Excerpted from: E. Study, *Geometrie der Dynamen*, B. G. Teubner, Leipzig, 1903.

Appendix.

General outline of a new method in kinematics.

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Now that we have reached the end of the path that we set out upon, we may perhaps allow ourselves a vision of a world of geometric forms that, up to now, we have only touched upon peripherally, and if not all of the evidence seems to deceive us, promises to be one of the most rewarding realms of geometrical research. A summary, whose significance we can only claim to sketch out, shall likewise serve to once more to focus one's attention to a greater degree on the main aspects of the matters that were treated up to now, and, through further applications, to shed light on the import of the theory that was developed.

A main concern of kinematics, which must be separated completely from the study of machines today, is defined by the study of (analytical) manifolds of ∞^r ($r = 1, \ldots, 6$) positions of a rigid body in space – i.e., of relative positions of this body relative to another one that one calls immobile, or, as one cares to say, imaginary. On the whole, this study is still only slightly developed, especially in the cases in which the *degree of mobility* of the body is greater than the indicated dimension r by two [*]). In many respects, concerning the problems that come from examples that are borrowed from technology and tentative investigations, one has yet to emerge from the problem of bringing some order to the details that feature in them [**]). However, one can, in this way, except for other theories that do not bear the cloak of kinematics, enter into a particularly important part of the differential geometry of space curves and curved surfaces, as Darboux has accomplished in his great work on this situation. In the following, once we have examined the infinitesimal mobility of a rigid body, as in the previous paragraphs, we shall now provide a further contribution to this part of theoretical mechanics that is also referred to as the "geometry of motion," and indeed to the study of the mobility of a rigid body in the finite, as well.

The soma and its coordinates.

We think of our rigid body as always being represented by a rectangular system of axes, namely, by three mutually perpendicular lines or rays that are associated with the three indices 1, 2, 3, and which can be coupled with a system of rectangular Cartesian coordinates. As a result of its motion, this figure can be brought into ∞^6 different positions. We call each of these positions a *soma*, and an arbitrary, but unambiguously chosen, one of them, a *protosoma*. Every other soma emerges from the protosoma by a *certain* motion, and it will be represented by the formulas that were developed in § 21 and § 25 when the coordinates with no accent relate to the protosoma and the ones with

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[*]) If $r > 2$ then it is known that one can, in addition to the demand that a rigid body can assume a prescribed manifold of positions, come to another one that is expressed by a non-integrable system of Pfaffian equations. The mobility in the infinitesimal that is called the "degree of freedom" will then also be less than r at a place that is in "general position." In the text, we will always address only the much simpler case in which these two numbers that one must distinguish by the terminology are equal to each other.

[**]) One confers, perhaps, the recently appearing article IV.3 of the Mathematischen Encyclopädie.

accents to the other somas, and the same figures that are rigidly coupled with the protosoma (points, lines or rays, planes, and loci of them, moreover) will be represented by means of all of these coordinates. *Thus, the parameters* (α, β) *of this motion can be regarded as coordinates of the soma.* We now reiterate the previously-used remark that the aforementioned formulas do not change in form at all, or only in an inessential way, when one increases the parameter β_i by the same multiple as the corresponding parameter α_i – insofar as one employs *not* Plückerian line coordinates, but our *ray coordinates* for the representation of the rays (pp. 174 et seq., pp. 220, no. 9). We can thus recapitulate the parameters of the dual couplings (pp. 220, no. 10). Then, in order to be able to use the symbol $\mathfrak{a}_i = \alpha_i + \beta_i\,\varepsilon$ again for the representation of arbitrary motions, we will ultimately introduce new symbols in their place:

As the coordinates of a soma, we can employ four (real) homogeneous, dual quantities:

$$X_0 = \mathfrak{X}_0 + \mathfrak{X}_{123}\cdot\varepsilon,\quad X_1 = \mathfrak{X}_1 + \mathfrak{X}_{23}\cdot\varepsilon,\quad X_2 = \mathfrak{X}_2 + \mathfrak{X}_{31}\cdot\varepsilon,\quad X_3 = \mathfrak{X}_3 + \mathfrak{X}_{12}\cdot\varepsilon, \tag{1}$$

or their scalar and vectorial coefficients, assuming that the former – thus, the quantities $\mathfrak{X}_0$, $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$ *– do not all vanish.*

Thus, the concept of a soma, which initially seemed to be purely formal, now seems to be an *extension of the concept of a real, proper ray* (pp. 200) here.

We now determine the motion with the dual parameters $\mathfrak{a}_i$ that makes the soma X coincide with some other soma Y. To that end, we let $\dot{\mathfrak{a}}$ ($\dot{A}$, resp.) denote a quaternion (bi-quaternion, resp.) of the form [*]):

$$\dot{\mathfrak{a}} = \mathfrak{a}_0\, e_0 + \mathfrak{a}_1\, e_1 + \mathfrak{a}_2\, e_2 + \mathfrak{a}_3\, e_3\,, \tag{2}$$

while $\dot{\bar{\mathfrak{a}}}$ ($\dot{\bar{A}}$, resp.) denotes the conjugate that comes about by changing the sign of $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, and then find, with no further assumptions, that:

$$\dot{A} = \dot{\bar{X}} : \dot{Y}\,. \tag{3}$$

That is, the desired motion will have the dual parameters:

$$\begin{aligned}
\mathfrak{a}_0 &= \alpha_0 + \beta_0\varepsilon = X_0Y_0 + X_1Y_1 + X_2Y_2 + X_3Y_3,\\
\mathfrak{a}_1 &= \alpha_1 + \beta_1\varepsilon = X_0Y_1 - X_1Y_0 - X_2Y_3 + X_3Y_2,\\
\mathfrak{a}_2 &= \alpha_2 + \beta_2\varepsilon = X_0Y_2 - X_2Y_0 - X_3Y_1 + X_1Y_3,\\
\mathfrak{a}_3 &= \alpha_3 + \beta_3\varepsilon = X_0Y_3 - X_3Y_0 - X_1Y_2 + X_2Y_1,
\end{aligned} \tag{4}$$

[*]) e_0, e_1, e_2, e_3 are the so-called units of quaternion theory, which are usually denoted by 1, i, j, k in the books on this subject, although the use of the sign i in this way obviously can only create confusion, which has already come to pass.

or finite redundant parameters (α, β), the first two of which α_0 , β_0 take on the values:

$$(\mathfrak{X} \mid \mathfrak{Y}) = \mathfrak{X}_0\mathfrak{Y}_0 + \mathfrak{X}_{01}\mathfrak{Y}_{01} + \mathfrak{X}_{02}\mathfrak{Y}_{02} + \mathfrak{X}_{03}\mathfrak{Y}_{03},$$

(5)
$$\begin{aligned}(\mathfrak{X}\mathfrak{X}) = \mathfrak{X}_0\mathfrak{Y}_{123} + \mathfrak{X}_{01}\mathfrak{Y}_{23} + \mathfrak{X}_{02}\mathfrak{Y}_{31} + \mathfrak{X}_{03}\mathfrak{Y}_{12}\\ \mathfrak{X}_{123}\mathfrak{Y}_0 + \mathfrak{X}_{23}\mathfrak{Y}_{01} + \mathfrak{X}_{31}\mathfrak{Y}_{02} + \mathfrak{X}_{13}\mathfrak{Y}_{03}.\end{aligned}$$

We can now deduce some important consequences from this. We say that two somata – or, as we would prefer to say, somas – are *parallel* when one of them is produced by the other one by means of a (Euclidian) translation. We call two somas *hemi-symmetric* when one of them is produced from the other one by a screw; finally, we call two somas *symmetric* when they can be exchanged with each other by an inversion (Umwendung) [*]). One now obtains immediately:

Two somas X, Y are parallel when the scalar components of their dual coordinates X_i, Y_i are proportional to each to each other, and conversely.

