is obtained immediately upon an *m*-fold application of the previous one.

Theorem 66. $X_1, \ldots, X_q, P_1, \ldots, P_q$ define a homogeneous group and $q < m$ then there is always a function X_{q+1} of degree zero that is in involution with our group. $X_1, \ldots, X_{q+1}, P_1, \ldots, P_q$ is, in turn, a new canonical group that envelops the given one.

The polar group of $X_1, \ldots, X_q, P_1, \ldots, P_q$ is then homogeneous, and consists of at least two parameters. It therefore contains at least one function of degree zero that satisfies our requirements.

Theorem 67. If $X_1, \dots, X_q, P_1, \dots, P_q$ is a homogeneous group then there are always further functions X and P of degree zero and one, resp., such that $X_1, \dots, X_n, P_1, \dots, P_n$ define a canonical homogeneous group that envelops the given one.

This theorem follows as corollary to the previous one.

Theorem XXI. *If two homogeneous groups whose distinguished functions are all of degree zero possess equally many parameters and equally many distinguished functions then there are always homogeneous contact transformations that take the one group to the other one.*

Proof. Let the parameters of one group be functions of $x_1, \dots, p_n$, and the other, functions of $x'_1, \dots, p'_n$, respectively. From the assumptions made, the two groups can assume the canonical forms:

$$X_1, \dots, X_{q+m}, P_1, \dots, P_q$$
$$X'_1, \dots, X'_{q+m}, P'_1, \dots, P'_q,$$

where the X, P are naturally functions of the x, p, and the X', P' are functions of the x', p'.

From the foregoing theorem, there are always further functions X, P (X', P', resp.) such that:

$$X_1, \dots, X_n, P_1, \dots, P_n \quad \text{and} \quad X'_1, \dots, X'_n, P'_1, \dots, P'_n,$$

in turn, define canonical homogeneous groups.

From Theorem X (proof), the $2n$ equations:

$$X_i = X'_i, \qquad P_i = P'_i$$

thus define a contact transformation. However, this takes the one group to the other one; it is, moreover, homogeneous, so the theorem is proved.

50. We now turn to the homogeneous groups with distinguished functions that are not all of degree zero.

Theorem 68. If $X_1, \dots, X_q, P_1, \dots, P_{q+m}$ define a canonical homogeneous group then there are always functions X_{q+1} such that $X_1, \dots, X_{q+1}, P_1, \dots, P_{q+m}$, in turn, is a canonical homogeneous group that envelops the given one.

The polar group of $X_1, \dots, X_q, P_1, \dots, P_q, P_{q+2}, \dots, P_{q+m}$ is homogeneous and contains P_{q+1}, which is not a distinguished function of it. Therefore (Theorem 62), this group contains functions of degree zero that yield:

$$(P_{q+1}, X_{q+1}) = 1,$$

and thus satisfy our requirements.

Theorem 69. If $X_1, \ldots, X_q, P_1, \ldots, P_{q+m}$ is a canonical homogeneous group then there are always functions $X_{q+1}, \ldots, X_{+m}$ such that $X_1, \ldots, X_{q+m}, P_1, \ldots, P_{q+m}$ defines a canonical homogeneous group that envelops the given one.

This theorem arises immediately from the m-fold application of the previous one.

Theorem 70. If $X_1, \ldots, X_q, P_1, \ldots, P_{q+m}$ define a canonical homogeneous group then there are always further functions X and P such that $X_1, \ldots, X_n, P_1, \ldots, P_n$ defines a canonical homogeneous group that envelops the given one.

This theorem follows by successive application of the previous theorems of this paragraph.

Theorem XXII. *If two homogeneous groups whose distinguished functions are not all of degree zero possess just as many parameters and distinguished functions then there is always a homogeneous contact transformation that takes the one group to the other.*

From our assumptions, our group can then assume one of the two canonical forms:

$$X_1, \ldots, X_q, P_1, \ldots, P_{q+m} \text{ and } X_1', \ldots, X_q', P_1', \ldots, P_{q+m}',$$

resp., where all X, P are functions of $x_1, \ldots, p_n$, and all X', P' are functions of $x_1', \ldots, p_1'$. Therefore, there are further functions X, P and X', P' such that:

$$X_1, \ldots, X_n, P_1, \ldots, P_n \text{ and } X_1', \ldots, X_n', P_1', \ldots, P_n'$$

is also a canonical homogeneous group. Thus, the $2n$ equations:

$$X_i = X_i', \qquad P_i = P_i',$$

in turn, define a homogeneous contact transformation that takes the two groups to each other.

