Theory

of rectilinear

Systems of light rays.

An extension

of the

Gaussian theory of the curvature of surfaces

by

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Table of contents

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Chapter I. Historical introduction.

1) Systems of light rays in a medium whose elementary wave surface is a sphere.

The first to investigate ray systems was Tschirnhausen in 1682. He examined the focal lines that arise under the reflection of parallel rays by a circle. However, the equation that he proposed for it was flawed, as was shown by the curators of the Paris Academy Cassini, Mariotte, and de la Hire. This first, fruitless search drew the attention of mathematicians to that kind of curve, which were soon recognized to be the key to all of the secrets of dioptrics and catoptrics. More generally, men like Bernoulli, l'Hôpital, and Carré treated a ray system in the plane that arose in such a way that parallel rays, or ones that originated from a point, were reflected or refracted from an arbitrary curve, and gave methods for the calculation of the caustic curve, which consists of two separate branches, in general ([1]). It has a complicated nature; however, Quetelet ([2]) showed that it could be developed from easily constructible epicycloids, which are ordinarily much simpler and serve the purpose well.

In 1810, Malus ([3]) first began to develop a theory of that sort of ray system in space, which Dupin ([4]), Hamilton ([5]), and Gergonne ([6]) built upon. The reflecting or refracting curve was now a surface, and the two branches of the focal line became two sheets of a focal surface that was contacted by all of the rays of the system, and it was upon this basis that all of dioptrics and catoptrics rested. The main result was first presented by Malus as the theorem:

When there is a surface to which all rays are perpendicular, they will preserve that property of being perpendicular to a surface, no matter how many times they are reflected or refracted from arbitrary surfaces.

As will emerge in what follows, those are, when translated into the language of undulation theory, systems that will be possible only in media whose characteristic elementary wave is the sphere. Most of the remaining theorems have a purely mathematical content; actually, all that is of interest to optics is the fact that *one can*

([1]) One will find the geometric construction of rays system of this type and their focal lines, which have been reflected or refracted by 1, 2, 3, or 4 spherical surfaces in the optical tables of Schellbach and Engel.
([2]) Mémoires de l'Acad. de Bruxelles III.
([3]) XIV letter of the Journal de l'École polyt.
([4]) *Développements de géométrie*, mémoires III, IV.
([5]) "Theory of Systems of Rays," Trans. of the Irish Acad. XV.
([6]) Annales de math. pur. et appl. XIV, XVI.

replace an arbitrary number of refracting or reflecting surfaces with a single one that produces the same effect as all of them together.

2) General systems of light rays in arbitrary media.

No matter how interesting that special case might be, what still remains of much greater importance for the theory is the investigation of rectilinear ray systems that possess no surfaces that intersect all of the rays perpendicularly in complete generality, and in a medium that is characterized by any of its elementary wave surfaces. This investigation was first carried out by Hamilton ([1]).

He began that quest with a general, third-order partial differential equation with 42 coefficients and 8 variables, namely, the starting point (x, y, z) of an arbitrary curved ray, its end point (x', y', z'), the color χ, and a function V that characterizes the medium. He succeeded in reducing the 42 coefficients to 10 of them that he regarded as given constants and divided into four groups. That gave him four problems:

In the first one, x, y, z, x', y', z' are given, but the color χ is variable, and he stipulated that the chromatic aberration of the various, infinitely-close rays of heterogeneous light should be brief.

He also dispatched the second problem, where χ is constant, so he was only dealing with homogeneous light, with a few words.

In the third problem, the color χ and starting point (x, y, z) are given, and he considered a ray system at its end point, after it was repeatedly refracted. Namely, he examined the surface that is perpendicular to that ray system at its endpoints, and found that its differential equation would fulfill the known integrability condition – so such a surface would exist – only when the elementary wave surface of the final medium, which he denoted by V, is a sphere, and the rays are also all perpendicular to a surface at the starting point.

This problem is Malus's Theorem, which we discussed above.

The fourth problem relates to the interrelationship between the tangents to the rays at the starting point and the tangents at the end point. Since Hamilton therefore considered tangents instead of curved rays, he was actually dealing with only rectilinear ray systems. In fact, a medium whose density or chemical nature varies according to any rule can be thought of as being decomposed into nothing but homogeneous layers of equal refractivity.

The ray system will be rectilinear in any of these layers ([2]). We shall also restrict ourselves to rectilinear, monochromatic light rays in anisotropic media.

([1]) "Supplements to an essay on the theory of Systems of Rays," Trans. of the royal Irish Acad., vol. XVI, pp. 7 and pp. 97.

([2]) Pogg. Ann., 1833, Bd. XXVIII, pp. 633 and Bd. XXIX, pp. 324, and furthermore, Phil. Mag., ser. III, vol. 2, pp. 284. Hamilton himself spoke about the content of his investigations in the Report of the first and second Meetings of the British Association for the Advancement of Science, pp. 545, in the following way:

"The general problem that I have proposed to myself in optics is to investigate the mathematical consequences of the law of least action: a general law of vision, in which is included, as it is well-known, all the particular conditions of reflexion and refraction, gradual and sudden, and the central idea from which my whole method flows is the idea of one radical and characteristic relation for each optical system

Perhaps it was due to the daunting investment of effort into the analytical apparatus and geometric tools that the theory was gradually forgotten since Hamilton. It was only in recent times that Kummer returned to it and published a theory of rectilinear, mathematical ray systems ([1]), in which the optically-possible ones were included as a special case.

He determined a ray through the cosines ξ, η, ζ of the angles that it made with the coordinate axes, and through those points (x, y, z) of the so-called initial surface of the ray system through which the ray system went. He then considered these six variables to be functions of two new independent variables u and v and, with Gauss, introduced the following relations for the differential quotients:

$$(1) \qquad dx = a\,du + a'\,dv, \qquad dy = b\,du + b'\,dv, \qquad dz = c\,du + c'\,dv.$$

From that, he defined the Gaussian functions A, B, C; D, E, F. Analogous to them, he set:

$$(2) \qquad d\xi = \mathfrak{a}\,du + \mathfrak{a}'\,dv, \qquad d\eta = \mathfrak{b}\,du + \mathfrak{b}'\,dv, \qquad d\zeta = \mathfrak{c}\,du + \mathfrak{c}'\,dv,$$

and likewise calculated $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$; $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ from this.

With the help of that and some other functions, he explored the theory of mathematical ray systems. We will speak about that subject in the mathematical part of this introduction.

At a later time, he also treated the question of which properties of light that actually occur in nature follow from the general theory of mathematical rays as optically-possible and how Malus's theorem follows as a special case of a more inclusive optical theorem, and presented it to the Berlin Academy of Science ([2]).

Any infinitely-thin, optical ray bundle inside of a homogeneous, transparent medium has the property that its two focal planes for the wave surfaces of light that belong to this

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of rays; that is, for each combination of straight or bent or curved paths along which light is supposed to be propagated according to the law of least action, this characteristic relation being different for different systems, and being such that the mathematical properties of the system can all be deduced from it."

"…In the relation contemplated by me, the related things are, in general, in number eight, of which, six are elements of position of two variable points in space, considered as visually connected, the seventh is an index of color, and the eighth, which I call the *characteristic function* – because I find that in the manner of its dependence on the seven foregoing are involved all the properties of the system…is the *action* between two variable points, the word "action" in the same sense as in the known law of vision, which has been already mentioned."

"I have assigned, for the variation of this characteristic, corresponding to any infinitesimal variations in the positions on which it depends, a fundamental formula, and I consider as *reducible to the study of this one characteristic function, by means of this one fundamental formula, all the problems of mathematical optics,* respecting all imaginable imaginations of mirrors, lenses, crystals, and atmospheres, and though among these problems of mathematical optics, it is not here intended to include investigations respecting the phenomena of interference, yet it is to perceive from the nature of the quality, which I called the characteristic, and which is the hypothesis of undulation, is the *time of propagation of light from one variable point to another* that the study of this function must be useful in such investigations also. My own researches, however, have been hitherto chiefly directed to the consequences of the law of least action, and to the properties of optical systems, and system, in general."

([1]) Crelle's Journal, Bd. 57.

([2]) Monatsberichte der Akad. der Wissenschaften zu Berlin, 30 July 1860.

medium, whose center will be assumed to lie on the axis of the ray bundle, will cut out two curves that intersect each other in conjugate directions. Any ray bundle that has this property will also be optically-representable.

We have had no proof of this apparently complicated theorem and its noteworthy corollaries up to now. In the following treatise, that will emerge as one aspect of a more general theorem whose other aspect offers a comparable sequence of interesting consequence, and it is only when one combines both aspects that one rises to simple generality.

More and more, optics and the theory of curved surfaces merge into each other, and the latter is already just as indispensable to optics as the infinitesimal calculus.

In the next paragraph, we would thus like to recall some of the lessons from the theory of surfaces that are indispensable to optics, and then go into the infinitely-thin ray bundles.

§ 2. Mathematical introduction.

3) The curvature of surfaces, according to Dupin.

It is well-known that the curvature of surfaces can be discussed, either with the help of Euler's principal radii of curvature, in which one pays especial attention to the normals to the surfaces, or by means of the Gaussian curvature, which relates to an auxiliary sphere with radius = 1, or finally, by applying the Dupin indicatrix, for which one starts with the properties of tangent planes. The Euler method is useful only for rays that are all perpendicular to a surface. Gauss's method would have immediately value only for media whose elementary wave surfaces are spheres, and admittedly allows a very appreciable generalization to other media. By contrast, the Dupin way of looking at things is of greatest importance for the theory of ray systems, precisely because it is connected with tangential planes, and one often does not have to base the tangential planes on the elementary wave surfaces of light. I will briefly explain that method ([1]).

If one draws a tangential plane through any point (x, y, z) of a curved surface then the tangential plane will cut out parallel plane curves whose degree will depend upon the nature of the curved surface. Among this family of parallel planes, however, the two planes that are infinitely-close to the tangential plane will be distinguished. Namely, the ones that continually cut out an infinitely-small conic section from the surface, which is called the *indicatrix* ([2]) of the point (x, y, z). At concave-concave points of the surface, where it is known that the inequality $s^2 - rt < 0$ for the two partial differential quotients r, s, t is true, the surface will lie completely on one side of the tangential plane. One of these two infinitely-close and parallel planes will therefore have no intersection at all with the surface, but only an imaginary one. However, as Dupin showed, the other one will cut out a real ellipse.

([1]) Charles Dupin, *Developpements de Géométrie*, mémoires I and II.