Two somas X, U are hemi-symmetric when the scalar components of their dual coordinates X_i, U_i satisfy the equation:

(6)
$$(\mathfrak{U} \mid \mathfrak{X}) = \mathfrak{U}_0\, \mathfrak{X}_0 + \mathfrak{U}_1\, \mathfrak{X}_1 + \mathfrak{U}_2\, \mathfrak{X}_2 + \mathfrak{U}_3\, \mathfrak{X}_3 = 0;$$

they are, moreover, symmetric when their dual coordinates X_i, U_i satisfy the synectic equation (or pair of equations):

(7)
$$\begin{aligned}(U\,X) = (\mathfrak{U} \mid \mathfrak{X}) + (\mathfrak{U}\,\mathfrak{X}) \cdot \varepsilon\\ = U_0X_0 + U_1X_2 + U_2X_2 + U_3X_3 = 0.\end{aligned}$$
(no. 5)

The inversion axis will then have the dual coordinates $\mathfrak{a}_1 : \mathfrak{a}_2 : \mathfrak{a}_3$ (no. 4).

If we say *pencil*, *congruence*, *sheaf* of (parallel) somas to express the concept of all somas that arise from any soma by means of a one, two, three-dimensional group of translations, resp., then we will immediately find an analytical representation for the simplest of manifolds that are defined by somas, moreover; e.g., the pencil will expressed in terms of any two elements X', X'' in the form $X = \sigma' X' + \sigma'' X''$ with the aid of two scalar ratios s $\sigma' : \sigma''$. The concepts of pencil, congruence, sheaf, resp., of parallel somas will then be analogues of the concepts of pencil, congruence, sheaf, resp., of parallel proper rays. When we add the obvious remarks that the pencil, congruence, and sheaf, resp., of somas exist in ∞^7, ∞^6, ∞^5 exemplars and that ∞^6 somas can be divided into ∞^3 of the ∞^3 sheaves, we can turn to the most important of the theorems cited, which is expressed by equation (7).

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[*]) Symmetric somas are thus always hemi-symmetric. We do not underestimate the awkwardness in such a terminology, but we have nothing better to offer.

The synectic bilinear equations (7) *differ from each other only by the number of variables in the equation between dual ray coordinates that represents the condition for the rectangular intersection of two proper rays, and by the appearance of dual quantities in the equation that describes the combined position of a point and a plane in the three-dimensional continuum of projective geometry.*

These facts obviously have a fundamental meaning:

It follows that they must define a special branch of geometry by means of a function-theoretic problem that will encompass the projective geometry of space, along with radial-projective geometry, and whose objects will be the figures that are composed of somas, and is thus a topic in the theoretical kinematics of a rigid body *).

The projective transformations of somas.

The problem to be posed will obviously be this one:

Find all analytical transformations of somas that always allow one to go from the concept of the somas that are symmetric to some soma to other somas of that sort.

Naturally, these transformations, which we will call *projective transformations of somas* – or, by a somewhat risky word construction, *somato-projective* – must define a group. Notions of equivalence will be defined by means of this group and its subgroups that must have a relationship with the elementary concept of the congruence of geometric figures that is similar to the equivalence notion of projective geometry. Furthermore, one would expect that the latter notion of equivalence, just like several of the new ones, will be easier to handle than the somewhat cumbersome equivalence of elementary geometry: One will collect the objects of kinematics into large classes so that one can propose properties of this structure that can subsume the elementary way of looking at things by a set of singularities.

The problem referred to can reduce to the problem that corresponds to the definition of radial-projective geometry, and the solution has a completely analogous form. Since we can still think of posing the bigger problem, *which one can construct geometrically from these transformations*, we shall elaborate.

We first remark that the group $\mathfrak{g}_7$ of similarity transformations defines a likewise seven-parameter continuous group whose object is the soma and whose transformations have the desired properties in a trivial way. Namely, a similarity transformation associates each of the ∞^6 different rectangular axis intersections (1, 2, 3) that characterize the various somas with a definite axis intersection (1′, 2′, 3′) in such a way that the original unit of length on each of the axes seems to be changed by a certain ratio k. The new unit of length may then be again reduced to the original one when one performs the

*) Obviously, one can, in a similar way, as we have shown for *one* rigid body, also represent systems of several bodies, each of which are found in a definite location. Our method thus includes all of kinematics. However, in this book, we consider only the positions of a single body.

perspective similarity transformation that belongs to the intersection point of the new axes and the value k^{-1}. What results is a new system of oriented axes that can also be derived from the first one by a completely determined motion. That is, each soma X will be uniquely associated with another soma X'. However, all of these "similarity transformations of somas" – or *somatic similarity transformations* – (which already represent a new concept in geometry) already emerge when one combines the ∞^6 motions (with somas as objects) with the one-parameter group of perspective similarity transformations of somas that leave any soma – e.g., the protosoma – at rest. One finds the following analytical representation for the transformations of the latter special group:

$$
\begin{aligned}
&\mathfrak{X}'_0 = \mathfrak{X}_0, \quad && \mathfrak{X}'_{01} = \mathfrak{X}_{01}, \quad && \mathfrak{X}'_{02} = \mathfrak{X}_{02}, \quad && \mathfrak{X}'_{03} = \mathfrak{X}_{03}, \\
&\mathfrak{X}'_{123} = k\mathfrak{X}_{123}, \quad && \mathfrak{X}'_{23} = k\mathfrak{X}_{23}, \quad && \mathfrak{X}'_{31} = k\mathfrak{X}_{31}, \quad && \mathfrak{X}'_{12} = k\mathfrak{X}_{12}
\end{aligned} \tag{8}
$$

($k \neq 0$; cf., pp. 239, no. 10).

We thus define a certain infinite group of transformations of somas by the requirement that the homogeneous, dual coordinates of the transformed somas should be homogeneous, synectic functions of the coordinates of the one being transformed. Among these *synectic transformations of the somas*, to which the similarity transformations that we just mentioned do not belong *), are the linear ones that are defined in the entire soma continuum:

$$
\begin{gathered}
X'_i = \mathfrak{a}_{i0} X_0 + \mathfrak{a}_{i1} X_1 + \mathfrak{a}_{i2} X_2 + \mathfrak{a}_{i3} X_3 \\
\{\mathfrak{a}_{i\chi} = a_{i\chi} + \alpha_{i\chi} \cdot \varepsilon, \quad | a_{00} a_{11} a_{22} a_{33} | \neq 0\}
\end{gathered} \qquad (i = 0, 1, 2, 3), \tag{9}
$$

which likewise have the previously-demanded properties in an obvious way. They define a group with 30 essential parameters that is analogous to the group of collineations in space; we thus call its transformations *dual-projective*.

The solution to the problem we posed may now be briefly formulated in the following way:

All projective transformations of somas arise from the composition of dual-projective ones with the similarity transformations of somas. They define a continuous group with 31 essential parameters, all of whose transformations are everywhere well-defined, single-valued, and continuous in the continuum of all somas.