Corollary. *The only properties of a homogeneous group that are independent of its form and remain unchanged under homogeneous contact transformations are:*

1) *The number r of parameters.*

2) *The number m of distinguished functions.*

3) *The number of distinguished functions of degree zero, which must be equal to either m or m* + 1.

Here, r is a positive whole number that cannot be greater than 2*n. Moreover, we have found in the foregoing Part that r – m must be a positive even number, and finally, that r+ m is equal to at most* 2*n.*

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51. There are no obstacles at all to extending the theory of paragraphs 14, 15, and 16 to homogeneous functions and homogeneous contact transformations.

This shows that the invariant relations between a homogeneous group and a homogeneous subgroup are determined completely by way of eight numbers. The first six of them define the individual invariant properties of each of the two homogeneous groups. The last two are the number ϖ of common distinguished functions and the number, ϖ or $\varpi - 1$, of common distinguished functions of degree zero.

Should one select a system in involution from a homogeneous group, then it would always be possible to arrive at a reduction in the order of the necessary integrations.

Should one decide whether r given homogeneous functions $H_1, \ldots, H_r$ can go to H_1', …, H_r', resp., by a homogeneous contact transformation, then one could always assume (§ 16) that all of the H_i, and likewise, all of the H_i', are mutually independent. A first requirement is that the corresponding functions of the two systems should be of the same degree. If this demand is fulfilled then one determines, as in § 16, the two groups:

$$H_1, \ldots, H_r, \ldots, H_\alpha \qquad \text{and} \qquad H_1', \ldots, H_r', \ldots, H_{\alpha'}'$$

that are determined by our groups. Here, α' must be equal to α, and furthermore, every (H_i', H_k') must be expressible in terms of the H_i' in the same way that the corresponding (H_i, H_k) are expressible in terms of the H_i. If all of these requirements are fulfilled then one recognizes that the desired transformation is possible, and indeed, it will obviously be a *homogeneous* transformation.

With that, all of the relations between $H_1, \ldots, H_r$ that remain invariant under arbitrary homogeneous contact transformations are found.

§ 25.

Reductions of the integration that are based on the foregoing developments.

The foregoing theories show how one can exploit the circumstances that arise in the integration of a partial differential equation:

$$F(z, x_1, \ldots, x_{n-1}, p_1, \ldots, p_{n-1}) = 0$$

to the best advantage. As I proved in § 17, 18, and 19, especially, I can restrict myself to the following.

52. I assume that a system in involution of degree zero:

$$N_1 = C_1, \ldots, N_q = C_q$$

is to be integrated, and that one knows a number of homogeneous functions $H_1, \ldots, H_r$ that satisfy all of the equations $(N_i, H_k) = 0$. If it were impossible to determine further functions H by means of the Poisson-Jacobi theorem then $N_1, \ldots, N_q, H_1, \ldots, H_q$ would define a homogeneous group. We now first consider the case in which this group contains distinguished functions that are not of degree zero, and then the case where all of the distinguished functions are of degree zero.

A. If the group $N_1, \ldots, N_q, H_1, \ldots, H_r$ contains, in addition to $N_1, \ldots, N_q$, m distinguished functions:

$$N_{q+1}, \ldots, N_{q+m-1}, H$$

that are not all of degree zero then one determines them (Theorem 60) by means of:

$$m-1,\; m-2,\; \ldots,\; 3,\; 2,\; 1,\; 1$$

operations. Thereupon, one poses the problem of integrating the system in involution:

$$N_1 = C_1, \ldots, N_{q+m} = C_{q+m}\,,$$

with $r - m$ homogeneous solutions $H_1, \ldots, H_{r-m}$ of the $q + m$ equations $(N_k\,, H) = 0$, in the simplest possible way. To this end, one exhibits the equations:

$$(A) \qquad (N_1\,, H) = 0, \ldots, (N_{q+m}\,, H) = 0, \qquad (H_1, N) = 0, \ldots, (H_{r-m}, N) = 0,$$

$$\sum_{k=1}^{n} p_k \frac{\partial N}{\partial p_k} = 0,$$

which must define a complete system. In fact, if the polar group of the group:

$$N_1, \ldots, N_{q+m}, H_1, \ldots, H_{r-m}$$

consisted of only functions of degree zero then this polar group would be identical with the totality of all distinguished functions $N_1, \ldots, N_{q+m}$. However, the integration of our system in involution would (Theorem XIII) already be considered to be achieved; we thus do not need to consider this case. One knows $m + q$ solutions $N_1, \ldots, N_{q+m}$ of the complete system (A); one then finds a further solution N_{q+m+1} by means of a:

$$2n - 2q - r - m - 1$$

operation. With that, everything is reduced to the integration of the system in involution:

$$N_1 = C_1, \ldots, N_{q+m+1} = C_{q+m+1}\,,$$

with $r - m$ solutions $H_1, \ldots, H_{r-m}$ of the system $(N_i\,, H) = 0$. One now proceeds in an analogous way, and determines a function N_{q+m+2} by means of a:

$$2n - 2q - r - m - 3$$

operation, etc., and finally determines a last function N by means of a 1-operation.

With that, from my previous argument (Theorem XIII), the integration process is concluded.

§ 26.

Completion of the theory of the Poisson-Jacobi theorem.

The Poisson-Jacobi theorem is capable of a completion that I will now give. First, I consider arbitrary functions of $x_1, \ldots, p_n$, and then homogeneous functions of x, p.

53. If φ_1 and φ_2 are solutions of the equation:

$$(f, \varphi) = 0$$

then the Poisson-Jacobi theorem says that (φ_1, φ_2) is also such a solution.

There are some related theorems, of which I will cite the following one, which originates with Laurent:

If $\varphi = \varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \varphi_k$ are any $2k$ solutions of the equation $(f, \varphi) = 0$ then:

$$\sum_{\lambda_1 \cdots \lambda_k} \sum \pm \frac{\partial \varphi_1}{\partial x_{\lambda_1}} \cdots \frac{\partial \varphi_k}{\partial x_{\lambda_k}} \frac{\partial \psi_1}{\partial p_{\lambda_1}} \cdots \frac{\partial \psi_k}{\partial p_{\lambda_k}}$$

is always one, too.

Mayer drew my attention to this theorem, and remarked that the same way would probably give those solutions that one could obtain by the successive application of the Poisson-Jacobi theorem. As a response to that, I can advise him of the theory of this paragraph.

Theorem 71. If all common solutions F of the equations:

$$(\Phi_1, F) = 0, \ldots, (\Phi_q, F) = 0$$

also simultaneously satisfy the equation:

$$(\Pi, F) = 0$$

then Π belongs to the group $\Phi_1, \ldots, \Phi_q$ that is determined by $\Phi_1, \ldots, \Phi_q, \ldots, \Phi_r$.

The common solutions of the given q equations are then the solutions of the complete system:

$$(\Phi_1, F) = 0, \ldots, (\Phi_r, F) = 0.$$

If one denotes them by $F_1, \ldots, F_{2n-r}$ then one must have:

$$(\Pi, F_1) = 0, \ldots, (\Pi, F_{2n-r}) = 0,$$

that is, Π must belong to the polar group of $F_1, \ldots, F_{2n-r}$. However, this polar group is just $\Phi_1, \ldots, \Phi_r$ itself.

This immediately yields the following remarkable theorem:

Theorem XXIII. *Of one knows any q solutions $\Phi_1, \ldots, \Phi_q$ of the equation:*

$$(F, \Phi) = 0,$$

and one finds a further solution Π from these solutions by any sort of operations that are entirely independent of the form of the function F then Π always belongs to the group that is determined by $\Phi_1, \ldots, \Phi_q$.

