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A new branch of geometry.

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The original concept of straight line in Euclidian space is the (in the modern terminology) *proper real* line, which is a locus of points that is unbounded in two directions and is representable by linear equations with *finite real* Cartesian coordinates. This concept is first proclaimed and then subjected to a double extension. By the creation of the concept of infinitely-distant points, the line will be converted into a closed, circular continuum that returns to itself. One is then dealing with an entirely new kind of so-called straight line by way of a special definition of the concept, namely, the improper or infinitely-distant lines. In that way, one would arrive at the fact that the totality of all "lines" in space can also be regarded as a *closed continuum* (with G. Cantor's definition). As is known, this totality is representable by the value system of six "homogeneous" quantities – viz., Plücker line coordinates $\mathfrak{X}_{ik}$ – that are linked by a quadratic equation:

$$\mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31} + \mathfrak{X}_{03}\,\mathfrak{X}_{12} = 0. \tag{1}$$

"Continuous" changes of position for a line will be defined (or should be defined; as a rule, one tacitly dismisses that fact) by means of *continuous* changes of their coordinates, under which merely proportional changes are considered to be trivial, and unbounded approaches to the meaningless system of values $\mathfrak{X}_{ik} = 0$, which remains excluded, might still occur. This explanation for the continuity agrees with the elementary explanation that originates in the use of Cartesian coordinates in the realm of proper (real) lines. However, it extends beyond this domain and implies that *any arbitrary* infinite set of "lines" will have well-defined accumulation points.

Finally, one can extend this definition of a closed, four-fold-extended line continuum by also allowing *complex* quantities as line coordinates, which adds a new conceptual structure to the previous ones (which we shall henceforth refer to as *real* straight lines), namely, *the imaginary line.* As is already true for the improper real line, moreover, it is no longer accessible to geometric intuition. The totality of all real and imaginary lines defines a closed continuum of twice the dimension of the former one.

We visualize the motivation for these conceptual constructs: With their help – but also in *only one* way – it is known that one can arrive at the statement of extended classes of geometric theorems in words of a simple sort. With good reason, one must develop a special interpretation of those properties of geometric figures that cannot be affected by a continuous group of ∞^{15} transformations, namely, the group of so-called collineations. Many intrinsically-distinct concepts grow out of that inescapable requirement, such as, e.g., the amalgamation of the concepts of "bundle of parallels" and "bundle of straight line through a point" into a common concept ([1]).

We do not particularly care to elaborate upon the algebraic-geometric theorem upon which the efficacy of the aforementioned conceptual structures rests. However, a specific formulation does seem useful to us. As far as the line continuum is concerned, it reads:

In the multiply (eight-fold, resp.) extended "Plückerian" line continuum, the real (real and imaginary, resp.) collineations (and likewise the so-called correlations) can be characterized as everywhere well-defined, single-valued, and continuous transformations.

This statement is remarkable due to the fact that it is characteristic of the present kind of extension of the original non-closed line manifold, and thus defines the Plücker line continuum.

It now emerges from the cited facts that *the modern concept of straight line (like that of point, line, etc) is indeed preferable within a certain circle of interest (and even practically indispensible), but it is in no way universally preferable, or even necessary.*

The truth of this statement can actually be self-evident. As a result, the current state of literature, which already superficially recognizes the general, customary use of the words "line geometry" (in place of "projective line geometry"), provokes one to expressly set down such thoughts. If we ignore some of the latest starting points that relate to non-Euclidian space, then there will exist in fact, as it seems, no other kind of algebraic "line geometry" than that of Plücker. The available Ansätze that are capable of development have not brought us to a clear understanding ([2]); indeed, no one also seems to have considered the possibility that one can grasp the basic concept of a straight line in a different way from the way that our forebears found appropriate.

Systems of mathematical concepts are creatures of our own creation. Within the barriers that are defined by the laws of thought, we can let them come into existence and die away with impunity. However, since we are a critical race – although not in *every* respect, moreover – we willingly divest ourselves of a large part of our creativity, and we must do that, must demand a *motivation* for new concepts that is not easy to take, when we do not do violence to the facts, when we understand each other, and would not like to capriciously open any arbitrary door and gate. Moreover, reasonable caution can also go too far. We also seek to get by with existing things when demands of a new sort are present and new forms might come to light, and we will even perhaps allow acts of God. Thus, the projective-geometric constructions have been known to establish a kind of

([1]) The bases for the introduction of "imaginary straight lines" are not peculiar to line geometry, and need only to be recapitulated here.

([2]) This is also true of an investigation of Johannes Petersen ("Nouveau principe pour études de géométrie des droites," Sitzber. der Kopenhagener Akademie, 1898, no. 6), that was recently made known to this author by way of a gracious mailing by its author. The content of this paper has several points of contact with the thoughts that are developed later in this paper. However, the basic concept of a "ray" is not found in it, and we cannot agree with a considerable part of Petersen's discussion, or only with some restrictions. The method that he employed is not by any means adequate for a proper treatment of the topic.

monopoly in algebraic geometry, and a strongly passive resistance to the concepts that one encounters in projective geometry itself that have their origins somewhere else, like the ones that were introduced by S. Lie.

Previous experiences do not therefore exactly encourage one to attempt to place them alongside the other traditional conceptual structures. However, the fact that projective geometry, especially the so-called synthetic kind, which is the geometry of Poncelet and Steiner, is ultimately only *one* approach to something *that has many facets*, and due to the fact that it cannot therefore be desirable to personally maintain only this specialized viewpoint when almost always *only* that approach is suggested, it might also be accepted – at least, in principle. One should not abandon hope that a less dogmatic way of looking at geometric concepts and a freer and more multifaceted treatment of geometric matters will ultimately arrive at a better foundation ([1]). The following discussion might perhaps contribute something towards bringing us closer to such a goal.

I.

We will (for the sake of brevity in presentation) begin with the definition of a new line continuum, and then define a group of transformations that has a relationship to this continuum that is similar to that of the group of collineations to the Plückerian line continuum. Since the use of one and the same word for different things results in many inconveniences, we will henceforth not speak of "straight lines," but of *rays*, and thus separate the meanings of the words "line" and "ray" that has been previously used by geometers as if they were synonymous. We thus begin with the concept of *proper, real ray*, which means precisely the same thing as the concept of "proper, real line," but shall be extended in another way. We connect the extension process itself with common notions, when possible.