We shall denote the formulas that are completely analogous to equations (5) on page 237 by no. (10) here, but we shall not write them down specifically; on the other hand, we would like to briefly discuss the system of invariant subgroups of our group, and assemble them into the systems that are analogous to the ones presented on page 393:

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*) Both types of transformations collectively generate a new group in which the group of synectic transformations is obviously invariant, and is included in such a way that the associated factor group will be finite and, in fact, of one parameter.

Similarity transformations	*Radial collineations*	*Somatic collineations*
$\begin{tikzcd} & \mathfrak{g}_7 \arrow[dl, dash] \arrow[dr, dash] & \\ \mathfrak{g}_6 \arrow[dr, dash] & & \mathfrak{g}_4 \arrow[dl, dash] \\ & \mathfrak{g}_3 & \end{tikzcd}$	$\begin{tikzcd} & G_{17} \arrow[dl, dash] \arrow[dr, dash] & \\ G_{16} \arrow[dr, dash] & & G_2 \arrow[dl, dash] \\ & G_8 & \end{tikzcd}$	$\begin{tikzcd} & \mathcal{G}_{31} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{G}_{30} \arrow[dr, dash] & & \mathcal{G}_{16} \arrow[dl, dash] \\ & \mathcal{G}_{15} & \end{tikzcd}$

The groups $\mathcal{G}_{31}$ and $\mathcal{G}_{30}$ have already been discussed. The group $\mathcal{G}_{16}$ is characterized by the conditions $a_{ii} = a_{\chi\chi}$, $a_{i\chi} = 0$ (i, $\chi = 0, 1, 2, 3$), and for the transformations of $\mathcal{G}_{15}$ one has $k = 1$, moreover. All of the transformations of $\mathcal{G}_{16}$ leave each individual sheaf of parallel somas at rest and permute its somas by means of a perspective similarity transformation. If *one* sheaf undergoes a translation then one finds that this will always be true; the transformation will then belong to the subgroup $\mathcal{G}_{15}$ .The transformations of this latter group are synectic and commutable, moreover. They are called *dual-projective translations of the somas*. (Cf., pp. 235, no. 3.) For each individual sheaf of parallel somas they reduce, as one sees, to ordinary translations. The same thing is true exactly for the similarly-defined infinite group of "synectic translations," which are, for that reason, especially easy to treat.

Analytic manifolds of somas.

We now consider any (real) analytical family or manifold of ∞^r somas, or thus an irreducible system of analytical conditions, that a rigid body in a certain position can assume as a result of this system of r degrees of mobility (in the finite). Of the r essential parameters, by which one can analytically express such a structure as a manifold in the neighborhood of a location (which is thus distinguished by it) in general position, a certain number ρ of them will also be essential for the ratios of the quantities $\mathfrak{X}_0$, $\mathfrak{X}_{01}$, $\mathfrak{X}_{02}$, $\mathfrak{X}_{03}$. We set $r = \rho + \rho'$ and remark that the ∞^r somas can then be divided into ∞^ρ manifolds of $\infty^{\rho'}$ parallel somas. Obviously, in the cases $r = 1, 2, 3, 4, 5, 6$ the number ρ' will have values that are at least 0, 0, 0, 1, 2, 3. We call the overshoot of ρ' above these minimal values the *degree of planarity* of the manifold before us, and we accordingly also speak of *aplanar*, *uniplanar,* and *triplanar* families of somas.

Among the manifolds that we just described, one now finds some families that seem to be of especial interest.

First, let $\rho = \rho'$, so r will be one of the numbers 2, 4, 6. Then, if (as is necessarily true in the case of $r = 6$) the dual coordinates X_i of a variable soma of the manifold can be expressed as synectic functions of ρ likewise dual parameters then we would like to call the manifold in question *synectic*.

It does not seem essential to go into the analytical criteria for the stated properties, since they are very easy to derive; however, it might be helpful to explain the usefulness of the concepts that were introduced by some likewise very simple, but important, applications.

Any aplanar analytic manifold of ∞^1 *somas lies in a single two-dimensional synectic manifold of somas.*

Any aplanar analytic manifold of ∞^2 *somas lies in a single four-dimensional synectic manifold of somas.*

When two aplanar analytic manifolds of ∞^3 *somas are mapped onto each other according to any analytic law, there is a single synectic transformation (or its opposite) of the somas that brings about this map precisely* [*]).

We will encounter a number of special cases of these theorems later on.

We further consider such manifolds of ∞^r somas whose coordinates $\mathfrak{X}_0, \ldots, \mathfrak{X}_{12}$ are of an especially simple type, namely, they can be expressed as *linear* homogeneous functions of $r + 1$ (or more) parameters. We call these figures *soma chains*, and we also include the ∞^6 individual somas as the case $r = 0$. One finds, e.g., the previously-considered pencil, congruence, and sheaf of somas, included amongst the individual r-dimensional soma chains that are planar of degree r. As one might expect that these naturally very special figures, which still encompass a great wealth of forms, will have relatively accessible geometric properties, we then pose the problem: *Classify the soma chains relative to the group* $\mathcal{G}_{31}$ *(and the group* $\mathcal{G}_{30}$*) and generate the chains of each individual class by geometric constructions.*

The solution to the first of these problems comes about in complete conformity with the reasoning that was presented in § 29; it results by representing the various types of soma chains through a *basis*. As we saw at that time, one also observe here that the representation of a chain by linearly-appearing parameters will not always be complete – that, on the contrary, certain somas of the chain can be necessarily omitted – and furthermore, that the number of the essential parameters for the ratios of the quantities $\mathfrak{X}_0, \ldots, \mathfrak{X}_{12}$ does not, in all cases, determine the dimension of the associated chain uniquely, and therefore that the transition from one parametric representation of the chain to another (likewise linear) one can never come about through only linear substitutions.

We commence with the solution of the stated problems by making some special observations that will then lead us to the construction of the transformations (9). *From now on, an r-dimensional chain will be denoted more briefly as "a* C_r*."*

[*]) One easily proves the corresponding theorem that pertains to the group of all real and imaginary analytic point transformations of an *n*-fold extended manifold. For that matter, similar theorems exist for other infinite groups.

Simplest reciprocity relations.

It will become clear from the statements that follow that it is preferable to double all somas, so that the entire continuum of somas will be covered with two layers, or, as we would like to say, *sheets*. Any projective transformation will then act on the somas X¸Y, Z, … of the first sheet, along with some other somas U, V, W, … on the second sheet, which will be referred to as contragredient or *discordant* to the former ones. Even better, one considers these transformations to be identical, but applied to different objects. Precisely as in ray geometry (pp. 232, *et. seq.*), this yields conceptual differences between (somatic) collineations and correlations, dual correlations and anti-collineations, etc. For the concept that subsumes all of the notions, the somatic projectivities will further yield the characteristic theorem:

Somatic projectivities leave invariant the parallelism of somas of the same sheet and the hemi-symmetric, as well as symmetric, position of somas on different sheets.

Thus, the properties that enter into the following self-evident statement will remain unperturbed under somatic projectivities:

"When two families of somas from different sheets each consist of all somas that are symmetric or hemi-symmetric to the somas of the other family then both families will be chains."