([2]) Cournot, *Théorie des fonctions* I, pp. 488. The indicatrix will result from the intersection of the surface with two infinitely-close planes, both of which are parallel to the tangent plane, and between which, it will be found.

If the surface is concave-convex, so $s^2 - rt > 0$, then it will lie on both sides of the tangential plane, and each of the two infinitely-close parallel planes will cut out a hyperbola. All that remains is the case in which $s^2 - rt = 0$. We are then dealing with a developable at the point (x, y, z). Only one of the two parallel planes that are infinitely-close to the tangential plane will enter the surface will cut out a real, infinitely-small parabola; only an imaginary intersection will exist for the other one.

These infinitely-small conic sections have all of the properties of ordinary conic sections and also possess conjugate diameters.

Any diameter of the indicatrix is tangent to the surface, and the conjugate diameters are conjugate tangents to the surface.

If we go from our tangential plane at (x, y, z) on the surface in any direction that is determined by dy / dx to a neighboring tangential plane then both of the planes will intersect in a line whose direction will be determined by $\delta y / \delta x$.

The directions that are determined by dy / dx and $\delta y / \delta x$ will now always be conjugate diameters of the indicatrix, and thus conjugate tangents. If r, s, t are the two partial differential quotients of the point (x, y, z) on the surface then Dupin gives the following equation for these conjugate directions:

$$\frac{dy}{dx} = -\frac{r + s\dfrac{\delta y}{\delta x}}{s + t\dfrac{\delta y}{\delta x}}. \tag{3}$$

4) The infinitely-thin mathematical ray bundle.

Later on, we will regard the ray system as a sum of individual, infinitely-thin, ray bundles whose properties we will examine, and then we will combine them into systems again. Here, we would thus like to clarify the term “the individual parts of an infinitely-thin ray bundle.”

An infinitely-thin ray bundle is any part of a ray system that consists of a well-defined ray – called the *axis* – along with all of its infinitely-close, neighboring ones. Kummer’s *base points for the shortest distance* from the axis to the neighboring rays (for Hamilton: foci by projection) all lie on a well-defined part of the axis that is bounded by the two *limit points of the shortest distance* (foci of extreme projection, resp.). The shortest distances at these two points will define a right angle with each other, and those two planes that go through one of these shortest distances at the limit points and the axis of the bundle will be called *principal planes* (planes of extreme projection, resp.). Furthermore, there will be, in general, two points on the axis of any infinitely-thin ray bundle – viz., the *focal points* – where a neighboring ray will intersect the axis. The two planes in which the axis is intersected by an infinitely-close ray will be the *focal planes* (Hamilton: planes of vergency). A plane that is perpendicular to the axis will cut a small, closed curve out of the bundle. This curve will degenerate into a line at the two focal points, namely, a *rectilinear cross-section* (guiding line, resp.), which all rays will go through, and which will lie in the focal plane. The angle between the rectilinear cross-sections will be measured by the angle between the focal planes. Since the entire ray system can be regarded as consisting of nothing but infinitely-thin bundles, the two rectilinear cross-sections will yield the elements of the two sheets of the focal surface.

Both sheets of the focal surface will be contacted by all rays of the system; any two infinitely-close rays will always intersect on them. Any plane that can be drawn through any two such rays will be a focal plane for a bundle, and likewise, a tangential plane to the focal surface. The family of first focal planes will contact the first sheet of the focal surface, and the contact planes to the second one will give the family of second focal planes. Thus, *the angle that the two sheets define with each other on their intersection curve will be measured by the angle between the focal planes at the individual points of the intersection curve.*

In addition to the ordinary, infinitely-thin ray bundles, there is yet another kind, which are called *principal rays*. Namely, whereas the axis of the ordinary bundle is met by only two of its infinitely-close rays – viz., the *focal rays* – for the principal rays, all of the infinitely-close rays will intersect the axis, and in fact at one and the same point, which will be called the *principal focus.* Any plane that goes through the axis will then become a focal plane. The angle between the two focal planes, which will be denoted by γ, ceases to possess a well-define value, and one will get:

$$\tan \gamma = \frac{0}{0} \tag{4}$$

as the characteristic feature of principal rays.

§ 3. Optical introduction.

5) Huyghens's principle ([1]).

As is known, one can present the spreading of light from a luminous point using Huyghens's principle in such a way that, as one assumes, it propagates in all directions from the luminous point as if were the center of a disturbance with a velocity that depends upon the elasticity of the medium around the luminous point. In a homogeneous medium, light will propagate with the same velocity in all directions, which is equal to v, and after t seconds, it will have spread from the luminous point S to the outer surface of a sphere of radius vt, and any line, such as SA, that connects a point A on the sphere surface to S will be called a *light ray*. After t' more seconds have elapsed, the light from S will have spread from S to the outer surface of a sphere of radius $v(t + t') = vt + vt'$. However, we will now get the same wave surface when we construct a sphere of radius vt' at any point of the spherical surface that was described at vt. This family of spheres will be enveloped by the desired spherical wave surface. In order to distinguish them, we will call any of the auxiliary spheres in that family an *elementary wave*; however, we will call the enveloping sphere of radius $v(t + t')$ a *principal wave.* Obviously, a light ray that goes through the point A' of the principal wave will also meet the center of those elementary waves that contact the principal wave at A'.

([1]) Christiani Hugenii, *Tractatus de Lumine*, Amstelodami, 1628.

If the medium is a uniaxial crystal then after t seconds the extraordinary rays from a disturbance at a luminous point S in the medium would have spread out to an ellipsoid of rotation ABC, and in order to know how far the light will have advanced after $t + t'$ seconds, one constructs an ellipsoid of rotation abc for this uniaxial medium at the time t' and locates one of them at every point of ABC, such that the elementary waves abc are all congruent and simiarly-oriented to each other, corresponding to the optical axis of the medium in question, such that the ellipsoid of rotation that envelopes this family of elementary waves will be the desired principal wave. One obtains the ray that goes through the point A' of this principal wave when one connects A' with the center of that elementary wave that contact the principal wave at A'; i.e., *it has a common tangent to it at* A'.

These elementary waves are purely mathematical constructions, without there being any necessity for ascribing any physical interpretation to them. Should they exist in reality, and should the points of the surface ABC be regarded as actual centers of optical disturbances from which light will spread out, such as it does from S, then one must first express that as a hypothesis. So many physicists seem to have been inclined to introduce this hypothesis that is therefore superflous; indeed, it seems to aggravate one's understanding.

How useful it is to consider a family of elementary waves instead of a principal wave can be illustrated easily by an example. A principal wave that is initially curved according to a well-defined mathematical rule is bent in such a way that it is broken into an irregular surface, or such that it has piece-wise suffered a slippage at an anistropic location in the medium. Light rays are now no longer straight lines from S to the individual points of the principal wave ABC. Moreover, for an infinitely-small time dt, I construct a family of infinitely-small elementary waves, and let the principal wave contact one of them at every point in order to obtain the tangents to the light rays at the endpoints.

6) Extension of Huyghens's principle.

That is how far this theory has been developed up to now. In order to go a step further, we direct our attention to a special case.

A plane principal wave OX comes in from an infinitely-distant luminous point. After t seconds, it is found at AB, such that $OA = vt$.

In a homogeneous medium, AB will contact a family of elementary spheres that have radius vt and their centers on OX. However, that will come down to the same thing as when I consider any other surface – e.g., $A'B$ – to be the locus of the centers of disturbance, and on it, also spheres, but described with different radii, according to the measure of time that the light has used along its path from the points of the surface $A'B'$ to those of the surface AB. At A', one will have $t = AA' / v$; at B, one will have $t = BB' / v$. At A', I construct a sphere with radius AA'; at B', I construct one with radius BB'. At the points between then, one will have other measures for time. We can call $A'B'$ the *initial surface of the ray system*, and obviously any ray can be determined when one knows the cosines ξ, η, ζ of its angles with the coordinate axes and the coordinates x, y, z of those points at which it enters the initial surface $A'B'$.

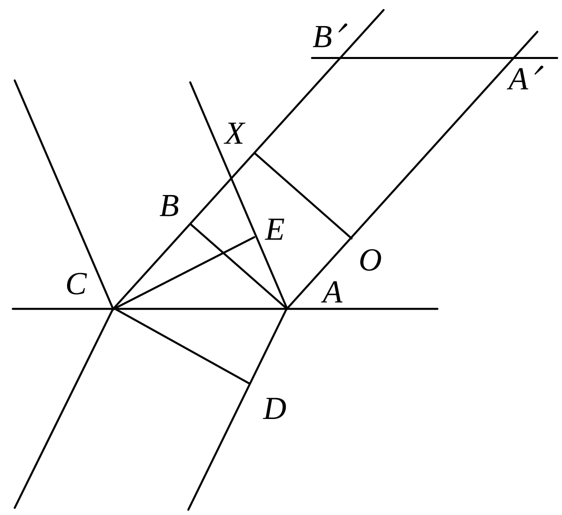

Figure 1.

Obviously, there is a host of surfaces like $A'B'$ that one can choose to be the initial surface.

Huyghens's principal has thus been extended in two steps. First, we had a luminous point, from which, light spread out in all directions simultaneously like a center of disturbance until it reached the wave surface. There was only one time and one center of disturbance. We then saw how the time indeed elapsed in it, but instead of one center of disturbance, an entire family of points would be regarded as centers of disturbances. Finally, we also obtained nothing but different times for the elementary waves when we chose the locus of the disturbance points to be, not the principal wave, but any other initial surface of the ray system. The method thus consists of taking things apart, and as it so frequently happens in mathematics, it will be possible by the introduction of a great degree of arbitrariness, to approach any given problem most closely in any special case by the determination of the arbitrary time and space quantities.

We return to our special case. The ray bundle $AA'B'B$ meets the surface AC and will be partly refracted and partly reflected. Let the refracted principal wave be CD and let the reflected one be CE. Just as we could also previously consider any curved surface that intersects the incident bundle, instead of the incident principal wave AB, to be the locus of disturbance centers, we will also be free now to regard, not only the principal waves CD and CE themselves, but, in fact, any other surface that intersects the refracted or reflected light bundle, as the geometric locus of the disturbance centers.

However, for some entirely specialized purposes, there is a surface that can be chosen to be the locus of disturbance centers after a measure of time in this case, because, in fact, it intersects all three bundles, and then possesses the property of being eligible for all three of them at once. That is the refracted surface CA. In this case, it is then simpler to choose, not the three principal waves, but this refracted surface CA to be the initial surface of the three ray bundles. However, the three families of elementary waves with their centers of disturbance on CA will always remain a purely geometrical tool with no physical existence.

