"Sulla teoria delle distorsioni elastiche," Rend. Accad. Lincei **23** (1914), 463-472.

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Session on 5 April 1914

P. BLASERNA, President

MEMOIRS AND NOTES

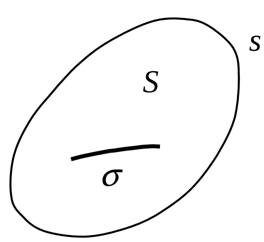
OF THE MEMBERS OR PRESENTATIONS BY MEMBERS

MECHANICS. – *On the theory of elastic distortions.* Note I by member CARLO SOMIGLIANA.

Translated by D. H. Delphenich

I.

The object of this note is to establish a very general method for the problem of the deformation of an elastic body in which one finds the addition or subtraction of thin layers of material along a datum surface, and the equilibrium that is established without the intervention of any external force.



Deformations of this type are being studied largely, and with brilliant success, by Volterra, who called them *distortions*. We shall also employ this terminology, while quickly pointing out, however, that in all of what follows it will assume a much broader significance than it had in the work of Volterra. Here is how we pose the problem of distortions:

Let S be the space occupied by an elastic body, and let s be the surface that bounds it. Now, let σ be a surface that we first assume, for simplicity, to be completely internal to the body; imagine that we have made a cut along it. Moreover, let the two faces of the cut be moved arbitrarily with respect to each other in such a way as to leave a small spatial gap, which is then assumed to be filled in with new material, or else in such a way as to penetrate the contiguous part of the body, in which case, we assume that we have suppressed the part that overlaps in such a way as to doubly fill the material of the body. In both of the cases, we assume that the edges of the cut have been soldered together or to the additional material.

Intuition says that a special state of tension and equilibrium must be determined in the body.

We proceed to study the question analytically while the ordinary theory of elasticity remains valid.

The conditions that must be satisfied along the cut σ are those of saying whether the corresponding points on the two faces of the cut are displaced in different ways to leave an opening in which the additional material enters, or determining the thin layer that must be suppressed. At an arbitrary point of σ, let u_σ, v_σ, w_σ denote arbitrary functions with the order of magnitude of elastic displacements u, v, w. Let ν, ν' be the normals to the two faces of s, while u_ν, v_ν, w_ν are the components of the elastic displacement for points of the face with normal ν and, analogously, $u_{\nu'}$, $v_{\nu'}$, $w_{\nu'}$ are the corresponding values on the other face. We must have:

$$(1_a) \qquad u_\nu - u_{\nu'} = u_\sigma, \qquad v_\nu - v_{\nu'} = v_\sigma, \qquad w_\nu - w_{\nu'} = w_\sigma .$$

Moreover, there must be no external forces, either volume or surface. For this reason, if we denote the coefficients of tension by the usual notation, where n is the interior normal to s, then we must have on this surface:

$$(2) \qquad X_n = 0, \qquad Y_n = 0, \qquad Z_n = 0,$$

and on the surface σ:

$$(1_b) \qquad X_\nu + X_{\nu'} = 0, \qquad Y_\nu + Y_{\nu'} = 0, \qquad Z_\nu + Z_{\nu'} = 0.$$

In order for equilibrium in the tensions along σ to exist it is necessary that the two vectors that represent the elastic tension on each face of σ must be in equilibrium.

More precisely, this condition must be satisfied on the two *displaced* faces of σ. However, as one always does in the theory of elasticity, one supposes that the displacement is very small, so that the same condition is satisfied on σ.

Now, it is easy to show that the stated condition is sufficient to specify the deformation of the body from the analytical point of view.

Let E be the function that represents the elastic energy and is, as one knows, a positive quadratic function of six coefficients of deformation:

$$x_x = \frac{\partial u}{\partial x}, \qquad y_y = \frac{\partial v}{\partial y}, \qquad z_z = \frac{\partial w}{\partial z},$$

$$y_z = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad z_x = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad x_y = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

for which one has:

$$2E = \frac{\partial E}{\partial x_x} x_x + \frac{\partial E}{\partial y_y} y_y + \cdots + \frac{\partial E}{\partial x_y} x_y;$$

in addition, we denote:

$$X_x = \frac{\partial E}{\partial x_x}, \qquad Y_y = \frac{\partial E}{\partial y_y}, \ \ldots, \ X_y = \frac{\partial E}{\partial x_y}.$$

In this condition, assuming that the indefinite equation of equilibrium:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0, \ \ldots$$

is satisfied, one immediately finds that:

$$\int_S 2E\,dS = -\int_\sigma (X_\nu u_\nu + Y_\nu v_\nu + Z_\nu w_\nu)\,d\sigma - \int_\sigma (X_{\nu'} u_{\nu'} + Y_{\nu'} v_{\nu'} + Z_{\nu'} w_{\nu'})\,d\sigma$$
$$-\int_s (X_n u + Y_n v + Z_n w)\,ds,$$

and thus if one assumes that (1_a, 1_b) and (2) are satisfied then one has:

(A) $$\int_S 2E\,dS = -\int_\sigma (X_\nu u_\sigma + Y_\nu v_\sigma + Z_\nu w_\sigma)\,d\sigma.$$

This relation proves what we asserted, since it, in fact, results that the resultant deformation of the difference of two deformations, which both satisfy ($1_{a,\,b}$), (2), annuls the expression for the elastic energy. It therefore represents a rigid displacement of the body. The deformation that satisfies the indefinite equations of equilibrium and the surface conditions ($1_{a,\,b}$), (2) is therefore uniquely determined, with the exception of a rigid motion, from which one may always abstract the question of elastic equilibrium.

These considerations remain valid regardless of whether the structure of the body is isotropic or anistropic from the elastic viewpoint.

We thus propose to establish what sort of distortions that arbitrary deformations produce in any elastic body whatsoever with a layer of discontinuity, in the previously indicated sense, when no external forces are present to establish equilibrium.

This consideration is extended immediately to the case in which σ meets the surface s.

II.