We provisionally call chains that are associated with these pairs, and which are very easy to determine, *reciprocal* to each other, before we make some later extensions of this concept. In each sheet, there will be six (three, resp.) classes of chains that are different under collineations, which on the first (second, resp.) type are paired with chains of the other sheet, and in total, there will be four (two, resp.) classes of pairs of such reciprocal chains, which are different under collineations *and* correlations.

We first consider the chains that are paired with somas by means of symmetric position.

For each pair of reciprocal chains, we present, as in § 29, a canonical form for the basis, for which we would, however, like to now avail ourselves of a notation that occupies less space. Namely, since only one of the units 1, ε will ever appear in a horizontal row of such a basis, we need only to count up the units that appear in the four vertical rows. The symbols that appear in [] refer to the classification of all chains that will be described later. The classes [2, C] and [4, C], together with [6], include the individual synectic chains.

$$[0], \qquad \left\{ \frac{1}{\varepsilon} 0\,0\,0 \right\}, \qquad \left\{ 0\, \frac{1}{\varepsilon}\, \frac{1}{\varepsilon}\, \frac{1}{\varepsilon} \right\}, \quad [4, \mathrm{C}].$$

In each sheet of the soma continuum there are ∞^6 *synectic four-dimensional soma chains, or, as we would like to say briefly, planar chains. Each of them is reciprocal to a certain soma of the other sheet.*

Through any three somas U_1, U_2, U_3 (of the second sheet), which are symmetric to no more than one soma, there goes *one* planar chain. If U is another soma of this chain then one always has:

(11) $$(U_1\ U_2\ U_3\ U) = 0,$$

and thus the chain itself and the reciprocal soma $X = \widehat{U_1 U_2 U_3}$ *). In order to also construct it, one first seeks the motions S_1, S_2, S_3 that cyclically permute the three somas, such that:

$$U_1\{S_3\}U_2\{S_1\}U_3\{S_2\}U_1 \text{ **)}.$$

After that, one decomposes these motions into twists (*Umwendungen*) around the three axes Y_1, Y_2, Y_3 in such a way that:

$$S_1 = \{Y_2, Y_3\}, \qquad S_2 = \{Y_3, Y_1\}, \qquad S_3 = \{Y_1, Y_2\}.$$

(Cf., pp. 8, 9) X will then be constructed from:

(12) $$U_1\{Y_1\}X\{Y_2\}U_2\{Y_2\}X\{Y_3\}U_3\{Y_3\}X\{Y_3\}U_1 \text{ ***)}.$$

The construction is determined just like the twist axes Y_1, Y_2, Y_3, and indeed this happens when $S(U_1\ U_2\ U_3\ V)$ does not vanish identically. If one is dealing with the opposite, easy to interpret, assumption then one comes to the cases that we will discuss next:

$$[2, C], \qquad \left\{\frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ 0\ 0\right\}, \qquad \left\{0\ 0\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\}, \qquad [2, C].$$

In any sheet, there are ∞^8 *two-dimensional synectic, or, as we will say,* linear *chains, which are reciprocal.*

One linear chain goes through any two non-parallel somas. If a linear chain has two non-parallel somas in common with a planar C_4 then it will lie in the C_4 completely. The linear chain lies on ∞^8 planar chains, and is already the complete intersection of two of them. In order construct the linear chain from two given somas and, in turn, the reciprocal chain, one sees the motion S that makes the first soma coincide with the second one. The two-parameter group of commuting motions in which S is included will then leave the two chains at rest, so its transformations will emerge from the first soma as all somas of the connecting chain. The twist around the ∞^2 (real) rays of the normal net of the axis that is common to all of these motions exchanges the two reciprocal chains. (Cf., pp. 60) – Only when the given somas are parallel can the construction become indeterminate: One then comes to the cases that we will discuss from now on:

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*) With the abbreviations that were used on page 126 and 127, as well as 313, *et seq.*

**) We avail ourselves of the Wiener notations that were already applied in § 1.

***) As far as we know, the construction and the theorem that is proved by it originates with C. Stephanos.

$$[1, B], \qquad \left\{\frac{1}{\varepsilon}\ \varepsilon\ 0\ 0\right\}, \qquad \left\{0\ \varepsilon\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\}, \qquad [3, E].$$

This case subsumes $2 \cdot \infty^2$ *pairs of reciprocal chains. A certain three-dimensional chain, or then a certain* C_3 *is, in fact, reciprocal to each sheaf of parallel somas.*

In order to construct this C_3, one seeks the planar ray complex whose rays are perpendicular to the direction of the translation that leaves the sheaf at rest. The C_3 twists around the (real) rays of this complex then take any one soma of the sheaf to all ∞^3 somas of the C_3 .

If three somas are present that determine no planar chain, but also belong to either a bundle of parallel somas or a linear chain, then *one* C_3 of the type described here will go through it. The construction of the reciprocal sheaf will then be obtained from the aforementioned.

Any of the C_3 thus found will be the locus of ∞^1 bundles of parallel somas and the partial intersection of ∞^1 planar chains, any two of which have a special reciprocal position (that is not difficult to describe more precisely) and determine C_3 . Likewise, ∞^2 linear chains, ∞^3 planar chains, and ∞^3 of the C_3 that were just described will go through the reciprocal sheaf:

$$[2, D], \qquad \left\{\varepsilon\ \varepsilon\ \frac{1}{\varepsilon}\ 0\right\}, \qquad \left\{\varepsilon\ \varepsilon\ 0\ \frac{1}{\varepsilon}\right\}, \qquad [2, D].$$

∞^3 *pairs of reciprocal chains: For each bundle of (parallel) somas there is another special bundle that is reciprocal to it.*

This easily yields the construction of the bundle that is reciprocal to a given one. Namely, both of them admit the same two-parameter group of Euclidian translations. This determines ∞^1 directions of translation, to which a certain direction is perpendicular, which defines a parallel bundle. All twists around rays of this bundle, and only such twists, permute the somas parallel to the reciprocal bundle. – Through each bundle go ∞^2 synectic C_4 and ∞^2 of the aforementioned special C_3 .

We now come to the reciprocal chain with somas in hemi-symmetric position. We call these chains, which take on a special meaning in the somatic-projective geometry due to their very small constant number, *distinguished chains.*

$$[3, F], \qquad \left\{\frac{1}{\varepsilon}\ \varepsilon\ \varepsilon\ \varepsilon\right\}, \qquad \left\{\varepsilon\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\}, \qquad [5, B].$$

$2 \cdot \infty^3$ *pairs of distinguished chains. One of them is an arbitrary sheaf of (parallel) somas, while the other one is the* C_5 *that is reciprocal to it.*

All somas of the C_5 arise from any soma of the sheaf by way of the ∞^5 screw motions. We cannot go further here into the remarkable geometry of these chains, which represents an analogy to the concept of "planar complex," and make only the comment regarding it that ∞^2, ∞^5, ∞^6 sheaves, bundles, and pencils of somas will lie in it, respectively, as well as ∞^5 C_3 that are reciprocal to the pencils and ∞^6, ∞^3 linear and planar chains, respectively. Any planar C_4 lies in *one* distinguished C_5, and will thus arise from the C_4 by way of the group of all translations.

$$[4, \mathrm{D}], \qquad \left\{\varepsilon\ \varepsilon\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\}, \qquad \left\{\frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ \varepsilon\ \varepsilon\right\}, \qquad [4, \mathrm{D}].$$

There are ∞^4 pairs of four-dimensional distinguished chains, which are pair-wise reciprocal.