We first remark that there are ∞^4 rays in real ray space, as well as ∞^4 pencils of parallels, and that the latter, and all geometric loci that they describe, are associated pair-

([1]) The fact that the conceptual structures of geometry have not kept pace with the tremendous production of details in that domain is a remark that must indeed be clear to any unbiased observer. To the causes of this phenomenon might belong the frequent customary non-consideration of the mental content of the closely-related algebraic and function-theoretic disciplines and the deep-seated practice of many geometers of formulating theorems that should be true "in general," i.e., they do *not at all* have a clear sense of what is also frequently presented as valid *in general* or will be deficient in its foundations. By this process, for which nevertheless any reference to bearers of very celebrated names of later lineage would certainly seem to be completely impermissible, but which, in our "critical" age, seems to be considered by many as a *justifiable peculiarity of geometry*, one naturally does not notice that extended groups of facts can be *correctly* represented only with the help of conventional concepts of a convoluted nature. Thus, scientific progress will be likewise be halted with the finality of a brick wall. We fondly quite a saying of the geometer v. Staudt, who referred to the need to remove exceptions to the rules by defining new concepts, and to the benefit that science would derive from such concepts: Is it not an assumption for the ultilization of this thought that one must concern oneself with the exceptions and the complexities that are frequently linked with them – at least, to the extent that is necessary – in order to clearly exhibit them? For example, R. Sturm, with some justification, lamented that geometry does put us in a position that would serve to address them. However, should the geometer not have to share some responsibility for this state of affairs? Is a thorough lack of precision not a much worse evil than the usual so-called errors that can mostly be easily recognized and corrected?

wise as *reciprocal figures*. The rays of "reciprocal" pencils of parallels intersect each other orthogonally. We now let a ray in the first of two such pencils go to infinity, and imagine the – on first glance, outlandish – notion that under passage to the limit the moving ray collapses to a "point," so, in fact, it goes to the infinitely-distant vertex ([1]) of the reciprocal pencil. In this way, we add ∞^4 improper rays to the ∞^4 real, proper rays, or, as we would like to say, *point rays*, which can be considered to be completely identical with the usual so-called points of the "infinitely-distant plane." The fact that such a notion has any meaning in the conventional way of presenting heterogeneous concepts rests upon a theorem that will be *fundamental* for all of what follows:

The totality of ∞^4 proper and ∞^2 point rays can be regarded as a closed continuum, and in this continuum the passage to the limit that was described, inter alia, is a continuous operation.

Namely, the two stated geometric structures can be represented together by a different sort of system of values of one and the same system of six variables $\mathfrak{X}_k$, $\mathfrak{X}_{kk}$, which are linked by the equation:

$$\mathfrak{X}_1\,\mathfrak{X}_{11} + \mathfrak{X}_2\,\mathfrak{X}_{22} + \mathfrak{X}_3\,\mathfrak{X}_{33} = 0. \tag{2}$$

These quantities, like the Plücker line coordinates, are *homogeneous*, but in another sense of the word: a substitution of the form:

$$\mathfrak{X}'_k = \rho\,\mathfrak{X}_k, \qquad \mathfrak{X}'_{kk} = \rho^2\,\mathfrak{X}_{kk}, \qquad (\rho \neq 0) \tag{3}$$

does not change the ray that is represented by *these* coordinates.

The ray that is thus described is *proper* when $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$ do not vanish simultaneously. If we then understand $\frac{x_1}{x_0}$, $\frac{x_2}{x_0}$, $\frac{x_3}{x_0}$ and $\frac{y_1}{y_0}$, $\frac{y_2}{y_0}$, $\frac{y_3}{y_0}$ to mean *rectangular* Cartesian point coordinates then the ray coordinates can be expressed with the help of the associated Plückerian line coordinates:

$$\mathfrak{X}_{01} = x_0\,y_1 - x_1\,y_0, \qquad \mathfrak{X}_{23} = x_2\,y_3 - x_3\,y_2 \qquad \text{(etc.)}$$

as follows:

$$\mathfrak{X}_1 = \mathfrak{X}_{01}, \qquad \mathfrak{X}_{11} = \begin{vmatrix} \mathfrak{X}_{02} & \mathfrak{X}_{31} \\ \mathfrak{X}_{03} & \mathfrak{X}_{12} \end{vmatrix}, \qquad \text{etc.} \tag{4}$$

Conversely, one has:

$$\mathfrak{X}_{01} = \mathfrak{X}_1, \qquad \mathfrak{X}_{23} = -\,(\mathfrak{X}_1^2 + \mathfrak{X}_2^2 + \mathfrak{X}_3^2)^{-1} \cdot \begin{vmatrix} \mathfrak{X}_2 & \mathfrak{X}_{22} \\ \mathfrak{X}_3 & \mathfrak{X}_{33} \end{vmatrix}, \text{ etc.} \tag{5}$$

([1]) The fact that we completely adopt the concept of "infinitely-distant point" here serves to ease the presentation, since this concept is quite familiar. Naturally, it is not necessary.

The ray being represented is a *point ray* when one has $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3 = 0$. The coordinates $x_0 : x_1 : x_2 : x_3$ of the associated point of the infinitely-distant plane are then the ratios:

(6) $$0 : \mathfrak{X}_{11} : \mathfrak{X}_{22} : \mathfrak{X}_{33} .$$

The system of values $\mathfrak{X}_k = 0$, $\mathfrak{X}_{kk} = 0$ remains excluded.

Continuous changes of position of a ray are defined by continuous changes of coordinates $\mathfrak{X}_k$, $\mathfrak{X}_{kk}$ (not, however, of the Plückerian coordinates $\mathfrak{X}_{ik}$). Therefore, unbounded approaches to the system of values $\mathfrak{X}_k = 0$, $\mathfrak{X}_{kk} = 0$ remain excluded, so coordinate changes that are *proportional* in the sense of equation (3) prove to be meaningless.

A closed continuum of rays will be defined by these determinations. *The new concept of continuity is identical with the elementary one in the realm of proper, real rays, and therefore also with the Plückerian one. However, when considered as a whole the ray continuum thus-defined is completely different from the former Plückerian line continuum.*

Whereas, e.g., the normal net of a proper line – viz., the totality of all proper lines that intersect it perpendicularly – can be extended to a closed continuum by means of ∞^1 improper lines, the *normal net of a proper ray*, which is defined by *rays*, contains only *one* point ray, when extended correspondingly.

