"Sur le tenseur d'impulsion-énergie," Mémoires Acad. Roy. de Belgique **18** (1940), 1-30.

On the energy-momentum tensor

BY

LÉON ROSENFELD

Liége

Translated by D. H. Delphenich

From the work [**1**] ([1]) of Lorentz, Hilbert, De Donder, F. Klein, and Weyl, one recognizes the close relationship that links the energy-momentum tensor of an arbitrary system of physical agencies, such as material particles or an electromagnetic field, to the gravitational field. In principle, that relation will automatically lead to a well-defined symmetric form for that tensor once one has given an expression for the Lagrange function of the system considered that is invariant under an arbitrary spatio-temporal coordinate transformation. However, on first glance, it might seem that this general procedure for constructing the energy-momentum tensor will present practical difficulties in it application, since it seems to necessitate special considerations in each case and calculations that depend upon the gravitational potentials; i.e., upon variables that are not directly related to the problem and whose influence is generally negligible from an empirical viewpoint. That is why one often prefers to resort to procedures that are less direct, but immediately applicable to a Lagrange function that is invariant under only the Lorentz group. Nevertheless, these procedures themselves also necessitate a special study in each case, notably, in order to insure that the desired tensor has a symmetric form [**2**], in such a way that no practical advantage will compensate for the small satisfaction that is derived from not taking into account the profound relationship that exists between the energy-momentum tensor and the general invariance of the Lagrange function.

Meanwhile, it seems that the problem is susceptible to a solution that presents none of the inconveniences that were just mentioned. Indeed, one confirms that upon taking into account the invariance properties of the Lagrange functions in a manner that is as complete as possible, one will arrive at a general expression for the energy-momentum tensor by a very simple and direct path whose "special" form (i.e., invariant under only the Lorentz group) can be specified immediately in every special case when one recognizes the "special" form of the corresponding Lagrange function. It is the deduction of that expression, under conditions of generality that encompass all of the cases that

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([1]) Numbers in square brackets refer to the bibliography on pp. 23.

have presented themselves in physics up to now, that forms the objective of these present study ([1]).

§ 1. The definition of the energy-momentum tensor

We begin by recalling how the energy-momentum tensor is introduced within the context of the deduction of the gravitational field equations when one starts with a Lagrange function. The gravitational field admits two equivalent specifications, both of which we will have to consider: With the first one, the field variables are the components of a metric tensor g_{lm} . With the second one [**3**], one introduces a "vierbein" at every point, or a set of four orthogonal vectors $h_j^{(\mu)}$ (μ = 1, 2, 3, 4), with whose help the metric tensor can be defined ([2]):

(1) $$g_{lm} = g_{(\mu\nu)}\ h_l^{(\mu)} h_m^{(\nu)} \qquad \text{in which} \qquad g_{(\mu\nu)} = \begin{cases} 0 & \mu \neq \nu, \\ +1 & \mu = \nu = 1, 2, 3, \\ -1 & \mu = \nu = 4. \end{cases}$$

Whenever we do not need to specify which choice of these two descriptions was made, we will denote the gravitational variables by Q_γ , where γ is a fixed ordinal number. Since these variables play only an auxiliary role for us, it will be pointless to imagine quantizing them, and we will treat them as simple parameters (i.e., "*c*-numbers") in all of the following calculations.

The Lagrange function of a system that is composed of a gravitational field and certain physical agencies that are capable of generating that field is the sum of a term that relates to the only the gravitational fields and another term that refers to the physical agencies and their interaction with the gravitational field. These two terms are both integrals of scalar densities $\mathcal{G}$ and $\mathcal{L}$ that depend upon the Q_γ and their derivatives over the spatio-temporal region that is occupied by the system. If we denote an arbitrary choice of independent combination of the Q_γ that is homogeneous of degree one by $Q_{\bar{\gamma}}$ then we will get the field equations of gravitation by specifying that the Lagrange function of the total system should be an extremum for arbitrary variations of the $Q_{\bar{\gamma}}$ that go to zero on the boundary. With the aid of an integration by parts, that condition:

$$\delta\int(\mathcal{G}+\mathcal{L})\,dw = 0 \qquad\qquad (dw = dx^1\,dx^2\,dx^3\,dx^4),$$

([1]) While the present paper (whose publication was delayed by various corrections) was already complete, some results that coincide essentially with the ones in § 5 were published by F. BELINFANTE [Physica, **6** (1939), pp. 887]. On the other hand, I was kindly informed that after having been made aware of my manuscript, he himself quite recently recovered his own results by a method that is quite analogous to the one that forms the objective of this paper, but with several interesting differences. Given the didactic form of my treatise, its present publication with no modifications might nonetheless offer some utility, since it presents a general view of a methodological aspect of a problem that Belinfante first published the solution to.

([2]) One adopts the usual convention of summing over any indices that appear twice in the same term.

will take the form:

$$\int \frac{\delta(\mathcal{G}+\mathcal{L})}{\delta Q_{\bar{\gamma}}} \delta Q_{\bar{\gamma}}\, dw = 0,$$

in which the coefficient of $\delta Q_{\bar{\gamma}}$ defines the "variational derivative" [**4**] of the density function $\mathcal{G} + \mathcal{L}$ with respect to $Q_{\bar{\gamma}}$; one deduces the field equations from that:

$$\frac{\delta \mathcal{G}}{\delta Q_{\bar{\gamma}}} = -\frac{\delta \mathcal{L}}{\delta Q_{\bar{\gamma}}}. \tag{2}$$

In this form, the left-hand side will contain only gravitational variables, while the right-hand side, which presents the role of the physical agencies as sources of the gravitational field precisely, will determine the "energy-momentum tensor density" of the system considered.

If the variables that specify the nature of this system do not involve spinorial quantities, such as the wave functions of elementary material particles of spin 1 / 2, then we can adopt the usual description of the gravitational with the help of the g_{lm}; otherwise, the function $\mathcal{L}$ will depend upon the vierbeins $h_j^{(\mu)}$ in a manner that is not rationally reducible to the g_{lm}, and it would be advantageous to choose these vierbeins to represent the field. In the former situation, equations (2) would give us the components of the energy-momentum tensor density directly, in the symmetric form:

$$\mathcal{T}^{lm} \equiv \mathcal{T}^{ml} = -\left(\frac{\delta \mathcal{L}}{\delta g_{lm}} + \frac{\delta \mathcal{L}}{\delta g_{ml}}\right). \tag{3}$$

In order to see how to generalize this definition to the case in which $\mathcal{L}$ depends essentially upon the $h_j^{(\mu)}$, it will suffice to observe that for any function (such as $\mathcal{G}$) that depends upon the $h_j^{(\mu)}$ only by the intermediary of the g_{lm}, one will have [**4**]:

$$\frac{\delta \mathcal{G}}{\delta h_j^{(\mu)}} = \frac{\delta \mathcal{G}}{\delta g_{lm}} \frac{\partial g_{lm}}{\partial h_j^{(\mu)}}. \tag{4}$$

Therefore, since from (1), one will have:

$$\frac{\partial g_{lm}}{\partial h_j^{(\mu)}} = g_{(\mu\nu)} \left\{\delta_l^j h_m^{(\nu)} + \delta_m^j h_l^{(\nu)}\right\}, \tag{5}$$

one can infer that:

$$\frac{\delta \mathcal{G}}{\delta h_j^{(\mu)}} h_i^{(\mu)} = \left(\frac{\delta \mathcal{G}}{\delta g_{jm}} + \frac{\delta \mathcal{G}}{\delta g_{mj}}\right) g_{mi}. \tag{6}$$

One then sees that in this case the energy-momentum tensor density will be given by:

(7)
$$\mathcal{T}^{\;j}_{\dots i} = -\frac{\delta\mathcal{L}}{\delta h_j^{(\mu)}}\,h_i^{(\mu)},$$

which is an expression that, from (6), will reduce to (3) when $\mathcal{L}$ contains the $h_j^{(\mu)}$ only by the intermediary of the g_{lm}. The condition of symmetry with respect to the indices is no longer satisfied identically for the quantity (7); we shall return to that fact later on.

