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Foundations and goals of analytical kinematics.

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Translated by D. H. Delphenich

Kind Sirs! As I was cordially invited by our chairman to speak to you on a freely-chosen topic, after some brief vacillation, I finally decided on kinematics. First, I have myself already been involved with this subject for quite some time, but then it also seems to me to be especially tempting to speak to a larger circle on a discipline that, as I believe, is still in the early stages of development, and might thus be of some natural interest to the younger academics. I thought of them, my younger colleagues, first and foremost in the preparation of this talk. Admittedly, I must also direct their attention to the difficulties that are rooted in such peculiarities of the subject, and even more so in the considerable multi-faceted nature that is nevertheless already present in it. However, I think I can count upon their indulgence. In the context of a single talk, they will certainly not expect more than a sketch whose single – and legitimate, moreover – purpose is not so much that of exhibiting a wealth of details, as much as allowing the broad lines to emerge through a blurring of this bewildering abundance, to the extent that it might be knowable at the present time and for the one who draws the map ([1]).

I.

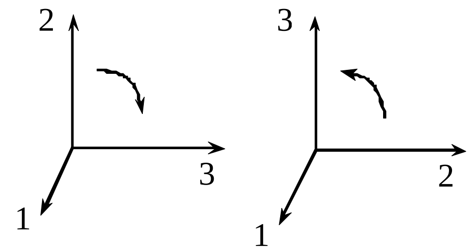
The soma as a spatial element.

I think of kinematics as a ***branch of geometry*** that deals, above all, with the positions that a rigid body can assume in its space. I will first assume that this space is *Euclidian*.

We can think of the mathematical concept of a rigid body as arising from the presentation of an empirically-given body by a process of abstraction, perhaps in the way that the concept of point might first come about. In fact, we disturb its boundary and let it fill up all of space; naturally, the impenetrability of the empirical body is also abandoned in this way. There is an essential distinction between what I would like to call *theoretical kinematics* and the study of machines, which has to do with bounded and mutually-impenetrable bodies. Such an abstraction is necessary, since the infinite manifold of boundaries stands in the way of presenting general laws, and also the deduction of advanced problems that are obtained in this way within the circle of consideration.

([1]) The talk that was actually given (in the span of an hour) was naturally even more abbreviated from what is communicated here.

Nevertheless, the filling of space by the individual positions of the rigid body must naturally remain mathematically tangible. This will be achieved by an axis cross that belongs to the body, and is therefore rigidly attached to all of its points; i.e., a moving coordinate trihedron. The position of the trihedron characterizes the position of the body completely. *I consider this position to be the* ***basic figure of kinematics****, as* ***its most important element,*** *and compare it to the figure of the point in ordinary geometry.* I have allowed myself a special word for it – viz., **soma** – and I would like to also used it today ([1]). The soma itself is then something simple and indivisible – i.e., an atom, like the point. However, points, curves, surfaces, etc., are rigidly attached to it. If the soma moves then these figures will move along with it. – One would arrive at the consideration of only one type of soma, but it would make it more convenient to express some facts if two neighboring ones were presented that would correspond to the two kinds of coordinate axis cross. They should be distinguished by the words *left* and *right*:



The objects in theoretical kinematics are then, above all, figures that consist of somas – left-handed, right-handed, or also somas of both kinds – and which can define either discrete sets or continua, namely, analytic continua of somas.

Although the program that this implies permits extensions, it can already encompass seemingly everything that the practitioners of kinematics were concerned with up to now. Frameworks and gears, rolling curves and ruled surfaces that grind against each other, the theory of the freedom of a rigid body in the infinitesimal and its realization by mechanisms, the moving trihedra of curves and surface theory, all of this falls within this viewpoint, and still other things can be subordinate to it. A set of interesting details is required at present through a not-insubstantial number of different methods. I remind you only of a main theorem that is in the center of numerous investigations, and in my language reads: *One can come to any left-handed or right-handed soma from any other one of the same kind by rotating around some axis and displacing in the direction of just that axis.*

In contrast to this abundance of viewpoints, one now poses the question of whether there is not a *general* method in kinematics by which one can bring all of these things into close connection, a method that is similar to analytic geometry, and thus, a method in which the *soma* enters in place of the point, and will be represented by coordinates that are comparable to the point coordinates x_1, x_2, x_3, which would naturally be coordinates with the simplest possible character, with which one could comfortably calculate, and which must be just as well-adapted to the nature of things as x_1, x_2, x_3 are to point geometry in Euclidian space. These coordinates must also help to give a rightful place to the *concept of a group*, which is completely absent in the kinematics of the engineer. Among other things, it must also allow one to write the transformation formulas without

([1]) *Geometrie der Dynamen,* (G. d. D), Leipzig, 1903, Appendix.

much calculation, by which one can go from one soma to a second one, from that one to a third one, and then also from the first one to the third one directly.

We can now obtain each soma from one of them that is determined once and for all – e.g., a right-handed soma – or ***protosoma***, by a *proper* or *improper* transformation (E or U) of the rectangular coordinates x_1, x_2, x_3, or in the language of geometry, by a ***motion*** or ***transfer.*** A motion allows a figure to emerge from any figure that is *congruent* to it, while a transfer will produce a figure that is *symmetric* to it. Thus, any *right-sided* soma will arise from a motion of the protosoma, while a *left-handed* soma will arise from a transfer of it. The number of constants for a motion or transfer is the same as that of a soma, namely, *six*. We say briefly: *There are* ∞^6 *left-handed somas and* ∞^6 *right-handed ones, as well as* ∞^6 *motions and* ∞^6 *transfers.*

The last-mentioned requirement comes from the following one, moreover: One shall *exhaustively* represent the possible systems of twelve coefficients in the transformation formulas for rectangular, Cartesian coordinates by parameters, and indeed, where possible, in such a way that *firstly*, the numerous relations between the twelve coefficients will be fulfilled identically, and that *secondly*, the parameters of the product of the of motions or transfers will be bilinear functions of the parameters of the given transformations.

We now come *in medias res* to the following theorem:

It is not possible to achieve the desired goals with six coordinates $x_1, \ldots, x_6$, *or seven homogeneous coordinates* $x_0 : x_1 : \ldots : x_6$ *(or also just to associate the totality of motions or right-handed somas with the system of values* $x_1 : \ldots : x_6 : x_0$ *in a one-to-one and invertible way).*

By contrast, both can be achieved with the use of eight homogeneous coordinates, between which one quadratic equation and one inequality exists ([1]).

Naturally, there will be infinitely many such systems of eight coordinates. However, they are all connected to each other by linear transformations. They will next be specialized in a suitable way; it will be shown how they can be connected to the coefficients of a proper or improper orthogonal transformation (E or U).

We would like to denote the eight coordinates or parameters of a motion (E) by the symbols:

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3 ,$$

and those of a transfer (U) by the symbols:

$$\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3 : \delta_0 : \delta_1 : \delta_2 : \delta_3 .$$

These parameters will then be linked by the relations:

$$(\alpha\beta) = \alpha_0\,\beta_0 + \alpha_1\,\beta_1 + \alpha_2\,\beta_2 + \alpha_3\,\beta_3 = 0, \tag{1}$$

$$(\alpha\alpha) = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0, \tag{2}$$

$$(\gamma\delta) = \gamma_0\,\delta_0 + \gamma_1\,\delta_1 + \gamma_2\,\delta_2 + \gamma_3\,\delta_3 = 0, \tag{3}$$

([1]) Math. Ann. **39** (1891), pp. 514, *et seq.*

(4) $$(\gamma\gamma) = \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \neq 0.$$

A motion (E) and a transfer (U), when composed with others of their kind will again give a motion or transfer by the following schema:

$$EE' = E''; \qquad EU' = U''; \qquad UE' = U''; \qquad UU' = E''.$$

In each of the four cases, the parameters of the product of two such transformations will then be *bilinear functions* of the parameters of the factors. Naturally, the result is not indifferent to the sequence of these factors.

II.

The algebraic apparatus.

I would now like to actually write down the transformation equations that one gets by using the parameters ($\alpha : \beta$) and ($\gamma : \delta$). Here, a complication admittedly arises for the completion of that task. These formulas will be clear and easy to handle only when one appeals to an abbreviated notation. The tool for this is the calculus of ***quaternions*** and the ***bi-quaternions*** that Clifford founded. However, to assume that this method of calculation should be familiar to any mathematician would be an unreasonable demand. Nevertheless, I cannot explain it rigorously, as I would then not arrive at my actual topic at all. However, perhaps I can still give a summary, merely suggestive, explanation of the things that matter here.