Much deeper-lying differences emerge when one also allows complex values for the ray coordinates, and thus defines another concept of *imaginary ray*, and an associated continuum. One then sees that the *proper*, imaginary rays are not rigorously identical to the proper, imaginary lines. One considers the special ruled surface that is known by the name of *cylindroid*. If one regards it as a locus of lines, and extends it by analytic continuation into the imaginary domain then one will obtain, *inter alia*, two of the imaginary generators that belong to the infinitely-distant plane. If, by contrast, one regards the same surface as a locus of rays then the continuation into the imaginary domain will also lead to only *proper* rays. Whereas in Plückerian line geometry the cylindroid decomposes into three separate (partially imaginary) surfaces when one lets its real lines go to a plane pencil, moreover, such a decomposition does not happen for the cylindroid that is considered to be the locus of real and imaginary *rays*. The possibility then exists in ray geometry – *but by no means also in line geometry* – of merging the concepts of "cylindroid" and "plane pencil of rays" (with proper vertex) into a single concept. We call this new concept, which will be employed later on, a *ray chain*.

As a further example that will be important in what follows, we mention the figure of all ∞^3 proper, real rays that are parallel to a plane. After we have extended this structure to a closed continuum that exists in the continuum of real or real and imaginary rays, we will call it a *planar ray complex*. One of them will generally be represented by an equation of the form:

$$\mathfrak{A}_1\, \mathfrak{X}_1 + \mathfrak{A}_2\, \mathfrak{X}_2 + \mathfrak{A}_3\, \mathfrak{X}_3 = 0.$$

If we extend this figure, which is considered to be a locus of proper lines, to a closed continuum that exists in the Plückerian line continuum then what will come about is a so-called special linear line complex or a line spray.

These two analytic structures prove to be endowed with very different properties. One would unsuccessfully attempt to map the one to the other in a birational and everywhere-continuous manner.

Indeed, *all* loci of rays likewise take on new properties. Two different real pencils of rays with proper vertices, e.g., will have, when duly extended, not just the connecting line between their two centers in common with each other, but two imaginary lines, in addition, and similar phenomena exhibit themselves at each step.

Just as the concept of Plückerian line continuum thus serves now to confer an unconditional validity on certain fundamental geometric theorems that underlie the elementary concept of exceptions, and thus to simplify further developments of a certain kind, similarly, the concept of ray continuum confers geometric truths to the same goal in relation to other and no less extended classes.

We clarify this immediately by two very elementary examples, which are, however, important for mainly that reason.

If one thinks of concepts like "parallelism" and "rectangular intersection" of proper, real rays as represented by algebraic relationships between ray coordinates then one can, conversely, employ the formulas obtained (which we will not go into here) as the *definition* of the likewise-named concepts in the case where one deals with imaginary rays. Following through on these thoughts leads to the knowledge that the following two theorems exist in line geometry, whose correlates in Plückerian line geometry are *not* true:

Any two real or imaginary non-parallel proper rays have a completely determined common normal.

Any real or imaginary pencil of parallels is reciprocal to a completely determined pencil of parallels.

With these preparations, we now go on to discussing a group of transformations that is linked with our ray continuum. We might perhaps interpret the three quantities $\mathfrak{X}_{kk}$ and the six products $\mathfrak{X}_i\,\mathfrak{X}_k$ together as homogeneous point coordinates in a "space" (i.e., a projective point continuum) of eight dimensions. What then arises are all points of an algebraic point manifold of order *six* that is mapped to the ray continuum on a single-valued, invertible, and everywhere-continuous way. This manifold M_4 admits a projective, continuous group of 17 parameters. Its transformations correspond to the transformations of a certain group G_{17} whose space element is the *ray* of the continuum under discussion, or it can be.

We call these latter transformations *radial-collinear* or *radial collineations.*

The radial collineations are everywhere well-defined, single-valued, and continuous in the ray continuum under discussion.

They have (when considered to be real continuous or also analytic transformations of only proper rays) the characteristic property that from the normal net to a proper ray – hence, from the ∞^2 rays that intersect one of them rectangularly – a figure of the same kind can always emerge.

All of these transformations will be represented analytically by equations of the following form:

$$\mathfrak{X}'_\kappa = a_{\kappa 1}\,\mathfrak{X}_1 + a_{\kappa 2}\,\mathfrak{X}_2 + a_{\kappa 3}\,\mathfrak{X}_3, \qquad (\kappa = 1, 2, 3)$$

(7)
$$\mathfrak{X}'_{11} = k \cdot \begin{vmatrix} \mathfrak{X}_{11} & \mathfrak{X}_{22} & \mathfrak{X}_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \sum_{i=1}^{3}\sum_{j=1}^{3} \begin{vmatrix} a_{2i} & a_{3i} \\ \alpha_{2j} & \alpha_{3j} \end{vmatrix} \mathfrak{X}_i \mathfrak{X}_j ,$$

etc. Thus, one assumes that $k \neq 0$, $| a_{11}\ a_{22}\ a_{33} | \neq 0$, and the matrix $\| \alpha_{ik} \|$ is determined from the matrix $\| a_{ik} \|$ only up to an arbitrary multiple. Proportional changes to the quantities a_{ik} , α_{ik} will not change the transformation being represented. There thus remain 9 + 9 + 1 – 1 – 1 = 17 essential parameters in formula (7).

We thus have the definition of a particular geometric discipline before us that the author refers to as *radial-projective geometry*: *This branch of geometry treats those properties of figures in the ray continuum under discussion that cannot be affected by arbitrary radial collineations.*

The cited definition is capable of some alteration, which we cannot go into completely, due to their principal meaning. Namely, instead of the indicated ray continuum, one can start from another one in which ∞^4 proper rays are extended, not by ∞^2, but by ∞^3 specially-defined "improper" rays. Similar to what was done before, one can define a group of ∞^{17} single-valued transformations, and they will have the same right to be called "radial collineations" as the ones that were defined before: As long as only *proper* rays come into question, the two groups will be completely identical. There is then yet another kind of "radial-projective geometry." We express the essential idea as:

There are two natural ray continua relative to radial collineations.

Either of the two is a subset of the one Plücker line continuum.

In what follows, only the first of these continua will be spoken of.

The fact that further ray continua that are "natural" in a similar sense do *not* exist, and thus that the two aforementioned ones are collectively *characterized* by the group G_{17} of permutations of proper rays, can be proved on the basis of more rigorous definitions and conclusions.