The relations $(1_{a,\,b})$ must be valid along all of the surface σ and comprise all of the mechanical conditions that relate to the distortions that must be added to the indefinite equations of equilibrium. (1_a) can therefore be differentiated along the tangent direction to s and one can thus deduce new necessary equality conditions.

Due to the invariance of the preceding relations with respect to position, as well as coordinates, in order to study the significance of $(1_{a,\,b})$ at a generic point of σ, one can likewise arrange that it be oriented in such a way that the positive direction of the z axis coincides with the direction of the normal ν at each point. Two arbitrarily-chosen orthogonal directions in a tangent plane to σ, which is always assumed to exist, also give the directions of the x and y axis.

One can thus differentiate (1_a) with respect to x and y. For brevity, we let $D[f]$ denote the jump that a function f experiences when it crosses σ in the direction of ν. If f_ν and $f_{\nu'}$ are the values of f on the two sides of σ, respectively, then one can say that:

$$D[f] = f_\nu - f_{\nu'} \,.$$

When (1_a) is then differentiated with respect to y, one finds:

$$(3) \qquad D[x_x] = \frac{\partial u_\sigma}{\partial x}\,, \qquad D[y_y] = \frac{\partial v_\sigma}{\partial y}\,, \qquad D[x_y] = \frac{\partial u_\sigma}{\partial y} + \frac{\partial v_\sigma}{\partial x}\,.$$

These equations determine the discontinuity that is experienced across the surface σ by three of the six coefficients of deformation; it is easy to see that (1_b) determines the discontinuities in the remaining three. In fact, one can write, taking into account the present orientation of the axes:

$$D[X_z] = 0, \qquad D[Y_z] = 0, \qquad D[Z_z] = 0,$$

and since X_z, Y_z, Z_z are independent linear functions of the six coefficients of deformation, the preceding equation, taking into account (8), can be put into the form:

$$D[a_{3i}\, z_z + a_{4i}\, y_z + a_{5i}\, z_x\,] = -\left[a_{1i}\,\frac{\partial u_\sigma}{\partial x} + a_{2i}\,\frac{\partial v_\sigma}{\partial y} + a_{6i}\left(\frac{\partial w_\sigma}{\partial y} + \frac{\partial v_\sigma}{\partial x}\right)\right], \quad i = 3, 4, 5,$$

where the a_{ih} are constants that represent the coefficients of the expression for energy, and since the same equations may also be written in the form:

$$(4) \qquad a_{3i}\, D[z_z] + a_{4i}\, D[y_z] + a_{5i}\, D[z_x] = -\left[a_{1i}\,\frac{\partial u_\sigma}{\partial x} + a_{2i}\,\frac{\partial v_\sigma}{\partial y} + a_{6i}\left(\frac{\partial u_\sigma}{\partial y} + \frac{\partial v_\sigma}{\partial x}\right)\right],$$

solving them with respect to $D[z_z]$, $D[y_z]$, $D[z_x]$ gives the desired discontinuity.

In the isotropic case these equations become:

$$D[X_z] = \mu\, D[x_z] = 0, \quad D[Y_z] = \mu\, D[y_z] = 0,$$

$$D[Z_z] = D[\lambda(x_z + y_y) + (\lambda + 2\mu)\, z_x] = 0,$$

where λ, μ are the isotropy constants.

We then have:

$$D[x_z] = 0, \qquad D[y_z] = 0,$$

$$D[z_z] = -\frac{\lambda}{\lambda+2\mu}\left(\frac{\partial u_\sigma}{\partial x}+\frac{\partial v_\sigma}{\partial y}\right).$$

We may then conclude *the stated condition with regard to the surface* σ*, that* ($1_{a,\,b}$) *completely determine the discontinuity that is experienced upon crossing this surface by all of the deformations:*

$$x_x \qquad y_y \qquad z_z \qquad y_z \qquad z_x \qquad x_y\,.$$

That this serves to establish the distortions, as Weingarten ([1]) did, that the six deformation coefficients remain continuous upon crossing the cut, *amounts to the definition of new conditions for a problem that is already physically and analytically determined*, that is, one must treat a special problem, which we will discuss from another point of view.

However, it is useful to observe that the continuity of all the deformation coefficients, and therefore those of the tension, leads one to consider a distribution of tension that contradicts all of the usual laws of equilibrium in the neighborhood of an arbitrary point that belongs to the cut, a distribution of tension that is analogous to the one around any ordinary point of the body, that is, as Volterra suggested, in such a way that it does not exist on the edge of the cut, and by the addition or subtraction of matter in the neighborhood of a point of the cut, even in the most general case that one has. In fact, this is completely respected by the surface elements tangent to σ, and for everything that crosses that surface, which one must regard as separated into two elements, one of which is situated on one side of σ and the other of which is situated on the other side, both of which are the same and subject to two equal and opposite tensions (due to the continuity along σ), as one usually has for all of the surface elements that are internal to an elastic body. These two portions of the element are therefore each separately in equilibrium, although they are generally subject to different tensions.

From the preceding, one may deduce the conditions for all of the six deformation coefficients (and therefore tension) to be continuous upon crossing the cut, that is, to verify the distribution that was supposed by Weingarten. These conditions, which are founded upon (3), (4), are obviously the following ones:

$$\text{(5)} \qquad \frac{\partial u_\sigma}{\partial x} = 0, \qquad \frac{\partial v_\sigma}{\partial y} = 0, \qquad \frac{\partial u_\sigma}{\partial y}+\frac{\partial v_\sigma}{\partial x} = 0.$$

([1]) *Sulle superficie di discontinuità nella teoria della elasticità dei corpi solidi.* Rend. Accad. dei Lincei, 1st sem., 1901.

Weingarten has given a geometrical interpretation of such conditions, which he himself found in a different form.

Meanwhile, we note that in these relations the x, y axes themselves are intended to belong to a triad whose axes vary from point to point along σ, in such a way that the z axis will always coincide with the normal to the surface.

The significance of (5) is then the fact that upon crossing the surface σ in an infinitesimal neighborhood of it one is subjected at any point to a displacement of the components u_σ, v_σ with neither linear elongation nor variation of the angle between two orthogonal directions; moreover, the surface element remains rigid. This led Weingarten to the condition that must apply to the two surfaces.