§ 2. Fundamental identities

Our goal now is to show how the direct calculation of the variational derivatives $\delta\mathcal{L}/\delta Q_{\bar{\gamma}}$ that enter into formulas (3) or (7) can be avoided by taking into account some fundamental identities that result from the invariance of the Lagrange function:

$$\int \mathcal{L}\,dw$$

under an arbitrary coordinate transformation. Since this is just a matter of dealing with well-known things, we shall rapidly recall the deduction of those identities after specifying the conditions that we shall impose upon ourselves and using a system of notation that will bring about certain auxiliary quantities that we shall make use of later on.

The variables that specify the physical agencies that we shall consider will be the components of certain tensors $(Q^{(\kappa)})$, $\kappa = 1, 2, \dots$. To fix ideas, we choose covariant components exclusively, and we denote them everywhere by Q_α, where the index α represents a set of values for the indices $i_1, i_2, \dots, i_n$ of the tensor, in addition to the ordinal number κ. We can add the gravitational variables Q_γ, which will define either a tensor $(Q^{(0)}) \equiv (g)$ or a system of four tensors $(Q^{(0,\,\mu)}) \equiv (h^{(\mu)})$ to this system of variables. We then denote the total system of variables Q_α and Q_γ by the collective notation Q_ω: An expression that contains a summation over the index ω (i.e., one in which that index appears twice) can therefore be decomposed into a sum of two analogous expressions that contain a summation over α and another over γ, respectively.

If we now consider an arbitrary infinitesimal coordinate transformation that is defined by:

(8)
$$\delta x^i = \bar{x}^i - x^i = \xi^i(x)$$

then, by virtue of the tensor character of the variables, the ("substantial") variation of the variable Q_ω will be given by an expression of the form:

(9)
$$\delta Q_\omega \equiv \bar{Q}_\omega(\bar{x}) - Q_\omega(x) = c^{\,j}_{\omega,i}\,\frac{\partial \xi^i}{\partial x^j}.$$

The auxiliary quantities $c^j_{\omega,i}$ are the components of a system of tensors that are attached to each type of variable ($Q^{(\kappa)}$) by the following definition:

(10) $$c^{(\kappa)\,j}_{i_1 i_2 \cdots i_n, i} = -\sum_{p=1}^{n} \delta^j_{i_p}\, Q^{(\kappa)}_{i_1 \cdots i_{p-1}\, i\, i_{p+1} \cdots i_n}\,.$$

respectively.

For example, for the gravitational variables, we have:

1) If we are concerned with the vierbein variables $h_i^{(\mu)}$:

$$c^{(0,\mu)\,j}_{l,i} = -\delta^j_l\, h_i^{(\mu)}.$$

2) If we are concerned with the metric tensor g_{lm} :

$$c^{(0)\,j}_{lm,i} = -(\delta^j_l\, g_{mi} + \delta^j_m\, g_{li})\,.$$

If we compare these expressions with formulas (3) and (7) then we will see that with either convention the energy-momentum tensor density can be written, with the aid of this notation:

(11) $$\mathcal{T}^{\;j}_{\cdots i} = \frac{\delta \mathcal{L}}{\delta Q_\gamma} c^j_{\gamma,i}\;,$$

and we shall see that it is precisely this expression that enters into the fundamental identities. We once more employ the "local" variations:

(12) $$\delta^* Q_\omega \equiv \bar{Q}_\omega(x) - Q_\omega(x) = \delta Q_\omega - \frac{\partial Q_\omega}{\partial x^j}\,\xi^j,$$

for which, one will obviously have:

(13) $$\delta^* \frac{\partial Q_\omega}{\partial x^j} = \frac{\partial}{\partial x^j}\;\delta^* Q_\omega.$$

The condition of the invariance of the Lagrange function under the transformation (8), which we assume to be satisfied for an arbitrary domain of integration, can be written as:

(14) $$\delta \int \mathcal{L}\, dw = \int \delta^* \mathcal{L}\, dw + \int \frac{\partial}{\partial x^j}(\mathcal{L}\,\xi^i)\, dw = 0.$$

We now assume that the function $\mathcal{L}$ depends upon only the variables Q_ω and their first derivatives:

$$Q_{\omega\,|\,j} \equiv \frac{\partial Q_\omega}{\partial x^j}.$$

Since the derivatives of the gravitational variables enter in only by the intermediary of the covariant derivatives of the other variables Q_α, it will suffice for us to formulate this hypothesis for just the Q_α, moreover. If we refrain from the quantization of these latter variables, as well as the Q_γ, then we will get:

$$\delta^*\mathcal{L} = \frac{\partial\mathcal{L}}{\partial Q_\omega}\delta^* Q_\omega + \frac{\partial\mathcal{L}}{\partial Q_{\omega|j}}\delta^* Q_{\omega|j}. \tag{15}$$

However, one will similarly verify that for all of the quantized fields Q_ω that have been introduced into physics, the Lagrange function will depend upon the Q_ω and $Q_{\omega\,|\,j}$ in a sufficiently simple manner that formulas (15) can be preserved, as long as one adds a "symmetrization" condition ([1]); i.e., one interprets the right-hand side as one-half the sum of the quantity in question and its Hermitian conjugate.