Just as one uses the sign i in algebra and function theory in order to couple two numbers x, y into a formal sum – viz., the complex number $z = x + y\,i$ – so will we employ a similar coupling of four real or ordinary complex numbers with the symbols e_1, e_2, e_3 . We thus define the *quaternions:*

$$\alpha = \alpha_0 + \alpha_1\,e_1 + \alpha_2\,e_2 + \alpha_3\,e_3\,,$$
$$\beta = \beta_0 + \beta_1\,e_1 + \beta_2\,e_2 + \beta_3\,e_3\,,$$

etc. The *product* of two such formal sums will be defined by the rules for quaternion calculation, namely:

$$\boxed{\alpha\alpha' = \alpha''}$$

will be clarified by the formulas:

(5) $$\boxed{\begin{aligned} \alpha_0\alpha'_0 - \alpha_1\alpha'_1 - \alpha_2\alpha'_2 - \alpha_3\alpha'_3 &= \alpha''_0, \\ \alpha_0\alpha'_0 + \alpha_1\alpha'_0 + \alpha_2\alpha'_3 - \alpha_3\alpha'_2 &= \alpha''_1, \\ \alpha_0\alpha'_2 + \alpha_2\alpha'_0 + \alpha_3\alpha'_1 - \alpha_1\alpha'_3 &= \alpha''_2, \end{aligned}}$$

which contain, in particular, the rules for calculating with the symbols e_1, e_2, e_3 :

$$e_1^2 = -1, \qquad e_2\, e_3 = e_1, \qquad e_3\, e_1 = e_2, \qquad \text{etc.}$$

Secondly, should two such sums – again by analogy with $x + i\,y$ – be combined once more:

$$\alpha + \varepsilon\,\beta = (\alpha_0 + \alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3) + \varepsilon\,(\beta_0 + \beta_1\, e_1 + \beta_2\, e_2 + \beta_3\, e_3)$$

then $\alpha + \varepsilon\,\beta$ will called a *bi-quaternion*.

In calculations, ε^2, unlike i^2, will not be set equal to – 1, but will be set equal to zero:

$$\boxed{\varepsilon^2 = 0.}$$

It follows from this that:

$$(\alpha + \varepsilon\,\beta)\,(\alpha' + \varepsilon\,\beta') = \alpha\alpha' + \varepsilon\,(\alpha\beta' + \beta\alpha'),$$

in which the quaternion products $\alpha\alpha'$, $\alpha\beta'$, $\beta\alpha'$ are calculated using the rule (5). One generally says that one equation in quaternions involves four equations in ordinary real or complex numbers, and one equation in bi-quaternions involves eight such equations.

From what was said, an association is already given, under which each (real) bi-quaternion $\alpha + \varepsilon\,\beta$ or $\gamma + \varepsilon\,\delta$ that satisfies the conditions (2) or (4) will correspond to a *well-defined* motion $(\alpha : \beta)$ or transfer $(\gamma : \delta)$, while, conversely, ∞^1 (mutually "proportional") bi-quaternions naturally belong to any motion or transfer.

The formulas for the ***composition*** of two of our transformations – whether motions or transfers – are now these ones:

$$(6) \qquad \boxed{\begin{aligned}(\alpha+\varepsilon\beta)(\alpha'+\varepsilon\beta') &= \alpha''+\varepsilon\beta'',\\ (\alpha+\varepsilon\beta)(\gamma'+\varepsilon\delta') &= \gamma''+\varepsilon\delta'',\\ (\gamma+\varepsilon\delta)(\alpha'-\varepsilon\beta') &= \gamma''+\varepsilon\delta'',\\ (\gamma+\varepsilon\delta)(\gamma'-\varepsilon\delta') &= \alpha''+\varepsilon\beta''.\end{aligned}}$$

The parameters of a product will be, as desired, *bilinear* functions of the parameters of the factors:

Furthermore, in order for us to not interrupt, we introduce two further symbols that have proved to be useful in the theory of quaternions, and are no less useful in the case of bi-quaternions. We set:

$$\boxed{\begin{gathered}S(\alpha+\varepsilon\beta) = \alpha_0+\varepsilon\beta_0,\\ V(\alpha+\varepsilon\beta) = (\alpha_1+\varepsilon\beta_1)e_1+(\alpha_2+\varepsilon\beta_2)e_2+(\alpha_3+\varepsilon\beta_3)e_3.\end{gathered}}$$

One will then have:

$$\alpha + \varepsilon\,\beta = S(\alpha + \varepsilon\,\beta) + V(\alpha + \varepsilon\,\beta).\ (^1)$$

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(1) $S(\alpha + \varepsilon\,\beta)$ is called the *scalar part* of the bi-quaternion, and $V(\alpha + \varepsilon\,\beta)$ is its the *vectorial component.*

With the help of this notation, the parameters of the transformation that belong to the inverses E^{-1} or U^{-1} of a motion E or a transfer U can be represented in the language of quaternions. This system of parameters is, in fact:

$$\begin{aligned} &\alpha_0 : -\alpha_1 : -\alpha_2 : -\alpha_3 : \;\; \beta_0 : -\beta_1 : -\beta_2 : -\beta_3 : \\ &\gamma_0 : -\gamma_1 : -\gamma_2 : -\gamma_3 : -\delta_0 : \;\; \delta_1 : \;\; \delta_2 : \;\; \delta_3 ; \end{aligned} \tag{7}$$

these are the bi-quaternions that arise:

$$\begin{aligned} (\alpha + \varepsilon\,\beta)^{-1} &= (\alpha\alpha)^{-1}\,\{S(\alpha + \varepsilon\,\beta) - V(\alpha + \varepsilon\,\beta)\}, \\ (\gamma - \varepsilon\,\delta)^{-1} &= (\gamma\gamma)^{-1}\,\{S(\gamma - \varepsilon\,\delta) - V(\gamma - \varepsilon\,\delta)\}. \end{aligned} \tag{8}$$

One comes to these bi-quaternions when one sets:

$$\alpha'' + \varepsilon\,\beta'' = 1$$

in the first and fourth of formulas (6).

III.

Motions and transfers.

We now consider the *objects* of the motion or transfer that is being performed to be:

1. ***Points***, and indeed ones that are endowed with ***masses*** or weights, and therefore ***mass points***. The mass of a point x is called x_0 ; its coordinates shall be called $\frac{x_1}{x_0}$, $\frac{x_2}{x_0}$, $\frac{x_3}{x_0}$. The four quantities x_0, x_1, x_2, x_3 will be linked to a quaternion $x_0 + x_1\, e_1 + x_2\, e_2 + x_3\, e_3$, and the *bi-quaternion:*

$$\boxed{Sx + \varepsilon Vx = x_0 + \varepsilon\,(x_1 e_1 + x_2 e_2 + x_3 e_3)}$$

will be derived from this.

2. ***Rods***, or ***straight lines with weights,*** i.e., ordered pairs of points $\overrightarrow{x, y}$ that will be displaced with the preservation of their separation distance along their connecting line; these are well-known figures in mechanics. One appeals to them in order to visualize *forces* that act upon a rigid body, and represent them by coordinates that are named:

$$X, Y, Z, L, M, N$$

in the usual notation, and are linked by the equation:

(9) $$X L + Y M + Z N = 0.$$

The ratios:

$$X : Y : Z : L : M : N$$

are the Plückerian coordinates of the line on which the rod lies. One can see from the relation (9) that one has the coordinates of a *dyname*. We use other symbols, in the sequence:

$$p_1, p_2, p_3, q_1, q_2, q_3,$$

such that the relation (9) goes to:

$$p_1 q_1 + p_2 q_2 + p_3 q_3 = 0,$$

and define the *bi-quaternion:*

$$\boxed{p + \varepsilon q = (p_1 q_1 + p_2 q_2 + p_3 q_3) + \varepsilon (q_1 e_1 + q_2 e_2 + q_3 e_3).}$$

3. ***Sheets***, which are ***planes with weights***. A plane has the equation:

$$(u\, x) = u_0\, x_0 + u_1\, x_1 + u_2\, x_2 + u_3\, x_3 = 0.$$

$u_0 : u_1 : u_2 : u_3$ are then the coordinates of the plane. We ascribe the four quantities u_0 , u_1 , u_2 , u_3 as the coordinates of a plane with a "weight." Its weight is then:

$$\sqrt{u_1^2 + u_2^2 + u_3^2}\ .$$

If y is any mass point then:

$$(u\, y) = u_0\, y_0 + u_1\, y_1 + u_2\, y_2 + u_3\, y_3$$

will also mean – up to sign – the distance from the point to the plane, multiplied by its weight and the mass of the point.