It is now convenient to add an observation of noteworthy importance.

We may, in fact, satisfy (5), besides in the aforementioned way, by supposing that the u_σ, v_σ are null at all points of the surface and attributing arbitrary values to the w_σ. The discontinuity is therefore conserved for only the normal component.

There thus exists a distortion that satisfies the Weingarten conditions (i.e., the condition that all of the coefficients of deformation must be continuous upon crossing σ) and which experiences a displacement normal to the cut for both copies of the corresponding points on each edge of it.

This distortion completely eluded Weingarten. I have already had occasion to point it out in a special case, the one in which σ is planar ([1]).

Naturally, this distortion must subsist under the preceding considerations, and the situation that we assumed for the two points defines a more general type of distortion, which, in the cited Communicazione al Congresso di Roma, we proposed to call *Weingarten*.

This result resolves, in a completely general way, the question of the conditions for a distortion to be Weingarten. The result that we obtained in the case of a planar cut extends in a direct and simple way to the case of a cut in an arbitrary surface ([2]).

III.

We now note the correspondence that exists between the problems of elastostatics and some problems in the theory of the potential, a correspondence that defined the basis for the brilliant research of Betti in the theory of elasticity.

In the theory of distortions this correspondence was interpreted by Volterra as an analogy between hydrodynamics and elastostatics. We shall see what its significance would be in the preceding results from this point of view.

We pose the problem of finding a harmonic function V in a space S, which has a null interior normal derivative on the surface s, and is regular in all of S, along with its first and second derivatives, except on a surface σ that is internal to S, and when crossing it,

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([1]) *Sulle deformazioni elastiche non regolari*. Atti del IV Congresso dei Matermatici, Roma, 1908.

([2]) We must note that in the same epoch in which Weingarten published his note, prof. Gebbia, in a memoir to the Annali di Matematica: *Le deformazioni tipiche dei corpi solidi elastici* (Ser. III, v. VII, 1902), considered the elastic deformation under the addition or subtraction of matter, with assumptions that partially coincide with the ones that we started with.

must have assigned discontinuities, while the normal derivative is continually conserved.

That is, one must have:

(6_a) $\Delta_2 V = 0$ in all of S,

(6_b) $\frac{\partial V}{\partial n} = 0$ on s,

(6_c) $V_\nu - V_{\nu'} = g,\quad \frac{\partial V}{\partial \nu} + \frac{\partial V}{\partial \nu'} = 0$ on σ,

where g is an arbitrary given function that is assumed to be regular at all of the points of σ.

It is easy to show that V, under the preceding conditions, is uniquely determined, except for an additive constant.

In fact, from (6_a) one has:

$$\int_S \Delta_1 V\, dS = -\int_s V \frac{\partial V}{\partial n} ds - \int_\sigma V_\nu \frac{\partial V}{\partial \nu} d\sigma - \int_\sigma V_{\nu'} \frac{\partial V}{\partial \nu'} d\sigma'.$$

and therefore, from ($6_{b,c}$):

$$\int \Delta_1 V\, dS = -\int_\sigma (V_\nu - V_{\nu'}) \frac{\partial V}{\partial \nu} d\sigma = -\int_\sigma g \frac{\partial V}{\partial \nu} d\sigma.$$

This relation is analogous to (A), and it immediately says that two functions that satisfy ($6_{a,b,c}$) cannot differ, except by a constant. The problem that is posed in the theory of the potential is analogous to the problem of distortions, as we have posed it in elastostatics.

Observe that whereas the second of (6_c) determines the behavior of the normal derivative of V while crossing σ, the first one, on the contrary, assigns definite discontinuities in the tangential derivatives. Indeed, assuming that the x, y axes are tangent planes to a point of σ, one finds that:

$$\frac{\partial V_\nu}{\partial x} - \frac{\partial V_{\nu'}}{\partial x} = \frac{\partial g_\sigma}{\partial x}, \qquad \frac{\partial V_\nu}{\partial y} - \frac{\partial V_{\nu'}}{\partial y} = \frac{\partial g_\sigma}{\partial y},$$

and upon differentiating once more with respect to x and y, one obtains:

$$\frac{\partial^2 V_\nu}{\partial x^2} - \frac{\partial^2 V_{\nu'}}{\partial x^2} = \frac{\partial^2 g_\sigma}{\partial x^2}, \quad \frac{\partial^2 V_\nu}{\partial y^2} - \frac{\partial^2 V_{\nu'}}{\partial y^2} = \frac{\partial^2 g_\sigma}{\partial y^2},$$
$$\frac{\partial^2 V_\nu}{\partial x \partial y} - \frac{\partial^2 V_{\nu'}}{\partial x \partial y} = \frac{\partial^2 g_\sigma}{\partial x \partial y}.$$

It results from this that the first equation (6_c) determines the discontinuity in three of the second derivatives. The second of (6_c), which may be written:

$$\frac{\partial V_\nu}{\partial z} = \frac{\partial V_{\nu'}}{\partial z},$$

when differentiated with respect to x and y, gives the discontinuities (which are null, in this case) in the other two:

$$\frac{\partial^2 V_\nu}{\partial x \partial z} = \frac{\partial^2 V_{\nu'}}{\partial x \partial z}, \qquad \frac{\partial^2 V_\nu}{\partial y \partial z} = \frac{\partial^2 V_{\nu'}}{\partial y \partial z},$$

and finally the discontinuity in its sixth derivative is determined by the Laplace equation:

$$\frac{\partial^2 V_\nu}{\partial z^2} - \frac{\partial^2 V_{\nu'}}{\partial z^2} = -\frac{\partial^2 g_\sigma}{\partial x^2} - \frac{\partial^2 g_\sigma}{\partial y^2}.$$

We may therefore conclude that the two relations (6_c) determine the discontinuities in all of the first and second derivatives of the function V.