First of all, by means of (13), we can put (15) into the form:

$$\delta^*\mathcal{L} = \frac{\delta\mathcal{L}}{\delta Q_\omega}\delta^* Q_\omega + \frac{\partial}{\partial x^j}\left(\frac{\partial\mathcal{L}}{\partial Q_{\omega|j}}\delta^* Q_\omega\right), \tag{16}$$

so, under our hypotheses, the variation derivative will reduce to:

$$\frac{\delta\mathcal{L}}{\delta Q_\omega} = \frac{\partial\mathcal{L}}{\partial Q_\omega} - \frac{\partial}{\partial x^j}\left(\frac{\partial\mathcal{L}}{\partial Q_{\omega|j}}\right). \tag{17}$$

If we substitute the expression (16) into (14) and use (12) and (9) then we will obtain the invariance condition:

$$\begin{aligned} &-\int\left\{\frac{\partial}{\partial x^j}\left(\frac{\delta\mathcal{L}}{\delta Q_\omega}c^j_{\omega,i}\right) + \frac{\delta\mathcal{L}}{\delta Q_\omega}Q_{\omega|i}\right\}\xi^i\,dw \\ &+\int\frac{\partial}{\partial x^j}\left\{\left(\mathcal{L}\,\delta^j_i - \frac{\partial\mathcal{L}}{\partial Q_{\omega|j}}Q_{\omega|i} + \frac{\delta\mathcal{L}}{\delta Q_\omega}c^j_{\omega,i}\right)\xi^i + \frac{\partial\mathcal{L}}{\partial Q_{\omega|j}}c^k_{\omega,i}\frac{\partial\xi^i}{\partial x^k}\right\}dw = 0. \end{aligned} \tag{18}$$

If one considers a transformation ξ^i that goes to zero on the boundary of the domain of integration then that will imply that the coefficient of ξ^i in the first integral of (18) must be annulled identically:

([1]) This "symmetrization" conventions applies to all of the following formulas. However, recall that it does not concern the gravitational variables.

(19) $$\frac{\partial}{\partial x^j}\left(\frac{\delta \mathcal{L}}{\delta Q_\omega} c^j_{\omega,i}\right) + \frac{\delta \mathcal{L}}{\delta Q_\omega} Q_{\omega|i} = 0 .$$

In order for the second integral to be annulled for an arbitrary choice of integration domain and transformation ξ^i, one will find that by annulling the coefficients of ξ^i, along with their first and second derivatives, individually, one will get two new groups of independent identities:

(20) $$\mathcal{L}\,\delta^j_i - \frac{\partial \mathcal{L}}{\partial Q_{\omega|j}} Q_{\omega|i} + \frac{\delta \mathcal{L}}{\delta Q_\omega} c^j_{\omega,i} + \frac{\partial \mathcal{R}^{kj}_i}{\partial x^k} \equiv 0$$

and

(21) $$\mathcal{R}^{jk}_i + \mathcal{R}^{kj}_i \equiv 0,$$

in which one has set:

(22) $$\mathcal{R}^{jk}_i = \frac{\partial \mathcal{L}}{\partial Q_{\omega|j}} c^k_{\omega,i} .$$

From the manner by which they were deduced, the fundamental identities (19), (20), (21) are covariant. For the sake of clarity in the following calculations, it will be advantageous to make that covariance enter into them formally by introducing the covariant derivatives, instead of the partial derivatives $Q_{\omega|j}$.

However, before we do that, we shall adopt the viewpoint of special relativity for the moment, in the name of orientation. If we compare the identity (20) with formula (11) and set $Q_{\gamma|j} = 0$ then we will see that this identity amounts to the calculation of $\mathcal{T}^{\,j}_{\cdots i}$ by operations that depend upon the non-gravitational variables Q_α, provided that one knows the tensor density $\mathcal{R}^{jk}_i$. On the other hand, the expression (22) for the latter shows that it contains the gravitational variables explicitly only by way of the derivatives $\partial\mathcal{L} / \partial Q_{\gamma|j}$, which immediately revert to the $\partial\mathcal{L} / \partial Q_{\alpha|j}$, since the $Q_{\gamma|j}$ enter in only by the intermediary of the covariant derivatives of the Q_α. The entire problem then comes down to eliminating the $\partial\mathcal{L} / \partial Q_{\gamma|j}$ from the $\mathcal{R}^{jk}_i$, and we shall see that this elimination is extremely easy. Since the consideration of spinorial variables would demand supplementary developments (which are entirely parallel to the foregoing ones, moreover) in regard to the invariance of the Lagrange function under orthogonal transformations of the vierbeins, we shall first treat separately the case in which one is concerned with only tensorial variables, properly speaking, for the sake of clarity.

§ 3. Tensorial variables

We now distinguish between the variables Q_α, which are purely tensorial, and which we shall collectively denote by Q_τ, and the spinorial variables that we shall call Q_σ. We temporarily assume that all of the variables of the system considered are of the former type. In that case, the expression for the covariant derivative $Q_{\tau|j}$, with our system of notation, is obtained easily by observing that a parallel displacement of the point (x) will be represented formulas of the type (8), (9), (12) when one imposes the condition:

$$\frac{\partial \xi^i}{\partial x^j} + \Gamma^i_{jl}\,\xi^l = 0$$

upon the ξ^i at that point, in which the Γ^i_{jl} are the components of the affine connection [**5**]. The covariant derivative $Q_{\tau|j}$ will then be defined to be the coefficient of ξ^i in the corresponding expression for $-\delta^* Q_\tau$; i.e.:

(23) $$Q_{\tau\|l} = Q_{\tau|l} + c^j_{\tau,i}\,\Gamma^i_{jl}\,.$$

If we describe the gravitational field with the help of the metric tensor then we will get, on the other hand, and with an analogous notation:

(24) $$Q_{\gamma\|l} = 0.$$

Finally, the tensorial densities that enter into the identities (19) and (20) by way of their partial derivatives will have the divergences:

(25) $$\left(\frac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,i}\right)_{\|j} = \frac{\partial}{\partial x^j}\left(\frac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,i}\right) - \frac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,k}\Gamma^k_{ji}$$

and

(26) $$\mathcal{R}^{kj}_{i\|k} = \frac{\partial \mathcal{R}^{kj}_i}{\partial x^k} - \mathcal{R}^{kj}_l \Gamma^l_{ki}\,,$$

respectively. In formula (26), one has taken into account the antisymmetric character of in $\mathcal{R}^{kj}_i$ its upper indices, which is expressed by the identity (21).

Formulas (23), (25), (26), and (22) allow us to write the identities (20) and (19) in the covariant form:

(27) $$\frac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,i} \equiv \frac{\partial\mathcal{L}}{\partial Q_{\omega|j}} Q_{\omega\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k}$$

(28) $$\left(\frac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,i}\right)_{\|j} + \frac{\partial\mathcal{L}}{\partial Q_\omega} Q_{\omega\|i} \equiv 0.$$

Formula (24) will give the desired expression for the energy-momentum tensor density by means of (11) and (24):

(29) $$\mathcal{T}^{\,j}_{\cdots i} \equiv \frac{\partial\mathcal{L}}{\partial Q_{\tau\|j}} Q_{\tau\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k} - \frac{\delta\mathcal{L}}{\delta Q_\tau} c^j_{\tau,i}\,.$$

In order to better emphasize the covariant form of this expression, we have written:

$$\frac{\partial \mathcal{L}}{\partial Q_{\tau\|j}} \text{ instead of } \frac{\partial \mathcal{L}}{\partial Q_{\tau|j}}.$$

As for (28), it will take on a simple, well-known, significance when one takes the field equation that the variables Q_τ must satisfy into account. In order to write down these equations, we shall define a set of independent variables $Q_{\bar{\tau}}$ with the help of the Q_τ, and in order to avoid any complication that might be due to the quantization of the variables, we shall assume that the $Q_{\bar{\tau}}$ are linear, homogeneous functions of Q_τ. The field equations will then be:

(30) $$\frac{\delta \mathcal{L}}{\delta Q_{\bar{\tau}}} = 0.$$

In addition, one will see that:

(31) $$\frac{\delta \mathcal{L}}{\delta Q_\tau} c^j_{\tau,i} = 0$$

and

(32) $$\frac{\delta \mathcal{L}}{\delta Q_\tau} Q_{\tau\|i} = 0,$$

by virtue of equations (30). From (11) and (24), the identity (28) will then mean that:

(33) $$\mathcal{T}^{\;j}_{\dots i\|j} = 0,$$

by virtue of the field equations; this is the general-relativistic expression of the conservation of energy-momentum.