Such a C_4 arises from any soma when one subjects it to the four-dimensional continuous group of motions that leave a certain direction at rest. The reciprocal C_4 – which belongs to the same group – will then arise from the same soma when one applies to it all screw motions around rays of the planar complex that are uniquely determined by any direction, or thus all motions that invert the stated direction. Any screw motions around any ray of the complex exchange both C_4 . The somas of such a C_4 are divided into ∞^2 linear C_4 ; the reciprocal C_2 fill up the reciprocal C_4 . Any linear chain lies in *one* such C_4, and thus arises from any one of them by way of the group of all translations.

The principal meaning of the distinguished chains is easy to recognize: They allow one to arrange the points, lines, and planes in space, uniquely or "projectively" throughout, into the following schema, which scarcely needs an explanation:

First sheet	*Image space*	*Second sheet*
Distinguished C_5	Point	Distinguished C_3
Distinguished C_4	Line	Distinguished C_4
Distinguished C_3	Plane	Distinguished C_5

This arrangement, which is also easy to present constructively, expresses the obvious isomorphism of the group $\mathcal{G}_{31}$, when extended to a so-called mixed group by the "absolute correlation" $X' = U$, $U' = X$, with the group of collineations and correlations in space.

Significance of the foregoing argument.

An immediate consequence of the considerations that were presented is, *inter alia*, the theorem:

The somatic-projective geometry of a linear or planar chain is identical with the radial-projective geometry of the normal net of a real ray or of the continuum of all real rays.

In fact, if one subjects any soma to all screw motions then the C_4 that thus arises will be mapped to the continuum of screw axes in a uniquely invertible way, and the chains of somas in the C_4 will correspond to chains of rays, and conversely *). If one permutes the somas of C_4 with each other somatic-projectively then the rays that are associated with them will be simultaneously permuted in a radial-projective way.

This shows, moreover, e.g., how the dual projectivities differ from the remaining somatic ones: Any four somas of a linear chain that lie in a certain sequence will have a certain *dual double ratio*, as long as no two of them are parallel (pp. 242). This remains unchanged only under the dual projectivities (pp. 243). There is thus a geometric content to the following obvious theorem, which needs no tedious explanation:

The synectic analytic transformations of the somas are the ones that are dual-projective in the infinitesimal.

This further implies the theorem:

Any five somas $X_0, \ldots, X_4$, no four of whom belong to a five-dimensional distinguished chain, may be made to coincide (in sequence) with five other ones $Y_0, \ldots, Y_4$ ($U_0, \ldots, U_4$) of the same nature by a single dual collineation (correlation).

Moreover, one can not only express this somatic projectivity analytically very easily (cf., pp. 246), but also almost as easily when one also, no quite so briefly, *constructs it geometrically* (cf., pp. 245).

It is, moreover, clear that one can place every well-defined theorem in the projective geometry of space that is expressed by construction next to a corresponding theorem of the much more inclusive somatic-projective geometry. However, it is clear that such associated theorems do not, by any means, need to have a form that reads the same way **), and that, in addition, as a rule, the somatic-projective theorem will be afflicted with a series of restrictions that are not necessary in the simpler cases (so they are always found to be *mechanical*, moreover, when the simpler theorem has been formulated exactly). This also illuminates the fact that known theorems with extensions of that type will not exhaust the facts of somatic-projective geometry to the furthest limits, and indeed the most interesting phenomena must be sought outside this circle of ideas. However, in spite of that, we have a very useful method in hand that will facilitate the discovery of new physical laws quite profoundly.

Classification and construction of the chains

We now enumerate all chains according to their dimension numbers, and give them by the still-not-mentioned geometric construction, which delivers either all individual members of a class, or, as would be sufficient from the foregoing representation,

*) All r-dimensional chains composed of real rays are summarized on pp. 326, …, 327. The additional "chains" that are denoted by stars there are naturally not necessary for the extension of the theory in the present discussion.

**) One observes that a product of dual quantities can vanish without the individual factors vanishing.

representatives of each class. We choose the basis, as we also have done before, to be as inclusive as possible in each case. It shows that chains with absolute invariants under $\mathcal{G}_{30}$ do not exist, and that all chains that are inequivalent under $\mathcal{G}_{30}$ also remain distinct from each other under transformations of $\mathcal{G}_{31}$. All chains can be regarded as limiting cases of the aplanar ones, which, in the cases:

$$r = 1, 2, 3, 4, 5,$$

will depend upon:

$$11, 14, 15, 14, 11$$

constants, respectively.

We further remark that one also always obtains a chain of somas when one subjects a single soma to all twists or all screw motions around a chain of rays.

[0]. *Null-dimensional chains.*

$$\left\{\frac{1}{\varepsilon}\ 0\ 0\ 0\right\} \qquad \infty^6 \qquad \text{The individual soma.}$$

[1]. *One-dimensional chains.*

A. $\{1\ 1\ 0\ 0\}$ ∞^{11}.

Any aplanar C_1 can be generated in ∞^2 ways by twisting a soma around the lines of a chain of rays. It will lie in a single linear chain. If the generating chain of rays is a planar pencil then the C_1 will admit all rotations around its principle axis. The various chains of rays (∞^2, or ∞^1 for the stated *rotational chains*) that are suitable, by construction, will all be coaxial and congruent to each other.

B. $\left\{\frac{1}{\varepsilon}\ \varepsilon\ 0\ 0\right\}$ ∞^7.

Naturally, these planar C_1 or pencils can also be derived from a single soma (that lies in the reciprocal C_3) by twisting with the help of parallel pencils; these parallel pencils define a net. (pp. 404).

[2]. *Two-dimensional chains.*

A. $\{0\ 1\ 1\ 1\}$ ∞^{14}.

Any aplanar C_2 lies in a single planar chain and arises from the soma that is reciprocal to it in a single way, when one subjects it to the twists around all rays of an *aplanar* congruence of chains.

B. $\{0\ \varepsilon\ 1\ 1\}$ ∞^{12}.

The uniplanar non-synectic C_2 . Any of these will be generated by twists, but in ∞^1 ways, and with the help of any *planar* chain congruence.

C. $$\left\{\frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ 0\ \ 0\right\} \qquad \infty^8.$$

The aforementioned synectic or linear chains, which can arise from any one of ∞^2 somas (viz., the reciprocal C_2) with the help of any normal net by twisting.

D. $$\left\{\frac{1}{\varepsilon}\ 0\ \ \varepsilon\ \ \varepsilon\right\} \qquad \infty^6.$$

The bundles that were already mentioned, which are pair-wise reciprocal.