That the function V must actually exist results immediately from the observation that from the conditions ($6_{b,\,c}$) one has:

$$\int_s \frac{\partial V}{\partial n} ds + \int_\sigma \frac{\partial V}{\partial \nu} d\sigma + \int_\sigma \frac{\partial V}{\partial \nu'} d\sigma = 0.$$

How one may determine it results from the following considerations: Set:

$$W = \frac{1}{4\pi} \int_\sigma g \frac{\partial \frac{1}{r}}{\partial \nu} d\sigma,$$

where r denotes the distance from a generic point of σ, and, letting U denote a new function to be determined instead of V, set:

$$V = U + W.$$

Let us look for the conditions that U must satisfy. From the well-known property of double layer potentials, if one has:

$$W_\nu - W_{\nu'} = g$$

on σ then one must have:

$$(7_a) \qquad U_\nu - U_{\nu'} = 0.$$

U will therefore be continuous upon crossing σ, and must likewise be continuous in its first and second derivatives. Indeed, from the second of (6_c) and the property of the normal derivative of W, one must have:

$$(7_b) \qquad \frac{\partial U_\nu}{\partial \nu} + \frac{\partial U_{\nu'}}{\partial \nu'} = 0.$$

Moreover, as the preceding procedure indicates, this relation, along with (7_a), results precisely in the continuity of all of the first and second derivatives. In other words, since the function W and its first and second derivatives have the same discontinuities as V on σ, the function U and its first and second derivatives must be continuous.

Therefore, U must be harmonic in S, as well as finite and continuous, along with its first and second derivatives; in addition, on the surface s it must satisfy the condition:

$$\frac{\partial U}{\partial n} = -\frac{\partial W}{\partial n},$$

while, on the other hand, one has:

$$\int_s \frac{\partial U}{\partial n} ds = -\int_s \frac{\partial W}{\partial n} ds - \int_\sigma \left(\frac{\partial W}{\partial \nu} + \frac{\partial W}{\partial \nu'}\right) d\sigma = \int_S \Delta_1 W \, dS = 0.$$

The problem thus reduces to the determination of a harmonic function that is regular in a space S when one has assigned the values of the normal derivatives on the surface s that defines its contour. Moreover, we note that such a function always exists.

However, we observe that some points of the line that forms the contour of s may present an infinity in the derivatives of the function W, and therefore those of V, as well, but such infinities tend to disappear if one supposes that the function g is null on the contour of σ, along with all of its tangential derivatives.

An observation of this sort regarding the elastic problem is found in my cited Communicazione al Congresso dei Matematici.

The problem of hydrodynamics that corresponds to the analytic problem that we considered is the following one:

In a space S with rigid walls there exists a fissure σ; from any surface element of this it an equal quantity of an incompressible fluid is emitted from one side of it and absorbed by the other in an unit of time. Determine the stationary, irrotational motion of the fluid that is produced.

It is clear that a stationary motion of this type must exist for any space S that is simply linearly connected. On the contrary, as is well known, no emission or absorption of fluid can occur in that case.

Strictly speaking, the stated problem for the velocity potential V is not determined directly from the function g, but the flow $\partial V / \partial \nu$ through the elements of σ, although one must then look for the relations that allow one to pass from g to $\partial V / \partial \nu$.

Some have then said that the case of constant g must then be excluded, since one must then effectively compare the infinite values of the first derivative of W along the contour line of σ. In this case, the problem of hydrodynamics becomes that of determining the motion of the fluid when a vortex line coincides with the contour of the surface σ. (Cf., Lamb, *Hydrodynamics*, art. 148).

The preceding process that reduced the determination of the function V to the so-called second Dirichlet problem suggests a procedure for solving, by an analogous path, the problem of the distortion of an elastic body. Such a concept was already applied substantially by Volterra in the course of his work.

In order to extend it to our case, we must find the deformation that corresponds in elastostatics to the double layer potential W. This question will occupy us in a later note.

"Sulla teoria delle distorsioni elastiche," Rend. Accad. Lincei **24** (1915), 655-666.

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MECHANICS. – *On the theory of elastic distortions.* Note II by member CARLO SOMIGLIANA.

Translated by D. H. Delphenich

The conditions that were established in the preceding note ([1]) for the surface of discontinuity σ were the following ones:

$$D[u] = u_\sigma \qquad D[v] = v_\sigma \qquad D[w] = w_\sigma \tag{1}$$

$$D[X_\nu] = 0 \qquad D[Y_\nu] = 0 \qquad D[Z_\nu] = 0. \tag{2}$$

They are sufficient, as we have seen, to determine the discontinuities when crossing the aforementioned surface in the six coefficients of deformations x_x , y_y , z_z , y_z , z_x , x_y . Now, one may see, moreover, that they likewise determine the discontinuities in all of the nine first derivatives of the three functions *u*, *v*, *w*. Furthermore, if one takes into account the equations of equilibrium – i.e., if one supposes that *u*, *v*, *w* satisfy these equations in all of the body – then it likewise results that one has the discontinuities in all of the second derivatives of this functions.

This result is of noteworthy interest. Indeed, one deduces the consequence that if two deformations are regular in all of the body, except when crossing σ, and both satisfy the conditions (1), (2) on that surface then the deformation that represents their geometric difference will be completely regular in all of the body. Indeed, for this deformation, the

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([1]) These Rendiconti, 1st sem., 1914, pp. 463.

discontinuities in the three functions that represent the components of the displacement, as well as those of their first and second derivatives, must be null. No other singularities may exist.

From this, it follows that if one finds an arbitrary triple of the regular functions *U*, *V*, *W* that satisfies the equations of equilibrium in all of the body, except when crossing the surface σ, on which it satisfies (1), (2), then the problem of determining the distortions may be reduced to that of determining the deformation of the body under the action of given surface forces. Indeed, let:

$$u = u' + U \qquad v = v' + V \qquad w = w' + W,$$

in which the new functions u', v', w' thus determined must have no singularities when crossing the surface σ. On the remaining surface of the body the total tension must be null and must satisfy the condition of producing tensions that are equal and opposite to the ones produced by the deformations *U*, *V*, *W*.

Now, we shall show how it is always possible to obtain such a triple of functions *U*, *V*, *W*. The problem of the distortions will then be that of discovering what sort of deformations result from given surface forces. This constitutes a method for the analytical resolution of the problem, but its major interest might perhaps be seen from the viewpoint of the existence of a solution to the problem of distortions.