We define the *quaternion*:

$$u = u_0 + u_1\, e_1 + u_2\, e_2 + u_3\, e_3$$

from the coordinates u_0 , u_1 , u_2 , u_3, and from this – *unlike before in the case of the point* – a *bi-quaternion:*

$$\boxed{Vu + \varepsilon Su = (u_1 e_1 + u_2 e_2 + u_3 e_3) + \varepsilon u_0.}$$

The figures:

mass-point, *rod*, *sheet*

thus each contain one more constant than the corresponding figures:

point, *line*, *plane.*

The *ratios* of the coordinates of the first three figures are the homogeneous coordinates of the last three, except that a *uniquely*-determined (i.e., special) bi-quaternion will belong to the first three figures.

With these preparations, we arrive at the representation of the motions and transfers in terms of the associated parameters (α : β) and (γ: δ) ([1]):

(10)
$$\boxed{\begin{aligned} Sx' + \varepsilon Vx' &= (\alpha + \varepsilon\beta)^{-1}(Sx + \varepsilon Vx)(\alpha - \varepsilon\beta), \\ p' + \varepsilon q' &= (\alpha + \varepsilon\beta)^{-1}(p + \varepsilon\, q)(\alpha + \varepsilon\beta), \\ Vu' + \varepsilon\, Su' &= (\alpha + \varepsilon\beta)^{-1}(Vu + \varepsilon\, Su)(\alpha - \varepsilon\beta), \end{aligned}}$$

(11)
$$\boxed{\begin{aligned} Sx' + \varepsilon Vx' &= (\gamma - \varepsilon\delta)^{-1}(Sx - \varepsilon Vx)(\gamma + \varepsilon\delta), \\ p' + \varepsilon q' &= -(\gamma - \varepsilon\delta)^{-1}(p - \varepsilon\, q)(\gamma - \varepsilon\delta), \\ Vu' + \varepsilon\, Su' &= (\gamma - \varepsilon\delta)^{-1}(Vu - \varepsilon\, Su)(\gamma + \varepsilon\delta). \end{aligned}}$$

These formulas are at the center of the theory that we are concerned with here; they are the main facts and the sources of all the advanced ones. In them, we have – and this is essential – *an* ***abbreviated form*** *for the transformation formulas of rectangular Cartesian coordinates*, and indeed, for the coordinates of points, lines, and planes, as well as mass-points, rods, and sheets. The formulas (10), (11) thus have – and this is also essential for the applications – the character of a certain *completeness*. On the right-hand side, one has the given variables (x_κ; p_κ, q_κ; u_κ), while the transformed ones are on the left and distinguished by primes. We interpret both of them in the same coordinate system; the formulas (10) then mean a *motion* whose parameters are the quantities (α : β); likewise, formulas (11) mean a *transfer*. The coordinates of the transformed figures depend upon those of the given ones *linearly*. If we calculate from the formulas, which must remain too laborious here and is often also superfluous in the applications, then that

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([1]) Cf., Math. Ann. **39** (1891), pp. 514-564. The equations that are published here are improved in form from the ones there, and extended in content. A brief derivation of formulas (10), (11) is the following: As is known, the rotations around the point (1 : 0 : 0 : 0) can be represented with the help of quaternions by the formulas:

$$x_0' = x_0 \quad (\text{or } Sx' = Sx) \qquad \text{and} \qquad Vx' = \alpha^{-1} \cdot Vx \cdot \alpha.$$

If one replaces these formulas with:

$$Sx' = Sx \qquad \text{and} \qquad Vx' = \alpha^{-1} \cdot Vx \cdot \alpha + x_0\, \xi,$$

where ξ means a vectorial quaternion (i.e., $S\xi = 0$), then one will have the representation for any motion. If one now sets $\xi = -2\, \alpha^{-1} \beta$ then the formulas:

$$Sx' = Sx \qquad \text{and} \qquad Vx' = \alpha^{-1} \{Vx \cdot \alpha - 2\, Sx \cdot \beta\}$$

will follow from $(\alpha\beta) = 0$, but they can be assembled into one:

$$Sx' + \varepsilon\, Vx' = (\alpha + \varepsilon\, \beta)^{-1} (Sx + \varepsilon\, Vx)\, (\alpha - \varepsilon\, \beta),$$

in which the parameters α_κ , β_κ, and the coordinates x_κ are distributed into different factors of a bi-quaternion product. A simple calculation will then yield the remaining formulas (10), (11). One will again come to equations (6) from (10) and (11).

will show that the coefficients of the linear functions to be defined will all be *fractions with the same denominator*:

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \qquad \text{or} \qquad \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 ,$$

which is non-zero, by assumption. *The numerators are then likewise entire rational functions of second degree in the parameters* α, β, *or* γ, δ. Conversely, if the coefficients of a transformation E or U are given then the ratios $\alpha : \beta$ or $\gamma : \delta$ will be determined ***uniquely*** by them. If $\beta_0 = 0$ then the motion (10) will be a ***rotation***. If $\alpha_0 = 0$ then it will be a ***screw***; i.e., it will consist of the composition of a rotation through an angle π with a displacement in the direction of the rotational axis. We will have to employ this concept soon, *which assumes a special place in kinematics, like that of the rotation*. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and also $\beta_0 = 0$ then one will have a rotation around an imaginary axis before one – i.e., a *displacement*.

How convenient it is to calculate with these formulas shall be illustrated by some simple examples.

The group of motions and transfers includes three kinds of involutory transformations that have primitive period two, namely, the *reflections through points, lines, and planes.*

We would like to assume that a point y, a *line* $(p^* : q^*)$, and a *plane* v are given. We would like to know how the coordinates of mass-points, rods, and aheets will be transformed under the associated reflections. These are problems that pose no actual difficulty using the ordinary methods of analytic geometry, but already require a considerable outlay of calculation. We simply have to suitably specialize our parameters, and indeed as follows:

$$\begin{aligned}
(\gamma : \delta) &= (y_0 : \ 0 : \ 0 : \ 0 : 0 : y_1 : y_2 : y_3) \\
(\alpha : \beta) &= (0 : p_1^* : p_2^* : p_3^* : 0 : q_1^* : q_2^* : q_3^*) \\
(\gamma : \delta) &= (0 : v_1 : v_2 : v_3 : v_0 : 0 : 0 : 0).
\end{aligned}$$

Another example: We imagine a motion that is not a displacement and decomposes into a rotation and a displacement in the direction of the rotation axis. The three quotients $\alpha_k : \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$ are then its direction cosines. Let 2Θ be the *rotation angle* and let $2H$ be the *step length* ("passo," magnitude of displacement) of this motion. One finds:

$$\boxed{\cot\Theta = -\frac{\alpha_0}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}, \qquad H = \frac{\beta_0}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}.} \tag{12}$$

One sees that – as we said – the equation $\alpha_0 = 0$ characterizes the *screws*, while $\beta_0 = 0$ characterizes the *rotations*.

IV.

Manifold of somas.

The application of the parameters $(\alpha : \beta)$ and $(\gamma : \delta)$ to continuous manifolds of somas now takes the form: The *protosoma*, as we said, which is thought of as a fixed *right-handed* trihedron, will be associated with a second *right-handed soma* by any motion $(\alpha : \beta)$ and a *left-handed soma* by any transfer $(\gamma : \delta)$; $(\alpha : \beta)$ and $(\gamma : \delta)$ are the *coordinates* of both somas. If, e.g., the coordinates $(\alpha : \beta)$ were represented as analytic functions of 0, 1, 2, 3, 4, 5 parameters that were essential to their *ratios* then a one-to-five-dimensional manifold of right-handed somas would arise; as we would like to say, an M_1, M_2, M_3, M_4, or M_5 of right-handed somas.