Now, observe that one can profit from (31) to simplify the definition (29) of the energy-momentum density without affecting the conservation theorem (33): Indeed, one can set:

(34) $$\mathfrak{t}^{\;i}_{\dots j} = \frac{\partial \mathcal{L}}{\partial Q_{\tau\|j}} Q_{\tau\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k},$$

instead of (29). However, the symmetry condition (3) is not necessarily satisfied identically by $\mathfrak{t}^{ij}$; that would only be a consequence of the field equations. Upon setting:

(35) $$d^{ij}_\omega = \tfrac{1}{2}\left(c^i_{\omega,k} g^{kj} - c^j_{\omega,k} g^{ki}\right)$$

in a general manner, from (3), and upon comparing (29) with (34), one can write:

(36) $$\mathfrak{t}^{ij} - \mathfrak{t}^{ji} = 2\,\frac{\delta \mathcal{L}}{\delta Q_\tau} d^{ij}_\tau.$$

The antisymmetric tensor d_ω^{ij} that is defined by (36), and which will soon play an essential role, takes on a very simple meaning in the case of *special* relativity, for which we can attribute constant values $\dot{g}_{lm}$ to the g_{lm}. Indeed, from (8) and (9), one sees that an infinitesimal Lorentz transformation will correspond to the ξ^i linearly and homogeneously:

(37) $$\xi^i = \varepsilon^i_{\cdot k} x^k ,$$

with the orthogonality condition:

(38) $$\varepsilon_{ik} + \varepsilon_{ki} = 0 \qquad (\text{in which } \varepsilon_{ik} = \dot{g}_{il}\varepsilon^l_{\cdot k}).$$

In that way, one will have:

(39) $$\dot{\delta} Q_\omega = \dot{d}_\omega^{ij} \varepsilon_{ji} .$$

It now remains for us to expression the $\mathcal{R}_i^{kj}$ that enter into (29) or (34), and are defined by (22), as functions of only the $\partial\mathcal{L} / \partial Q_{\tau\|j}$. One can achieve the elimination of the $\partial\mathcal{L} / \partial Q_{\gamma|j}$ immediately when one observes that from (35) one will have:

(40) $$d_\gamma^{ij} = 0$$

for the components Q_τ of the metric tensor. Indeed, one concludes that the quantities:

(41) $$\mathcal{D}^{k;\,ij} \equiv \tfrac{1}{2}(\mathcal{R}_l^{ki} g^{lj} + \mathcal{R}_l^{kj} g^{li}) = \frac{\partial\mathcal{L}}{\partial Q_{\tau\|k}} d_\tau^{ij}$$

no longer depend explicitly upon the gravitational variables. On the other hand, by virtue of the antisymmetry property (21), the $\mathcal{R}_l^{ki}$ can be expressed in terms of only the $\mathcal{D}^{k;\,ij}$ in the form of:

(42) $$\mathcal{R}_l^{ki} g^{li} = \mathcal{D}^{\,j;\,ki} - \mathcal{D}^{\,i;\,jk} + \mathcal{D}^{k;\,ij}.$$

One should observe that the explicit form of the energy-momentum density that is given by (29) or (34), (41), (42) was obtained by using the general consequence of the invariance of the Lagrange function as expressed by the fundamental identities (20), (21), while the third identity (19) led to the conservation theorem (33). Similarly, the extension of the preceding considerations to the case in which the Lagrange function contains spinorial variables must necessitate the use of identities that would result from the invariance of that function under orthogonal transformations of the vierbeins. That is what we shall show in the next section.

§ 4. Spinorial variables

We shall assume that the variables of the system under consideration are both tensorial, which will be denoted by Q_τ, and spinorial, which will be called Q_σ, while we take the vierbeins $h_i^{(\mu)}$ to be the gravitational variables Q_γ. An infinitesimal orthogonal transformation of these vierbeins will be represented by:

(43) $$\delta h_i^{(\mu)} = d_{i(\lambda\nu)}^{(\mu)}\,\varepsilon^{(\lambda\nu)} \qquad (\varepsilon^{(\lambda\nu)} + \varepsilon^{(\nu\lambda)} = 0),$$

in which the coefficients $d_{i(\lambda\nu)}^{(\mu)}$ are related to the $d_i^{(\mu)jk}$ that were defined by the general formula (35) by the simple relations:

(44) $$d_{i(\lambda\nu)}^{(\mu)} = g_{(\lambda\rho)}g_{(\nu\alpha)}h_j^{(\rho)}h_k^{(\sigma)}d_i^{(\mu)jk};$$

explicitly, one has:

(45) $$\begin{cases} d_i^{(\mu)jk} = \frac{1}{2}[\delta_i^k h^{(\mu)j} - \delta_i^j h^{(\mu)k}], \\ d_{i(\lambda\nu)}^{(\mu)} = \frac{1}{2}[\delta_\lambda^\mu g_{(\nu\alpha)}h_i^{(\alpha)} - \delta_\nu^\mu g_{(\lambda\alpha)}h_i^{(\alpha)}], \end{cases}$$

The spinorial variables will submit to variations of the form:

(46) $$\begin{cases} \delta Q_\sigma = d_{\sigma(\lambda\nu)}\varepsilon^{(\lambda\nu)}, \\ \text{with } d_{\sigma(\lambda\nu)} = -d_{\sigma(\nu\lambda)} \end{cases}$$

under such a transformation.

If Q_σ represents, in particular, the set Ψ of two spinors whose rank corresponds to an elementary particle of spin 1 / 2 then one will have [**3**]:

(47) $$d_{(\sigma)\lambda\nu} = -\tfrac{1}{8}\,(\gamma_{(\lambda)}\,\gamma_{(\nu)} - \gamma_{(\nu)}\,\gamma_{(\lambda)})\,Q_\sigma,$$

in which the matrices $\gamma_{(\lambda)}$ satisfy the relations:

(48) $$\gamma_{(\lambda)}\,\gamma_{(\nu)} + \gamma_{(\nu)}\,\gamma_{(\lambda)} = -\,2g_{(\lambda\nu)}\,.$$

One has, in turn:

(49) $$d_{\sigma'(\lambda\nu)} = \tfrac{1}{8}\,Q_{\sigma'}(\gamma_{(\lambda)}\,\gamma_{(\nu)} - \gamma_{(\nu)}\,\gamma_{(\lambda)})$$

for the adjoint variable $Q_{\sigma'} \equiv \Phi = \Psi^\dagger\beta$ (in which $\Psi^\dagger$ is the Hermitian conjugate of Ψ and β^{-1} is a matrix that transforms the $\gamma_{(\lambda)}$ into their Hermitian conjugates).