Indeed, from the research of Lauricella, Cosserat, and Korn ([1]), the existence of the solution to the problem of deformations under given surface forces may be regarded as having been proved, if only in a general way. The problem of distortions can reduce to this other problem, and therefore it might admit a solution. The question can be considered solved in this way.

We now carry out the proof of the property that we originally stated.

The formulas that give the discontinuities in the coefficients of deformation, with the canonical orientation of the axes, are:

$$(3) \qquad D[x_x] = \frac{\partial u_\sigma}{\partial x} \qquad D[y_y] = \frac{\partial v_\sigma}{\partial y} \qquad D[x_y] = \frac{\partial u_\sigma}{\partial y} + \frac{\partial v_\sigma}{\partial x}$$

$$(4) \qquad D[x_z] = \alpha \qquad D[y_z] = \beta \qquad D[z_z] = \gamma,$$

where α, β, γ are linear functions that one finds in the right-hand sides of the first three equations. In the isotropic case:

$$\alpha = 0 \qquad \beta = 0 \qquad \gamma = -\frac{\lambda}{\lambda + 2\mu}\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right).$$

Now, to this condition, which is satisfied on σ, one may add that following one, which is obtained by differentiating (1) tangentially:

([1]) See, in particular, A. Korn, *Solution générale du problème d’équilibre dans la théorie de l’élasticité dans the cas où les efforts sont donnés à la surface.* Annales de la Faculté des Sciences de l’Université de Toulouse, 2nd ser., t. X, 1908.

(5)
$$\begin{cases} D\left[\dfrac{\partial w}{\partial x}\right]=\dfrac{\partial w_\sigma}{\partial x} & D\left[\dfrac{\partial w}{\partial y}\right]=\dfrac{\partial w_\sigma}{\partial y} \\ D\left[\dfrac{\partial u}{\partial y}\right]=\dfrac{\partial u_\sigma}{\partial y} & D\left[\dfrac{\partial v}{\partial x}\right]=\dfrac{\partial v_\sigma}{\partial x} \end{cases}$$

The first two of equations (3), the third of (4), and the latter four ultimately determine the discontinuities in seven of the derivatives of the functions u, v, w; i.e., all of them, except for $\partial u / \partial z$, $\partial v / \partial z$. However, from the first of equations (4), taking into account (5), one finds that:

(6)
$$D\left[\frac{\partial u}{\partial z}\right]=\alpha-\frac{\partial w_\sigma}{\partial x} \qquad D\left[\frac{\partial v}{\partial z}\right]=\beta-\frac{\partial w_\sigma}{\partial y}.$$

The question of how one obtains the discontinuities of the first derivatives is therefore resolved.

For the second derivatives, one may immediately observe that if one supposes that the discontinuities in the first derivatives of one of the functions – e.g., u – are known, as they are, in fact, then the discontinuities in the five second derivatives that contain one or no differentiations with respect to z may be obtained immediately by tangential differentiation. What remains for u is only the desired discontinuity in $\partial^2 u / \partial z^2$.

One may do likewise with the derivatives of the other two functions v, w. Everything then comes down to the search for the discontinuities in $\partial^2 u / \partial z^2$, $\partial^2 v / \partial z^2$, $\partial^2 w / \partial z^2$.

Now, the three equations of equilibrium are linear with constant coefficients in the second derivatives of u, v, w. This is why the knowledge of the discontinuities in all of the remaining second derivatives may lead to the discontinuities in the three second derivatives with respect to z.

In the isotropic case, the formulas that determine this discontinuity are the following ones:

$$\mu D\left[\frac{\partial^2 u}{\partial z^2}\right]=-D\left[\lambda\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}\right)+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right]$$
$$=-\frac{2\lambda\mu}{\lambda+2\mu}\frac{\partial}{\partial x}\left(\frac{\partial u_\sigma}{\partial x}+\frac{\partial v_\sigma}{\partial y}\right)-\mu\left(\frac{\partial^2 u_\sigma}{\partial x^2}+\frac{\partial^2 u_\sigma}{\partial y^2}\right)$$

$$\mu D\left[\frac{\partial^2 v}{\partial z^2}\right]=-\frac{2\lambda\mu}{\lambda+2\mu}\frac{\partial}{\partial y}\left(\frac{\partial u_\sigma}{\partial x}+\frac{\partial v_\sigma}{\partial y}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\mu D\left[\frac{\partial^2 v}{\partial z^2}\right]=-D\left[\lambda\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 w}{\partial x\,\partial z}\right)+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right],$$

in which:

$$(\lambda+\mu)D\left[\frac{\partial^2 w}{\partial z^2}\right] = -\lambda D\left[\frac{\partial^2 u}{\partial x\,\partial z}+\frac{\partial^2 u}{\partial y\,\partial z}\right] - \mu D\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right],$$

and from (6), taking isotropy into account, $\alpha = 0$ and $\beta = 0$:

$$(\lambda+\mu)D\left[\frac{\partial^2 w}{\partial z^2}\right] = (\lambda-\mu)\left(\frac{\partial^2 w_\sigma}{\partial x^2}+\frac{\partial^2 w_\sigma}{\partial y^2}\right).$$

II.

The deformations U, V, W that we have discussed at the beginning of the preceding section, which are subject only to the conditions (1), (2) on the surface σ, and which are regular in the remaining part of the body, are not uniquely determined. However, for our purposes, it is enough to find any one of the infinitude of deformations that satisfy these conditions.

Imagine an indefinitely extended elastic body, and in it, a surface of discontinuity σ. We may look for the deformation that is produced in such a body by the discontinuity that we have on σ, adding the condition that the deformation must vanish at infinity; i.e., from the physical point of view, the deformation produced by the infiltration or suppression of a thin layer of matter along the surface σ.

