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The infinitesimal contact transformations of mechanics.

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On various occasions, I have already emphasized the importance of the general concept of *infinitesimal contact transformation.* The purpose of the following Note is, first, to point out some interesting examples that come about by simply computing with such transformations. At the same time, I wish to direct attention to the remarkable character of *a noteworthy category of infinitesimal transformations that play a role in mechanics.*

1.

An infinitesimal transformation in the variables $x_1, \dots, x_n, p_1, \dots, p_n$:

$$\delta x_\kappa = \xi_\kappa(x_1, \dots, x_n, p_1, \dots, p_n)\ \delta t,$$
$$\delta p_\kappa = \pi_\kappa(x_1, \dots, x_n, p_1, \dots, p_n)\ \delta t,$$

is a homogeneous contact transformation when it fulfills the condition equation:

$$\frac{\delta}{\delta t}\sum p_\kappa dx_\kappa = 0 = \sum \pi_\kappa dx_\kappa + \sum p_i d\xi_i\ .$$

This implies [1]) the fact that the ξ_κ and π_κ possess the form:

$$\xi_\kappa = \frac{\partial H}{\partial p_\kappa}, \qquad \pi_\kappa = -\frac{\partial H}{\partial x_\kappa},$$

and the fact that H is homogeneous of first order with respect to the $p_1, \dots, p_n$.

Therefore, the equations:

$$\delta x_\kappa = \frac{\partial H}{\partial p_\kappa}\,\delta t, \qquad \delta p_\kappa = -\frac{\partial H}{\partial x_\kappa}\,dt \qquad (\kappa = 1, 2, \dots, n),$$

[1]) Archiv for Mathematik og Naturvidenskab, Bd. 2, Christiana 1877; cf., also Math. Annalen, Bd. VIII, 1874, pp. 239.

$$p_1 \frac{\partial H}{\partial p_1} + \ldots + p_n \frac{\partial H}{\partial p_n} - H = 0$$

determine the most general infinitesimal homogeneous contact transformation in the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$.

If one sets, e.g., $n = 3$ and:

$$H = \sqrt{p_1^2 + p_2^2 + p_3^2}$$

then one obtains an infinitesimal contact transformation of the three-fold extended space x_1, x_2, x_3:

$$\delta x_k = \frac{p_\kappa}{\sqrt{p_1^2 + p_2^2 + p_3^2}}\, dt, \qquad \delta p_k = 0,$$

which obviously fulfills the relations:

$$\frac{\delta x_1}{p_1} = \frac{\delta x_2}{p_2} = \frac{\delta x_3}{p_3}, \qquad \delta p_k = 0,$$

$$\sqrt{\delta x_1^2 + \delta x_2^2 + \delta x_3^2} = \delta t.$$

It then takes all surface elements $x_1, x_2, x_3, p_1, p_2, p_3$ of the space x_1, x_2, x_3 *the same distance* δt *along the normal* to the element in question.

Our contact transformation is then an infinitesimal parallel transformation: It takes any surface into an infinitely close parallel surface.

We now pose the problem of finding the most general infinitesimal homogeneous contact transformation:

$$\delta x_\kappa = \frac{\partial H}{\partial p_\kappa}\, \delta t, \qquad \delta p_\kappa = -\frac{\partial H}{\partial x_\kappa}\, dt$$

in the n-fold extended space of $x_1, \ldots, x_n$ that takes any element x, p along a direction $\delta x_1, \ldots, \delta x_n$ that is perpendicular to the element in question such that one has the equations:

$$\frac{\delta x_1}{p_1} = \frac{\delta x_2}{p_2} = \ldots = \frac{\delta x_n}{p_n}.$$

This problem finds its analytical expression in the equations:

$$\frac{\frac{\partial H}{\partial p_1}}{p_1} = \frac{\frac{\partial H}{\partial p_2}}{p_2} = \ldots = \frac{\frac{\partial H}{\partial p_n}}{p_n}$$

$$p_1 \frac{\partial H}{\partial p_1} + \ldots + p_n \frac{\partial H}{\partial p_n} - H = 0,$$

the first of which says that H depends upon the quantities:

$$p_1^2 + \cdots + p_2^2, \qquad x_1, \ldots, x_n,$$

while the last one shows that H has the form:

$$H = \Omega(x_1, \ldots, x_n) \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}\,. \tag{1}$$

With that, we know an *extended category of infinitesimal homogeneous contact transformation of the n-fold extended space that possesses the property that any element proceeds in the direction of the associated normal.*

If such an infinitesimal transformation were repeated infinitely often then this would generate ∞^1 finite contact transformations that would define a one-parameter group. If all of the transformations of this group were performed on a manifold:

$$W(x_1, x_2, \ldots, x_n) = 0$$

then this would generate a family of ∞^1 manifolds:

$$\pi(x_1, \ldots, x_n) = \text{const.}$$

whose orthogonal trajectories are the orbits [1]) of the infinitesimal contact transformations in question.