The invariance of the Lagrange function under the transformations (43), (46) is expressed by the condition:

$$\int\left\{\frac{\delta\mathcal{L}}{\delta h_i^{(\mu)}}\delta h_i^{(\mu)} + \frac{\delta\mathcal{L}}{\delta Q_{\sigma|j}}\delta Q_\sigma\right\}dw + \int\frac{\partial}{\partial x^j}\left\{\frac{\partial\mathcal{L}}{\partial h_{i|j}^{(\mu)}}\delta h_i^{(\mu)} + \frac{\partial\mathcal{L}}{\partial Q_{\sigma|j}}\delta Q_\sigma\right\}dw = 0,$$

and will give rise to two independent groups of identities:

(50)
$$\begin{cases} \dfrac{\delta \mathcal{L}}{\delta h_i^{(\mu)}} d_{i(\lambda\nu)}^{(\mu)} + \dfrac{\delta \mathcal{L}}{\delta Q_\sigma} d_{\sigma(\lambda\nu)} = 0, \\ \dfrac{\delta \mathcal{L}}{\delta h_{i|j}^{(\mu)}} d_{i(\lambda\nu)}^{(\mu)} + \dfrac{\delta \mathcal{L}}{\delta Q_{\sigma|j}} d_{\sigma(\lambda\nu)} = 0. \end{cases}$$

From (44), one has:

$$h^{(\lambda)j} h^{(\nu)k} d_{i(\lambda\nu)}^{(\mu)} = d_i^{(\mu)jk} .$$

If we likewise set (except for a sign difference whose meaning will emerge later on):

(51)
$$h^{(\lambda)j} h^{(\nu)k} d_{\sigma(\lambda\nu)} = - d_\sigma^{(h)jk}$$

[the index (h) has been added in order to avoid any confusion with the quantity d_σ^{jk} that was defined by (35)] then we can put the identities (50) into the equivalent form:

(52)
$$\begin{cases} \dfrac{\delta \mathcal{L}}{\delta h_i^{(\mu)}} d_i^{(\mu)jk} + \dfrac{\delta \mathcal{L}}{\delta Q_\sigma} d_\sigma^{(h)jk} = 0, \\ \dfrac{\delta \mathcal{L}}{\delta h_{i|j}^{(\mu)}} d_i^{(\mu)lk} + \dfrac{\delta \mathcal{L}}{\delta Q_{\sigma|j}} d_\sigma^{(h)lk} = 0. \end{cases}$$

Thanks to (45) and the definition (7) of $\mathcal{T}_{\cdot\cdot i}^{j}$, the first identity in (52) will give us:

(53)
$$\mathcal{T}^{jk} - \mathcal{T}^{kj} \equiv 2 \frac{\delta \mathcal{L}}{\delta Q_\sigma} d_\sigma^{(h)jk} .$$

If we once more assume that the independent spinorial variables $Q_{\bar{\sigma}}$ are linear, homogeneous functions of the Q_σ then we will see that we can write:

(54)
$$\frac{\delta \mathcal{L}}{\delta Q_\sigma} d_\sigma^{(h)jk} = 0,$$

by virtue of the field equations:

(55)
$$\frac{\delta \mathcal{L}}{\delta Q_{\bar{\sigma}}} = 0.$$

Formula (53) then shows [**3**] that the energy-momentum tensor density $\mathcal{T}^{jk}$ is once more symmetric in the indices j, k; if that is not true identically, then it will at least be a consequence of the field equations (55).

Now, if we return to the identities (19) and (20) then, first of all, we can give them the following covariant form [which is analogous to (27), (28)]:

(56) $$\begin{cases} \dfrac{\delta \mathcal{L}}{\delta Q_\omega} c^j_{\omega,i} \equiv \dfrac{\partial \mathcal{L}}{\partial Q_{\omega|j}} Q_{\omega;j} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k}, \\ \left(\dfrac{\delta \mathcal{L}}{\delta Q_\omega} c^j_{\omega,i}\right)_{\|j} + \dfrac{\delta \mathcal{L}}{\delta Q_\omega} Q_{\omega;j} \equiv 0, \end{cases}$$

in which we have set:

(57) $$Q_{\omega;l} = Q_{\omega|l} + c^j_{\omega,i}\,\Gamma^i_{jl}$$

for all of the variables, and we have taken (25), (26) into account, in addition. As far as the purely tensorial variables are concerned, naturally, $Q_{\tau;l}$ is nothing but the covariant derivative $Q_{\tau\|l}$ that was defined by (23). However, for spinor variables, the covariant derivative $Q_{\tau\|l}$ will differ from $Q_{\tau;l}$ by a term [**3**] that originates in the fact that a parallel displacement of the vierbeins is equivalent to an orthogonal transformation that affects the Q_σ. If parallel displacement is defined by a vector ξ^i (that is stationary at the point considered) then, from (43), this equivalence will be expressed in the form:

$$\xi^{(\lambda\nu)} d^{(\mu)}_{i(\lambda\nu)} = -\,h^{(\mu)}_{i;j}\xi^j;$$

hence, one can infer the parameters $\xi^{(\lambda\nu)}$ of the equivalent orthogonal transformation by means of (45):

(58) $$\begin{cases} \xi^{(\lambda\nu)} = \Gamma^{(\lambda\nu)}_j \xi^j, \\ \text{with} \quad \Gamma^{(\lambda\nu)}_j = -\tfrac{1}{2}\left\{h^{(\lambda)}_{i;j} h^{(\nu)i} - h^{(\nu)}_{i;j} h^{(\lambda)i}\right\}; \end{cases}$$

it will then result that the covariant derivative of Q_σ is:

(59) $$Q_{\sigma\|l} = Q_{\sigma;l} + d_{\sigma(\lambda\nu)}\,\Gamma^{(\lambda\nu)}_l.$$

The first identity (50) will then give:

$$\frac{\delta \mathcal{L}}{\delta h^{(\mu)}_l} h^{(\mu)}_{l;i} + \frac{\delta \mathcal{L}}{\delta Q_\sigma} d_{\sigma(\lambda\nu)} \Gamma^{(\lambda\nu)}_l \equiv 0,$$

so

$$\frac{\delta \mathcal{L}}{\delta h^{(\mu)}_l} h^{(\mu)}_{l;i} + \frac{\delta \mathcal{L}}{\delta Q_\sigma} Q_{\sigma;i} \equiv \frac{\delta \mathcal{L}}{\delta Q_\sigma} Q_{\sigma\|i};$$

likewise, if one starts with the second identity in (50) then one will get:

$$\frac{\delta\mathcal{L}}{\delta h^{(\mu)}_{l|j}} h^{(\mu)}_{l;i} + \frac{\delta\mathcal{L}}{\delta Q_{\sigma|j}} Q_{\sigma;i} \equiv \frac{\delta\mathcal{L}}{\delta Q_{\sigma\|j}} Q_{\sigma\|i} \ .$$

If one substitutes these results in (56) then one will get:

(60)
$$\begin{cases} \dfrac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,i} \equiv \dfrac{\partial\mathcal{L}}{\partial Q_{\alpha\|j}} Q_{\alpha\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k}, \\ \left(\dfrac{\delta\mathcal{L}}{\delta Q_\omega} c^j_{\omega,i}\right)_{\|j} + \dfrac{\partial\mathcal{L}}{\partial Q_\alpha} Q_{\alpha\|i} \equiv 0, \end{cases}$$

in which, the symbol Q_α encompasses all of the tensorial Q_τ and spinorial Q_σ variables. From (11), the first formula (60) can be written:

(61)
$$\mathcal{T}^{\ j}_{\cdots i} \equiv \frac{\partial\mathcal{L}}{\partial Q_{\alpha\|j}} Q_{\alpha\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k} - \frac{\delta\mathcal{L}}{\delta Q_\alpha} c^j_{\alpha,i} \ ,$$

which then provides a direct generalization of (29).