[3]. *Three-dimensional chains.*

A. $$\{1\ \ 1\ \ 1\ \ 1\} \qquad \infty^{15}.$$

The aplanar C_3, which have the highest possible number of constants for a chain of somas – viz., 15 – enjoy especially beautiful properties. *One* such chain goes through five somas, no four of which belong to a distinguished C_5, which is implied by the following geometric construction: Any such C_3 has *one* soma in each sheaf of somas, and each of them can be taken to any other one by *one* dual-projective translation. The entire theory of the geometric addition of aplanar chain congruences (pp. 390, *et seq.*), with all of the consequences that are connected with them, can be extended to these chains of somas. For example, each of these chains is a closed continuum, and each of them is the carrier of a quaternary domain. Any of them admits a 15-parameter simple group of dual collineations, which (i.e., its discordant) likewise leaves a second chain of the same type at rest: the *reciprocal* congruence to the other sheet. The division of the chains of lower dimension, and likewise the type, like the two reciprocal chains alternately determine from each other, will be represented most conveniently by a simultaneous map of two chains to the ordinary space of projective geometry:

C_3 in the first sheet	*Image space*	*C_3 in the first sheet*
Soma	Point	Aplanar C_2
Aplanar C_1	Line	Aplanar C_1
Aplanar C_2	Plane	Soma

Somas that face each other in the first and third column are symmetric. From this, one immediately deduces the construction of the reciprocal chain. The two reciprocal C_3 coincide in the chosen canonical form; the associated ∞^4 C_4 are all rotation chains. In order to find such a special aplanar C_3, one only has to subject any soma to the motions (rotations) of the three-parameter group that leaves a real point at rest. There are ∞^6 such special C_3, which defines an importance class of chains, especially for mechanics.

B. $\{\varepsilon \ 1 \ 1 \ 1\}$ ∞^{14}.

These are all uniplanar C_3 that are included in a planar chain. Each of them lies in a certain distinguished C_3 . Any soma of the sheaf that is reciprocal to this can be carried to all of the somas of the C_3, either by certain twists whose axes fill a chain complex or by all twists whose axes fill an aplanar congruence of chains, and is, in fact, independent of the choice of soma in the sheaf. In particular, the aforementioned C_3 – namely, its ∞^{11} – may be presented in such a way that one subjects a soma to all twists around rays of an aplanar congruence of chains, and indeed each of these C_3 can be generated in ∞^{11} ways. Among these particular C_3, we find special ones that can be constructed with the help of ray bundles, and these C_3 can also be found in an intuitive way, e.g., in such a way that one subjects any soma to all reflections in the planes of space. (Cf., pp. 183 and pp. 560.) The chains of lower dimension that are included in C_3 will become evident quite simply under the map of C_3 to the real planes that thus given.

The aforementioned chains C_3 are pair-wise *reciprocal* to each other. In fact, such a chain contains ∞^2 uniplanar C_2, to whose somas, any pencil of parallel somas will be symmetric. These ∞^2 pencils fill up the reciprocal C_3, from which the first one arises by the process described. In the example given, the two reciprocal C_3 coincide, and two somas that lie in them are then symmetric when the associated planes are perpendicular to each other.

C. $\left\{0 \ \frac{1}{\varepsilon} \ 1 \ 1\right\}$ ∞^{13}.

Each of these uni-planar C_3 lies in a planar chain, and arises from the soma that is reciprocal to it by twists around rays of an (aplanar) chain complex. The singular ray of this complex corresponds to a "singular" soma in C_3. The somas of C_3 that lie in a bundle of somas that are non-singular, but parallel to them, eliminate a linearly-appearing parameter from the representation [*]).

The C_3 include linear chains that pierce them in singular somas. The C_2 that are reciprocal to these C_2 describe a new C_3 that lies in the planar chain that is reciprocal to the singular ray; we call this C_3 , which will be represented in the canonical form by:

$$\left\{\frac{1}{\varepsilon} \ 0 \ 1 \ 1\right\},$$

reciprocal to the former one.

D. $\{1 \ 1 \ \varepsilon \ \varepsilon\}$ ∞^{11}.

The subdivision of the chains of lower dimensions in these biplanar C_3 can be made intuitive in a manner that is similar to the way that we mapped their somas to the planes (or points) of an R_3 in B. The planes of a certain pencil thus correspond to no somas. The C_3 include, *inter alia*, ∞^1 bundles of parallel somas, and the bundles that are reciprocal to them fill up the present *reciprocal* C_3, which belongs to the stated basis in the case that was discussed.

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[*]) Some of the chains that will be enumerated later on also have similar singularities.

A special pair of reciprocal – in fact, coincident – C_3 of the present class arises when one subjects any soma to the three-dimensional groups of motions that leave the parallel planes of a pencil individually at rest. More inclusive, but still not exhaustive, is the generation of a C_3 of the class in question with the help of the twists of a soma around the rays of a planar congruence of chains.

E. $$\left\{0\ \varepsilon\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\} \qquad \infty^7.$$

The aforementioned biplanar C_3 that are reciprocal to pencils.

F. $$\left\{\frac{1}{\varepsilon}\ \varepsilon\ \varepsilon\ \varepsilon\right\} \qquad \infty^3.$$

The triplanar C_3, the sheaves or three-dimensional distinguished chains.

[4]. *Four-dimensional chains.*

For the sake of brevity, we omit the given of certain ways of generation from the following chains, which can be found to fit the pattern of the constructions that were previously described, and communicate only those constructions that are based upon a demonstrable "reciprocity" between these chains and the ones that were already constructed in each case.

A. $$\left\{\frac{1}{\varepsilon}\ 1\ 1\ 1\right\} \qquad \infty^{14}.$$

The aplanar C_4 . They are *reciprocal* to the aplanar C_2 . Any of them includes, in fact, ∞^2 aplanar C_1, which lie in any synectic C_2 or linear chain, and whose reciprocals fill up the C_4 .

B. $$\left\{\frac{1}{\varepsilon}\ \varepsilon\ 1\ 1\right\} \qquad \infty^{12}.$$

The uniplanar non-synectic C_4 . They are *reciprocal* to the uniplanar non-synectic C_2. Any of them includes, in fact, ∞^1 planar C_1; the C_3 that are reciprocal to them fill up the C_4 .

C. $$\left\{0\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\} \qquad \infty^6.$$

The synectic C_4 or planar chains, which are reciprocal to the individual somas.

D. $$\left\{\varepsilon\ \varepsilon\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\} \qquad \infty^4.$$

The biplanar or distinguished C_4 that are pair-wise reciprocal to each other.

[5] *Five-dimensional chains.*

A. $$\left\{1\ 1\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\} \qquad \infty^{11}.$$

The aplanar C_5 are *reciprocal* to any aplanar C_1 . The C_5 arises from them when one subjects their somas to all ∞^4 twists, and will then be described by ∞^1 planar chains.

B. $$\left\{\varepsilon\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\} \qquad \infty^{7}.$$

The planar or distinguished C_5, which are reciprocal to the pencils of parallel somas.

[6]. *The six-dimensional chain.*

$$\left\{\frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\right\} \qquad \infty^{0}.$$

The totality of all somas, a synectic chain.

We have thus established the fact that geometric ways can be given for generating all chains, and most of them, in turn, will have a very simple character.

The chains of somas that have been subdivided into twenty classes will now be examined more closely in a manner that is similar to the way that we examined the chains of rays.

The reciprocity theorem for chains and its extension.

In the foregoing discussion, there was a theorem that was pointed out as *extremely noteworthy*, especially in view of the fact that we explained the concept of reciprocity that entered into it in different ways in the individual cases.

If one ignores the totality of all somas then one can bring any soma chain on the first sheet to one on the second one in such a way that any two associated – or "reciprocal" – chains will determine each other alternately by a geometric construction that is invariant under somatic projectivities.

The dimension numbers of the chains that are paired in this way extend to one of the numbers 4, 6, 8.