It is now very noteworthy that the infinitesimal contact transformations of the form (1) *play a preeminent role in mechanics.*

The motion of a system of points will be, in fact, defined by a partial differential equation of first order:

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial W}{\partial x_n}\right)^2 - 2U(x_1, \ldots, x_n) - 2h = 0$$

that can be written in the following way:

$$p_1^2 + \cdots + p_n^2 - 2(U + h) = 0. \tag{2}$$

Now, for a long time, I have remarked that the integration of any partial differential equation:

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[1]) If all transformations of a one-parameter group of contact transformations performed on an element then this produce ∞^1 locations whose point locus will be referred to in the text as the *orbit* of the infinitesimal transformation in question.

$$\varphi(x_1, \ldots, x_n, p_1, \ldots, p_n) - a = 0$$

can be achieved by considering the infinitesimal contact transformation whose characteristic function is $\varphi - a$, and which determines the associated one-parameter group of contact transformations. In the present case, however, an important simplification enters into the picture that brings equation (2) into the form:

$$\frac{\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}}{\sqrt{2(U+h)}} = 1.$$

Here, the left-hand side is homogeneous of first order in $p_1, \ldots, p_n$; it thus represents an infinitesimal *homogeneous* contact transformation that belongs to the aforementioned general category.

Some interesting conclusions may be drawn from this remark. On the other hand, it would be simple to extend the present considerations to the case in which the coordinates x_κ were coupled by given relations.

2.

We now pose the problem (in three-fold extended space) of finding all infinitesimal contact transformations that take lines of curvature into other such curves.

If we denote the partial derivatives of a function f with respect to x_1, x_2, x_3 by p_1, p_2, p_3, as usual, then the curvature lines of the surface $f(x_1, x_2, x_3) = 0$ can be defined by the differential equation:

$$dp_1\,(p_2\,dx_3 - p_3\,dx_2) + dp_2\,(p_3\,dx_2 - p_1\,dx_3) + dp_3\,(p_1\,dx_2 - p_2\,dx_1) = 0,$$

or by the equivalent one:

$$\Delta = \begin{vmatrix} dp_1 & dp_2 & dp_3 \\ p_1 & p_2 & p_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} = 0.$$

Our problem can therefore be formulated in the following way:

Find all infinitesimal homogeneous contact transformations:

$$\delta x_\kappa = \frac{\partial H}{\partial p_\kappa}\,\delta t, \qquad \delta p_\kappa = -\frac{\partial H}{\partial x_\kappa}\,dt$$

that leave the systems of equations:

(3) $$\Delta = 0, \qquad p_1\,dx_1 + \ldots + p_n\,dx_n = 0$$

invariant.

The homogeneous function H must then be chosen in such a way that the expression:

$$\frac{\delta\Delta}{\delta t}$$

vanishes as a result of the system of equations (3).

By evaluating it, one comes to:

$$\frac{\delta\Delta}{\delta t} = -\begin{vmatrix} d\frac{\partial H}{\partial x_1} & d\frac{\partial H}{\partial x_2} & d\frac{\partial H}{\partial x_3} \\ p_1 & p_2 & p_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} - \begin{vmatrix} dp_1 & dp_2 & dp_3 \\ \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} & \frac{\partial H}{\partial x_3} \\ dx_1 & dx_2 & dx_3 \end{vmatrix} + \begin{vmatrix} dp_1 & dp_2 & dp_3 \\ p_1 & p_2 & p_3 \\ d\frac{\partial H}{\partial x_1} & d\frac{\partial H}{\partial x_2} & d\frac{\partial H}{\partial x_3} \end{vmatrix}.$$

This expression is a homogeneous function of second degree in the differentials dp_1, dp_2, dp_3, dx_1, dx_2, dx_3 whose coefficients depend upon only the x, p. An equation:

$$\text{(4)} \qquad \frac{\delta\Delta}{\delta t} = \lambda\Delta + \sum_\kappa p_\kappa dx_\kappa \left(\sum_\kappa \alpha_\kappa dx_\kappa + \sum_\kappa \beta_\kappa dp_\kappa\right)$$

is valid identically in the aforementioned differentials, in which λ, α_1, …, β_3 denote functions of x, p. Both sides of this equation are homogeneous functions of second degree in the differentials dx, dp. Therefore, if the corresponding coefficients on the left-hand and right-hand sides were computed and set equal to each other then this would yield some relations that would give differential equations after eliminating the auxiliary quantities λ, α, β that determine H.