The field equations (30) and (55) have the consequence that if one takes (54) and (59), in particular, into account then:

(62)
$$\begin{cases} \dfrac{\delta\mathcal{L}}{\delta Q_\alpha} c^j_{\alpha,i} = 0, \\ \dfrac{\delta\mathcal{L}}{\delta Q_\alpha} Q_{\alpha\|i} = 0, \end{cases}$$

which will permit us to conclude the conservation theorem (33), as before. Thanks to (62), one can further simply the definition (61) by setting the energy-momentum tensor density equal to:

(63)
$$\mathrm{t}^{\ j}_{\cdots i} \equiv \frac{\partial\mathcal{L}}{\partial Q_{\alpha\|j}} Q_{\alpha\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k} \ .$$

By virtue of (35) and (53), one will then have:

(64)
$$\mathrm{t}^{ji} - \mathrm{t}^{ij} = 2\,\frac{\delta\mathcal{L}}{\delta Q_\alpha} s^{ji}_\alpha ,$$

if one sets:

(65)
$$\begin{cases} s^{ji}_\tau = d^{ji}_\tau, \\ s^{ji}_\sigma = d^{ji}_\sigma + d^{(h)\,ji}_\sigma, \end{cases}$$

and from (62) and (54), relation (64) will express the idea that the symmetry of the tensorial density $\mathfrak{t}^{ji}$ is a consequence of the field equations.

As for the tensorial density $\mathcal{R}^{kj}_{i\|k}$, it will always be calculated from formula (42), but one will now have:

$$\mathcal{D}^{k;\,ij} = \frac{\partial \mathcal{L}}{\partial Q_{\omega|k}} d^{ij}_{\omega},$$

instead of (41), or furthermore, from the second identity in (52), and with the notation in (65):

$$\mathcal{D}^{k;\,ij} = \frac{\partial \mathcal{L}}{\partial Q_{\alpha\|k}} s^{ij}_{\alpha}, \tag{66}$$

which is a formula to which the gravitational variables will no longer contribute explicitly.

If one assumes the viewpoint of *special relativity* then one can immediately confirm that the s^{ij}_{α} constitute the natural generalization of the $\dot{d}^{ij}_{\alpha}$ that appear in formula (39) to the case in which there are spinorial variables. In particular, one can set:

$$\dot{h}^{(\mu)}_{i} = \delta^{\mu}_{i},$$

in which, the orthogonal transformation of the vierbeins will be identical to the spatio-temporal Lorentz transformation that was defined (37). If one takes (39), (46), and (51) into account then one will see that in this case the s^{ij}_{α} that were given by (65) will reduce to the coefficients of the variation:

$$\dot{\delta} Q_{\alpha} = \dot{s}^{ij}_{\alpha} \varepsilon_{ji}, \tag{67}$$

which (by reason of the choice of covariant tensorial components for the variables Q_{α}) corresponds to the Lorentz transformation that is *contragredient* to the transformation of the spatio-temporal variables (37). [It is this contragredient relationship that was at the root of the – sign that was introduced into formula (51)].

§ 5. Summary and applications

Formulas (63), (42), (66) express the general result to which we will arrive for a system that is described by tensorial and spinorial variables Q_{α}, and whose Lagrange function is assumed to contain the Q_{α} and their first (covariant) derivatives $Q_{\alpha\,\|\,j}$ (in addition to certain supplementary restriction that would be necessitated by the quantization of the variables). If we regroup these formulas then we will see that the energy-momentum tensor density of such a system can be calculated by starting with the Lagrange function – viz., the density $\mathcal{L}$ – with the aid of the formula:

(I) $$\mathfrak{t}^{j}_{\cdots i} \equiv \frac{\partial \mathcal{L}}{\partial Q_{\alpha\|j}} Q_{\alpha\|i} - \mathcal{L}\delta^j_i - \mathcal{R}^{kj}_{i\|k}$$

by means of:

(II) $$\mathcal{R}^{kj}_{l} g^{li} = \mathcal{D}^{j;\,ik} - \mathcal{D}^{i;\,kj} + \mathcal{D}^{k;\,ji},$$

(III) $$\mathcal{D}^{k;\,ij} \equiv \tfrac{1}{2}(\mathcal{R}^{ki}_{l} g^{lj} - \mathcal{R}^{kj}_{l} g^{li}) = \frac{\partial \mathcal{L}}{\partial Q_{\alpha\|k}} s^{ij}_{\alpha},$$

in which the s^{ij}_{α} are certain very simple linear combinations of the Q_α. The components $\mathfrak{t}^{ji}$ are symmetric in the indices by virtue of the field equations for the Q_α, and these same equations will likewise imply the conservation theorem:

(IV) $$\mathfrak{t}^{j}_{\cdots i\|j} = 0.$$

One will observe that, except for the term $-\mathcal{L}\delta^j_i$, the expression for $\mathfrak{t}^{j}_{\cdots i}$ will depend upon only the derivatives of the Lagrange function with respect to the derivatives of the variables Q_α.

Since formulas (I)-(III) do not exhibit the manner by which the Lagrange function depends upon the gravitational variables explicitly, they are immediately adapted to the usual case in which one desires to calculate the expression for the energy-momentum tensor in the context of *special* relativity by starting with the "special" form (i.e., invariant under only the Lorentz group) of the Lagrange function.