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Chapter II. Systems of light rays that occur in nature.

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§ 4. A light ray.

7) Geometric construction of a light ray by means of principal and elementary waves.

One can think of mathematical ray systems as being resolved into nothing but infinitely-thin ray bundles. We shall now temporarily not consider such a system and not look for the properties that makes it into a light bundle that actually occurs in nature, but we shall now consider a single light ray, and indeed, in this number, a light ray shall be constructed with guidance from the optical part of the introduction in regard to the individual examples of the methods that were presented. In the next number, its equations shall be presented and the general functions for a mathematical ray that were mentioned in the introduction will be specialized to the case of an optical ray. The examination of an infinitely-thin ray bundle of light can first begin in the next paragraph.

We choose an entirely arbitrary curved surface to be the principal wave, and an elementary wave that is as general as possible for the infinitely-small time *dt*, except that the elementary wave, as would follow from its nature, must be a closed surface and contain a center of disturbance in its interior. At every point of the principal wave, one of the elementary waves will contact congruent and similarly-oriented surfaces. Any line that connects any point *A* of the principal wave with the disturbance center of the elementary wave that contacts the principal wave at *A* – which will then *have the same tangential plane with it at A* – will be a light ray. In doubly-refracting (or even multiply-refracting) media we will consider each sheet of the elementary wave, and for polychromatic light, we will consider each color by itself.

Since all elementary waves that contact the principal wave are not only similar, but also similarly-oriented, that will allow us to introduce a simplification that will have great significance for what follows.

Namely, we replace the family of elementary waves with a single one that is congruent and similarly-oriented to the other ones, and has its center of disturbance at the coordinate origin *O*. Let the elementary waves that contacted the principal wave at *A* up to now be ones whose centers of disturbance we have shifted to *O*. The principal and elementary waves will then no longer have a common tangential plane at *A*, but the tangential plane to the elementary wave will only be parallel to the other one at *A* and will contact the elementary wave at *A′*. – The construction of a light ray in this way will take the following form: *One draws a tangential plane to the principal wave at A, and also a tangential plane that is parallel to it on the elementary wave whose center of disturbance is at O, and which contacts the latter plane at A′, so the light ray will be a line through A that is parallel to the radius of the elementary wave OA′.*

The analogy between this new way of looking at things and Gaussian curvature comes to mind, and in fact, the entire method can be regarded as an extension of it, which will be move to the foreground later on. If the elementary wave is a sphere then we will have its curvature; the rays will then become normals to the principal wave.

We then have one principal wave and only a single elementary wave. However, such gross analogies between them can also present themselves that one must never exchange one with the other, but always establish that the rays must belong to the principal wave and the radius vectors, which are parallel and corresponding to them, must belong to the elementary waves. If the center of disturbance lies on one of the rays then, in this case, it will indeed coincide with its corresponding radius of the elementary wave. However, the rays that are infinitely-close to it are still only parallel to its corresponding radius, and do not all possibly go through the center of disturbance.

8) Analytical equations for a light ray as a function of the principal and elementary waves.

The equations for a light ray shall now be presented. It goes through the point (x', y', z') and defines angles with the coordinate axes whose cosines are ξ, η, ζ, so its equations will be:

(5) $$\frac{X - x'}{\xi} = \frac{Y - y'}{\eta} = \frac{Z - z'}{\zeta},$$

if X, Y, Z are the running coordinates. If the point (x', y', z') lies on the principal wave $z = F(u, v)$ then one will have:

(6) $$x' = u, \quad y' = v, \quad z' = F(u, v),$$

and the tangential plane that contacts the principal wave at (x', y', z') will be:

(7) $$Z - z' = P(X - x') + Q(Y - y'),$$

if X, Y, Z are the running coordinates, and P, Q denote the first differential quotients of the principal wave. The tangential plane that corresponds to this will contact the elementary wave at the point (x, y, z), and let its equation be, by analogy with the above:

(8) $$Z - z' = p(X - x') + q(Y - y').$$

Since the planes that are represented by equations (7) and (8) are parallel, one must have:

(9) $$Q = q, \quad P = p,$$

which are then condition equations for the points (x, y, z) and (x', y', z') to correspond to each other. Moreover, since the radius ρ of the elementary wave at (x, y, z) has the same direction cosines ξ, η, ζ as the ray at x', y', z' itself, one will then have the equations:

$$(11)\qquad \left\{\begin{aligned} \xi &= \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \\ \eta &= \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \\ \zeta &= \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}. \end{aligned}\right.$$

By substituting these values for ξ, η, ζ into the equations of the light ray in (5), one will get:

$$(11)\qquad \frac{X-x'}{x} = \frac{Y-y'}{y} = \frac{Z-z'}{z},$$

which are equations that depend upon only the principal and elementary waves.

It still remains for us to calculate the Gaussian functions that we already mentioned in the mathematical part of the introduction for this special case. If known that Gauss ([1]) introduced the following relations into analysis:

$$dx = a\,du + a'\,dv, \qquad dy = b\,du + b'\,dv, \qquad dz = c\,du + c'\,dv,$$

and further:

$$(12)\qquad A = bc' - b'c, \qquad B = ca' - c'a, \qquad C = ab' - a'b,$$

and

$$E = a^2 + b^2 + c^2, \qquad F = aa' + bb' + cc', \qquad G = a'^2 + b'^2 + c'^2.$$

When equation (6) is differentiated, that will give:

$$dx' = du, \qquad dy' = dv, \qquad dz' = P\,du + Q\,dv.$$

A comparison of this formula with the one in (12) will yield:

$$a = 1,\ a' = 0; \quad b = 0,\ b' = 1; \quad c = P,\ c' = Q.$$

With the help of this, one will find that:

$$A = -P, \qquad B = -Q, \quad C = 1; \quad E = 1 + P^2, \quad F = PQ, \quad G = 1 + Q^2.$$

If we differentiate formulas (10) then that will produce the similar Kummer values that were also mentioned already:

$$\mathfrak{a} = \frac{C(y^2+z^2)+Axz}{(x^2+y^2+z^2)^{3/2}}, \qquad \mathfrak{a}' = \frac{-x(Cy-Bz)}{(x^2+y^2+z^2)^{3/2}},$$

([1]) Gauss, *Disquisitiones generales circa superficies curvas.*

$$\mathfrak{b} = \frac{C(x^2+z^2)+Ayz}{(x^2+y^2+z^2)^{3/2}}, \qquad \mathfrak{b}' = \frac{-y(Cx-Bz)}{(x^2+y^2+z^2)^{3/2}},$$

$$\mathfrak{c} = \frac{A(x^2+y^2)+Cxz}{(x^2+y^2+z^2)^{3/2}}, \qquad \mathfrak{c}' = \frac{-P(x^2+y^2)+Cyz}{(x^2+y^2+z^2)^{3/2}}.$$

These are the values of the functions that appeared in the Kummer treatise on mathematical ray systems when they are specialized in the way that is necessary for optical ray systems.

§ 5. An infinitely-thin bundle of light rays.

9) Corresponding directions, in general.

In the previous paragraphs, we were concerned with the construction and equations of a light ray. It is now easy to examine two infinitely-close rays, and indeed, should the one of them be the axis of an infinitely-thin ray bundle; the other one can be any ray that is infinitely-close to it. If the equation of a light ray is, as above:

(11) $$\frac{X-x'}{x} = \frac{Y-y'}{y} = \frac{Z-z'}{z},$$

for the two corresponding points (x', y', z') and (x, y, z) on the principal and elementary wave, respectively, then an infinitely-close ray of the system will have the equations:

$$\frac{X-(x'+dx')}{x+dx} = \frac{Y-(y'+dy')}{y+dy} = \frac{Z-(z'+dz')}{z+dz}.$$

dy' / dx' will determine the direction of the principal wave, when one goes from the intersection point (x', y', z') of the axis of the bundle to the intersection point $(x' + dx', y' + dy', z' + dz')$ of a neighboring ray, and dy / dx will determine the corresponding direction on the elementary wave by which one will go from (x, y, z) to $(x + dx, y + dy, z + dz)$.

I have proved two theorems about these directions on the principal and elementary wave that are determined by dy' / dx' and dy / dx in an analytical way in a treatise ([1]) "Ueber allgemeine Strahlensysteme des Lights in verschieden Medien." Namely, as in the introduction "if $\delta y' / \delta x'$ determines the direction that is conjugate to dy' / dx', and $\delta y / \delta x$ determines the direction that is conjugate to dy / dx then the directions that are determined by $\delta y' / \delta x'$ and $\delta y / \delta x$ will be parallel; i.e., one will have $\delta y' / \delta x' = \delta y / \delta x$."

([1]) Meibauer, Zeitschrift für Mathematik un Physik, VIII, Jahrg. 1863.

"Conversely, if one knows that $\delta y' / \delta x' = \delta y / \delta x$ then the directions that are determined by dy' / dx', and dy / dx will correspond to each other." I went on to the second partial differential quotients by differentiating equations (9), which yielded:

$$r\,dx + s\,sy = R\,dx' + S\,dy', \qquad s\,dx + t\,dy = S\,dx' + T\,dy'.$$

With the help of these equations, dy / dx can be expressed in terms of dy' / dx', and if one replaces dy' / dx' in that expression with its values in the Dupin expression:

$$(3) \qquad \frac{dy'}{dx'} = -\frac{R + S\dfrac{\delta y'}{\delta x'}}{S + T\dfrac{\delta y'}{\delta x'}}$$

then that will yield:

$$(13) \qquad \frac{dy}{dx} = -\frac{r + s\dfrac{\delta y'}{\delta x'}}{s + t\dfrac{\delta y'}{\delta x'}}.$$

However, since the Dupin equation gives:

$$(3) \qquad \frac{dy}{dx} = -\frac{r + s\dfrac{\delta y}{\delta x}}{s + t\dfrac{\delta y}{\delta x}}.$$

a comparison of these equations will yield that:

$$\frac{\delta y}{\delta x} = \frac{\delta y'}{\delta x'}.$$

Due to the importance of this theorem for what follows, a simple proof of it shall be given that is linked to geometric considerations.