For the study of machines, which mostly deals with compulsory (*zwangsläufig*) processes of motion, the most important case is that of a one-dimensional manifold of somas. One then deals with only one parameter, which one might care to interpret as the measure of time *t*. Except for limiting cases, there then exists an instantaneous screw axis at each moment; the locus of these axes in a fixed soma is a ruled surface. A second such surface exists in a moving soma, and through the so-called *grinding* of the second surface on the first one, a process of motion will come about – still ignoring the limiting cases – in which the behavior can be described in a simpler way. ∞^5 manifolds of somas will be generated simultaneously in this way, since any soma can be rigidly coupled to the second surface. Everything can ensue in a conveniently computable way with the help of the parameter $(\alpha : \beta)$. The screw that takes place in the time element *dt* belongs to the bi-quaternion:

$$\text{(13)} \qquad \begin{aligned} \alpha^* + \varepsilon\beta^* &= (\alpha + \varepsilon\beta)^{-1}\{(\alpha + \varepsilon\beta) + (\alpha' + \varepsilon\beta')\,dt\} \\ &= 1 + (\alpha + \varepsilon\beta)^{-1}(\alpha' + \varepsilon\beta')\,dt. \end{aligned}$$

The associated *rotation angle* $2\,d\Theta$ and the associated *step length* $2\,dH$, e.g., are given by the formulas:

$$\text{(14)} \qquad \begin{aligned} d\Theta &= -\frac{\sqrt{(\alpha\alpha)(\alpha'\alpha') - (\alpha\alpha')^2}}{(\alpha\alpha)}\,dt\,, \\ dH &= -\frac{(\alpha'\beta')}{\sqrt{(\alpha\alpha)(\alpha'\alpha') - (\alpha\alpha')^2}}\,dt\,, \end{aligned}$$

and if one would like to assume that $(\alpha\,\alpha) = \text{const.} = 1$ then one will obtain even simpler expressions. However, the apparent motion of the fixed soma for the observer that sits on the moving soma will be found simply by replacing the bi-quaternion $\alpha + \varepsilon\beta$ or $S(\alpha + \varepsilon\beta) + V(\alpha + \varepsilon\beta)$ with the bi-quaternion $S(\alpha + \varepsilon\beta) - V(\alpha + \varepsilon\beta)$, and thus, replacing the motion parameters:

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3$$

with the parameters of the opposite motion:

$$\alpha_0 : -\alpha_1 : -\alpha_2 : -\alpha_3 : \beta_0 : -\beta_1 : -\beta_2 : -\beta_3 .$$

Generally, one advances in any manifold of somas to a neighboring soma by a *rotation* when the Monge equation:

$$(d\alpha\, d\beta) = d\alpha_0\, d\beta_0 + d\alpha_1\, d\beta_1 + d\alpha_2\, d\beta_2 + d\alpha_3\, d\beta_3 = 0 \tag{15}$$

is fulfilled; naturally, this rotation can also be a displacement. It can happen that a manifold of somas has this property *everywhere*, and that will define an interesting problem that we shall speak about later. One can, e.g., also conveniently express the *kinetic energy* of a moving soma that is rigidly endowed with mass points, and exhibit the differential equations of its motion under the influence of given forces ([1]).

For more-than-one-dimensional manifolds of somas, (13) yields differential equations that are similar to the Codazzi equations of surface theory; we would also like to cite these equations, since they are meaningful for applications. Let s and t be any two parameters upon which the soma coordinates a, b depend. If one then sets:

$$(\alpha + \varepsilon\,\beta)^{-1} \frac{\partial}{\partial s}(\alpha + \varepsilon\,\beta) = a_s + \varepsilon\, b_s ,$$

$$(\alpha + \varepsilon\,\beta)^{-1} \frac{\partial}{\partial t}(\alpha + \varepsilon\,\beta) = a_t + \varepsilon\, b_t$$

then one will get:

$$\frac{\partial}{\partial t}(a_s + \varepsilon\, b_s) - \frac{\partial}{\partial s}(a_t + \varepsilon\, b_t) = (a_s + \varepsilon\, b_s)\,(a_t + \varepsilon\, b_t) - (a_t + \varepsilon\, b_t)\,(a_s + \varepsilon\, b_s). \tag{16}$$

However, such circumstances are already of a relatively well-developed kind. In kinematics, the geometry of somas, like the geometry of points, defines foundations of an *algebraic* nature. Here, as there, one will do well to begin with the study of the *simplest* manifolds of somas, just as one begins with the study of straight lines and planes in the geometry of points. In fact, there are figures in kinematics that can be very comparable to the lines and planes of point geometry. The lecturer called them ***chains*** and examined them systematically ([2]). Here, we shall satisfy ourselves with the consideration of some examples. Such chains of various dimensions will arise when one reflects a soma in all lines of a cylindroid, all normals to a certain line, and all lines in space, and when one rotates it around a fixed point, or subjects it the reflections in all planes or points of space.

This and related figures will be linked to each other by remarkable laws, *one* of which, we would now like to get to know.

([1]) Journal des Mathématiques, **7** (1911), 97-112. The use of bi-quaternions is avoided in this.

The possibility of much more far-reaching applications is opened up in a book by E. and F. Cosserat: *Théorie des Corps deformables*, Paris, 1909.

([2]) See the definition, G. d. D, pp. 563.

V.

Kinematics and projective geometry.

Certainly one of the simplest figures of kinematics is the ***chain of rotations***, which is a soma-M_1 that is generated from a soma by rotation around a line. We would therefore also like to calculate the ***chains of displacements***, which arise from displacing a soma in a given direction. The consideration of these figures likewise provides us with the insight that *kinematics encompasses all of the projective geometry of our space.* In fact, if we now rotate a right-handed soma in all possible ways *around a fixed point* then a soma-M_3 will arise in which two somas can obviously be linked to each other by a rotation. We will obtain such an M_3 when we set $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$. However, we can interpret the parameters $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$ as point coordinates in space and the chains of rotations will be mapped onto straight lines ([1]). Any theorem in the projective geometry of space will furnish a property of the M_3 . A simple enumeration shows that ∞^6 such M_3 are present; we would also like to calculate certain limiting figures for them that arise when the center of rotation goes to infinity. However, we can provide yet a second kind of figure that has completely similar properties. If we now subject a *left-handed* soma to all reflections in *planes* of space then an M_3 of right-handed somas will again arise, and any two somas in it can be coupled by a chain of rotations in it ([2]). A limiting case of such an M_3 will arise when we subject a left-handed soma to all reflections in points (or a right-handed one to all displacements). These M_3 also exist in ∞^6 exemplars.

Kinematics thus encompasses projective geometry in three-dimensional spaces, and in fact, in several different ways: These spaces appear in kinematics as an extract of geometry inside certain three-dimensional chains of somas. However, another situation is of much greater significance, as a result of which the geometry in the continuum of all ∞^6 – e.g., right-handed – somas will have the closest relationship to ***projective geometry in the complex domain*** in our space ([3]).

We would now like allow a right-handed soma to admit all rotations around a line and all displacements along that line. A two-dimensional chain of somas M_2 will arise in that way, whose *axis* will be the line. One can generate the same M_2 in another way, namely, when one subjects a suitable soma to all *reflections* in lines that cut the axis perpendicularly. However, if one reflects a (right-handed) soma in *all* lines in space then an M_4 will arise. I would like to call an isolated soma an M_0 . One then sees almost immediately that there are:

$$\infty^6 M_0 , \qquad \infty^6 M_2 , \qquad \infty^6 M_4$$

of the kind that was just described.

If we next consider the extension of projective geometry in space into the complex domain then we will likewise find three kinds of manifolds of complexes – i.e., real *or* imaginary figures that appear in just the same dimensions and numbers of individuals:

([1]) Cf., C. Stéphanos, Math. Ann. **22** (1883), 299-367.
([2]) G. Königs, *Leçons de Cinématique*, Paris, 1897, pp. 239-241.
([3]) G. d. D., Foreword and pp. 555, *et seq.*

We will have $\infty^{2 \cdot 3}$ complex points M'_0, $\infty^{2 \cdot 4}$ complex lines M'_2, each of which is the locus of ∞^2 complex points, and finally, once more, $\infty^{2 \cdot 3}$ complex M'_4, each of which carries ∞^4 complex points. One will then have the well-known law: Two M'_0 determine an M'_2, three M'_0 that belong to no M'_2 determine an M'_4, etc.

Laws that are entirely similar – but not without exceptions – exist between the figures M_0, M_2, M_4 ***of kinematics.***

On these grounds, the lecturer has called the M_2 and M_4 *chains of lines and planes*, resp.

As an example of an exceptional case let us cite: Two M_0 will determine no M_2 when one of the M_0 can be taken to the other one by a displacement. *As such,* ***for each*** *theorem in projective geometry, when extended to the complex domain, there is, despite this exceptional case, a kinematic counterpart. The group of all* ∞^{30} *complex collineations is analogous to a group of* ∞^{30} *transformations that permute somas amongst themselves* ([1]).