We have already thoroughly classified the pairs that correspond to dimension sums four and eight (pp. 565-567). The remaining ones are summarized here:

[1 A]	$\{0\ 0\ 1\ 1\}$	$2 \cdot \infty^{11}$	$\left\{\frac{1}{\varepsilon}\ \frac{1}{\varepsilon}\ 1\ 1\right\}$	[5 A],
[2 A]	$\{0\ 1\ 1\ 1\}$	$2 \cdot \infty^{14}$	$\left\{\frac{1}{\varepsilon}\ 1\ 1\ 1\right\}$	[4 A],
[2 B]	$\{0\ \varepsilon\ 1\ 1\}$	$2 \cdot \infty^{12}$	$\left\{\frac{1}{\varepsilon}\ \varepsilon\ 1\ 1\right\}$	[4 B],
[3 A]	$\{1\ 1\ 1\ 1\}$	∞^{15}	$\{1\ 1\ 1\ 1\}$	[3 A],
[3 B]	$\{\varepsilon\ 1\ 1\ 1\}$	∞^{14}	$\{\varepsilon\ 1\ 1\ 1\}$	[3 B],
[3 C]	$\left\{0\ \frac{1}{\varepsilon}\ 1\ 1\right\}$	∞^{13}	$\left\{\frac{1}{\varepsilon}\ 0\ 1\ 1\right\}$	[3 C],
[3 D]	$\{1\ 1\ \varepsilon\ \varepsilon\}$	∞^{12}	$\{1\ 1\ \varepsilon\ \varepsilon\}$	[3 D].

Is there now no deeper basis for these relations? The answer sounds completely similar to that of the corresponding question of ray geometry, which we have summarily discussed (pp. 410, *et seq.*):

All chains may be regarded as the geometric loci of bundles of parallel somas of the one type or the other. The reciprocal chains of corresponding loci will then be described by reciprocal bundles, and they will thus obtain the same dimension number in this way.

For the correct interpretation of this assertion, one must observe that a chain can be determined by several kinds of loci of soma bundles that will not all be useful for the description of the chain.

The reciprocity theorem that was thus proved obviously extends far into the geometry of chains. One can then naturally introduce, in principle, the soma bundle as a spatial element in a manner that is similar to the way that we have singled out the analogous concept of the pencil of parallels (§ 34). Indeed, one can represent this figure in exactly the same way as the latter by means of a system of "homogeneous" coordinates Ω; Ξ_0, …, Ξ_3, Φ_0, …, Φ_3, of which, the last eight are coupled by a bilinear equation. Loci of somas of similar types will now be paired, as we did before with loci of rays, and one can also examine analytic transformations of the paired bundle, which can then change the dimensions of the manifolds of somas, as a rule. These transformations define an infinite group of contact transformations in the continuum of somas.

The so-called duality principle that is associated with ordinary projective geometry takes the form of a special case of the reciprocity of geometric figures that is spoken of here.

We must content ourselves with this interpretation of a situation, that is delightful, but not by any means simple, and in any event, not treated briefly.

The parasoma.

Up to now, we have limited the concept of soma in such a way that it completely covered the basic notion of kinematics, namely, the notion of the position of a rigid body in Euclidian space. Although we have first made small steps into the projective of somas, it has already become necessary for us to add a restriction to the theorems that were presented that is not in the analytical nature of things, but in the limits of the viewpoint that is distinguished, and has its origin in a certain incompleteness of the tool that was applied. It is clear that one can advance to many deeper-lying and more general theorems only when one extends the manifold of somas to a closed and invariant algebraic one, or, as we will say for that purpose, *natural continuum*, and simultaneously introduces "imaginary" somas. We would now like to speak on the first point, and also do so only partially. We will explain a new concept that we call a *parasoma*. *From now on, the figures that we simply called somas up to this point will be called* actual *somas, and we will use the word "soma" itself for both the actual somas and parasomas.*

We next go from the coordinates "of the first kind" that we employed up to now for an actual soma to ones "of the second kind," which we clarify as follows:

$$\mathcal{X}_0 = \mathfrak{X}_0\,, \qquad \mathcal{X}_1 = \mathfrak{X}_{01}\,, \qquad \mathcal{X}_2 = \mathfrak{X}_{02}\,, \qquad \mathcal{X}_3 = \mathfrak{X}_{03}\,,$$

$$\mathcal{X}_{01} = \begin{vmatrix} \mathfrak{X}_0 & \mathfrak{X}_{01} \\ \mathfrak{X}_{123} & \mathfrak{X}_{23} \end{vmatrix}, \qquad \mathcal{X}_{02} = \begin{vmatrix} \mathfrak{X}_0 & \mathfrak{X}_{02} \\ \mathfrak{X}_{123} & \mathfrak{X}_{31} \end{vmatrix}, \qquad \mathcal{X}_{03} = \begin{vmatrix} \mathfrak{X}_0 & \mathfrak{X}_{03} \\ \mathfrak{X}_{123} & \mathfrak{X}_{12} \end{vmatrix}, \tag{13}$$

$$\mathcal{X}_{23} = \begin{vmatrix} \mathfrak{X}_{02} & \mathfrak{X}_{03} \\ \mathfrak{X}_{31} & \mathfrak{X}_{12} \end{vmatrix}, \qquad \mathcal{X}_{31} = \begin{vmatrix} \mathfrak{X}_{03} & \mathfrak{X}_{01} \\ \mathfrak{X}_{12} & \mathfrak{X}_{23} \end{vmatrix}, \qquad \mathcal{X}_{12} = \begin{vmatrix} \mathfrak{X}_{01} & \mathfrak{X}_{02} \\ \mathfrak{X}_{23} & \mathfrak{X}_{12} \end{vmatrix}.$$

This now implies the new notion of parasoma that is first defined analytically in a manner that is similar to the way that one defines the notion of point ray (pp. 258, *et seq.*) in ray geometry: The *parasoma* will be described by the remaining coordinates of the second type that have been excluded up to now, which satisfy the equations $\mathcal{X}_i = 0$ and:

$$\mathcal{X}_{01}\mathcal{X}_{23} + \mathcal{X}_{02}\,\mathcal{X}_{31} + \mathcal{X}_{03}\,\mathcal{X}_{12} = 0. \tag{14}$$

It likewise follows that the manifold of all ∞^6 actual somas and ∞^4 parasomas defines a *closed continuum* that can be mapped in a uniquely invertible way without any singular points to, e.g., an algebraic point manifold M_6 that lives in a planar space of 15 dimensions and admits a projective group that is holomorphically isomorphic to $\mathcal{G}_{31}$ (cf., pp. 278, *et seq.*, as well as § 32), and can be exhaustively represented as the partial intersection of quadratic manifolds. Finally, it must be remarked that under the somatic collineations the parasomas of the first and second sheets behave precisely like the lines in the Plücker line continuum, and coordinates of the parasomas of both sheets will be transformed contragrediently.

What geometric concept that is analogous to that of the bundle of parallels that is associated with the "point ray" in radial-projective geometry can now be linked with the analytical concept of parasoma that we just described?