Among the terms on the right-hand side of equation (4), one finds none that have the form $\nu\, dp_i\, dp_j$. One then obtains, with no further assumptions, six of the equations that are free of α, β, and λ, which can be written, with the use of the abbreviation:

$$\frac{\partial H}{\partial p_\kappa} = H_\kappa,$$

in the following way:

$$\text{(5)} \qquad \begin{cases} \frac{\partial}{\partial p_1}(p_2 H_3 - p_3 H_2) = 0, \\ \frac{\partial}{\partial p_2}(p_3 H_1 - p_1 H_3) = 0, \\ \frac{\partial}{\partial p_3}(p_1 H_2 - p_2 H_1) = 0, \end{cases}$$

$$(6)\qquad \begin{cases} \dfrac{\partial}{\partial p_1}(p_1H_3 - p_1H_3) + \dfrac{\partial}{\partial p_2}(p_2H_3 - p_3H_2) = 0, \\ \dfrac{\partial}{\partial p_2}(p_1H_2 - p_2H_1) + \dfrac{\partial}{\partial p_3}(p_3H_1 - p_1H_3) = 0, \\ \dfrac{\partial}{\partial p_3}(p_2H_3 - p_3H_2) + \dfrac{\partial}{\partial p_1}(p_1H_2 - p_2H_1) = 0. \end{cases}$$

The first three give, by integration:

$$(7)\qquad \begin{cases} p_2H_3 - p_3H_2 = K_{32}(p_2, p_3, x_1, x_2, x_3), \\ p_3H_1 - p_1H_3 = K_{13}(p_2, p_3, x_1, x_2, x_3), \\ p_1H_2 - p_2H_1 = K_{21}(p_2, p_3, x_1, x_2, x_3), \end{cases}$$

and it is therefore obvious that K_{ij}, like H itself, must be of first order in the p; by replacing the values thus found, equations (6) take on the form:

$$\frac{\partial K_{13}(p_3, p_1, x_1, x_2, x_3)}{\partial p_1} + \frac{\partial K_{32}(p_2, p_3, x)}{\partial p_2} = 0, \quad \frac{\partial K_{21}}{\partial p_2} + \frac{\partial K_{13}}{\partial p_3} = 0, \qquad \frac{\partial K_{32}}{\partial p_3} + \frac{\partial K_{21}}{\partial p_1} = 0,$$

and will thus be fulfilled in the most general way by expressions of the form:

$$K_{13} = \mu_{13}(x)\, p_1 - \mu_{21}(x)\, p_3\,,$$
$$K_{21} = \mu_{21}(x)\, p_2 - \mu_{32}(x)\, p_1\,,$$
$$K_{32} = \mu_{32}(x)\, p_3 - \mu_{13}(x)\, p_2\,.$$

As a result, equations (7) go to the following ones:

$$\frac{H_1 + \mu_{21}}{p_1} = \frac{H_2 + \mu_{32}}{p_2} = \frac{H_3 + \mu_{13}}{p_3},$$

which can be written in the following way when one introduces an auxiliary variable ρ:

$$(8)\qquad \begin{cases} H_1 = \rho\, p_1 - \mu_{21}, \\ H_2 = \rho\, p_2 - \mu_{32}, \\ H_3 = \rho\, p_3 - \mu_{13}. \end{cases}$$

Here, H_1, H_2, H_3 are the differential quotients of H with respect to p_1, p_2, and p_3; thus, the known integrability conditions give the equations:

$$\frac{\frac{\partial\rho}{\partial p_1}}{p_1} = \frac{\frac{\partial\rho}{\partial p_2}}{p_2} = \frac{\frac{\partial\rho}{\partial p_3}}{p_3},$$

such that ρ must have the form:

$$\rho = P(p_1^2 + p_2^2 + p_3^2, x_1, x_2, x_3) .$$

If this value of ρ were replaced in (8) then we would obtain three equations for the determination of H, and since, on the other hand, we know that H is homogeneous of first degree in the p, we can set:

$$H = \Omega(x_1, \ldots, x_n) \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2} + \sum_{\kappa=1}^{3} \xi_\kappa(x_1, x_2, x_3) p_\kappa .$$

We can now decompose our problem into two simpler problems.