Yet another example might be given:

One allows a soma to be rotated around a certain line and displaced in the direction of that line, so it moves in a describing M_2, and adds *any* of the following three demands:

1. A point of the soma shall remain in a fixed plane.
2. A plane of the soma shall go through a fixed point.
3. The soma shall emerge from a certain soma by reflecting in any generator of a *cylindroid* or *pencil of planes* whose axis is the line. (There are some obvious qualifications that emerge, which state that the point and plane that are employed may *not* have certain special positions.)

The point of a soma that moves in that way will describe an ellipse, and its plane will envelop a cone of rotation or limiting cases of such figures. There are ∞^{11} of these soma-M_1, which I call ***one-dimensional chains***. *If one subjects any of them to the aforementioned group of transformations then one will always again obtain an* M_1 *that is described in the same way.* ([2])

Such an M_1 has its counterpart in projective geometry in the form of a locus M'_1 of complex points that is well-known and considered more closely in projective geometry, namely, the ***figure of the Staudt chain.***

Although the Staudt chains seem to be quite dissimilar to the figures that are depicted in kinematics, both of them have closely-related properties. There exists a passage to the limit by which one can derive the properties of **soma chains** M_1 *from those of Staudt chains.*

([1]) Here, we count complex constants. The real transformations in the stated group of kinematics define two separate families, so there are then "$2 \cdot \infty^{30}$" of them. Similarly, later on, when one would like to consider real constants, one will set $4 \cdot \infty^{28}$, instead of $2 \cdot \infty^{30}$.

([2]) The rotational chains belong to these M_1, except for the displacement chains.

This is also especially true for the characteristic property of Staudt chains that consists in the fact that any four of their points define real double ratios ([1]).

The multi-dimensional chains of projective geometry also have their kinematic counterparts in just those figures that were called *chains* here ([2]).

The close relationship between projective geometry and kinematics, of which only suggestions can be made here, also extends to the *metric geometric of the soma continuum*. This will also be explained by an example.

Let X, Y be bi-quaternions that belong to two right-handed somas, e.g.:

$$X = X_0 + X_1\, e_1 + X_2\, e_2 + X_3\, e_3\,, \qquad X_k = a_k + \varepsilon\, b_k\,.$$

The product $X^{-1}\, Y$ then gives the parameters of the motion that takes X to Y. Let 2Θ and $2H$ be the rotation angle and step length of this motion. By analogy with $\cos(\vartheta + i\,\eta)$, one can now define a function:

$$\cos(\Theta + \varepsilon H) = \cos\Theta - \varepsilon \cdot \sin\Theta \cdot H.$$

One then finds that ([3]):

$$\boxed{\cos(\Theta + \varepsilon H) = \frac{X_0 Y_0 + X_1 Y_1 + X_2 Y_2 + X_3 Y_3}{\sqrt{X_0^2 + X_1^2 + X_2^2 + X_3^2}\sqrt{Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2}}.} \tag{17}$$

This formula, which is very familiar to every mathematician with a different meaning, thus appears in kinematics in a certainly quite unexpected way, and with a new content. The fact that one can infer a whole set of consequences from this fact alone scarcely needs to be stated. Theorems of non-Euclidian geometry furnish theorems of kinematics by a *mechanically*-defined passage to the limit.

Kinematics itself seems to be an extension of non-Euclidian geometry that therefore also takes on an immediate meaning for Euclidian space.

VI.

Continuation: A kinematic analogue to projective line geometry.

However, kinematics has room for ***a second analogue to projective geometry***, among many other ones ([4]). While the first one led us to a group of ∞^{30} transformations that

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([1]) G. d. D., pp. 244, 331, 568.

([2]) G. d. D., pp. 563, *et seq.*

([3]) G. d. D., pp. 585.

([4]) G. d. D., pp. 580, *et seq.* The material there can be treated only very briefly, corresponding to the plan of the book. The following brings many new things. – The existence of analogies between kinematics and Plücker's line geometry was also known to the Swiss mathematician R. de Saussure. He attempted to exploit such ideas, but with inadequate tools and without investing sufficient care.

permute somas and certain manifolds of somas amongst themselves, the second one belongs to a group of $2 \cdot \infty^{30}$ transformations ([1]).

We recall that one can also regard the Plückerian line coordinates as coordinates of the points of a quadratic manifold M_4^2:

$$p_1\, q_1 + p_2\, q_2 + p_3\, q_3 = 0, \tag{18}$$

which lies in a space of *five* dimensions, and that precisely this interpretation of line coordinates immediately opens up a royal road to numerous results in projective geometry. One needs only to translate the results from the theory of quadratic manifolds into another language. We then interpret the soma coordinates as point coordinates in a quadratic manifold M_6^2 in a space of seven dimensions whose equation is just the relation:

$$\alpha_0\, \beta_0 + \alpha_1\, \beta_1 + \alpha_2\, \beta_2 + \alpha_3\, \beta_3 = 0. \tag{19}$$

The images of somas, whose distribution is space is difficult to imagine, then lie next to each other like points on a second-order ruled surface and are presented almost intuitively. The aforementioned group of kinematics has the group of $2 \cdot \infty^{28}$ *collineations* that leave M_6^2 at rest for its image.

Admittedly, an essential difference exists between the line and soma coordinates. Namely, whereas the line continuum, as we regard it today, is *closed* and leads to a *gapless* map of the lines in space to the points of a M_4^2, a soma can disappear when one extends it to infinity. Coordinate systems for which:

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0,$$

and thus, for which:

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0,$$

since only real figures will be considered here, do not correspond to any soma. However, things would happen the same way in line geometry if the imaginary lines were still not fictitious; we must then exclude the coordinate systems for which:

$$p_1 = p_2 = p_1 = 0.$$

In mechanics, we then know only coordinates *X*, *Y*, *Z*, *L*, *M*, *N* of forces, but still not those 0, 0, 0, *L*, *M*, *N* of force-pairs. Here, as there, the filling in of this hole is *possible*, and it is also *necessary* if one would like to come to know simple algebraic laws. I shall be brief, and thus give only the theorem that will provide the definition of the analogue to an imaginary line (i.e., a ***pseudo-soma***):

If one subjects any (right-handed) soma to all screwing motions ([1]) *then a soma-*M_5 *will arise. Its image on* M_6^2 *will be a cone* M_5^2, *namely, the intersection of* M_6^2 *with one of*

([1]) $4 \cdot \infty^{28}$ transformations in the real domain. See the remark on page 13.

its linear tangential spaces R_6 . *The contact point of this space* R_6 *– viz., the vertex of the cone* M_5^2 *– will be any point with coordinates of the form:*

$$0 : 0 : 0 : 0 : \beta_0^* : \beta_1^* : \beta_2^* : \beta_3^* .$$

If the soma that is employed for the construction has the coordinates:

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3$$

then the proportion will exist:

(20) $$\beta_0^* : \beta_1^* : \beta_2^* : \beta_3^* = \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 .$$

Obviously, the soma-M_5 thus-constructed is *the complete analogue to the figure of all real lines that are parallel to a plane*, and the starting soma is analogous to any line that is perpendicular to this plane. One obtains a simple figure that can represent the (right-handed) pseudo-soma in the form of three ordered, mutually-perpendicular directions that correspond to the positive directions of the three axes of any of the ∞^3 somas:

$$\beta_0^* : \beta_1^* : \beta_2^* : \beta_3^* : * : * : * : *.$$

Naturally, I cannot go further into this very important point. Therefore, as we will now do, it shall be subjected to restrictions. *Now, only those (analytic) loci of points on* M_6^2 *will be considered that do not traverse the entire linear manifold* $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$, *and also only when their points do not belong to just that manifold.*

We will now pick some of the simplest relations that the map described yields by contrasting them with figures in M_6^2, and indeed, with *linear* point-manifolds in M_6^2, that correspond *constructively* to the figures that are described in kinematics. The somas to be mapped shall be *right-handed* throughout.

1. There are ∞^9 *straight lines* R_1 in M_6^2. They will correspond to the *chains of rotations* that were described already.

2. There are also ∞^9 *planes* R_2 in M_6^2. Kinematic counterpart: One subjects a *left*-handed soma to all reflections in the planes of a bundle, or in a limiting case, all reflections in the points of a plane.

3 and 4. There are ∞^{12} "left-handed" three-dimensional *linear spaces* R_3 in M_6^2 and ∞^6 right-handed linear spaces R_3.