In that case, the covariant derivatives will reduce to the corresponding partial derivatives, and the quantities s^{ij}_{α} will take on an especially simple meaning: They are the coefficients that determine the variations:

(V_1) $$\dot{\delta} Q_\alpha = \dot{s}^{ij}_{\alpha}\, \varepsilon_{ji}$$

of the variables Q_α under an infinitesimal Lorentz transformation:

(V_2) $$\left\{ \begin{array}{c} \dot{\delta} x^i = \varepsilon^i_{\cdot k} x^k, \\ \varepsilon^i_{\cdot k} = \dot{g}^{il}\varepsilon_{lk},\ \dot{g}^{il} = \begin{cases} 0 & i \neq l, \\ +1 & i = l = 1,2,3, \\ -1 & i = l = 4, \end{cases} \\ \varepsilon_{il} + \varepsilon_{li} = 0. \end{array} \right.$$

By reason of the antisymmetry of $\mathcal{R}^{kj}_{i}$ in the indices *k*, *j*, the divergence of the term $\mathcal{R}^{kj}_{i\|k}$ that appears in the tensor density $\mathfrak{t}^{j}_{\cdots i}$ can be written in a general manner:

(68) $$\left(\mathcal{R}^{kj}_{i\|k}\right)_{\|j} = \tfrac{1}{2}\mathcal{R}^{kj}_{h} R^{h}_{ijk}\,,$$

in which R^{h}_{ijk} is the Riemann-Christoffel tensor. This expression will go to zero in the case of *special* relativity (as one sees immediately, moreover, since it will reduce to the second partial derivatives of the antisymmetric components). In that case, one can then omit the divergence $\mathcal{R}^{kj}_{i\|k}$ from the definition (I) of $\mathfrak{t}^{j}_{\cdot\cdot\cdot i}$ without affecting the validity of the conservation theorem (IV). Conversely, a special process of inquiry into the expression for $\mathfrak{t}^{j}_{\cdot\cdot\cdot i}$ in special relativity that starts with simply the condition that the desired expression must satisfy equation (IV) by virtue of the field equation (IV) will lead only to the addition of the term $\mathcal{R}^{kj}_{i\|k}$. Meanwhile, if one omits it then the resulting expression for $\mathfrak{t}^{ji}$ will no longer possess the essential property of symmetry in the indices j, i (by virtue of the field equations, in general).

If one always remains in the context of *special* relativity then one will see that the term $\mathcal{R}^{kj}_{i\|k}$ has no influence on the definition of the *total energy-momentum vecto*r of the system:

(69) $$G^{i} = -\int \mathfrak{t}^{4i}\,dv \qquad (dv = dx^1\,dx^2\,dx^3).$$

By contrast, it *will* affect the expression for the *total moment of momentum:*

(70) $$M^{ij} = -\int (x^{i}\mathfrak{t}^{4j} - x^{j}\mathfrak{t}^{4i})\,dv \qquad (i, j = 1, 2, 3)$$

in an essential manner. Indeed, if one sets:

(71) $$\mathcal{P}^{\alpha} \equiv \frac{\partial\mathcal{L}}{\partial Q_{\alpha|4}}$$

then the definition (69) (in which the – sign was chosen in order to conform to the usual conventions) will give ([1]):

(72) $$\mathbf{G} = -\int \mathcal{P}^{\alpha}\,\mathrm{grad}\,Q_{\alpha}\,dv$$

for the *total momentum*, and:

(73) $$G_4 = \int \left\{\mathcal{P}^{\alpha} Q_{\alpha|4} - \mathcal{L}\right\} dv$$

for the *total energy*.

However, due to the presence of the divergence term $\mathcal{R}^{kj}_{i\|k}$ in the form (I) of the energy-momentum tensor, the definition (70) will lead very naturally to a distinction between "orbital moment":

(74) $$\mathbf{M}_o = -\int \mathcal{P}^{\alpha}(\mathbf{x}\wedge\mathrm{grad}\,)\,Q_{\alpha}\,dv\,,$$

([1]) One sets $x_4 = ct$. The expressions for **G** and **M** will then represent the corresponding physical quantities, but multiplied by c.

which corresponds directly to the momentum density that enters effectively into the expression (72) and an "intrinsic moment" or "spin":

(75)
$$M_s^{ij} = \int \left\{ x^i \frac{\partial(\dot{\mathcal{R}}_l^{k4}\dot{g}^{lj})}{\partial x^k} - x^j \frac{\partial(\dot{\mathcal{R}}_l^{k4}\dot{g}^{li})}{\partial x^k} \right\} dv\,,$$

which is due to the divergence term precisely. Of course, this separation of the total moment of momentum into two terms:

(76)
$$M^{ij} = M_o^{ij} + M_s^{ij}$$

has a direct physical meaning only for physical agencies that are endowed with inertia so that one could attach a system of reference that is at rest with respect to it. With the help of an integration by parts, the intrinsic moment (75) can be put into the simpler form:

$$M_s^{ij} = \int \left\{ \mathcal{R}_l^{j4}\dot{g}^{li} - \mathcal{R}_l^{i4}\dot{g}^{lj} \right\} dv\,,$$

or furthermore, from (III):

(77)
$$M_s^{ij} = 2 \int \mathcal{P}^{\alpha} s_{\alpha}^{ij}\, dv\,.$$

A discussion of the passage to the Hamiltonian form and the quantization of the variables with the help of (71), (73) would go beyond the scope of the present article. We shall then content ourselves with several complementary remarks on those subjects, while confining ourselves to the "regular" case in which the formulas (71) that relate to the independent variable $Q_{\bar{\alpha}}$ do not give rise to identities between those variables and their canonically-conjugate momenta $\mathcal{P}^{\bar{\alpha}}$. One can observe that if the independent variables $Q_{\bar{\alpha}}$ are not homogeneous, linear combination of the Q_{α} then the summations over all of the components Q_{α} that enter into the formulas of the paragraph can be replaced with corresponding summations over the independent variables $Q_{\bar{\alpha}}$. If one then introduces suitable commutation relations between the $Q_{\bar{\alpha}}$ and the $\mathcal{P}^{\bar{\alpha}}$, while remaining in the "regular" case in which we have placed ourselves, then one will immediately recover the well-known connection between the total momentum operator and the group of translations in formula (72), and the analogous connection between the total moment of momentum and the rotation group in formula (76) [along with (74) and (77)]. One will then verify (with no difficulty, moreover) that the components of the orbital moment, on the one hand, and those of spin on the other will be related amongst themselves by commutation relations that are characteristic of the group of rotations, while each component of the orbital moment will commute with each component of the spin; these properties serve to justify the distinction between orbital moment and spin for physical agencies that have finite mass.

§ 6. Examples

In conclusion, we shall treat, by way of examples, the case of elementary particles of spin 1 / 2 and then that of fields of spin 1, such the fields of vector mesons or the electromagnetic field.

A. – Spin $\frac{1}{2}$

With the notations that were introduced above in the occasion of formulas (47), (48), (49), we can take the Lagrange density function to be:

(78) $$\mathcal{L} = -\text{Re}\left\{\frac{\hbar c}{i}\Phi\gamma^j\Psi_{\|j} + \Phi\mu\Psi\right\},$$

when we set $\gamma^i = h^{(\nu)j}\,\gamma_{(\nu)}$ and denote the real part of any quantity by Re. The operator μ relates to the mass of the particle considered in the usual way. In order for us to recover the usual representation, we choose $\beta = \gamma^4$ and introduce the current matrices:

(79) $$\alpha^j = \gamma^4\gamma^j$$

and the spin matrices:

(80) $$\sigma^j = i\,\gamma^k\gamma^l \qquad (j,\,k,\,l \text{ are permuted cyclically}),$$

while, from the choice of β, the γ^j (j = 1, 2, 3) will be skew-Hermitian, while the matrices β, α^j, σ^j, and μ will be Hermitian.