We make the following further remarks: In addition to the rays themselves, there is a second kind of figure in the ray space that depends upon four (real or complex) constants that are permuted under radial collineations by a group that is holohedrally-isomorphic to it, namely, the *pencil of parallels.* Just like the rays, they can be considered to the *basic structure of geometry* and can be chosen to be the space element to some advantage. It can also be represented by *coordinates* in a simple way: For that, one uses seven quantities Ω, Ξ_i, Φ_k that are coupled by the equation:

$$\Xi_1\,\Phi_1 + \Xi_1\,\Phi_1 + \Xi_1\,\Phi_1 = 0, \tag{8}$$

and are *homogeneous* in the sense that substitutions of the form:

$$\Omega' = \sigma\cdot\tau\cdot\Omega, \qquad \Xi_i' = \sigma\cdot\Xi_i\,, \quad \Phi_k' = \tau\cdot\Phi_i \qquad (\sigma,\ \tau\neq 0) \tag{9}$$

are meaningless. The quantities Ξ_i and the quantities Φ_k cannot vanish simultaneously. The pencil that is reciprocal to the pencil (Ω; Ξ_i, Φ_k) will have the coordinates:

$$\Omega' = -\,\Omega; \qquad \Xi_i' = \Phi_i\,, \qquad \Phi_k' = \Xi_k\,. \tag{10}$$

The manifold of the ∞^4 ("proper") pencils of parallels can also be extended to a closed continuum by creating "improper pencils of parallels," in which the likewise extended (i.e., by analytic continuation) radial collineations are well-defined, single-valued, and continuous.

There are four continua of pencils of parallels that are natural under the radial collineations.

Just as the group of ∞^{17} radial-collinear permutations of proper rays is, in a certain way, *defined* by any of the two natural continua of rays, so are they also *defined* by any of the four natural continua of pencils of parallels.

II.

We shall now deal with the illustration of the foregoing ideas by applications, so we are up against the difficulty that is always present wherever a concept of an extended and exotic nature has been given, and because of that, we come to another difficulty in the presentation. The real geometric loci, ruled surfaces, congruences, and complexes that are considered in Plückerian line geometry are to be extended to closed continua everywhere in a different way from the usual one, and imaginary figures of a different kind will then appear. The fact that there are analytic structures that are pair-wise identical and can even coincide completely in one's intuition, but still have completely different properties, undoubtedly constitutes a difficulty for the representation, as well as the understanding, of the new discipline ([1]). Everything in ray geometry will then be organized from a different viewpoint than the one that one used for line geometry: Figures are regarded as *equivalent* that can be taken to each other by *radial* collineations. The establishment of a special *terminology* then becomes necessary if one would not like to get by with awkward paraphrases. Finding suitable expressions is, however, not an

([1]) It might be useful to point out a related – and due to its greater simplicity, easier to understand – phenomenon that already appears in older investigations: We recall the various kinds of extension of the elementary point continua in projective geometry and in the geometry of inversion.

easy thing to do today, especially due to the fact that the majority of mathematicians are entirely loath to extend the terminological apparatus with no basis. Finally, it is mostly very awkward to employ common words with a different meaning, such as we did with the word “ray” and the ones that were derived from it. To find a form for the presentation that accounts for the factual and also, at the same time, historical hindsights might be virtually impossible under these circumstances. We beg that a sympathetic critic will overlook these difficulties for now.

We thus begin by putting a previously-cited theorem into a more convenient form.

It follows from the fact that a radial collineation can always emerge from the normal net of a real ray (but not necessarily the axis of the second net from the first one) that these transformations belong together pair-wise, so any two of them are *discordant*, as we would like to say. We thus double the entire manifold of rays, pencils of parallels, etc., cover it with two sheets, and speak of, e.g., rays $\mathfrak{X}$ in the first sheet of the ray space and rays $\mathfrak{A}$ in the second one. We can then say:

If two proper rays $\mathfrak{X}$, $\mathfrak{A}$ *cut each other at a right angle then this property will not be affected by discordant radial collineations, and the radial collineations will be characterized by this property.*

A new family of ∞^{17} transformations arises from permuting the two sheets ($\mathfrak{X}' = \mathfrak{A}$, $\mathfrak{A}' = \mathfrak{X}$) that one can call *radial correlations*.

The construction of the common normal to two non-parallel rays of the same sheet is an *invariant* construction under radial collineations (and correlations) when one calculates the normal to the other sheet. The fact that two rays of both sheets lie over each other is, by contrast, *not* an invariant property.

The similarity transformations of Euclidian space are radial collineations in a trivial way, when one considers real rays to be their objects, and correspondingly continues them into the continuum of real and imaginary *rays*. They are the ones that coincide with them discordantly, the ones that are permutable under “absolute correlations” $\mathfrak{X}' = \mathfrak{A}$, $\mathfrak{A}' = \mathfrak{X}$, or finally the ones that do not affect the overlay of two rays $\mathfrak{X}$, $\mathfrak{A}$.

In order to come to a clear insight into the relationship between Euclidian geometry and the radial-projective geometry that it is subordinate to, we briefly consider the invariant, continuous groups, first, the group G_{17} of radial collineations, and then the group g_7 of similarity transformations. Both systems of groups can already be referred to completely by their parameter numbers according to the following diagram:

$$\begin{tikzcd}[column sep=small] & G_{16} \arrow[dr, dash] & & & & g_6 \arrow[dr, dash] & \\ G_{17} \arrow[ur, dash] \arrow[dr, dash] & & G_8 & & g_7 \arrow[ur, dash] \arrow[dr, dash] & & g_3 \\ & G_9 \arrow[ur, dash] & & & & g_4 \arrow[ur, dash] & \end{tikzcd}$$

Here, g_6 means the group of Euclidian motions. On the basis of things that we cannot go into here, we refer to the corresponding subgroup ($k = 1$) G_{16} of G_{17} as the *group of dual collineations*.

g_4 is the group of perspective similarity transformations that leave all points of the infinitely-distant plane individually unchanged. Analogously, G_9 is the group of all radial collineations that leave all point rays individually unchanged.

g_3 is the three-parameter group of displacements that is defined by the intersection of the groups g_6, g_4 of commuting transformations. Analogously, G_8 is the intersection of the groups G_{16}, G_9, so it is, in turn, a group of commuting transformations. We call its ∞^8 transformations *dual displacements.*

The following theorem now becomes understandable:

The groups g_7, g_6, g_4, g_3, whose transformations have rays for their objects, are contained in the groups G_{17}, G_{16}, G_9, G_8, respectively. They will be defined by all transformations of the latter group that leave a certain imaginary ray congruence fixed, namely, the "absolute congruence."

That congruence consists of ∞^1 pencils of parallels, namely, all (proper) pencils of parallels that are reciprocal to it. The ∞^1 point rays of this pencil define the so-called imaginary absolute conic section of Euclidian geometry.