As Dupin proved (see the Introduction), the two tangential planes at (x, y, z) and $(x + dx, y + dy, z + dz)$ will intersect in a direction that is conjugate to dy / dx, from which, one will determine $\delta y / \delta x$. The same is true for the tangential planes at (x', y', z') and $(x' + dx', y' + dy', z' + dz')$; its line of intersection will also be determined by $\delta y' / \delta x'$. However, these two intersections will arise in two corresponding pairs – i.e., parallel planes – so they must be parallel, and one must have:

$$\frac{\delta y}{\delta x} = \frac{\delta y'}{\delta x'},$$

which was to be proved. The directions that are determined by $\delta y / \delta x$ and $\delta y' / \delta x'$ do not correspond, in general. In a similar way, one can prove that, conversely, dy / dx and dy' / dx' will correspond when $\delta y / \delta x = \delta y' / \delta x'$.

We then have the theorems:

1) *If one has any pair of corresponding directions on the principal and elementary waves then the directions that are conjugate to them will be parallel.*

2) *If one has any pair of parallel tangents at corresponding points, one of which is on the principal wave, and the other of which is on the elementary wave, then the directions that are conjugate to them will correspond to each other.*

These two theorems seem to have not been noticed, up to now. I first published then in my inaugural dissertation ([1]).

By means of their reciprocity, any theorem that one concludes from them for the principal wave and its rays will find an analogue for the elementary wave and its corresponding radius vectors. When one is, perhaps, treating a theorem on the relationship between rays and the curvature of the principal wave then its analogue must address the curvature of the elementary wave and its relationship to the radius vectors, but not its relationship to the rays. For example, the theorem that all rays in a homogeneous medium must be perpendicular to the principal wave will have the trivial analogue: The radii are perpendicular to the sphere. The rays fall into the domain of the principal wave, but the radii fall into the domain of the elementary wave.

10) Corresponding directions in focal planes.

Concerning the corresponding directions that were treated in the two theorems above, the most interesting one is the one by which one goes from the axis of the bundle to the focal ray, which (see above) cuts the axis at a focal point, and which will correspond to that direction.

The origin of our rectangular coordinate system already lies at the center of disturbance. We now draw the Z-axis parallel to the axis of the ray bundle, such that it coincides with the radius, which corresponds with the axis of the bundle. The rest of the coordinate system will remain temporarily undetermined.

Once that is established, we turn to the focal plane. It will go through the axis of the bundle and the focal ray. The plane that corresponds to the focal plane will go through the Z-axis and the radius that is parallel to the focal ray. Therefore, *the focal plane is parallel to its corresponding plane.* If dy' / dx' denotes the direction in which one goes from the axis of the bundle to the focal ray and dy / dx denotes the corresponding one then dy' / dx' and dy / dx will determine the positions of two parallel planes, and one will have the following formula for the focal plane:

([1]) Meibauer, *De generalibus et infinite tenuibus luminus fascibus, praecipue in chrystallis*, Berlin, 1861. (Published by Lauderits)

$$\frac{dy}{dx} = \frac{dy'}{dx'}.$$

In words: *If the direction that is determined by* dy' / dx' *lies in a focal plane then the corresponding direction on the elementary wave will be parallel to it.*

From Theorem 1, one first had $\frac{\delta y}{\delta x} = \frac{\delta y'}{\delta x'}$. However, if $\frac{dy}{dx} = \frac{dy'}{dx'}$ then secondly, from Theorem 2, the directions that are determined by $\frac{\delta y}{\delta x}$ and $\frac{\delta y'}{\delta x'}$ must henceforth also correspond. Now, $\frac{\delta y}{\delta x}$ and $\frac{\delta y'}{\delta x'}$ have the same properties as the values $\frac{dy}{dx}$ and $\frac{dy'}{dx'}$, and that will just as well determine the position of a focal plane, namely, the second one. As a result, we will obtain the following two fundamental theorems:

3) *The two focal planes intersect each other on the principal wave in conjugate directions.*

4) *The planes that correspond to the focal planes intersect each other on the elementary wave in conjugate directions.*

For the positions of the focal planes, one can present the equations:

$$\text{(14)} \qquad \left\{ \begin{aligned} \frac{dy}{dx} &= -\frac{r + s\,\frac{\delta y}{\delta x}}{s + t\,\frac{\delta y}{\delta x}}, \\ \frac{dy'}{dx'} &= -\frac{R + S\,\frac{\delta y'}{\delta x'}}{S + T\,\frac{\delta y'}{\delta x'}}. \end{aligned} \right.$$

It is now easy to derive the theorem that Kummer presented to the Berlin Academy of Science that was mentioned in the introduction. In fact, it is only necessary to transfer the coordinate origin, along with the center of disturbance to the axis of the ray bundle. The focal planes will then coincide will the planes that correspond with them, and the coordinate axis will lie on the axis of the bundle. In this special case, *the focal planes will also intersect on the elementary wave in conjugate directions* ([1]).

([1]) Hamilton seems to have already known not only this theorem, but also Theorem 3. At least, one will find the following passage in the Transactions of the Royal Irish Acad., vol. XVIII, pp. 122:

"Thus, we are led to consider a series of waves or action surface V_1 similar and similarly-placed, and determined in shape, but not in size, by the uniform medium [ROM: our family of waves], and then to seek the limiting surface of this set, which osculates to the given surfaces V [ROM: our principal wave], and it follows that the conjugate planes of vergency [ROM: our focal planes] in a uniform medium are *conjugate planes of each medium surface* V_1 [ROM: elementary wave], *and also of the surface* V [ROM: principal wave], *determined by the whole combination.*"

In general, the principal and elementary wave can have only a first-order contact, such that one will have $P = p$, $Q = q$. However, the second-order partial differential quotients do not naturally need to be equal, in turn. Furthermore, the indicatrices will be at corresponding points of them, and the conjugate diameters of the one are in no way parallel to the all of the conjugate diameters of the other one. However, it is henceforth conceivable that at least one pair of such diameters is parallel under the corresponding indicatrices. The theorem above shows that *there is only one such pair, and that this pair of conjugate diameters will lie in both of the focal planes.* This theorem is true, not merely for the principal and elementary waves, but also for any two arbitrary curved surfaces, after one strips away the optical accessories from the theory of surfaces.

11) The angle between focal planes.

The main value that the Theorems in 3) and 4) have – and we will go into this after the present number – consists of the fact that they give one the means to express the angle γ between the focal planes as a function of the principal and elementary waves.

In our coordinate system, the focal planes are perpendicular to the xy coordinate plane, and dx' / dy' is therefore the trigonometric tangent to the angle α that the first focal plane makes with the xz coordinate plane. Likewise, $\delta x' / \delta y'$ is the trigonometric tangent to the angle β that the second focal plane makes with the xz-plane, the following expressions arise:

$$\frac{dy}{dx} = \frac{dy'}{dx'} = \tan\alpha, \qquad \frac{\delta y}{\delta x} = \frac{\delta y'}{\delta x'} = \tan\beta, \qquad \angle\gamma = \angle\beta - \angle\alpha,$$

and it follows that:

$$\tan\gamma = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta}. \tag{15}$$

If we replace dy' / dx' and $\delta y' / \delta x'$ with the values $\tan\alpha$ and $\tan\beta$ in equations (14), which represent the positions of the focal planes as functions of the principal wave and elementary wave, then that will give:

$$\tan\alpha = -\frac{R + S\tan\beta}{S + T\tan\beta}, \tag{16}$$

$$\tan\beta = -\frac{r + s\tan\beta}{s + t\tan\beta}. \tag{17}$$

The three variables α, β, γ can be determined from equations (15), (16), (17). Later on, we will define an equation in the single variable $\tan\gamma$ by eliminating $\tan\alpha$ and $\tan\beta$.

Since $\angle\,\alpha$ and $\angle\,\beta$ have entirely analogous meanings, one of them can be omitted. By eliminating tan β from (15), (16), (17), two new equations in only two variables will arise. Namely, (15) and (16) will yield:

$$\tan\gamma = -\frac{T\tan^2\alpha + 2S\tan\alpha + R}{S\tan^2\alpha + (R-T)\tan\alpha - S}, \tag{18}$$

and from formulas (15) and (17), one will get:

$$\tan\gamma = -\frac{t\tan^2\alpha + 2s\tan\alpha + r}{s\tan^2\alpha + (r-t)\tan\alpha - s}. \tag{19}$$

These two equations shall be examined in the next chapter, or rather just one of them, since one can read off the result for the other one by analogy. The one will give the connection between the angle γ between the focal planes and the elements R, S, T of the principal waves; the other one will link γ with the elementary wave.

For any theorem about the elementary wave, there will appear an analogous one that pertains to the principal wave, although the form in which those analogous theorems are expressed can deviate very much from each other.

I have determined the axis of an infinitely-thin ray bundle at the point (x', y', z') of the principal wave, and if I know the $\angle\,\alpha$ that its first focal plane subtends with the xz coordinate plane then the angle between the focal planes can be computed for this well-define ray bundle by means of equation (18). For any other value of α, I will get another ray bundle that possesses another angle γ, and if tan α runs through all possible values from 0 to ∞ then all possible ray bundles will be produced for these particular values of R, S, T; i.e., this particular point of the principal wave.

The geometric process by which one can typify the generation of all possible ray bundles using the first method then consists of a rotation of the first focal plane around the axis of the bundle.

In the next paragraphs, we will examine which ray bundles are possible for a given point of the elementary wave. This question will reduce to the discussion of the nature and evolution of the function tan γ in equation (19) when tan α varies.

Chapter III. The focal surface of a ray system, according to the first method.

§ 6. The dependency of the focal surface upon the properties of elementary waves.

12) The maximum angle between focal planes.

As we did before when we went from simple light rays to ray bundles, we will now rise from that to general ray systems of light. The focal planes are then (see the Introduction) tangential planes to the two sheets of the focal surface, and γ means the angle by which the sheets intersect, which will also be of importance to the general theory of curved surfaces when one considers the elementary wave as the generalized Gaussian curvature and the principal wave as the initial surface of the ray system.

The first question that one must actually answer will be whether principal rays are even possible when $\tan \gamma = 0 / 0$; the answer will read $r = 0$, $s = 0$, $t = 0$. It will then emerge that either equation (18) or (19) will be suitable for the discussion of principal rays. The second method will then give some indication about it.