It is easy to see that such a deformation is uniquely determined, and is given immediately, in the isotropic case to which we are limited at the moment, from the general formulas for the integral representation of the components of the elastic displacement. Indeed, if we suppose that the volume and surface forces in these formulas are null then the integrals relating to those forces disappear, and the remaining integrals (in case they are extended over the surface σ, and for the components of the displacement that accompany it, one takes u_σ, v_σ, w_σ, the components of the discontinuity) gives precisely the deformation that satisfies the requested conditions. These integrals correspond, in the formulas of elastostatics, to the double layer potential function of Green's formula, and one may effectively prove that it satisfies, other than the general conditions, the conditions (1), (2), as well, by a process that that we indicated on another occasion ([1]). Such a process is very simple, but it is based upon a passage to the limit that may give rise to objections, and not exhibit the properties of the singular parts that accompany such total integrals.

Recall the proof of these results by a direct procedure that is independent of any passage to the limit.

Consider the integrals:

$$A = \int_\sigma u_\sigma \frac{\partial r}{\partial \nu} d\sigma, \qquad B = \int_\sigma v_\sigma \frac{\partial r}{\partial \nu} d\sigma, \qquad C = \int_\sigma w_\sigma \frac{\partial r}{\partial \nu} d\sigma,$$

([1]) *Sul problema statico di Maxwell*. Memoirs of the R. Accademia dei Lincei, vol. VII, 1908.

where r is the distance from a generic point (x, y, z) of space to the center of the surface element $d\sigma$. This integral may be considered as a biharmonic double layer potential, which differs from the corresponding Newtonian potential by the substitution of the function r with $1/r$. Now, let:

$$\Omega = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$$

and set:

$$4\pi(u_1, v_1, w_1) = \alpha \frac{\partial \Omega}{\partial(x, y, z)} + \Delta_2(A, B, C). \tag{7}$$

It is easy to verify that for:

$$\alpha = -\frac{\lambda + \mu}{\lambda + 2\mu}$$

the preceding expressions satisfy the equations of equilibrium:

$$(\lambda + \mu)\frac{\partial \theta}{\partial(x, y, z)} + \Delta_2(x, y, z) = 0,$$

where:

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

It is regular in all of space, except on the surface σ, and at infinity it goes to zero like the Newtonian double layer potential.

In order to study the discontinuities in the expression (7) and those of the corresponding components of the deformation and tension on the surface, I will refer to some results that will be published soon in the Atti della R. Accademia delle Scienze di Torino. From them, it results that the second derivatives of the biharmonic double layer potential that are continuous contain one or no derivatives in the direction of the normal to the active layer. Supposing that the axes have the canonical orientations, one then has for the second derivatives with respect to the normal:

$$D\left[\frac{\partial^2 A}{\partial z^2}\right] = 8\pi u_\sigma, \qquad D\left[\frac{\partial^2 B}{\partial z^2}\right] = 8\pi v_\sigma, \qquad D\left[\frac{\partial^2 C}{\partial z^2}\right] = 8\pi w_\sigma.$$

From this, it follows immediately that we have formulas for the discontinuities in u_1, v_1, w_1:

$$D[u_1] = 2u_\sigma \qquad D[v_1] = 2v_\sigma \qquad D[w_1] = 2(\alpha + 1)w_\sigma. \tag{8}$$

In order to obtain the discontinuities in the components of the deformations, it is convenient to recall the formulas that give the discontinuities in the third derivatives of the biharmonic double layer potential. By means of these formulas, and supposing that the x and z are tangent to the two lines of curvature on the surface σ at the point considered, and denoting the respective radii of curvature by R_1 and R_2, one finds that:

$$D[x_x] = -2\alpha\frac{w_\sigma}{R_1} + 2\frac{\partial u_\sigma}{\partial x}$$

$$D[y_y] = -2\alpha\frac{w_\sigma}{R_2} + 2\frac{\partial v_\sigma}{\partial y}$$

$$(8)\qquad \begin{cases} D[z_z] = 2\alpha\left(\dfrac{\partial u_\sigma}{\partial x} + \dfrac{\partial v_\sigma}{\partial y}\right) + 2\alpha\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) w_\sigma \\ D[y_z] = -4\alpha\dfrac{v_\sigma}{R_1} + 2(2\alpha+1)\dfrac{\partial w_\sigma}{\partial y} \\ D[z_x] = -4\alpha\dfrac{u_\sigma}{R_1} + 2(2\alpha+1)\dfrac{\partial w_\sigma}{\partial x} \\ D[x_y] = 2\alpha\left(\dfrac{\partial u_\sigma}{\partial y} + \dfrac{\partial v_\sigma}{\partial x}\right). \end{cases}$$

From these formulas, it results that:

$$D[x_x + y_y + z_z] = D[\theta] = 2(\alpha+1)\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right).$$

The discontinuities relative to the coefficients of tension are obtained quite easily by means of the preceding formulas, if one recalls that:

$$X_x = \lambda\theta + 2\mu x_x, \qquad Z_x = \mu y_z, \qquad \text{etc.}$$

One then finds:

$$D[X_x] = 2(\alpha+1)\lambda\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right) + 2\mu\left(2\frac{\partial u_\sigma}{\partial x} - 2\alpha\frac{\partial w_\sigma}{\partial R_1}\right)$$

$$D[Y_y] = 2(\alpha+1)\lambda\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right) + 2\mu\left(2\frac{\partial v_\sigma}{\partial y} - 2\alpha\frac{\partial w_\sigma}{\partial R_2}\right)$$

$$D[Z_z] = [2(\alpha+1)\lambda + 4\mu\alpha]\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right) + 4\mu\alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right) w_\sigma .$$

The three remaining formulas then result by multiplying the last three of (8) by μ.