In parallel pencil coordinates, the absolute congruence will be represented by the equations:

$$(12) \qquad \Omega = 0, \qquad \Xi_1^2 + \Xi_2^2 + \Xi_3^2 = 0, \qquad \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0,$$

and in ray coordinates, by the equations:

$$(13) \qquad \mathfrak{X}_1^2 + \mathfrak{X}_2^2 + \mathfrak{X}_3^2 = 0, \qquad \mathfrak{X}_{11} : \mathfrak{X}_{22} : \mathfrak{X}_{33} = \mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3 .$$

The following theorem is useful:

All transformations of the groups G_9 and G_{17} will be found by composing transformations of the group G_8 and G_{16} with such a one-parameter group of perspective similarity transformations (any one-parameter subgroup of g_4 that does not belong to g_4).

The problem of exhibiting all (real) radial collineations with elementary tools can, on these groups, lead back to the construction of the *dual* collineations. However, this can be accomplished very simply:

Any four rays, no three of which belong to the same planar complex (pp. 5), *can go to any other four rays that have the same property under a single dual collineation.*

In order to find an arbitrary (proper) ray of this association, one needs only to repeatedly apply a single construction, namely, drawing the common normal between two non-parallel rays of the same sheets.

One indeed notes the main interpretation of the situation that the auxiliary construction can also be *nowhere*-indeterminate in the imaginary domain.

We now turn to the consideration of ray loci.

We call an analytic locus of ∞^3 rays a *ray complex*, an analytical locus of ∞^2 rays, a *congruence*, an analytical locus of ∞^3 rays, a *volume* (*Band*) ([1]). Here, we consider only those congruences and ray volumes that do not consist of point-rays entirely. A volume of parallel rays is called a *cylinder*. The simplest non-cylindrical volumes are the *chains* (pp. 5)

Any chain is radially projective to any other one, and in particular, dually projective, as well.

The chains are carriers of binary domains, and they are related to each other *projectively* by the radial projective relationship. All rays of a chain are perpendicular to a certain ray – viz., the *principal axis* of the chain – in whose normal net the chain therefore lies.

A completely determined chain goes through three rays, no two of which are parallel, in the normal net to a proper ray.

One encounters the following remarkable theorem, *inter alia*, for the chains:

If two chains lie in such a way that no ray of one of them is perpendicular to the principal axis of the other then a triple of distinct chains will belong to them.

The congruence of all rays that are perpendicular to the rays of two united chains of a triple is also the one that one might get by choosing two chains from the triple.

The fundamental significance of the concept of chain emerges from a theorem whose basic idea goes back to Hamilton:

If the proper rays of an analytic congruence cannot be distributed on a cylinder then the common normals between a ray of the congruence in general position and its neighboring rays will define a chain.

Naturally, we must clarify precise just what we mean by a ray of the congruence in "general position." Here, where we are dealing with just one orientation, we shall start with that fact.

The last theorem leads to some especially remarkable families of ray congruences when one also brings the closed congruences into the circle of consideration.

If two analytic congruence of rays have such a relationship to each other that inside a regular domain of any ray of the one congruence is intersected perpendicularly by ∞^1 *rays* ([2]) *of the other one, and conversely, then the following cases are possible:*

([1]) A volume of rays that belongs to the complex $\mathfrak{X}_1^2 + \mathfrak{X}_2^2 + \mathfrak{X}_3^2 = 0$ cannot be considered to be a *surface*, i.e., *not* as a locus of ∞^2 points or planes.

([2]) I. e., of an analytic continuum with a (complex) parameter.

Either the one congruence lies in a planar complex and is arbitrary, moreover, and the other one is a bundle of parallels, or both congruences can be described by pencils of parallels and these pencils, and therefore the congruences themselves, are reciprocal to each other, or thirdly, each of the two congruences consists of all common normals to the rays of the other one.

The congruences of the latter family are especially remarkable: They depend upon only one *finite* number. We call then *aplanar chains of congruences.*

One encounters the following theorems with them:

There are ∞^8 *aplanar chain congruences that, like the given ones, are pair-wise reciprocal to each other. A single aplanar chain congruence goes through any four rays, no two of which lie in a planar complex. Any aplanar chain congruence can be taken to any other one by a single dual displacement.*

A special pair of such reciprocal congruences is a doubly-covered pencil of rays with a proper vertex. Thus, all other pairs will arise from this simple figure by discordant, dual displacements. The rays of any pair can be associated with the points and lines of a plane in a single-valued and invertible way in such a way points and lines in united position will correspond to rays that intersect in a mutually orthogonal way. The rays of the one congruence that intersect a ray of the reciprocal congruence perpendicularly *always* define a chain. Any aplanar chain congruence consists *entirely* of proper rays; each of them has a single ray ([1]) in *each* bundle of parallels.

Any analytic transformation of rays that permutes the aplanar chain congruences is radial-projective.

In the next theorems, we will consider the briefer expressions in terms of only *real* figures, but remark that this restriction is not essential.

The rays of any aplanar chain congruence remain individually fixed under all of the transformations of a one-parameter continuous group of G_9 *. Conversely, any such group that does not belong to the subgroup* G_8 *will define an aplanar chain congruence.*

For the rays of an arbitrary bundle of parallels, any of the aforementioned transformations reduce to a perspective similarity transformation of constant expansion ratio that leaves the rays of the bundle in the congruence fixed.

If the aforementioned ratio has the value –1 then the transformation will be involutory; the distance between two mutually associated rays are then bisected by the rays of the congruence that are parallel to them. We call this special transformation a *reflection* in the aplanar chain congruence.

([1]) All of these theorems are also true in the imaginary domain. In this simple form, they are peculiar to radial-projective geometry.

Any involutory transformation of the group G_9 – thus, any involutory radial collineation that leaves all point-rays fixed – is a reflection in an aplanar chain congruence.

Any dual displacement – hence, any transformation of G_8 – can (in ∞^8 ways) be generated by the composition of two reflections in aplanar chain congruences.

We further emphasize that:

If one divides the distances between parallel rays of two arbitrary aplanar congruences by a constant ratio that is different from positive unity then the locus of the rays thus constructed will be a new aplanar congruence.

A geometric calculus involving aplanar chain congruences resides in these theorems that is akin to the Möbius point calculus.

The aplanar chain congruences also have very distinguished metric properties. If one constructs the common normals between any two real generators of the same kind of a hyperboloid then that will produce a subset of the real rays of an aplanar chain congruence: If one ignores the bundles of rays then they are, *in the real domain*, identical with the line congruences that Waelsch considered quite rigorously under the name of transversal congruences ([1]). The cited theorems, which can be augmented even more, present properties of these figures that were apparently entirely unknown up to now.