We have already given the construction for the corresponding families of somas, as well. However, we must now establish what sort of three-dimensional spaces should be called "left-handed" and which should be called "right-handed." We will determine that the *left-handed* R_3 correspond to the soma-M_3's that are generated by rotation. The image of the M_3 of all pseudo-somas ($\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$) will then likewise be a *left-handed*

([1]) See the definition on page 9.

R_3 . However, all R_3 that can be derived from a left-handed soma by the process of reflection that was described above will be *right-handed*.

All linear manifolds in M_6^2 are now already exhausted with the enumerated figures. However, manifolds M_k^2 ($k = 1, \ldots, 5$) – e.g., the planar sections of M_6^2 – can likewise be generated.

VII.

The Ribacour problem.

As an application of the theory that was sketched out, we might demonstrate the response to a question that was posed already in the context of the Monge equation:

$$d\alpha_0\, d\beta_0 + d\alpha_1\, d\beta_1 + d\alpha_2\, d\beta_2 + d\alpha_3\, d\beta_3 = 0,$$

which we will refer to as the *Ribacour problem:*

How are the (analytic) soma manifolds M_k , in which any two consecutive somas can be linked by a chain of rotations, to be generated kinematically?

The images of M_k that correspond to values $k = 1, 2, \ldots$ must be curves, surfaces, etc. in M_6^2 whose tangents all traverse M_6^2 itself. One now finds that one must have $k \leq 3$ ([1]), and the assumption that $k = 3$ is satisfied by only the *linear* R_3 in M_6^2 that were described already.

Trivially, all M_1, M_2 whose images lie in such R_3 will then have the desired property. Their exhaustive enumeration, classification, and construction, which encounters no difficulty, might be omitted, for sake of brevity.

What will then remain are certain other M_1 and M_2 . The M_1 arise when one lets any ruled surface that is not a cone roll upon a second ruled surface, and indeed, a rectilinear bending surface. A soma that is rigidly coupled to the moving surface will describe any of the desired M_1; the isometric relationship of one surface to the other will be evoked by either a motion or a transfer, but it cannot be a relationship of congruence or symmetry.

We now come to the non-trivial soma-M_2's (e.g., M_2 of right-handed somas) of the desired kind whose exhaustive enumeration defines the interesting, and likewise difficult, part of our problem.

[1]. One lets two surfaces that are related to each other by *isometry*, but neither congruent nor symmetric, roll upon each other in such a way that corresponding points and line elements coincide at each position of the moving surface, and also all possible positions of that kind will be assumed. If the two surfaces are rectilinear then their generators might not correspond. The image of a soma-M_2 that is generated by a moving soma in this way is a *non-rectilinear* surface in M_6^2 whose tangential planes all lie on M_6^2.

([1]) See G. Koenigs, *Leçons de Cinématique*, Paris, 1897, pp. 239-241.

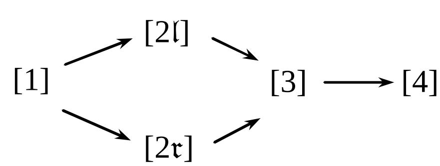
[2$\mathfrak{l}$]. One lets two curves that are isometrically related to each other, but do not have equal curvature at every corresponding point, roll one upon the other in any way, and lets the moving curve rotate around the common tangent to both curves in each position. The image of a soma-M_2 that is generated by a moving soma is a rectilinear M_2 on M_6^2 that has the property that any two consecutive generators can be linked by a *left-sided* R_3, *but not* by a right-handed R_3 .

[2$\mathfrak{r}$]. One lets a rectilinear bending surface roll upon a rectilinear surface that is not developable. One thinks of a *left-handed* soma as fixed in the moving surface. In each of the ∞^1 positions thus obtained, one now reflects (the moving surface and) the soma in all tangential planes to the fixed surface, and thus, in the planes through the generators of this surface. An M_2 of *right-handed* somas arises that has the desired property. The image of this soma-M_2 is a rectilinear M_2 in M_6^2 that has the property that any two consecutive generators can be linked by a *right-handed* R_3, *but not* a left-handed R_3 .

[3]. One considers two curved lines that are isometrically related to each other, but not by congruence or symmetry, such that they have equal curvatures at corresponding points; thus, the tangential surface to one of them appears as the bending surface to the tangential surface of the other (while the rectilinear generators correspond to each other). One then lets the one curve roll upon the other one (or the tangential surface to the one on the tangent surface to the other). At each of the ∞^1 positions thus obtained one lets the moving curve and a (right-handed) soma that is fixed in it rotate around the common tangent to both curves. A soma-M_2 of the desired kind will then arise. Its image is a rectilinear M_2 in M_6^2 in which any two consecutive generators can be coupled by a left-handed R_3, *as well as* a right-handed R_3 . {Cf., the cases [2$\mathfrak{l}$] and [2$\mathfrak{r}$]}. This image is, in fact, the tangent surface to a curve that runs in M_6^2 whose osculating planes all likewise lie in M_6^2.

[4]. One lets any (right-handed) soma rotate around all tangents to a skew curve. The image of the soma-M_2 thus-obtained is a rectilinear M_2 in M_6^2, namely, a cone whose tangential planes all lie in M_6^2.

One recognizes, with no further assumptions, how the figures that were enumerated last can be regarded as limiting cases of the first-discussed ones:



However, the solutions to the Ribacour problem for the stated figures are still not exhausted with them, if one, as we did here, omits the previously-defined *trivial* limiting cases. Namely, it can happen that *any* soma in a soma-M_2 can be coupled to a suitably-chosen neighboring soma by a chain of displacements; i.e., that among the tangents to the image M_2 at any point there is always one of them that meets the R_3 for which $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$. One then adds some further figures. As one finds, they are (except for the "trivial" cases; cf., *supra*) all limiting cases of the families [2ℓ] and [4]. They can be described as follows:

[5]. One lets the second of two non-cylindrical and non-symmetric rectilinear surfaces roll upon the first one, where one of them can arise from the other one by bending, with preservation of straight lines (1). In each of the ∞^1 positions thus-obtained, one subjects the moving surface and a (right-handed) soma that is fixed in it to all displacements in the direction of the common normal to the two consecutive associated generators of the fixed surface (in the direction of the line of striction in the contact line of the two surfaces).

Each soma-M_2 that can be described in this way at all can be generated in this manner in infinitely many ways; in particular, with the help of developable surfaces. Their image will be either [2ℓ*] the tangent surface to a certain curve that lies in M_6^2 or [4*] a certain cone. If the second case appears then the soma-M_2 can be constructed in an even simpler way: One subjects a (right-handed) soma to all *screws* (cf., pp. 9) whose axes are the tangents to a non-planar curve or the lines of a non-cylindrical cone.

The non-trivial solutions to the Ribacour problem are exhausted with the enumerated figures.

What Ribacour himself presented as the solution to his problem for soma-M_2, and what others then sought to prove (2), can only amount to an entirely coarse approximation to reality.

All of the desired soma-M_2, or, in the terminology of the French geometer, all "déplacements à deux paramétres pour lesquelles les mouvements élémentaires sont toujours des rotations (*)," shall, in fact, arise from isometrically-related surfaces that roll upon each other, and also the converse shall always be true. The fact that one thus formulates assertions that already contradict elementary and well-known facts is, in any event, also explained by the otherwise widespread, but highly questionable, practice of paying no heed whatsoever to the "trivial" exceptional cases. Errors in logic that the consideration of just these cases has uncovered immediately can very easily remain unnoticed in that way.

VIII.

A kinematic reciprocity theorem.

Kind sirs, let us once more consider the formulas (6), (10), and (11). These formulas, in fact, lead to still more beyond the ones that we have arrived at up to now.

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(1) It is not excluded that the generators of the one surface define a pencil of planes (and the other ones then define a cone).

(2) Darboux, *Théorie des Surfaces. I*, pp. 66-73. G. Koenigs, *loc. cit.*, pp. 236-239.

(*) Translator's note: "two-parameter displacements for which the elementary motions are all rotations."

We have set $\varepsilon^2 = 0$ in all calculations. However, similar arguments can also be made under more general assumptions. It suffices to assume that $\varepsilon^2 = 1$ and $\varepsilon^2 = -1$. In place of the inequality $(\alpha\alpha) \neq 0$, one will have to employ one or the other inequality:

$$(\alpha\alpha) + (\beta\beta) \neq 0, \qquad (\alpha\alpha) + (\beta\beta) \neq 0.$$

[In all three cases, one will thus have the inequality $(\alpha\alpha) + \varepsilon^2 (\beta\beta) \neq 0$].