The deformation that one may construct by solving the question posed results from the composition of the deformation that was considered in (7) with another one that we shall study. Set:

$$\varphi = \int_\sigma \left(u_\sigma \frac{\partial a}{\partial \nu} + v_\sigma \frac{\partial b}{\partial \nu} + w_\sigma \frac{\partial c}{\partial \nu} \right) d\sigma$$

$$\psi_1 = \int_\sigma \left(v_\sigma \frac{\partial c}{\partial \nu} - w_\sigma \frac{\partial b}{\partial \nu} \right) d\sigma \qquad \psi_2 = \int_\sigma \left(w_\sigma \frac{\partial a}{\partial \nu} - u_\sigma \frac{\partial b}{\partial \nu} \right) d\sigma$$

$$\psi_3 = \int_\sigma \left(u_\sigma \frac{\partial b}{\partial \nu} - v_\sigma \frac{\partial a}{\partial \nu} \right) d\sigma ,$$

in which a, b, c are the coordinates of the surface element $d\sigma$. These four functions are Newtonian potentials. From this, it follows that the result satisfies the equations of equilibrium if one takes:

(7*a*) $$4\pi(u_1, u_2, u_3) = (2\alpha + 1) \text{ grad } \varphi + \text{rot}(\psi_1, \psi_2, \psi_3).$$

Supposing, as usual, that the axes have the canonical orientations, the discontinuities in these expressions are determined immediately from the formulas for the discontinuities in the first derivatives of the potential function on the surface.

In these formulas, one has:

$$D\left[\frac{\partial \varphi}{\partial x}\right] = 0 \qquad D\left[\frac{\partial \varphi}{\partial y}\right] = 0 \qquad D\left[\frac{\partial \varphi}{\partial z}\right] = -4\pi w_\sigma$$

$$D\left[\frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z}\right] = -4\pi w_\sigma \qquad D\left[\frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x}\right] = -4\pi v_\sigma , \qquad D\left[\frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y}\right] = 0.$$

By means of these formulas, one finds that:

$$D[u_2] = -u_\sigma , \qquad D[v_2] = -v_\sigma , \qquad D[w_2] = -(2\alpha + 1)\, w_\sigma .$$

If one compares these formulas with (8) then one immediately finds that the deformation, which is obtained from the components of the two deformations considered, satisfies the following conditions on the surface:

(1) $$D[u_1 + u_2] = u_\sigma \qquad D[v_1 + v_2] = v_\sigma \qquad D[w_1 + w_2] = w_\sigma ;$$

i.e., precisely the conditions (1) that must be satisfied for the desired deformation.

The calculation of the discontinuities in the components of the deformation requires a knowledge of the discontinuities in the second derivatives of the potential functions on the surface. These discontinuities are known (see: Poincaré, *Théorie du potential newtonien*, chap. VI), and the formulas that one obtains in our case, always with the canonical orientation of the axes, are the following ones for the components of the deformation relative to u_2, v_2, w_2:

$$
\begin{aligned}
D[x_x] &= \ \ 2\alpha\frac{w_\sigma}{R_1} - \frac{\partial u_\sigma}{\partial x} \\
D[y_y] &= \ \ 2\alpha\frac{w_\sigma}{R_2} - \frac{\partial w_\sigma}{\partial y} \\
D[z_z] &= -2\alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)w_\sigma + \left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right) \\
D[y_z] &= \ \ 4\alpha\frac{v}{R_2} - 2(2\alpha+1)\frac{\partial w_\sigma}{\partial y} \\
D[z_x] &= \ \ 4\alpha\frac{u}{R_1} - 2(2\alpha+1)\frac{\partial w_\sigma}{\partial x} \\
D[x_y] &= -\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right),
\end{aligned}
$$

from which it follows that:

$$D[\theta] = 0.$$

From these formulas, as well as (8), one may deduce the discontinuities in the components of the deformation relative to the deformation that results from the composition $u_1 + u_2$, $v_1 + v_2$, $w_1 + w_2$. In this case, one finds that:

$$
\begin{aligned}
D[x_x] &= \frac{\partial u_\sigma}{\partial x} & D[y_z] &= 0 \\
D[y_y] &= \frac{\partial v_\sigma}{\partial y} & D[z_x] &= 0 \\
D[z_z] &= (2\alpha+1)\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right) & D[y_z] &= \frac{\partial u_\sigma}{\partial y} + \frac{\partial v_\sigma}{\partial x},
\end{aligned}
$$

and thus:

$$D[\theta] = (2\alpha+1)\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right).$$

From this formula it results immediately that the components of the tension relative to a surface element that belongs to the surface σ are:

$$
\text{(II)} \qquad
\begin{aligned}
& D[X_z] = 0 \qquad D[Y_z] = 0 \\
& D[Z_z] = [2\lambda(\alpha + 1) + 2\mu(2\alpha + 1)]\left(\frac{\partial u_\sigma}{\partial x} + \frac{\partial v_\sigma}{\partial y}\right) = 0,
\end{aligned}
$$

when one recalls the value of the constant α.

It thus remains to prove that the components of the tension relative to a surface element of σ are continuous upon crossing that surface, or, in other words, that these elements are not subject to any tension as a result of the deformation of the body. However, this is precisely the significance of conditions (2), which are then satisfied by the deformation $u_1 + u_2$, $v_1 + v_2$, $w_1 + w_2$.

One may observe that the deformations (7), (7_a) go to zero at infinity as ordinary potential functions do, and have no other singularities except for the ones that were defined on the surface σ.

We may then conclude that the deformation that represents the composition of these two deformations and satisfies the conditions (I), (II) on σ represents the deformation of an indefinite elastic medium that is produced by the infiltration or suppression of a thin layer of material that defines the discontinuities u_σ, v_σ, w_σ.

By means of this deformation, as we have seen since the outset, any problem of distortions relative to a surface of discontinuity σ may be converted into a problem of deformation under given surface forces, save for the limitation relative to the border of the cut that it be internal to the body, as we discussed in Note I.