The next thing shall be to examine whether the function $\tan \gamma$ can assume, perhaps, a maximum value under the variation of α, and whether and when it increases to infinity. The denominator on the right in formula (19) will vanish:

$$\tan^2 \alpha + \frac{r-t}{s} \tan \alpha - 1 = 0. \tag{20}$$

This is the condition for one to have $\tan \gamma = \infty$, so the two focal planes are perpendicular to each other ([1]). It follows from (20) that:

$$\tan \alpha = \frac{t - r \pm \sqrt{(r-t)^2 + 4s^2}}{2s}, \tag{21}$$

which will give the position that the first focal plane must have in order for one to have $\gamma = \pi / 2$. Since the square root is always real, there will always be two such positions for the first focal plane. However, since the last term in (20) is – 1, the product of the two roots of $\tan \alpha$ will be equal to – 1. The two positions of the first focal plane, where the second one is perpendicular to it, will themselves be perpendicular to each other; the two focal planes will only be switched with each other. Therefore, at any point of the

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([1]) The expression (20) also occurs in the theory of principal intersections. Thus, the condition $\left(\frac{dy}{dx}\right)^2 + \frac{r-t}{s}\frac{dy}{dx} - 1 = 0$ for the principal planes to be perpendicular to each other was well-known to Monge in his *Application de l'Analyse à la Géométrie*. Just as the normal to the point in question in the principal intersection was intersected by the neighboring ones there, here, in the general theory, the bundle axis and the focal rays on the principal wave, which are generally skew, will intersect in the focal planes. For Monge, the z-axis was parallel to the normal at the point in question; here, it is parallel to the axis of the light bundle.

elementary wave there will always be just one position of the planes that correspond to the focal planes where they are also perpendicular to each other.

One will have tan $\alpha = 0 / 0$ in (21) when:

$$s = 0, \quad r = t. \tag{22}$$

If ρ_1 and ρ_2 are the principal radii of curvature then it is known, from Euler, that for $s = 0$:

$$\rho_1 = \frac{1}{r}, \qquad \rho_2 = \frac{1}{t},$$

and it will follow from (22) that:

$$\rho_1 = \rho_2 .$$

Therefore, if the elementary wave is a sphere or the point to which it is applied is an umbilical point then there will be no well-defined position of the first focal plane, where $\gamma = \pi / 2$, since all of the light bundles will have the angle $\gamma = \pi / 2$. It emerges from this that: The focal planes of all ray bundles in homogeneous media, and even the ordinary ray bundles in uniaxial crystals, and finally, the ray bundles whose axes run parallel to the optical axes in uniaxial crystals, will be perpendicular to each other ([1]). The rotational axis will meet an ellipsoid of rotation, which is the extraordinary elementary wave of a uniaxial crystal, at an umbilical point. From what was said above, one can derive the theorem that was found previously for the entire ray system: The focal surfaces of all possible ray systems in homogeneous media will intersect at right angles. They will do the same thing in uniaxial crystals for the ordinary ray systems. (If the focal planes are perpendicular to each other then they will coincide with Kummer's principal planes, and the focal points will coincide with the limit points of the shortest distance. As a result, if the focal surfaces are perpendicular to each other then they will coincide with the limit surfaces.) For the theory of surfaces, the spherical elementary wave is the true Gaussian curvature, which the principal wave is the initial wave of a ray system. The rays that are generated in this way will be normal to the initial surface, whose two surfaces of principal curvature centers will intersect perpendicularly.

13) The minimum angle between focal planes.

The function tan γ in (19) will possess a minimum, but not a maximum. The numerator of the right-hand side might vanish, so:

([1]) This theorem was first found by Hamilton:

"When the medium is ordinary, as well as uniform, then the osculating surfaces V_1 [ROM: elementary waves] are spheres, and the directions of extreme osculation [ROM: the directions, along which the "principal planes" of the principal wave will intersect] are the rectangular directions of the lines of curvature on the surface V [ROM: principal wave], which is now perpendicular to the rays; in this case, therefore, and more generally when a ray in a uniform medium corresponds to an umbilical point on the medium surface V_1, the planes of vergency [ROM: focal planes] cut that surface and the surface V to which it osculates in two rectangular directions."

$$t \tan^2 \alpha + 2s \tan \alpha + r = 0,$$

or:

$$\tan \alpha = \frac{-s \pm \sqrt{s^2 - rt}}{t}. \tag{23}$$

The position of the first focal plane will thus be determined, while the second one will coincide with it. There will always be two such positions when $s^2 - rt > 0$, so the point of the elementary wave will then be concave-convex. As is known, there are then four places that are bounded by four circles on the Fresnel wave surface ([1]).

The two focal planes that coincide in one ([2]) will intersect the elementary wave along the direction of the infinitely large radius of curvature, or what amounts to the same thing, in the direction of the asymptotes to the hyperbolic indicatrix, which one can derive from Dupin ([3]).

However, if $s^2 - rt = 0$ then it is well-known that one will be dealing with a point that can be osculated by a developable surface. Since $s^2 - rt > 0$ inside of the four circles on the Fresnel wave surface, and is less than zero outside of them, one will have $s^2 - rt = 0$ on the circles themselves. However, one will find internal conical refraction on those circles. If one would like to study the properties of focal surfaces with internal conical refraction then one would have to set $s^2 - rt = 0$. From (23), $\tan \alpha$ will then be equal to $-s / t$ and $\tan \gamma = 0 / 0$. This 0 / 0 suggests only an apparent indeterminacy, and therefore no principal rays will appear here. This case shall be treated more specifically in a later section.

Finally, if have $s^2 - rt < 0$ in (23) – i.e., the point of the elementary wave is concave-concave – then $\tan \alpha$ will be imaginary. No ray bundle will then exist at those points whose focal planes coincide, and all of the focal surfaces that they generate will have two sheets.

In this case – where $\tan \gamma$ cannot vanish – a minimum calculation will be necessary. Therefore, in:

$$\tan \gamma = -\frac{t \tan^2 \alpha + 2s \tan \alpha + r}{s \tan^2 \alpha + 2(r - t) \tan \alpha - s} \tag{19}$$

([1]) Hamilton, Irish Acad., XVII.
Plücker, "Discussion de la forme générale des ondes lumineuses," Crelle's Journal, Bd. XIX.

([2]) Hamilton described this case as follows (Irish Acad., XVII, pp. 85):

"The two planes of vergency close up in one place. The two vergencies [ROM: focal rays] reduce themselves to a single vergency, corresponding to this single plane, and the two guiding lines [ROM: rectilinear cross-section] reduce themselves to a single guiding line."

([3]) Dupin, *Développements de Géométrie:*

"In order to know what values of ψ [ROM: $\tan \alpha$, here] one must start with in order for the radii of curvature to be positive when ψ varies in one sense, and negative when ψ varies in the opposite sense, one needs only to suppose that:

$$r + 2s\,\psi + t\,\psi^2 = 0$$

or

$$\psi = \frac{-s \pm \sqrt{s^2 - rt}}{t}.$$

[ROM: and that is the same equation as (23).]

one must set the partial differential quotient of tan γ with respect to tan α equal to zero.

$$(2t \tan \gamma + 2s)\,[s \tan^2 \alpha + (r - t) \tan \alpha - s] - (2s \tan \gamma + r - t)(t \tan^2 \alpha + 2s \tan \alpha + r) = 0,$$

or

$$\tan \alpha = \frac{s(r+t) \pm \sqrt{s^2(r+t)^2 + [r(r-t)+2s^2][t(r-t)-2s^2]}}{t(r-t)-2s^2},$$

or more simply:

$$\tan \alpha = \frac{s(r+t) \pm \sqrt{(rt-s^2)+[(r-t)^2+4s^2]}}{t(r-t)-2s^2}. \tag{24}$$

The sign of the quantity under the square root sign will depend upon only $rt - s^2$. For concave-concave points, there will therefore always be two positions of the first focal plane where γ is a minimum, which will be determined from the values in (24). In order to find this minimum, the expression for tan α in (24) should be substituted into equation (19). However, in order to arrive at an understandable result, one should make use of a coordinate change in the latter, which was also done by Dupin, Hamilton, and Monge.

Up to now, only the *Z*-axis was parallel to the axis of the ray bundle; the *xz* and *yz* coordinate planes could then rotate around it. We would lie to arrange that:

$$s = 0.$$

The position of the first focal plane will be determined by (21), in which $\angle\,\gamma$ will be a right angle. The same equation will give the value of $\pi / 2$ for α when $s = 0$. The first focal plane will lie in the *yz*-plane. In the new coordinates, the mutually-perpendicular focal planes will be the *xz* and *yz* coordinate planes. However, since the mutually-perpendicular focal planes lie on the principal planes, the principal planes will play the role of coordinate planes for $s = 0$, and that is the basis upon which Hamilton emphasized that one should choose the principal planes to be coordinate planes.

For $s = 0$, equation (19) turns into:

$$\tan \gamma = -\frac{r + t \tan^2 \alpha}{(r-t)\tan \alpha}, \tag{25}$$

and (24) will become:

$$\tan \alpha = \pm\sqrt{\frac{r}{t}}. \tag{26}$$

The two equations, when combined, will give:

$$\tan \gamma = -\frac{\pm 2\sqrt{rt}}{r-t} = \frac{2\sqrt{\rho_1 \cdot \rho_2}}{\rho_1 - \rho_2},$$

when one sets $\rho_1 = 1 / r$, $\rho_2 = 1 / t$, as the desired minimum of the angle γ.

As is known, one will always have:

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan\alpha^2},$$

and it will follows, upon substituting $\tan\alpha = \pm\sqrt{r/t}$, that:

$$\tan 2\alpha = -\frac{\pm 2\sqrt{rt}}{r-t},$$

so

$$\tan 2\alpha = \tan\gamma,$$

and

$$\angle 2\alpha = \angle\gamma.$$

The smallest value of γ is then found to be 2α. If one observes formula (26) then one will also have:

$$\tan\frac{\gamma}{2} = \sqrt{\frac{r}{t}} = \sqrt{\frac{\rho_2}{\rho_1}}$$

for any coordinate system, and *that light bundle will have the smallest* $\angle\gamma$ *for which* $\tan\gamma/2$ *equals the square root of the quotient of the principal radii of curvature.*

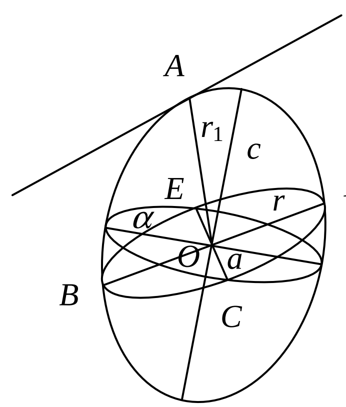

Figure 2.