The momenta that are conjugate to Ψ and $\Phi = \Psi^\dagger\beta$ are:

(81) $$\mathcal{P}_\Psi = -\frac{1}{2}\frac{\hbar c}{i}\Phi\beta, \qquad \mathcal{P}_\Phi = \frac{1}{2}\frac{\hbar c}{i}\beta\Psi.$$

One will then get:

(82) $$\begin{aligned} G_i &= -\int\left\{\mathcal{P}_\Psi\Psi_{\|i} + \Phi_{\|i}\mathcal{P}_\Phi\right\}dv \\ &= \text{Re}\,\frac{\hbar c}{i}\int\Phi\beta\Psi_{\|i}\;dv \\ &= \int\Psi^\dagger p_i\Psi\;dv, \end{aligned}$$

for the total momentum when one introduces the operator:

(83) $$\mathbf{p} = \frac{\hbar c}{i}\,\text{grad}.$$

Likewise, the total energy will become:

$$G_4 = \int\left\{\mathcal{P}_4\Psi_{\|i} + \Phi_{\|4}\mathcal{P}_\Phi - \mathcal{L}\right\}dv$$

(84) $$= \int \Psi^\dagger \left\{ \sum_{j=1}^{3} \alpha^j p_j + \beta\mu \right\} \Psi \; dv.$$

As for the moment of momentum, it will split into an orbital moment:

(85) $$\mathbf{M}_0 = \int \Psi^\dagger (\mathbf{x} \wedge \mathbf{p}) \Psi \; dv$$

and spin:

$$M_s^{ij} = 2 \int \left\{ \mathcal{P}_\Psi \, s_\Psi^{ij} + s_\Phi^{ij} \, \mathcal{P}_\Phi \right\} dv.$$

From (47), (48), (49), (51), and (65), one will have:

$$2 \; s_\Psi^{ij} = \tfrac{1}{2} \gamma^i \, \gamma^j \, \boldsymbol{\Psi},$$

$$2 \; s_\Phi^{ij} = \tfrac{1}{2} \boldsymbol{\Phi} \gamma^j \, \gamma^i,$$

so

$$M_s^{ij} = \tfrac{1}{2} \mathrm{Re} \int \mathcal{P}_\Psi \gamma^i \gamma^j \Psi \; dv,$$

or finally, with the notation in (80):

(86) $$\mathbf{M}_s = \frac{\hbar c}{2} \int \Psi^\dagger \boldsymbol{\sigma} \Psi \, .$$

B. – Spin 1.

We take the covariant components Q_i of the quadri-vector potential to be the variables; i.e., the components of the vector potential **U** and the scalar potential with the sign inverted – V. We introduce the field tensor:

(87) $$F_{ik} = \frac{\partial Q_k}{\partial x^i} - \frac{\partial Q_i}{\partial x^k} \, ,$$

which represents two spatial vectors **H** and **E** that are defined by:

(88) $$\begin{cases} H_i = F_{kl} & (i, k, l \text{ are permuted cylically}), \\ E_i = F_{i4} & i = 1, 2, 3, \end{cases}$$

respectively.

We set the Lagrange density function equal to:

(89) $$\begin{aligned} \mathcal{L} &= -\tfrac{1}{4} F^{ik} \, F_{ik} - \tfrac{1}{2} \kappa^2 \, Q^i \, Q_i \\ &= -\tfrac{1}{2} (\mathbf{H}^2 - \mathbf{E}^2) - \tfrac{1}{2} \kappa^2 (\mathbf{U}^2 - \mathbf{V}^2), \end{aligned}$$

in which κ is, as one knows, a constant that relates to the mass of the particle that is associated with the field in question.

From (89), the momenta that are conjugate to the components of **U** are the corresponding components of **– E**, while the momenta that conjugate to **V** are identically zero. It then results from this that the field equations that expresses the annulment of the variational derivative $\delta\mathcal{L} / \delta\mathbf{V}$ will reduce to form that contains no temporal derivatives:

(90) $$\operatorname{div}\mathbf{E} + \kappa^2\,\mathbf{V} = 0.$$

The total energy is easily put into the form:

(91) $$G_4 = \tfrac{1}{2}\int \{(\mathbf{H}^2 + \mathbf{E}^2) + \kappa^2(\mathbf{U}^2 + \mathbf{V})\}\,dV - \int V(\operatorname{div}\mathbf{E} + \kappa^2\mathbf{V})\,dv\,.$$

One knows that from the viewpoint of the quantization of the variables, one can take (**U**, **– E**) to be canonical variables, but that two essentially different cases will present themselves, according to whether κ is non-zero or zero. In the former case, one must then consider equation (90) to be the definition of the operator **V**, and the last integral in (91) will be zero identically. In the latter case (which is the case of electrodynamics), equation (90), when reduced to div **E** = 0, will be an "accessory condition," by virtue of which, the last term of (91) will go to zero.

As for the total momentum and the orbital and intrinsic moments of momentum, they will take the following forms, which are independent of κ:

(92) $$\mathbf{G} = \sum_{i=1}^{3}\int E^i \operatorname{grad} U_i\,dv\,,$$

(93) $$\mathbf{M}_o = \sum_{i=1}^{3}\int E^i (\mathbf{x}\wedge\operatorname{grad})\,U_i\,dv\,,$$

(94) $$\mathbf{M}_s = \int \mathbf{E}\wedge\mathbf{U}\,dv\,.$$

Instead of (92), one can write:

(95) $$\begin{aligned}\mathbf{G} &= \int \mathbf{E}\wedge\operatorname{rot}\mathbf{U}\,dv + \int(\mathbf{E}\operatorname{grad})\,\mathbf{U}\,dv \\ &= \int \mathbf{E}\wedge\mathbf{H}\,dv - \int \mathbf{U}\operatorname{div}\mathbf{E}\,dv\,,\end{aligned}$$

which will reduce to the familiar expression for the Poynting vector for a pure electromagnetic field, by virtue of the accessory condition that div **E** = 0. If one starts with (93) and (94) then one will likewise find the corresponding form of the total moment of momentum:

(96) $$\mathbf{M} = \int \mathbf{x}\wedge(\mathbf{E}\wedge\mathbf{H})\,dv - \int(\mathbf{x}\wedge\mathbf{U})\operatorname{div}\mathbf{E}\,dv\,.$$

In order to establish this result in the case of an electromagnetic field, it is nonetheless necessary to return to the original form (75) for the term $\mathbf{M}_s$. In this case, one will see that, by virtue of the condition that div $\mathbf{E}$ = 0, the expression (96) will possess the property of gauge invariance precisely ([1]), while the separation into orbital moment and spin no will longer have any unambiguous meaning.

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The present work was dome (in April 1939) at the Princeton Institute for Advanced Studies, in the course of a study-voyage to the United States that was made possible thanks to a subsidy from the Belgian-American Educational Foundation.

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([1]) If we consider an electromagnetic field that interacts with a system of charges with density ρ then the accessory condition will become div $\mathbf{E} - \rho = 0$. The last terms on the right-hand side of (95) and (96) will then combine with the supplementary terms that relate to the momentum and the moment of momentum of the system of charges, respectively, in such a manner that the resulting expressions will possess gauge invariance precisely.

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