We call one limiting case of an aplanar chain congruence a *planar chain congruence.* The ray field that belongs to an arbitrary *real* proper plane is one such congruence; any other "planar" chain congruence is dual-projective to this special figure. There are ∞^7 planar chain congruences. Each of them is the locus of ∞^1 pencils of parallels. The pencils that are reciprocal to them trace out a second planar chain congruence that is *reciprocal* to the first one, which coincides with the first one in the case of the ray field.

The real rays of an arbitrary real planar congruence arise from either those of a chain by the displacements that are perpendicular to the its principal axis or, in the most general case, from the tangents to a parabola by perspective similarity transformations whose center lies perepndicularly over the focal point of the parabola.

An abundance of remarkable properties belongs to a genre of ray congruences that belong partly to the second of the three families that were defined, but partly to the first one. We explain:

An analytic ray congruence is called synectic when the common normals between a ray of the congruence in general position and its neighboring rays lie in a single pencil of parallels.

The following theorem, *inter alia*, is true of these congruences:

([1]) The *focal surfaces* of these congruences, with the wealth and beauty of their properties (among which one finds relationships with the theory of elliptic functions), indeed take the background to no known genre of special surfaces of higher order.

Depictions of such a focal surface were presented at the meeting by the student W. Vogt in Greifswald.

Any analytic volume that is defined by proper rays and is not a cylinder lies in a single synectic congruence.

Any synectic congruence will be defined by ∞^1 *pencils of parallels. The pencils that are reciprocal to it will trace out a new congruence. If the given congruence does not lie in a planar complex then the second congruence will also be synectic and "reciprocal" to the first one.*

In the other case, the second congruence will be a bundle of parallels, so it will thus *not* be synectic and will *not*, by definition, be available to the first one.

Among the different kinds of synectic congruences that get mapped to each other analytically, there is an especially interesting one, for which, we shall likewise use the word *synectic*. The "synectic" maps of synectic congruences encompass, firstly, the *dual*-projective ones, secondly, motions, which arise when one associates rays (inside a suitably-regular domain) of both congruences with each other that cut one and the same ray of a third synectic congruence orthogonally. The most general map of two synectic congruences, which we shall call synectic ([1]), arises by composing two kinds of associations, in which each of them is employed at most *once*.

One now has the theorem:

If two synectic congruences are synectically related to a third one then they will be related to each other synectically.

Thus, if the common normals between any sort of analytically-associated rays of the first and second congruence define a new synectic congruence (through which, the association itself is then defined), and if the same thing is true in relation to the second and third congruence then it will also be the case for the first and third congruence, *or* one can bring about such a position relationship by subjecting the third congruence to a dual collineation.

The normal net of a proper ray and the absolute congruence belong to the synectic congruences.

The absolute congruence belongs to a genre of rational ray congruences that are all radially-projective to each other, as well as dually-projective, and all of their irreducible degeneracies (whose components are not synectic without exception) shall be called *conical congruences*. They have dual-projective (metric, resp.) properties that are closely related to those of the conic sections in plane projective (non-Euclidian, resp.) geometry.

One has, e.g., the following theorem:

If two normal nets of proper rays are dual-projectively related to each other in such a way that no ray of the one is parallel to the ray in the other one that corresponds to it then the locus of common normals between associated proper rays of an irreducible conic congruence and all proper rays of this congruence will be found in this way.

The ∞^1 point rays of the congruence are obtained, in turn, by analytic continuation.

([1]) The synectic relationship can be explained more simply analytically.

The principal axes of the generating normal nets belong to the conical congruence, and are two arbitrary proper rays in them that are not parallel: The congruence will be projected from any two of them by a dual-projective normal net that is suitable for the generation of the congruence.

One can then formulate, *inter alia*, a theorem that is completely similar to Pascal's or Brianchon's theorem of projective geometry, and also has an entirely similar significance. One can define a concept of confocal or concyclic conic congruences, and exhibit analogues to Ivory's theorem and the elliptic and Lamé coordinates, etc.

The synectic congruences take on a special interest mainly due to their relationship to Euclidian geometry.

As long as they are not parallel to rays of the absolute congruence, the rays of a synectic congruence are identical with the normals to a family of parallel developable surfaces. Conversely, the normals to any non-planar analytic developable surface lie in a synectic congruence ([1]).

One must indeed observe that the concept of synectic congruence includes rays of a congruence and entire congruences that belong to the complex $\mathfrak{X}_1^2 + \mathfrak{X}_2^2 + \mathfrak{X}_3^2 = 0$ and that therefore a sufficiently-encompassing definition of these congruences *cannot* be based upon the last theorem. As a further probe of such metric relationships, we state the following theorem:

One associates analytic curves that are not generators inside of regular regions of two analytic non-planar developable surfaces according to some arbitrary analytic law. One measures off varying distances r, r′ from the corresponding points on the associated generators, and establishes that:

$$r : r' = \pm\, d\sigma : d\sigma',$$

where $d\sigma$ and $d\sigma'$ mean the corresponding angles between consecutive normals to the surface (along the corresponding curves).

In that way, the two surfaces, and with them, their normal congruences, will be mapped to each other in two ways.

Of the associations between rays of two synectic congruences thus constructed, one of them (but never the other) is always synectic, and the relationship between these congruences is, moreover, a completely arbitrary synectic one.

That is, if one constructs the common normals between associated rays of the congruence *then one will again obtain the normal congruence to a family of developable surfaces* (or a subset of their rays). If that is not the case then it will suffice to perform a

([1]) The tangent surface to a curve with minimal lines as its generators or a cone of that type would *not* be "developable."

dual-collinear transformation (e.g., something that is already a suitable motion) from the congruence in order to remove the exceptional case.

We finally consider some *complexes* of rays.

Any algebraic ray complex can be represented by a single homogeneous equation $\mathfrak{F}(\mathfrak{X}_k, \mathfrak{X}_{kk}) = 0$ (whose left-hand side can be put into a certain form with the help of identity (2), moreover) ([1]), where $\mathfrak{F}$ means an entire rational function.

The simplest ray complexes beyond the planar ones (whose properties we must pass over completely) are the *quadratic* ones, whose equations have the form:

$$\mathfrak{A}_1\,\mathfrak{X}_{11} + \mathfrak{A}_2\,\mathfrak{X}_{22} + \mathfrak{A}_3\,\mathfrak{X}_{33} + \sum \mathfrak{P}_{ik}\,\mathfrak{X}_i\,\mathfrak{X}_k = 0.$$

If:

$$\sum \mathfrak{A}_i\,\mathfrak{A}_k\,\mathfrak{P}_{ik} \neq 0 \qquad (i, k = 1, 2, 3)$$

then we call such a complex *regular*.