As an example, we choose the ellipsoid of rotation of extraordinary rays in a uniaxial crystal. Such an ellipsoid of rotation is presented in Fig. 2. Let *c* be the rotational semi-axis and let α be the semi-minor axis. Amongst all of the ray bundles that are possible for the point *A*, the one that possesses the smallest $\angle\gamma$ will be the one for which $\tan\gamma/2 = \sqrt{\rho_1/\rho_2}$. Let the ellipse *BCDE* be the intersection of the ellipsoid whose semi-major axis is *r* and semi-minor axis is *a* with a plane that is parallel to the tangential plane at *A*. Now, since it is known that $\dfrac{\rho_1}{\rho_2} = \dfrac{a^2}{r^2}$, one will get the equation:

$$\tan\frac{\gamma}{2} = \frac{a}{r}.$$

However, from the laws of undulation theory, a and r will be the oscillation amplitudes of the ordinary and extraordinary light rays at A, and of the light bundles that are possible at A, the one that has the small $\angle\,\gamma$ will be the one for which $\tan \gamma / 2$ equals the quotient of the oscillation amplitudes on the bundle axis AO.

Since $\tan \gamma / 2 < 1$, ρ_1 / ρ_2 must be an (*ächter*) fraction; if $\rho_1 / \rho_2 > 1$ then one will take ρ_2 / ρ_1 .

For a point of the equator on the ellipsoid of rotation, one will have:

$$\sqrt{\frac{\rho_2}{\rho_1}} = \frac{a}{c} = \tan\frac{\gamma}{2}.$$

The minimum of γ will get smaller as a / c gets smaller; i.e., as the birefringent energy of the crystal gets larger.

§ 7. The dependency of focal surfaces on the properties of principal waves.

14) The angle γ as a function of the principal wave.

As we did with equation (19), from now on, we shall proceed with (18). The previous paragraph answered the question: "Which bundles are possible for a given point of the elementary wave?" The theorem that we arrived at is suitable only in special cases for deriving theorems for the entire rays system from it, because only the radii of the elementary waves, but not the rays, have an immediate connection with those waves. However, for the investigation of a line bundle at a point of the principal wave it will be possible to combine the bundles into a ray system and the rectilinear cross-sections into focal surfaces. At the same time, we will cast a glance towards the general theory of surfaces for which the elementary wave is the extended curvature model. Now, if the elementary wave remains unconsidered then the various ray systems will be examined that can belong to one and the same principal wave when they are found in other media.

If the denominator on the right-hand side of:

$$\tan\gamma = -\frac{T\tan^2\alpha + 2S\tan\alpha + R}{S\tan^2\alpha + 2(R-T)\tan\alpha - S} \tag{18}$$

vanishes then the analogue of the previous expression will be:

$$\tan\alpha = \frac{T - R \pm \sqrt{(R-T)^2 + 4S^2}}{2S}.$$

At any point of the principal wave there will always exist positions – but generally only a single one – for which the focal planes are mutually perpendicular, and amongst all of the ray systems that are possible for a certain principal wave, according to the variety of the media, there will be, in general, only one of them whose focal surfaces intersect rectangularly.

For $S = 0$, $R = T$, one will have $\tan\alpha = 0 / 0$, and if P_1 and P_2 denote the principal radii of curvature then one will find, in turn:

$$P_1 = P_2 ,$$

which is the condition for spherical curvature. Spherical principal waves – which might exist in a medium wherever one desires – possess only focal surfaces that intersect at right angles. If one regards a purely mathematical principal wave as an initial surface for the ray system then one can say: If the initial surface is a sphere then so are all of the focal surfaces that are generated by a ray system with an arbitrary curvature model and intersect perpendicularly. Conversely, it is known that when the sphere is the curvature model any initial surface will generate ray systems with mutually-perpendicular center-of-curvature surfaces.

If the value for $\tan\gamma$ vanishes in the denominator then:

$$\tan\alpha = \frac{-S \pm \sqrt{S^2 - RT}}{S}.$$

At any concave-convex point of the principal wave there will be two light bundles for which the two focal planes coincide, and that single focal plane will go through one of the two asymptotes to the indicatrix. As is known, these asymptotes will be the straight lines that are possible on concave-convex surfaces. Amongst all of the ray systems of a concave-convex principal wave, there will always exist two that possess a focal surface with only one sheet. This can be expressed mathematically as: *For any concave-convex initial surface, one can find two curvature models that generate ray systems that have simple focal surfaces. The tangential planes to this focal surface will cut out the two systems of directions that exist on the concave-convex initial surface that have vanishing curvature.*

With the help of a minimum argument, one gets:

$$\tan\alpha = \frac{S(R+T) \pm \sqrt{(RT - S^2)[(R-T)^2 + 4S^2]}}{T(R-T) - 2S^2},$$

from which, it will follow that ray systems with such simple focal surfaces for concave-convex principal waves are impossible.

15) Ray systems with imaginary focal surfaces.

Before we go on to the second method, we shall, conversely, develop tan α as a function of tan γ from equation (18). That will permit a glimpse into the seemingly-esoteric essence of rays that emanate from focal surfaces.

$$\text{(18)} \qquad \tan\gamma = -\frac{T\tan^2\alpha + 2S\tan\alpha + R}{S\tan^2\alpha + 2(R-T)\tan\alpha - S}$$

gives:

$$\tan\alpha = \frac{(T-R)\tan\gamma - 2S \pm \sqrt{[(R-T)\tan\gamma + 2S]^2 - 4(S\tan\gamma + T)(R - S\tan\gamma)}}{2(S\tan\gamma + T)},$$

or

$$\tan\alpha = \frac{(T-R)\tan\gamma - 2S \pm \sqrt{(R+T)^2\tan^2\gamma + 4(S^2 - RT)(1+\tan^2\gamma)}}{2(S\tan\gamma + T)},$$

so:

$$\tan\alpha = \frac{(T-R)\tan\gamma - 2S \pm \dfrac{1}{\cos\gamma}\sqrt{[(R+T)^2\sin^2\gamma + 4(S^2 - RT)}}{2(S\tan\gamma + T)}.$$

For any value of γ, this equation will produce two values of α. In fact, the second value will give the position of the second focal plane. If the quantity under the square root sign vanishes then there will be only value for α, and the two focal planes will coincide. When that happens will depend merely upon the curvature of the surfaces, but not on γ. The sign of the quantity under the square root sign will then be determined by just $S^2 - RT$.

If $S^2 - RT$ takes on a sufficiently large value then α will become imaginary, and one is dealing with light bundles with imaginary, rectilinear cross-sections. *Such light bundles are thus possible only at concave-convex points.* Also, $S^2 - RT$ cannot approach zero very closely, since the concave-convex character of the surface must be strongly imprinted on it; namely, it is when $\angle\gamma$ it large that ray systems with no focal surfaces can be possible. At most, $\angle\gamma$ should still refer to a ray system with a one-sheeted focal surface.

Chapter IV. The focal surfaces of a ray system, according to the second method.

§ 8. 16) The interrelationships between principal and elementary waves.

The equations:

$$\tan\gamma = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta},$$

$$\tan\alpha = -\frac{R + S\tan\beta}{S + T\tan\beta},$$

$$\tan\alpha = -\frac{r + s\tan\beta}{s + t\tan\beta}$$

have provided the value for tan γ in (18) and (19). In this chapter, an equation shall be discussed that contains only one of the three angles. At its conclusion, we would also like to cast a fleeting glance towards the equation that only involves $\angle\ \alpha$. The last two equations can then be brought into the form:

$$R + S(\tan\alpha + \tan\beta) + T\tan\alpha\tan\beta = 0,$$
$$r + s(\tan\alpha + \tan\alpha) + t\tan\alpha\tan\alpha = 0,$$

or

$$\tan\alpha + \tan\beta = -\frac{Tr - RT}{Ts - St},$$
$$\tan\alpha \cdot \tan\beta = \frac{Sr - Rs}{Ts - St}.$$

The first equation can assume the following form:

$$\tan\gamma = \pm\frac{\sqrt{(\tan\alpha + \tan\beta)^2 - 4\tan\alpha\tan\beta}}{1 + \tan\alpha\tan\beta}.$$

If one substitutes the two values above for tan α + tan β and tan α · tan β into this then it will become:

$$\tan\gamma = \pm\frac{\sqrt{(Tr - Rt)^2 - 4(Sr - Rs)(Ts - St)}}{(Ts - St) + (Sr - Rs)},$$

or

(27)
$$\tan\gamma = \pm\frac{\sqrt{(Tr + Rt - 2Ss)^2 - 4(S^2 - RT)(s^2 - rt)}}{s(T - R) - S(t - r)}.$$

It is this equation, most especially, that is suited to the task of discussing the inter-relationships between principal and elementary waves, as well as their common effect on the ray bundle. As a result, the evolution of the function tan γ can be investigated in two ways, namely, one can first vary *R*, *S*, *T* and keep *r*, *s*, *t* constant. One is then dealing with a definite point of the elementary wave, while the principal wave takes on all possible forms. Secondly, *R*, *S*, *T* can remain constant, while *r*, *s*, *t* vary. Indeed, if the elementary wave is not able to assume different forms then it will be well-defined for any medium, and also it cannot rotate if it has a fixed position for its center; however, one can rotate the entire medium, along with the elementary waves that are established in it – e.g., a crystal – and preserve the principal waves that emerge in it. This geometric process is more intuitive than the rotation of the first focal plane around its axis.

Admittedly, in that variation of the second partial differential quotients, one must proceed with some foresight. At the singular points, these functions will then become discontinuous, and the equations for the indicatrix will lose their validity. Fortunately, however, the focal planes will intersect on both surfaces along conjugate directions, and when the elementary wave has failed in its duty, one will take the principal wave. Due to the complete symmetry in the equation above, it is only necessarily to perform the calculation once; the result can then be immediately omitted from the other operations.

§ 9. The second method.

17) The principal rays.