As far as the other three components of tension of the deformation considered are concerned, from the preceding formulas, one immediately deduces that:

$$D[X_x] = 2\lambda(\alpha+1)\left(\frac{\partial u_\sigma}{\partial x}+\frac{\partial v_\sigma}{\partial y}\right)+2\mu\frac{\partial u_\sigma}{\partial x}$$
$$D[Z_y] = 2\lambda(\alpha+1)\left(\frac{\partial u_\sigma}{\partial x}+\frac{\partial v_\sigma}{\partial y}\right)+2\mu\frac{\partial v_\sigma}{\partial y}$$
$$D[X_y] = \frac{\partial u_\sigma}{\partial y}+\frac{\partial v_\sigma}{\partial x}.$$

Thus, there are discontinuities in general, but continuity results when the components of the discontinuities satisfy the conditions:

$$\frac{\partial u_\sigma}{\partial x} = 0 \qquad \frac{\partial v_\sigma}{\partial y} = 0 \qquad \frac{\partial u_\sigma}{\partial y}+\frac{\partial v_\sigma}{\partial x} = 0.$$

This conclusion is perfectly consistent with the general properties that we proved in the preceding Note, when one has established conditions for a distortion to satisfy the Weingarten definition; viz., to have continuity on the surface of discontinuity in all six components of the tension.

It therefore seems that by means of the preceding considerations the principles of the theory of elastic distortions take on a general and definitive aspect, and that the research of Volterra, who has fortunately created that theory, will result in a closer relationship with the general theory of elasticity.

III.

It still remains for us to respond to an objection that must be resolved before any use of it can be made.

Given a relation of the form:

$$f_\nu - f_{\nu'} = g$$

that defines the discontinuity in a function f along a surface σ, we made use of the relations that are deduced by tangential derivation on that surface. Now, due to the regularity of the functions f_ν, $f_{\nu'}$, g one sees that without a doubt, along with the preceding relation, everything that is deduced by tangential derivation must subsist, assuming that this is possible. However, one may not likewise be sure that these relations represent the discontinuity of the corresponding derivatives in the function f, since one might then have that the derivatives that are calculated on the surface σ do not coincide with the limits of the derivatives calculated outside of σ when they come indefinitely close that surface. Nevertheless, in most cases this coincidence is actually verified ([1]).

It will not be easy to directly discuss the validity of the following procedure outside of our case, but one may see that the results at which we arrived may also be established without that process of derivation. One does not consider the elastic problem, but limits oneself to the corresponding problem of potential theory that we considered at the end of Note I. The difficulty and the method of overcoming it are substantially identical in the two cases; formally, the question is presented much more simply in the case of the potential.

The problem to consider will be that of determining a regular harmonic function V in a space S that is bounded by a surface s and has an internal cut σ, with the conditions that one must have:

1. On the surface s: $\dfrac{\partial V}{\partial n} = 0$;
2. On the surface σ: $V_\nu - V_{\nu'} = g$, $\dfrac{\partial V}{\partial \nu} + \dfrac{\partial V}{\partial \nu'} = 0$,

where g will be a given function of the points of σ.

For the validity of the relations:

$$\int_S \Delta_1 V\, dS = -\int_S (V_\nu - V_{\nu'}) \frac{\partial V}{\partial \nu} dS = -\int_S g \frac{\partial V}{\partial \nu} d\sigma$$

from which one derives the uniqueness of the solution to the problem that was posed, it is enough that the second derivative of V on S exists and is integrable.

([1]) For example, the equations that one deduces from the relations that determine the discontinuities of a double layer potential:

$$W_\nu - W_{\nu'} = 4\pi g, \qquad W = \int_\sigma g \frac{\partial (1/r)}{\partial \nu} d\sigma,$$

when tangentially differentiated:

$$\frac{\partial W_\nu}{\partial x} - \frac{\partial W_{\nu'}}{\partial x} = 4\mu \frac{\partial g}{\partial x}, \qquad \frac{\partial W_\nu}{\partial y} - \frac{\partial W_{\nu'}}{\partial y} = 4\mu \frac{\partial g}{\partial y},$$

actually given the discontinuities in the tangential derivatives $\partial W / \partial x$, $\partial W / \partial y$ of the double layer potential (Cf., Poincaré, *Théorie du potential newtonien*).

In order to determine the V, we agree to set:

$$V = U + W \qquad \text{where} \qquad W = \frac{1}{4\pi}\int_\sigma g\,\frac{\partial(1/r)}{\partial\nu}\,d\sigma$$

assuming that g satisfies the conditions that were defined in Note I, in order that the derivative of W present no infinity on the contour of σ.

The conditions the U must satisfy as a consequence of the relations that W verify on σ:

$$W_\nu - W_{\nu'} = 4\pi g, \qquad \frac{\partial W}{\partial\nu} + \frac{\partial W}{\partial\nu'} = 0,$$

are then:

$$U_\nu - U_{\nu'} = 0, \qquad \frac{\partial U_\nu}{\partial\nu} + \frac{\partial U_{\nu'}}{\partial\nu'} = 0,$$

while the condition on s:

$$\frac{\partial V}{\partial n} = 0,$$

gives:

$$\frac{\partial U}{\partial n} = -\frac{\partial W}{\partial n}.$$

In addition, U must be harmonic. From this condition it is easy to deduce that U must be regular, along with its derivatives, when one crosses the surface σ.

Indeed, consider the space S to be bounded, along with the surface s, by the two edges of the cut σ, and if one represents U by Green's formula then one finds:

$$U = \frac{1}{4\pi}\int_s\left(U\,\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n}\right)ds + \frac{1}{4\pi}\int_\sigma (U_\nu - U_{\nu'})\frac{\partial(1/r)}{\partial n}\,d\sigma + \frac{1}{4\pi}\int_\sigma\left(\frac{\partial U}{\partial\nu} + \frac{\partial U}{\partial\nu'}\right)\frac{d\sigma}{r}\;;$$

to be precise, the condition that U must satisfy on σ is:

$$U = \frac{1}{4\pi}\int_s\left(U\,\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n}\right)ds\,.$$

Now, this integral has no relationship with the surface σ so U must have no singularity on that surface.

We may then conclude that U has no singularity in the space S and for that reason the determination of it is reduced to that of a regular, harmonic function when one is given the values of the normal derivative on the surface that bounds the field.

As is well known, this function exists in general, and therefore one may conclude that V exists, as well. In addition, since the function V is uniquely determined from the conditions posed, this leads to the consequence that on the surface σ the discontinuities in all of its derivatives of arbitrary order are completely determined, as well.

The existence of a formula in elastostatics that is analogous to that of Green permits us to immediately transport the preceding considerations to the problems of the theory of distortions that was studied previously.