54. This theorem is no longer correct when the function F is subjected to certain restrictions. I will consider the important case in which F is a homogeneous function, and develop a corresponding theory for it.

We have previously seen (Theorem 54) that any homogeneous group that contains a function Φ must likewise contain the function:

$$\sum_{k=1}^{n} p_k \frac{\partial \Phi}{\partial p_k},$$

and that, conversely (Theorem 55), a group $\Phi_1, \ldots, \Phi_r$ is homogeneous when any:

$$\sum_{k=1}^{n} p_k \frac{\partial \Phi_i}{\partial p_k}$$

can be expressed in terms of the Φ. As a result, I can speak about the homogeneous group that is determined by a given function, and likewise about the homogeneous group that is determined by several given functions.

Theorem 72. Let F be a homogeneous function, let Φ be any function that is related to it by way of:

$$(F, \Phi) = 0,$$

and finally, let $\Phi_1, \ldots, \Phi_r$ be the homogeneous group that is determined by Φ. All equations $(F, \Phi_k) = 0$ are then true.

This theorem is a consequence of a previous one (Theorem 58), namely, that the equation:

$$(F, \Phi) = 0$$

implies the following one:

$$\left(F, \sum_{k=1}^{k-n} p_k \frac{\partial \Phi}{\partial p_k}\right) = 0,$$

when coupled with the Poisson-Jacobi theorem.

Theorem 73. Let F be a homogeneous function, and let $\Phi_1, \ldots, \Phi_q$ be given functions that are each in involution with F – i.e., they yield $(F, \Phi_k) = 0$. If one then denotes the homogeneous group that is determined by $\Phi_1, \ldots, \Phi_q$ by $\Phi_1, \ldots, \Phi_q, \ldots, \Phi_q$ then the new functions Φ are also in involution with F.

This theorem is a consequence of the foregoing one, when coupled with the Poisson-Jacobi theorem.

Theorem 74. If all of the common *homogeneous* solutions F of the equations:

$$(\Phi_1, F) = 0, \ldots, (\Phi_q, F) = 0$$

likewise satisfy the relation:

$$(\Pi, F) = 0$$

then Π belongs to the homogeneous group $\Phi_1, \ldots, \Phi_q, \ldots, \Phi_q$ that is determined by $\Phi_1, \ldots, \Phi_q$.

The common homogeneous solutions F to the given q equations are then the homogeneous solutions to the complete system:

$$(\Phi_1, F) = 0, \ldots, (\Phi_r, F) = 0.$$

There are $2n - r$ such solutions $F_1, \ldots, F_{2n-r}$ that define the polar group of $\Phi_1, \ldots, \Phi_r$ precisely.

We then have the theorem:

Theorem XXIV. *If one knows any q solutions* $\Phi_1, \ldots, \Phi_q$ *to the equation:*

$$(F, \Phi) = 0,$$

in which F denotes a homogeneous function, and one finds a further solution Π *by way of any sort of operations that are independent of the form of the homogeneous function F then* Π *also belongs to the homogeneous group that is determined by* $\Phi_1, \ldots, \Phi_q$.

Finally, one can develop an analogous theory for the case in which F is homogeneous of degree zero. In that way, one would obtain some remarkable results. Namely, one would, *a priori*, recognize the possibility of some simplifications to the integrations that I arrived at in § 25 in a different way.

In closing, I shall pose a general problem:

Let $H_1, \ldots, H_\alpha$ be homogeneous functions of degree one that have such a reciprocal relationship that any (H_i, H_k) can be expressed as a linear function with constant coefficients of the H:

$$(H_i, H_k) = \sum_{\omega=1}^{\alpha} C_{\omega}^{ik} H_{\omega} .$$

I then say that all H define a *transformation group*, and thus consider all linear functions of the H that have the form:

$$d_1 H_1 + d_2 H_2 + \dots + d_\alpha H_\alpha$$

as being equivalent to the H themselves. I now ask what the properties are of a given transformation group that remain invariant under homogeneous contact transformations. I have found that there are a *limited number* of types of transformation groups. Further research is necessary in order to clarify the precise sense, validity, and meaning of this assertion.

Christiania, 5 July 1874.