Before we begin a deeper investigation, the case where:

$$\tan \gamma = \frac{0}{0}$$

shall be singled out. We square the equation above:

$$\tan^2 \gamma = \frac{(Tr + Rt - 2Ss)^2 - 4(S^2 - RT)(s^2 - rt)}{[S(r-t) - s(R-T)]^2},$$

and set the numerator and denominator of it, when differentiated with respect to *R*, *S*, *T* equal to 0. The denominator gives:

$$2\,[S\,(r - t) - s\,(R - T)]\,(r - t - s + s) = 0,$$

or

$$S\,(r - t) - s\,(R - T) = 0.$$

The numerator gives:

$$2\,(Tr + Rt - 2Ss)\,(r + t - 2s) + 4\,(s^2 - rt)(R + T - 2S) = 0.$$

These two condition equations for the principal rays will be satisfied when:

1) $R = 0, S = 0, T = 0,$

or when:

2) $R = r, S = s, T = t.$

Had one differentiated with respect to r, s, t then the last formula would still be true, and in place of the one before it, one would have the following ones:

3) $r = 0, s = 0, t = 0,$

which we will again discard, since they refer to a plane elementary wave or turning points on it, which are not present. The first group of equations is independent of r, s, t, so it will be true for any medium. It states that *in any medium, plane principal waves will possess only principal rays.* If the middle group of equations is true then the *principal and elementary waves will have second-order contact*, and the fact that principal rays will arise is obvious. The two corresponding points will then have congruent indicatrices with conjugate diameters, of which each pair on the one indicatrix will be parallel to a pair on the other one. All of the planes that go through the bundle axis will then be focal planes.

The following cases might serve to explain this: In homogeneous media, the principal wave must be a sphere in order for the elementary wave to have second-order contact. Therefore, *in homogeneous media, the ray bundles are planar, and spherical principal waves are all principal rays.* The ray bundles of spherical principal waves have rectangular focal planes, in general, but in homogeneous media they will turn into principal rays. On the four distinguished circles of the Fresnel elementary wave, where conical refraction exists, it is known that they will be osculated by a developable surface, and one can say: *Amongst the ray bundles that admit internal conical refraction, one will find principal rays whose corresponding point on the principal wave will be planar or osculated by a developable.*

Finally, in complete generality, if the initial surface of the ray system has a second-order contact with the curvature model then it will possess a principal ray at that point. If two surfaces are congruent then a system of principal rays with a principal focal point will arise.

18) The common influence of the principal and elementary wave on the focal surface of a ray system.

If one lets the denominator on the right-hand side of:

$$\tan \gamma = \pm \frac{\sqrt{(Tr + Rt - 2Ss)^2 - 4(S^2 - RT)(s^2 - rt)}}{s(T - R) - S(t - r)} \tag{27}$$

vanish, in order to learn when $\angle \gamma = \pi / 2$, then:

$$S\,(r - t) - s(R - T) = 0$$

would yield nothing new, and it will be satisfied by $s = 0$, $r = t$ or $S = 0$, $R = T$. Whether $\angle\,\gamma$ can further equal zero will depend upon whether it is possible to satisfy:

$$4\,(S^2 - RT)\,(s^2 - rt) = (Tr + Rt - 2Ss)^2.$$

Since the right-hand side is always positive, $(S^2 - RT)$ and $(s^2 - rt)$ must always have the same sign. Therefore, the two focal planes can coincide only when the principal and elementary wave have the same type of curvature, and indeed, they must both be concave-convex, since, as was shown above, such light bundles cannot be present at concave-convex points, or else $\angle\,\gamma$ would possess a minimum value at that place. In order to find this minimum, the value of $\tan^2\gamma$ will be differentiated with respect to R, S, T, while r, s, t are kept constant:

$$\begin{gathered}[2\,(Tr + Rt) - 2Ss]\,(r + t - 2s) + 4\,(s^2 - rt)\,(R + T - 2S)]\,[S(r - t) - s(R - T)]^2 \\ + 2\,[S\,(r - t) - s(R - T)]\,(r - t)\,[(Tr + Rt - 2\,Ss)]^2 \\ - 4\,(S^2 - RT)\,(s^2 - rt) = 0.\end{gathered}$$

The common factor of $2[S(r - t) - s(R - T)]$ can be removed from this, since it adds nothing new.

One treats a definite point (r, s, t) of the elementary wave through which the Z-axis goes. As in the first method, one rotates the coordinate system around the Z-axis so that one gets $s = 0$. The equation can then be brought into the form:

$$(Tr - Rt)^2 - [t\,(r - t)\,R - r\,(r - t)\,T]\,S = 0,$$

which will give the conditions:

$$S = 0,\ Tr = Rt$$

for γ to have a minimum.

If R, S, T were taken to be constant and differentiated with respect to r, s, t, and one were to make $S = 0$ by a change of coordinates then one would get:

$$s = 0,\ Tr = Rt.$$

Once more, let the principal radii of curvature be $\rho_1 = 1 / r$, $\rho_1 = 1 / t$, $P_1 = 1 / R$, $P_2 = 1 / T$, so $Tr = Rt$ will take on the form:

$$\frac{P_1}{P_2} = \frac{\rho_1}{\rho_2}.$$

Of all the light bundles that are possible at a point of the principal wave, the one that possesses the smallest angle between its focal planes will have a corresponding point on the elementary wave that possesses principal radii of curvature that are proportional to the corresponding values on the principal wave. In other words, the principal and elementary waves must be similar at corresponding points. *All the ray systems of light that are possible for a concave-convex principal wave will have focal surfaces that intersect each other under the smallest angle that is generated by an elementary wave that is similar to the principal wave. In other words: If initial waves and curvature* *models are similar, concave-convex surfaces then a ray system will come about whose focal surfaces intersect with the smallest possible angle.*

Before we leave this section, we shall eliminate tan β from:

$$\tan\,\alpha = -\,\frac{R + S\tan\beta}{S + T\tan\beta}$$

and

$$\tan\beta = -\,\frac{r + s\tan\alpha}{s + t\tan\alpha}.$$

That will produce:

$$\tan\,\alpha = -\,\frac{R(s + \tan\alpha) - S(r + s\tan\alpha)}{S(s + \tan\alpha) - T(r + s\tan\alpha)}$$

or

$$\tan\,\alpha = -\,\frac{Tr - Rt \pm \sqrt{(Rt - Tr)^2 - 4(St - Ts)(Rs - Sr)}}{2(St - Ts)}$$

or, when rearranged:

$$(28)\qquad \tan\,\alpha = -\,\frac{Tr - Rt \pm \sqrt{(Rt + Tr - 2Ss)^2 - 4(S^2 - RT)(s^2 - rt)}}{2(St - Ts)}.$$

The two values of this equation will give the positions of the two focal planes. The quantity under the square root sign is the denominator in the expression for tan γ in (27). If tan $\gamma = 0$ then the square root will vanish, and the focal planes will coincide. The sign of the quantity inside the square root sign will depend upon $(S^2 - RT)(s^2 - rt)$. Ray systems with imaginary focal surfaces can appear when the curvature of the principal and elementary waves of the same type is concave-convex.

Chapter V. The focal surface for the two conical refractions.

§ 10. 19) Internal conical refraction.

It still remains for us to research the properties of focal planes under conical refraction. It is known ([1]) that when a light ray falls on six-parameter or four-parameter crystals, it will generally split into two, one of which will follow Snell's law of sines, while the other one – viz., the extraordinary ray – will be determined by the law that was first proposed by Huyghens. One also holds this law to be valid for two-parameter, one-parameter, as well as for two and one-parameter crystals, so Fresnel started from the hypothesis that the elasticity of the osculating media being unequal along the three crystal axes would explain the fact that the elementary wave of a crystallized medium is either a sphere or an ellipsoid of rotation, as Huyghens would prefer, or a fourth-order surface that consists of two sheets, and whose contact point with the tangential planes will determine the directions of the two sheets. If a crystal possesses a direction around which its surfaces are grouped equivalently – i.e., if it has a crystallographic principal axis – then Fresnel also assumed that the elasticity is the same in all of the directions that are perpendicular to the principal axis. The equation of the elementary wave is then decomposable into two quadratic factors, one of which is the equation of a sphere, while the other one is that of an ellipsoid of rotation. Thus, Huyghens's law is derived from the general solution. However, in that way, there will be two cases that Fresnel foresaw. Namely, its elementary wave possesses four funnel-shaped indentations, with a singular point at the innermost one. The tangent plane at such a singular point degenerates into a second-degree contact cone. Furthermore, these four funnel-shaped indentations will be bounded by a circle in which a singular tangential plane will contact the elementary waves.

As we did already, the above formulas can be applied to conical refraction, assuming that the necessary foresight has been applied to any possible discontinuities. Internal conical refraction comes about on those four circles. The indicatrix is a parabola on the circles, a hyperbola inside of them, and an ellipse outside of them. One will find no jump in the surface on the circles, since the surface itself is continuous, so its partial differential quotients will also be continuous functions, and the theorems that were found up to now can be applied provisionally with no misgivings.

The principal rays are have already been dealt with above. In order to examine the remaining light bundles that are possible under internal conical refraction, one applies the following formula:

$$\tan\gamma = -\frac{t\tan^2\alpha + 2s\tan\alpha + r}{s\tan^2\alpha + 2(r-t)\tan\alpha - s}, \tag{19}$$

in which, $\angle\,\gamma = \pi/2$ for:

$$\tan\alpha = \frac{t - r \pm\sqrt{(r-t)^2 + 4s}}{2s}, \tag{21}$$

([1]) Lloyd, Phil. Mag. 1833, pp. 112.

and $\angle\gamma = 0$ for:

$$\tan\alpha = \frac{-s \pm \sqrt{s^2 - rt}}{t}. \tag{23}$$

By substituting $s^2 - rt$ into (21), one will get:

$$\tan\alpha = \frac{t}{s} = \frac{s}{t}. \tag{29}$$

However, by substituting formula (23), one will get:

$$\tan\alpha = -\frac{s}{t} = -\frac{r}{s}. \tag{30}$$

According to Dupin, on any developable surface, and therefore also on the four circles of Fresnel's elementary wave, $\tan\alpha = -s/t$ will determine the direction in which the curvature possesses an infinitely large radius of curvature, and to which *all of the remaining tangents to the points that are possible will be conjugate* ([1]). At the points of the four circles, the directions in which the radii of curvature become infinite will be tangent to the circles. Every tangent to one of the circles will then have the property that it is conjugate to all of the remaining possible tangents at its contact point with the Fresnel elementary wave, or *at the points of the circles, one of two arbitrary conjugate tangents will also be certainly tangent to the circle.* Formula (30) will give the position at which the focal planes coincide, and all of the other possible planes that go through the bundle axis will be the second planes to it.

In order to be certain of this, we would not like to carry out the calculations blindly, but return to the original physical considerations and the construction of a light ray.