In both cases, $\varepsilon^2 = 1$ and $\varepsilon^2 = -1$, the formulas can likewise be interpreted geometrically quite simply. If $\varepsilon^2 = 1$ then we will have the group of $2 \cdot \infty^6$ proper and improper (real) orthogonal transformations of four variables before us, and thus, the group of *motions and transfers* in a Euclidian space of dimension four that leave a (real) point (0, 0, 0, 0) at rest. The other assumption $\varepsilon^2 = -1$ has recently taken on a special interest, as a result of arguments that have their roots in physics in the bold ideas of Lorentz. This assumption leads to the *geometry of Minkowski space.*

I would like to treat only the assumption that $\varepsilon^2 = 1$ and further consider only *points,* and also only ones whose four Cartesian coordinates x_0, x_1, x_2, x_3 satisfy the equation:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, \tag{21}$$

which then lie on a *three-dimensional cone.* The geometry of this manifold is a type of non-Euclidian one, namely, three-dimensional *spherical geometry*.

Kinematics in these spherical spaces must already be of interest to us because we do not know whether we do not perhaps live in such a space. However, in addition, they, like all of spherical geometry, take on a highly theoretical meaning through an intrinsic symmetry that is foreign to kinematics in Euclidian space. Euclidian space is, however, a limiting case – i.e., a degeneracy – of spherical space, just as the plane is a limiting case of a sphere. Therefore, spherical geometry can also lead to many insights into Euclidian space that would not be quite as easily accessible to us. The same thing is true in kinematics.

If we go down a path that runs parallel to the one that is trodden in Euclidian kinematics then we will find that our first method – viz., to present a kinematical analogue to projective geometry – does not, however, yield anything new beyond the second one ([1]). We then arrive, in turn, at the geometry on the quadratic manifold M_6^2 and the group of $2 \cdot \infty^{28}$ collinear transformations of it. That cannot be surprising, since precisely the same thing is true for projective geometry itself, which is likewise independent of the hypotheses that one cares to make on the nature of space. However, spherical kinematics does teach us something that can very easily be overlooked.

We consider an ordered pair of points $\overrightarrow{x, x'}$ in spherical space, and the point $\overrightarrow{-x, -x'}$ that is *diametrically* opposite to it, and combine these figures into a *double pair.* We remark that a motion or transfer that associates the point x with the point x' must also associate the point $-x$ with the point $-x'$. We would like to say that the motion or

([1]) The case $\varepsilon^2 = -1$ $\{\varepsilon = i\}$ leads to a kinetic interpretation of projective geometry in the complex domain.

transfer is then *united* with the double pair. However, we can introduce the *ratios* of eight quantities ξ_k, η_k as coordinates of a double pair, which are coupled by the equation:

(22) $$(\xi\,\eta) = \xi_0\,\eta_0 + \xi_1\,\eta_1 + \xi_2\,\eta_2 + \xi_3\,\eta_3 = 0.$$

We need only to set:

(23) $$x_k = \xi_k + \eta_k\,, \quad x'_k = \xi_k - \eta_k\,.$$

Here, we then have our quadratic equation once more, only with a different meaning, and also from this point onward we likewise once more enter the realm of kinematics. Namely, when we likewise use the symbols ξ', η', y, like the symbols ξ, η, x, it will become:

(24) $$(x\,y) - (x'\,y') = 2\,\{(\xi\,\eta') + (\eta\,\xi')\}.$$

On the left here, one finds the difference between the cosines of two spherical distances:

$$\cos(x, y) - \cos(x', y').$$

If this expression vanishes then the point-pair x, y will be congruent – or what amounts to the same thing – symmetric to the point-pair x', y', and likewise the point-pair $-x$, $-y$ will be naturally related in the same way to the point-pair $-x'$, $-y'$. We would like to say that the two double pairs of points are then ***isometric*** to each other. However, the right-hand side of our equation (24) provides a meaning for the fact that the two points (ξ, η) and (ξ', η') on M_6^2 can be coupled by a line that runs completely on M_6^2. It is now obvious that all double pairs that are united by a motion or transfer in spherical space are isometric to two of them. The image of all of these double pairs is then necessarily a *linear* R_3 in M_6^2.

The details of this argument, which I would not like to trouble you with, lead to a sequence of theorems, among which, the following one might be emphasized:

I. *The manifold of* ∞^6 *– e.g., left-handed –* R_3 *in* M_6^2 *can be mapped birationally and without singularities onto the manifold of all points in just that* M_6^2 *(and indeed also in the complex domain).*

II. *Each theorem in projective geometry on* M_6^2 *is associated with five other ones (that are not necessarily different from the first one). Under the transition from one such theorem to the remaining ones of the same group, the concepts of:*

left R_3 , *point,* *right* R_3

will be permuted in all possible ways, and indeed, such that the figures in united position will again go to other such figures, and above all, projective properties will again go to other ones ([1]). *In particular, pencils of lines in* M_6^2 *will again correspond to other ones.*

Naturally, a point and an R_3 are said to be united when the point lies in the R_3 . However, it must still be clarified what one means by the "united position" of a left and a right R_3 . Now, a left and a right R_3 will always have *at least one* point in common. If they have more than one point in common then they will have all points of a *plane* (i.e., an R_2) in common ([2]), and we will then say that they are united.

The content of theorems I, II is thoroughly comparable to the group of facts in projective geometry that are summarized under the name of the *principle of duality*. However, while that principle always presents two theorems as equivalent, here, we have six that belong together. They will be permuted amongst themselves by a *group of* $6 \cdot \infty^{28}$ *single-valued transformations* ([3]) whose theory is easiest to understand when one introduces the ∞^9 lines in M_6^2 as spatial elements.

In order to be able to briefly represent what this implies for kinematics, we will still need to define some further concepts. It can happen that a motion and a transfer in spherical space permute all points whose coordinates satisfy a linear, homogeneous equation:

$$u_0\, x_0 + u_1\, x_1 + u_2\, x_2 + u_3\, x_3 = 0,$$

i.e., all points of a certain spherical surface (viz., the principal sphere), in the same way. We would then like to say that the motion and the transfer are *united*.

One now has the remarkable theorem (viz., the reciprocity theorem):

III. *The real figures:*

left R_3, ***point,*** ***right*** R_3

in M_6^2 *can be simultaneously associated with the (real) figures:*

motion, ***double pair of points,*** ***transfer****,*

resp., of spherical geometry ***without exception*** *in an invertibly single-valued and continuous way such that figures in united position will again correspond to other ones.*

Here, a motion or transfer, when regarded as a "figure," is the totality of all associated point-pairs $\overrightarrow{x, x'}$. The fact that the sequence of the last three structures is arbitrary was already asserted in II. We now perform a passage to the limit and find:

([1]) As implied in G. d. D., pp. 583.

([2]) Göttinger Nachrichten, 1912, pp. 20, remark 2. *One* left and *one* right R_2 will go through any plane in M_6^2.

([3]) In the real domain, there are $12 \cdot \infty^{28}$. See the remark on pp. 13.

IV. *A theorem that otherwise reads the same as Theorem* III *also exists in Euclidian kinematics, except that the concept of (ordered) point-pair appears in place of the concept of double pair of points, and singular places for the map already appear in the real domain.*

These singular places can be eliminated by defining special concepts that are similar to the concept of the "imaginary point."