The real, proper rays of a real complex of that kind are identical with the real, proper lines in a *special* (Plückerian) quadratic line complex. If $\sum \mathfrak{A}_i\,\mathfrak{A}_k\,\mathfrak{P}_{ik} = 0$, but not $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_3 = 0$, then we call the complex a *chain complex*.

The following statement pertains to quadratic complexes in general:

If no two of five proper rays are parallel, and no four of them belong to a planar ray complex, and no aplanar chain congruence can couple all five rays, moreover, then they will define a sixth ray uniquely.

Any quadratic complex that goes through five of six rays will also contain the last one.

The possibilities that the last ray will go to a point ray or will be undetermined are excluded by the assumptions of this theorem. However, it is not excluded that it might coincide with one of the given ones, and it will then be a double ray of the figure.

The regular quadratic ray complex and its limiting case – viz., the chain complex – have the property that is characteristic of the totality of both figures that they make a normal net in general position pierce a chain. Any regular, quadratic complex is dual-collinear to each of the other ones, and any chain complex is dual-collinear to each of the other ones.

The chain complex consists of all the normals to the rays of a chain.

It can be mapped onto a projective point-continuum R_3 *in such a way that the points of each plane in* R_3 *that do not go through a specified point will correspond to the rays of an aplanar chain congruence that is contained in a complex in a one-to-one way.*

([1]) As is known, a theorem that reads similarly exists in Plückerian line geometry.

The chain complex consists of a group of ∞^{10} *radial collineations, and under the stated map they will go to the ten-parameter projective group of* R_3 *that leaves a line element fixed.*

Normal nets in special position have pairs of pencils of parallels in common with a regular quadratic or chain complex. The axes of this normal net define a *planar* ray complex. If the two pencils of parallels coincide then the locus of the associated axes is a congruence, namely, the *singularity congruence* of the complex.

The regular quadratic ray complex is determined uniquely by its singularity congruence.

The congruence of tangents to all circles that lie on a paraboloid of rotation, when extended by a point-ray, is such a real singularity congruence. If one links consecutive parallel rays of this congruence by a pencil of parallels then the reciprocal pencil will trace out the associated complex.

Any regular, quadratic ray complex is dual-projective to this one, in particular.

The singularity congruences of the regular, quadratic complexes, whose properties we likewise cannot go into in more detail, assume a distinguished place among all possible ray congruences, just like the aplanar chain congruences:

Any analytic ray congruence that includes ∞^2 *ray chains is either identical with one of the* ∞^8 *aplanar chain congruences or with one of the* ∞^8 *singularity congruences of regular, quadratic complexes, or with one of the* ∞^7 *planar chain congruences, or finally with one of the* ∞^4 *normal nets of proper rays.*

The last two families of congruences can be regarded as partial intersections of the first two; the normal nets include ∞^3, and thus, infinitely many, kinds of ∞^2 chains.

The aplanar complexes, the chains complexes, and the regular, quadratic complexes can be characterized by similar theorems.

New viewpoints for the examination of structures that were considered before already, and in part, also structures of an entirely new kind, will come about when we consider the *pencil of parallels* to be the main space element. Any algebraic *complex* of such pencils – i.e., any analytic manifold of ∞^3 pencils of parallels – can, by a suitable definition of the continuum of pencils of parallels, be represented *purely* by a single homogeneous equation $F(\Omega; \Xi, \Phi) = 0$, which can be put into a certain form with the help of the identity (8), moreover. We shall consider only those analytic complexes that are not fixed by all translations. We call them *regular*, and the other ones, *singular.*

Any regular analytic complex of pencils of parallels is invariantly linked to either a ray complex whose proper rays cannot be distributed on a pencil of parallels or with a ray congruence whose rays cannot be distributed on a cylinder.

Conversely, each such analytic ray complex and each such analytic congruence determines a regular complex of pencils of parallels.

We clarify the nature of this closely-related relationship, which is based upon the concept of an envelope, by an *example*. If we call the complexes whose equations can be described in the form:

$$q\,\Omega - \sum q_{ik}\,\Phi_i\,\Xi_k = 0$$

bilinear then we will have the theorem:

Any regular ($q \neq 0$) *bilinear complex of pencils of parallels consists of all pencils that contain any ray of a certain aplanar chain congruence.*

The reciprocal complex (locus of ∞^3 *reciprocal pencils of parallels) likewise belongs to the reciprocal chain congruence.*

Conversely, *any* pair of reciprocal, aplanar, chain congruences defines a pair of reciprocal, regular, bilinear complexes of pencils of parallels.

Finally, the following question, which is of fundamental importance, shall be posed:

Under what circumstances do the ∞^3 *rays of an analytical sequence of* ∞^1 *pencils of parallel define a synectic congruence?*

In order to be able to cast the answer, which reads very simply, into a geometric form, we remark that a line element in the infinitely-distant plane is linked with any (proper) pencil of parallels: The point-ray of the pencil of parallels is the point of the element, and the line of the element has the point-ray of the reciprocal pencil for its absolute pole. The desired condition now consists in the idea that the line element thus constructed must define a *union*, and indeed one *that is not a line*, so it consists of the ∞^1 line elements of a point or an arbitrary *curved* analytic curve.

III.

The use of the concept described can indeed be shown to be sufficient by the various cited examples; we can thus regard our problem as perhaps being solved. However, to acknowledge the meaning that radial-projective geometry takes on as a system of geometry, some facts must, however, be still brought into consideration that lie outside the circle of ideas in which have been moving up to now. We next draw attention to the presence of certain *infinite groups* that subsume the radial or dual collineations. One has the following theorem:

There is an infinite group of analytic ray transformations that take synectic congruences to other ones (in a domain of regular behavior).

This group contains a subgroup that does not affect synectic relationships between synectic congruences, and this group, in turn, leaves invariant a subgroup whose transformations can emerge from other synectic congruences that are synectically related to them.

The second of the stated infinite groups encompasses the radial collineations, while the third one encompasses the dual collineations.

The finite transformations of all three groups can be represented by explicit formulas. The interesting one of them is the last one, which we call the *group of synectic ray transformations*. One has, *inter alia*, the following theorem for it:

If the rays of two analytic congruences cannot be distributed on a cylinder then any arbitrary analytic map of one congruence to the other will determine a single synectic transformation that provokes just that map.