Namely, if we remark that the quantity H and its differential quotients enter into the condition equation (4) *linearly* and *homogeneously* then we immediately recognize that the most general value of H can be represented as the sum of two particular values of this quantity, which we find when we, on the one hand, set Ω equal to zero and, on the other, set ξ equal to zero.

As is known, the problem of finding the most general infinitesimal *point* transformation:

$$H = \xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3$$

that leaves the differential equation of the curvature lines invariant subsumes the problem that LIOUVILLE treated of determining all conformal point transformations of ordinary space.

We therefore need to find only the most general infinitesimal homogeneous contact transformation of the special form:

$$H = \Omega(x_1, \ldots, x_n) \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}$$

that takes curvature lines to other ones. If we enter these special values of H in (4) then we obtain the relations:

$$\frac{\partial^2\Omega}{\partial x_1^2} = \frac{\partial^2\Omega}{\partial x_2^2} = \frac{\partial^2\Omega}{\partial x_3^2}, \qquad \frac{\partial^2\Omega}{\partial x_i\,\partial x_j} = 0,$$

for the determination of Ω, which show that H possesses the form:

$$\{a(x_1^2 + x_2^2 + x_3^2) + 2b_1x_1 + 2b_2x_2 + 2b_3x_3 + c\}\sqrt{p_1^2 + p_2^2 + p_3^2} . \tag{9}$$

If one sets, e.g.:

$$a = b_1 = b_2 = b_3 = 0, \qquad c = 1$$

then one gets:

$$H = \sqrt{p_1^2 + p_2^2 + p_3^2}\ ;$$

the corresponding transformation is an infinitesimal parallel transformation. On the other hand, if one sets:

$$a = 1, \qquad c = b_1^2 + b_2^2 + b_3^2$$

then H takes on the form:

$$H' = [(x_1 + b_1)^2 + (x_2 + b_2)^2 + (x_3 + b_3)^2]\sqrt{p_1^2 + p_2^2 + p_3^2}\ ,$$

or, after introducing the new variables:

$$x'_\kappa = \frac{x_\kappa + b_\kappa}{(x_1 + b_1)^2 + (x_2 + b_2)^2 + (x_3 + b_3)^2}\ ,$$

the form:

$$H' = \sqrt{p_1'^2 + p_2'^2 + p_3'^2}\ .$$

The infinitesimal transformation H' is therefore similar to an infinitesimal parallel transformation by means of a transformation through reciprocal radii.

If one takes an arbitrary infinitesimal transformation H of the form (9), *determines the associated one-parameter group, and performs of the all transformations on a completely arbitrary surface then one obtains* ∞^1 *surfaces that belong to an orthogonal system. Thus, formula* (9) *delivers the most general infinitesimal contact transformation of this kind.*

One obtains a more general method of construction for orthogonal systems when one regards the coefficients a_1, b_1, b_2, b_3, c in (9) as functions of a parameter.

I will exhibit the connection between my older method for the construction of orthogonal systems and the problems of mechanics in more detail on another occasion.

3.

If a differential equation of second order admits a continuous group of transformations then from my older investigations only the following cases are possible:

1) The group is similar to the eight-parameter group:

(I) $$p, \quad q, \quad xq, \quad xp, \quad yp, \quad yq, \quad x^2p + xyq, \quad xyp + y^2q,$$

which leaves only *one* differential second order invariant, namely, $y'' = 0$.

2) The group is similar with the three-parameter group:

(II) $p, \quad 2xp + yq, \quad x^2 p + xyq,$

which leaves invariant the ∞^1 differential equations of second order:

$$y^3\, y'' = A = \text{const.}$$

3) The group is similar to the group:

(III) $p + q, \qquad xp + yq, \qquad x^2 p + y^2 q,$

which leaves invariant the ∞^1 differential equations of second order:

$$(x - y)\, y'^{-3/2}\, y'' + 2(y'^{1/2} + y'^{-1/2}) = a = \text{const.}$$

4) The group is similar to the group:

(IV) $p, q, xp + (x + y)\, q,$

which leaves invariant the ∞^1 differential equations of second order:

$$y'' + k\, e^{-y'} = 0 \qquad (k = \text{const.})$$

5) The group is similar to the group:

(V) $p, q, xp + c\, yq, \qquad c \neq 0, \quad c \neq \infty, \quad c \neq 1,$

which leaves invariant the ∞^1 differential equations of second order:

$$y''\, y'^{\frac{2-c}{c-1}} = a = \text{const.}$$

6) The group is similar to the group:

(VI) $p, q,$

which leaves invariant any second-order differential equation of the form:

$$y'' - f(y') = 0.$$

7) The group is similar to the group:

(VII) $p, xp + yq,$

which leaves invariant any second-order differential equation of the form:

$$y'' - y\, f(y') = 0.$$

8) The group is similar to the group p that leaves invariant any second-order differential equation of the special form:

$$y'' - f(y, y') = 0.$$

We would now like to assume, in particular, that a second-order differential equation of the special form:

$$y'' - f_3(x, y)\, y'^3 + f_2(x, y)\, y'^2 + f_1(x, y)\, y' + f(x, y) = 0 \tag{10}$$

is present. Since any point transformation takes a differential equation of this form to another such equation, we can likewise conclude that the group of such equations never possesses the form (IV), and that it possesses the form (I) only when the differential equation in question can be converted into $y'' = 0$ by a point transformation. On the other hand, if the group of a differential equation (10) possesses the form (III) then this differential equation can take on the form:

$$(x - y)\, y'' + 2y'^2 + 2y' = 0,$$

and consequently, also the form $y'' = 0$. Finally, if the group of a differential equation (10) possesses the form (V) then the constant c fulfills one of the four equations:

$$\frac{c-2}{c-1} = 3, \qquad \frac{c-2}{c-1} = 2, \qquad \frac{c-2}{c-1} = 1, \qquad \frac{c-2}{c-1} = 0.$$

From the foregoing, among the three corresponding values:

$$c = \frac{1}{2}, \qquad c = 0, \qquad c = \infty, \qquad c = 2,$$

only the two obviously equivalent values:

$$c = \frac{1}{2}, \qquad c = 2$$

come under consideration. Therefore, it is easy to see that the corresponding differential equation of second order:

$$y'' - a = 0$$

can take on the form $y'' = 0$ by a suitable choice of variables.

This simple argument, which can be carried further, gives, *inter alia*, the following theorem:

If the family of geodetic curves on a surface whose curvature is not constant admits more than two independent infinitesimal transformations then these transformations generate a three-parameter group that possesses the canonical form p, $2xp + yp$, x^2p +

xyq. *In the inconvenient cases, the determination of the family of curves in question requires the integration of a Riccati equation of first order. The three-parameter group always leaves invariant a family that consists of* ∞^1 *geodetic curves.*

4.

If an r-parameter continuous group of *real* transformations of a simple manifold x is present then one sees, when one fixes a real point $x = a$, that the corresponding ∞^{r-1} real transformations define a *real* subgroup. In this way, one finds ∞^1 real $(r - 1)$-parameter subgroups.

From this, it follows that any real group of the simple manifold x can take on one of the three forms:

$$p; \qquad p, xp; \qquad p, xp, x^2 p$$

by a *real change of variables.*

If a real group of point transformations of a plane leaves only *one* differential equation of first (second, resp.) order invariant then this differential equation is real; if it leaves two and only two differential equations of first order invariant then the transformations are either both real or conjugate imaginary.

When one connects these remarks with my general determination of all r-parameter groups of a plane then one finds, with no new computations, canonical forms – so to speak – into which all real r-parameter groups of point transformations of a plane can be brought by real changes of variables.

Only for the case in which the group in question leaves invariant two and only two families of curves $\varphi(x, y) = a$, $\psi(x, y) = b$ that are pair-wise conjugate imaginary do new canonical forms appear. One finds them when one exhibits all real groups of point transformations that take circles to circles; one achieves this objective even more quickly when one replaces the x, y in the groups that I presented (Math. Ann., Bd. XVI, pp. 524, *C*) with the new variables $\mathfrak{x}$, $\mathfrak{y}$ by the substitution:

$$x = \mathfrak{x} + i\mathfrak{y}, \qquad y = \mathfrak{x} - i\mathfrak{y},$$

and thus demands that the r-parameter group that emerges should include r real independent infinitesimal transformations.

Similar arguments give all *real* infinite continuous groups of point transformations of the plane, all real projective continuous groups of the plane, etc.

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