We would like establish the sequence that was given in Theorem III. For Euclidian space, we then obtain the following association of pairs of *real* points x, x' (of mass one) with certain points in M_6^2 in place of the association (23):

$$\boxed{\begin{gathered}\xi_0 = 1,\ \eta_0 = \tfrac{1}{4}\{x_1'^2 + x_2'^2 + x_3'^2 - x_1'^2 - x_2'^2 - x_3'^2\},\\ \xi_1 = \frac{x_1 + x_1'}{2},\ \xi_2 = \frac{x_2 + x_2'}{2},\ \xi_3 = \frac{x_3 + x_3'}{2},\\ \eta_1 = \frac{x_1 - x_1'}{2},\ \eta_2 = \frac{x_2 - x_2'}{2},\ \eta_3 = \frac{x_3 - x_3'}{2}.\end{gathered}} \tag{25}$$

Isometric point-pairs now also again correspond to points on M_6^2 that can be connected by a line that lies in M_6^2. Isometric pairs of curves or surfaces will, however, correspond to the same kinds of curves and surface on M_6^2, as we considered in the context of the Ribacour problem, and indeed in *both* cases (23) and (25). We would like to consider only surface pairs, and among them, only ones in the first-given family [1]. One finds that their images in M_6^2 will be permuted amongst each other by not only the $2 \cdot \infty^{28}$ collineations of M_6^2, but also by the remaining $4 \cdot \infty^{28}$ single-valued transformations that we spoke of. We then have the theorem:

V. *The* $6 \cdot \infty^{28}$ *single-valued transformations that belong to the quadratic manifold* M_6^2 *evoke transformations in spherical, as well as Euclidian, spaces that generally take pairs of surfaces that are related by isometries, but not by congruence or symmetry, to other such surfaces. For surface pairs with non-rectilinear images that is indeed without exception. Surface pairs that contact in corresponding pairs of surface elements (viz., facets), and also have corresponding line elements in common there, will again correspond to other such surfaces.*

The subgroup of $2 \cdot \infty^{28}$ transformations of the group that is thus distinguished was given by P. Stäckel ([1]) for the case of Euclidian space. *If follows further that one can derive points, curves, and surfaces in a non-Euclidian space from such things in a Euclidian space, and conversely.* A special transformation of that kind, from which, one then obtains the remaining ones, is given for the case of spherical space:

([1]) Comptes Rendus, t. CXXI, 1895, pp. 396, and Jahresbericht der Deutschen Mathematikervereinigung, Bd. 14, 1905, pp. 507-516.

$$\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 = x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2\}$$

by the formula:

(26) $$\boxed{\xi_k = \frac{x_k}{x_0 + x_0'}, \; \xi_k' = \frac{x_k'}{x_0 + x_0'},} \qquad (k = 1, 2, 3),$$

in which ξ_k, ξ_k' means ordinary rectangular point coordinates, moreover. If we use the symbols x and ξ, along with the symbols y and η then that will yield, in fact:

(27) $$\sum_{k=1}^{3} (\xi_k' - \eta_k')^2 - \sum_{k=1}^{3} (\xi_k - \eta_k)^2 = \frac{1}{x_0 + x_0'} \cdot \frac{1}{x_0 + x_0'} \cdot \left\{ \sum_{k=0}^{3} x_k' y_k' - \sum_{k=0}^{3} x_k y_k \right\}.$$

Naturally, this transformation is endowed with singular places. When one omits them, it will take congruent or symmetric surface pairs to other such surface pairs. Just as the difference between Euclidian and non-Euclidian geometry becomes meaningless under the group of collineations, so does the difference between Euclidian and non-Euclidian kinematics vanish under the analogous groups of $2 \cdot \infty^{28}$ and $6 \cdot \infty^{28}$ transformations.

IX.

Further outlook.

What I have spoken to you of, kind sirs, is certainly only a fragment of a fragment. I have been able to treat only some of the simplest examples, and also only summarily, and have, in fact, been forced to strongly curtail the formal apparatus, whose proper development is not free of complications. At least, what was presented might help one realize that there are methods in kinematics that have great import that are on a par with the classical methods of analytic geometry, and which subsume them, in addition.

Kinematics contains a tremendous wealth of general algebraic and analytic problems, as well as also special forms that seem to be worthy of consideration, and await only the mathematician who would like to possess this wealth. If I were to ask that you follow me down a longer path then I would be able to show you, e.g., the surprising way by which the theory of a much-investigated figure – viz., the eight intersection points of three second-order surfaces – is connected with geometry on the manifold M_6^2, and thus, with kinematics, and how deeper-lying results can be derived from it. I must deny myself that, although you might invite me to find a place for some remarks that open up a broader perspective.

The latter association of motions, point-pairs, and transfers in a non-Euclidian or Euclidian space with left-handed linear spaces, points, and right-handed linear spaces (of highest dimension), resp., in a quadratic manifold:

$$\sum \xi_k \eta_k = 0$$

can be extended to an indeterminate dimension.

The law by which this extension of our program proceeds can be expressed, if only incompletely, in certain dimensions, of which, the first of these are:

$$\begin{array}{cccccc} n = & 1 & 2 & 3 & 4 & 5 \\ & (1,2,1) & (3,4,3) & (6,6,6) & (10,8,10) & (15,10,15). \end{array}$$

The third case is the one that I had the honor of entertaining you with. We now see that this case precisely, and thus kinematics in a three-dimensional space, assumes a *special place* in the entire sequence.

The space of three dimensions is characterized by the kinematical reciprocity theorem that was proved (Theorems III, IV).

We remark in passing that our space has such peculiarities, along with others. Thus, the known reciprocity between dynames and infinitesimal motions is also restricted to three-dimensional (Euclidian or non-Euclidian) space ([1]).

The case n = 2, whose theory is naturally contained in the case n = 3, leads to an M_4^2 in a five-dimensional space and thus, to projective geometry in ordinary space. I have had the pleasure of knowing that some of the younger mathematicians have followed me down this path. The late Jos. Grünwald and W. Blaschke have studied kinematics in the Euclidian plane from this viewpoint ([2]). Motions, point-pairs, and transfers correspond to points, straight lines, and planes, resp., and except for singular points, the converse is also true. In the case of spherical geometry ([3]), it is even true in general. Naturally, one can, however, also put somas in place of motions, here. The lines in space will then be associated with *chains of rotations*. Recently, an American mathematician – viz., E. Kasner ([4]) – come upon these ideas with no knowledge of the work of the aforementioned. He called a chain of rotations a *turbine*, and his theorem that these turbines (after a suitable extension of the manifold that they define!) can be permuted by a group that is isomorphic to the group of collineations in space is included as a special case of the one that you heard of today.

The next assumptions (n = 4, 5) also have a certain, if also somewhat deviated, relationship to the geometry of our space, and indeed to *projective* geometry and to Lie's *sphere geometry*.

The structures that are analogous to the point-pair are, in the first case, straight lines (number of constants = 8), and in the second case, linear complexes (number of constants = 10). A suitable analytic tool for arbitrary values of the number n also exists already. It

([1]) G. d. D, pp. 119.

([2]) J. Grünwald, Sitzungsber. der Wiener Akademie, **120**, IIa (1911), 677-741. W. Blaschke, Zeitschr. f. Mathematik u. Physik, **60** (1911), 61-91. Cf., also, Study, *Vorlesungen über Geometrie, I,* Leipzig, 1911, pp. 120, 121.

([3]) Jahresber. d. D. Mathematiker-Vereinigung, **11** (1902), pp. 320, 321, and American Journal of Mathematics, **19** (1906), 116-159.

([4]) Am. J. Math. **33** (1910?), 193-202. Cf., the figure in G. d. D. on pp. 588.

is included in the profound investigations of R. Lipschitz on the *sums of squares* ([1]). The Lipschitz process consists in calculating with certain complex quantities. They reduce to the quaternions and bi-quaternions in the cases $n = 3, 4$. An (*exhaustive*) parameter representation of our group of $6 \cdot \infty^{28}$ transformations can also be achieved in this way.

Let me conclude, kind sirs, with a methodological remark. I have spoken to you almost exclusively of *real* figures, corresponding to the current limits of science. However, that is true of kinematics, although it cannot be said of geometry, at all. The introduction of *imaginary* figures is *vital* when one would like to arrive at an understanding of certain much more encompassing laws and to a simple form of expression for other ones. In my opinion, an *analytical* method must already stand at the center of theoretical kinematics upon the basis of this. However, the methods of analytic geometry also have far greater significance, so it appeals to briefer chains of logic and the possibility of verification, whose absence from synthetic geometry was the source of many errors.

Analysis currently commands powerful tools, and one notices a gratifying solemnity in the work of its exponents, thanks to the influence of Weierstrass. Not long ago, the same thing was true of geometry, *and often not even where* an analytical method was preferred. For kinematics, a not-insubstantial danger lay in this, and such apprehensions will be reinforced when one sees how far the criticism sometimes falls short of its task in this domain.

To err must always be allowed, especially considering the brittleness of a subject that places high demands upon the imagination of geometers and also requires the construction of special methods, so occasional oversights can certainly be excused. However, let us be careful that the humanly-inescapable exception will not become the rule, and that kinematics will remain altogether exempt from dilettantism, which so often, on the contrary, makes the study of geometric papers such a pleasure.

Res severa verum gaudium.

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([1]) Bonn, 1886. See also Cartan, *Encyclopédie des Sciences Mathématiques* (1890), v. I, pp. 463-465.