One can not only prove the existence of this transformation, but one can also represent it analytically, and indeed very simply, when the given congruences themselves are represented in a suitable way.

As we go on to further developments that are connected with the ones here – *inter alia*, an *extension of the theory of the conformal map to ray geometry* – we turn to the consideration of certain *relationships between radial-projective geometry and other geometric disciplines.* The study of the manifold connections between the different branches of mathematical science defines one of the most appealing topics in mathematical research, and should not be neglected, as happens with most geometers, unfortunately. Naturally, only some summary facts can be dealt with here.

We have already spoken of the (highly developed) relationships between our ray geometry and Plücker's line geometry. However, there exists yet a second connection between radial-projective geometry and Plücker's circle of ideas: This connection comes about when one considers the space element to be, not the straight line, but a structure that depends upon five constants, namely, the line complex or *thread.*

If one subjects the manifold (that is mappable onto the projective point continuum of an R_5) of threads (not collinearly in the ordinary sense, but) *to those linear transformations of the six homogeneous thread coordinates that take coaxial threads to other ones then the axes of this thread will be permuted with each other in a radial-collinear way, and indeed, in the most general way.*

This relationship also has an intricate character. In order to represent it clearly, one must conceptualize the notion of the "axis of a thread" in the imaginary domain in a different way than is usual.

Furthermore, a certain analogy emerges in our consideration between the composition of the group G_{17} and the composition of the group g_7 of similarity transformations (cf. no. II). Here, much more than a superficial agreement is present. Admittedly, we cannot adequately deal with the nature of things without developing an associated formal apparatus. However, a brief outline might also be of some interest:

A relationship exists between Euclidian geometry in space and radial-projective geometry that is similar to the one between projective geometry on the line and the projective geometry on the plane.

(Moreover, in both cases, one deals, not only with the advance from the first term to the second one in an infinite series of geometric disciplines, but also with a deeper-lying connection between parallel series.)

The following structures, *inter alia*, can be compared:

The groups G_{17}, G_{16}, G_9, G_8	The group g_7, g_6, g_4, g_3
The aplanar chain congruence *and* the regular, bilinear complex of pencils of parallels	The proper point in Euclidian space
The proper ray *and* the pencil of parallels.	The (imaginary, proper) plane, which contacts the absolute conic section.

Further analogies between radial-geometry and *plane* projective geometry must be mentioned in our exposé. These formal agreements have their basis in the fact that one can extend plane projective geometry to a geometry in a *four-fold*-extended manifold. Indeed, this can happen in several ways. If we consider groups of transformations of a fourfold-extended analytic manifold to be equivalent when they are similar to each other, in Lie's terminology, then the following theorem can be formulated:

The group of real, dual collineations, with the rays of the first and second sheet as space elements, is a limiting case of the real and imaginary collineations of a plane, with the (real or) imaginary point and the imaginary lines as space elements.

We again give *some* of figures that we can set down in parallel to each other:

A pair of points (lines) in both planes	Ray in the first (second) sheet
Pair of projective point sequences (or pencils of lines)	Ray chain
Collineations between two planes	Pair of reciprocal aplanar chain congruences
Correlation between two planes	Quadratic ray complex
Point of one plane and line in the other	Pencil of parallels
Pair of analytic curves.	Pair of reciprocal synectic congruences.

The study of such passages to the limit can obviously have a very elevated heuristic value.

We dare to assert that the method of research that is thus indicated belongs to the most fruitful ones known to geometry.

Therefore, the author is indebted to, *inter alia*, the investigations of C. Segre, who made meaningful progress in bi-ternary projective geometry that was of great utility. The details of such relationships are, however, not always easy to understand; they are still not widely exploited ([1]). Great caution is required in its application. One must also beware of mixing the peculiarities of the kinds of geometry that are defined in the limiting cases with the attempts that are doomed from the outset to understand *all* of their theorems as limiting cases of so-called general ones.

Systems of complex quantities can be employed in the study of many – but by no means all – of the relationships that were mentioned, and indeed not only the common complex quantities, but also other ones, namely, the author's *dual* (hypercomplex) *quantities* that are constructed from two units whose rules of multiplication are given by the formulas:

$$\varepsilon^2 = +1, \qquad \varepsilon^2 = 0, \qquad \varepsilon^2 = -1.$$

However, these dual quantities have other properties than the common complex quantities, and they can never be overlooked. For example, if one finds that the irreducible conic congruences (at least, as far as their *proper* rays are concerned) are representable in terms of equations of the same form with the help of those complex quantities and the use of special coordinates that are suitable to them then one must beware of extending such a theorem and its corollaries beyond its true domain of validity. It is completely incorrect that *any* theorem of plane projective geometry can be carried over to radial-projective geometry in the manner that we have presented in several examples. For example, there is a larger manifold of conic congruences that are different under radially-projectivities than the manifold of projectively-distinct conic sections; a conic congruence can decompose into *three* or still more irreducible components, etc. ([2]).

In conclusion, we shall establish some further applications of some of the cited ideas.

The *concept of a natural continuum* ([3]), which was described in our sketch only by an example and also, as we know very well, only in a deficient way, also has meaning for other, and indeed important branches of geometry, to which elementary and non-Euclidian geometry belong. Of no less modest significance is the *employment of general kinds of homogeneous coordinates*, which have had only a very restricted use up to now.

Other geometric disciplines that are defined by finite groups of birational transformations that are likewise of interest can also be treated in a similar way to radial-projective geometry. Some of them have a relationship to radial-projective geometry itself that is similar to Plückerian line geometry. Among these disciplines whose foundation would be a worthwhile problem of future research, just like the construction

([1]) One finds further discourse relevant to this in the author's paper: "Über Nicht-Euklidische und Liniengeometrie," Festschrift of the Greifswald philosophical faculty, 1900. That article shall be published in the next issue of the Jahresberichte. Cf., on this, also Joh. Petersen, "Géométrie des droites dans l'espace non euclidien," Kopenhagener Akademieberichte, 1900, pp. 306, *et seq.*

([2]) We see that these remarks allow improper applications that have already been made in complex quantities.

([3]) On this subject, cf. the author's paper: "Die Elemente zweiter Ordnung in der ebenen projektive Geometrie," Leipz. Ber. (1901), pp. 338, *et seq.*

of radial-projective geometry, we mention the extension of the circle geometry of Möbius and Lie.

The majority of the theorems that were communicated without proof here will established soon in the second edition of the author's *Geometrie der Dynamen*.

Hamburg, 23 September 1901.