The tangential plane that corresponds to the singular tangential plane *ABtt* to the Fresnel elementary wave will contact the principal wave at *C*, so the one point *C* will correspond to all of the points of the contact circle $\alpha\alpha'\alpha''$, and one must draw parallels through *C* to all of the radius vectors of the contact circle – e.g., $CD \neq O\alpha$, $CD' \neq O\alpha'$, $CD'' \neq O\alpha''$ – in order to obtain the ray cone of internal conical refraction. Any side of the ray cone will be the axis of an infinitely-thin ray bundle. One ray will not distribute infinitely many rays into a ray cone, since its large light intensity would then be

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([1]) Dupin, Mémoire II, art. VII and IX:

"When one of the radii of curvature of the surface is infinite, that radius will be that of a unique cross-section that is conjugate to all of the other normal sections. Therefore, all of the lines that one can then draw by starting at the point of contact of the tangent plane will be conjugate to the unique line, which presents a second-order contact with the surface at the tangent point:

$$(\tan\alpha =)\ \psi = -\frac{r}{s} = -\frac{s}{t}; \quad s^2 - rt = 0$$

is the condition equation that the second-order coefficients *r*, *s*, *t* must satisfy in order for the surface to have one of its curvatures equal to zero. In this case, the line of curvature is osculated by a line and the general surface will be osculated by a developable surface, with that line for one edge."

unexplainable, but infinitely many ray bundles can combine into a light cone. For the bundle axis *CD*, the first focal plane *CD* $t'\,t'$ will be parallel to the corresponding plane $O\alpha\,t\,t$, as the second focal plane might be.

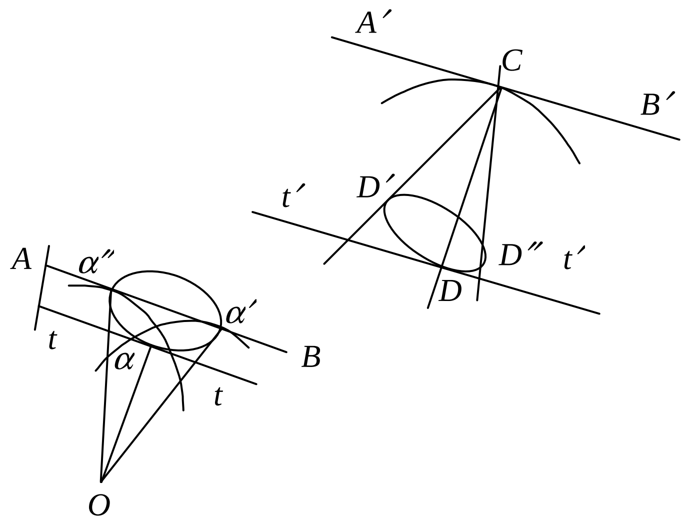

Figure 3.

All of the planes that correspond to the first focal plane will always go through the tangent to the contact circles, and will envelop the cone of singular radius vectors $O\ \alpha\ \alpha'\ \alpha''$. *The first focal planes themselves will envelop the ray cone of internal conical refraction CDD′D″*, and thus the principal wave can take on any shape that it wants. The first sheet of the focal surface and the ray cone of internal conical refraction will have the first focal planes as common tangential planes, and one can say that *in biaxial crystals the first sheet of any arbitrary ray system will run in the ray cone of internal conical refraction*, independently of the principal wave, upon whose form alone, the position of the second focal plane will be determined.

20) External conical refraction.

r, *s*, *t* will, in fact, become discontinuous at a singular point of the Fresnel elementary wave. Indeed, equation (18) and the theorems that one infers from it will preserve their validity, because they are independent of the elementary wave. However, it should likewise go back to the physical construction of the light rays.

At a singular point *S* of the Fresnel elementary wave, there are infinitely many tangential planes; that is, there is a contact cone *SAB*. Now, in order to get the rays that correspond to the singular radius vector *OS*, one must also draw infinitely many tangential planes to the principal wave *A′B′C′* that are parallel to the previous ones, or instead of them, a cone *S′A′B′* that is similar and similarly-oriented to the cone *SAB* that can contact the principal wave. It contacts the principal wave on the curve *A′F′B′D′*. Inverse to what happens with internal conical refraction, the one singular point *S* of the elementary wave will correspond to a curve *A′F′B′D′* of points on the principal wave.

Any parallel to *OS* that goes through one of the points of *A′F′B′D′* will be a light ray, and therefore, under external conical refraction in crystals, the singular radius *OS* will correspond to a ray cylinder whose axis *S′O′* ≠ *SO*. Any line of the cylinder will be the axis of an infinitely-thin ray bundle. The rays that are infinitely-close to one such line on the cylinder will be parallel to the axis and will thus meet it at infinity, so the cylinder will contact a focal plane along any line, or *the one family of focal planes will envelop the ray cylinder. The one sheet of the focal surface will run out of the ray cylinder.*

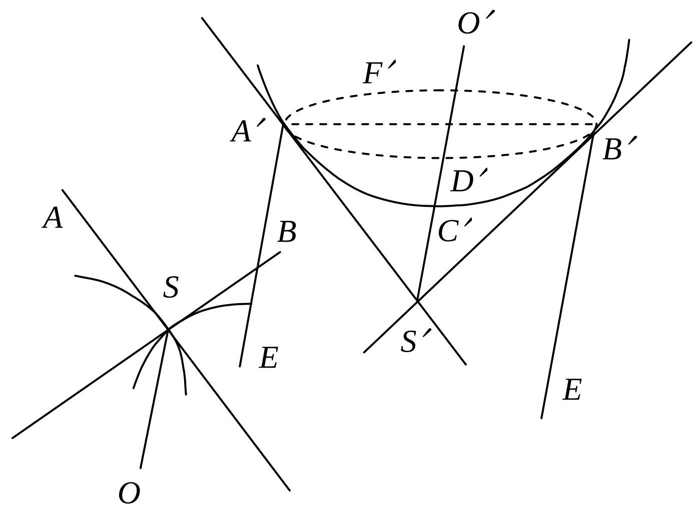

Figure 4.

Since the ray cylinder cuts the principal wave along the curve *A′F′B′D′*, the first focal plane will also cut the principal wave along a direction that is tangent to *A′F′B′D′*. The second focal plane must go through the direction that is conjugate to that tangent. The following theorem was stated in the mathematical introduction in § 2, no. 3: If any surface (the principal wave *A′B′C′*, resp.) is contacted by a developable surface (the cone *S′A′B′*, resp.) then the tangents to the contact curve *A′F′B′D′* will be conjugate to the lines of the cone. Thus, the first focal plane will go through the tangents to *A′F′B′ D′*, and the second one will go through the lines of the contact cone *S′A′F′B′D′*. For example, two focal planes will go through *A′S′*, and let their bundle axis be *A′E*. *A′E* ≠ *S′O′*, which is the axis of the ray cylinder, and *S′O′* will then lie in the second focal plane *E A′S′O′* of the axis *A′E*. Thus, any other second focal plane will also go through the cylinder axis *S′O′*, and *the second focal planes of the ray cylinder of external conical refraction will go through its axis.*

If one imagines all of the tangent planes to a cone that is always pointed, and ultimately degenerates into a line, then the planes will all go through that line. One can regard the axis of the ray cylinder as such a line through which the two focal planes will go, and *the second sheet of the focal surface will then run through a vertex that ends in the axis of the ray cylinder.*

The internal conical refraction then advances outside of the crystal as a ray cylinder, while external conical refraction inside of a crystal is connected with a cylinder of infinitely-thin ray bundles, which does not seem to have noticed up to now.

§ 11. Results.

21) For optics.

1. One obtains the light ray at any point A' of the principal wave when one draws a tangential plane through A' and a tangential plane to the elementary wave that contact it at A and is parallel to the latter plane. The ray at A' will then be parallel to the radius at A.

2. If one has any pair of corresponding directions on the principal and elementary wave then the directions that are conjugate to them will be mutually parallel, and conversely. Since the corresponding directions will be likewise parallel for the focal planes, the focal planes will intersect on the principal wave; the planes that correspond to them on the elementary wave will intersect in conjugate directions.

3. In homogeneous media, as well as everywhere on spherical principal waves, the focal surfaces will intersect perpendicularly. This theorem can also be expressed as: For any arbitrary principal wave, there is only one ray system with focal surfaces that intersect rectangularly, namely, when the principal wave is found in a homogeneous medium. For any medium, there is only one such ray system, namely, for a spherical principal wave.

4. Otherwise, the sheets of the focal surfaces will intersect in skew angles, as a rule; the two sheets can coincide only inside the four contact circles on the Fresnel elementary wave. Even then, light rays with no focal surface can appear.

5. All plane waves possess principal rays, and furthermore, all spherical light waves in homogeneous media, and finally, one has those principal rays for the light bundles that are possible for internal conical refraction whose corresponding point on the light wave has a parabola for an indicatrix.

6. For internal conical refraction, one sheet of the focal surface runs in the light bundle cone of internal conical refraction; the other sheet is independent of the form of the light wave. For external conical refraction, there is a ray cylinder inside the crystal that runs out of the one sheet of the focal surface, while the other sheet will have a vertex that lies on the axis of the cylinder. The light wave will therefore have no influence.

22) For the theory of surfaces.

1. The theorems above can be stripped of their physical character when one switches the principal wave with the initial surface of the ray system and endows the elementary wave with the name of the generalized Gaussian curvature model.

2. Among all of the possible conjugate directions at a point of the initial surface, there will be, in general, only one pair that is parallel to its corresponding directions on the curvature model.

3. The rays will be perpendicular to the initial surface when it is either a sphere or a cone, or the curvature model is a sphere. The focal surfaces will then coincide with the limit surfaces of the shortest distance, and will intersect rectangularly.

4. In general, focal surfaces consist of two sheets. It is only for concave-convex initial surfaces that one can find two curvature models that generate a ray system with a

one-sheeted focal surface. The tangential planes to these two one-sheeted focal surfaces will cut out two systems of directions on the initial surface that will have vanishing curvature.

5. For concave-concave initial surfaces, there is a minimum of the angle γ by which the sheets intersect, namely, when tan $\gamma / 2$ is equal to the square root of the quotient of the principal curvature radii on the curvature model.

6. There also exist ray systems with no focal surfaces, which are possible only when the initial surface and curvature model are simultaneously concave-convex.

7. If the initial surface and the curvature model are congruent, or if they at least have congruent curvatures at isolated points then principal rays will arise. If both of them are similar then the $\angle \gamma$ by which the sheets of the focal surface intersect will be a minimum, such that tan $\gamma / 2$ will now also be equal to the square root of the quotients of the principal radii of curvature on the initial surface.