The answer to this question emerges precisely as it did in the simpler cases that we mentioned for greater ease of understanding. We consider the planar chain, which is reciprocal to an actual soma $\mathcal{X}$, and thus includes all of the actual somas $\mathcal{U}$ that are symmetric to $\mathcal{X}$. We now let $\mathcal{X}$ go to infinity in such a way that a certain parasoma emerges from this, and in the limiting case, we obtain the equations:

$$\begin{array}{rrrrl} * & -\mathcal{X}_{01}\mathcal{U}_1 & -\mathcal{X}_{02}\mathcal{U}_2 & -\mathcal{X}_{03}\mathcal{U}_3 & = 0, \\ \mathcal{X}_{01}\mathcal{U}_0 & * & -\mathcal{X}_{12}\mathcal{U}_2 & +\mathcal{X}_{31}\mathcal{U}_3 & = 0, \\ \mathcal{X}_{02}\mathcal{U}_0 & +\mathcal{X}_{12}\mathcal{U}_1 & * & -\mathcal{X}_{23}\mathcal{U}_3 & = 0, \\ \mathcal{X}_{03}\mathcal{U}_0 & -\mathcal{X}_{31}\mathcal{U}_1 & -\mathcal{X}_{23}\mathcal{U}_2 & * & = 0. \end{array} \tag{15}$$

However, these equations obviously determine a distinguished C_4 (which lies in the distinguished C_5 that is represented by only the first equation). *We can then say that the somas of each four-dimensional distinguished chain are "symmetric" to a well-defined soma (on the other sheet).* One also comes to the response to the question that was just posed with no difficulty by extending the concept of "symmetric" using a geometric argument. Moreover, a further consequence is that any two parasomas (of different sheets) are said to be *symmetric* to each other. We will refer to them as at least *doubly-symmetric* when each of them is associated with the pertinent extended distinguished C_4 that is determined by the other one. This comes about when each of the two C_4 has a sheaf in common with the reciprocal of the other one or when the corresponding lines of the line continuum intersect each other:

$$\mathcal{X}_{01}\,\mathcal{U}_{01} + \mathcal{X}_{02}\,\mathcal{U}_{02} + \mathcal{X}_{03}\,\mathcal{U}_{03} + \mathcal{X}_{23}\,\mathcal{U}_{23} + \mathcal{X}_{31}\,\mathcal{U}_{31} + \mathcal{X}_{12}\,\mathcal{U}_{12} = 0. \tag{16}$$

Finally, we will then call two parasomas *triply-symmetric* when the associated C_4 are reciprocal to each other or when the corresponding lines coincide:

$$\mathcal{X}_{01} : \mathcal{X}_{02} : \mathcal{X}_{03} : \mathcal{X}_{23} : \mathcal{X}_{31} : \mathcal{X}_{12} = \mathcal{U}_{23} : \mathcal{U}_{31} : \mathcal{U}_{12} : \mathcal{U}_{01} : \mathcal{U}_{02} : \mathcal{U}_{03}\,. \tag{17}$$

(Cf., pp. 263) All of these relations are invariant under $\mathcal{G}_{31}$. We shall pursue such considerations no further, but highlight its most essential aspects in the following theorems:

The manifold of ∞^6 actual somas will be extended by the addition of the ∞^4 parasomas to continuum that is natural relative to the group $\mathcal{G}_{31}$. The individual points of this continuum will be associated, in a uniquely invertible way, with the ∞^6 four-dimensional synectic (planar) chains and the ∞^4 four-dimensional distinguished chains.

Somatic-projective geometry reduces to Plücker line geometry when only parasomas are considered (cf., pp. 568).

It is perhaps worth mentioning that, first of all, three somas that are symmetric to a parasoma must have a special relation in regard to position, while the corresponding statement for two rays already appears in ray geometry.

Some important further consequences can be found in the following theorems:

The connection between somatic-projective geometry in a planar chain and radial-projective geometry in a ray continuum that was previously proved for actual figures also applies to the extension of these continua to parasomas and point-rays.

Under the transition from a variable actual soma to a well-defined parasoma, the planar chain that is reciprocal to the soma will decompose into the distinguished chain that is associated with the parasoma and the continuum of all parasomas.

We will make no further use of the parasoma in the present sketch; for that reason, we would like to completely pass over this basic notion.

Group-theoretic notions.

If one introduces any sort of non-distinguished chain as a spatial element then one will always obtain a group that is *holomorphically*-isomorphic to the group $\mathcal{G}_{31}$ that can be represented as a projective group in many ways. The following theorem relates to some particularly interesting examples of these groups:

If one introduces the three-dimensional aplanar chain of somas as a spatial element then a primitive space group R_{15} will arise from $\mathcal{G}_{31}$ that takes the form of a projective group for a suitable choice of coordinates and then becomes identical with a group that can be derived from the adjoint of the general projective group of ordinary space by a certain process of extension. (Cf., pp. 393, *et seq.*)

Another group is correlative to this group, which one arrives at when one maps the bundle of somas to a suitably chosen point-manifold M'_6. (Cf., pp. 420, *et seq.*)

It is further noteworthy that not only two essentially different groups can be confirmed by their connection to the group $\mathcal{G}_{31}$ in the space R_6, but also in the space R_7. One obtains one of the latter groups by the introduction of a pencil of somas or the chains that are reciprocal to them; the other one is included in a group of 32 parameters that is completely analogous to the group Γ_{13} that was discussed on page 240. The leads us to a closely related question:

Can one obtain a geometric figure that behaves in relation to the soma the way that the winding does to the ray?

One finds that the answer to this is in the affirmative: There are, in fact, (at least) two such figures, which depend upon seven constants and can be represented by eight homogeneous coordinates. One comes to one of them, which can perhaps be referred to as a "hypersoma," when one remarks that the group of motions can be extended to a seven-parameter group of contact transformations by the dilatations, which commute with them, and that this can still be represented by our parameter (α, β) while using the same formulas to express the connection [*]). Spheres of equal radius (as well as sign) will then enter into hypersomas in place of the points of the individual soma. This somewhat complicated figure, which for purely geometric considerations also seems to have as little as possible in common with the simple thread, still proves to have properties that are analogous to it in many details. To these properties also belongs the fact that the hypersoma can be related to not only to a geometry of somas that is comparable to radial-projective geometry, but also to another type of geometry that can be placed alongside Plückerian line geometry, and defines its immediate generalization. We would like to consider this new situation now, but from another viewpoint.

The pseudo-conformal transformations of somas.

In the following suggestion regarding a further type of "geometry of somas," we will, in order to avoid monotony, proceed along a different train of thought than the one that we have followed up to now.

We now make the assumption that the parameter (α, β) of the motion that produces an actual soma $\mathfrak{X}$ from the protosoma as described by equation (9), pp. 176, thus employs eight homogeneous quantities – in the *ordinary* sense of the words – as coordinates that are coupled by the quadratic equation $\frac{1}{2}(\mathfrak{X}\,\mathfrak{X}) = 0$, or:

$$\mathfrak{X}_0\,\mathfrak{X}_{123} + \mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31} + \mathfrak{X}_{03}\,\mathfrak{X}_{12} = 0. \tag{18}$$

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[*]) With the help of systems of complex quantities that include the quaternions, one can *conveniently* represent a series of enveloping groups and combine them. Thus, the similarity transformations in spaces of four or three dimensions (Papers from the Chicago Congress, New York 1896, pp. 379), and naturally all groups that are composed in the same way, can be represented under an eleven-parameter mixed group of contact transformations that subsumes the group described in the text. The similarity transformations of ordinary space have been recently treated by Combebiac in this way (*Calcul des Triquaternions,* Thése, Paris, 1902).

It might be permissible for us to set down an especially handy way of writing the latter general formulas: In non-homogeneous form, they will be given by the quaternion equations:

$$x' = \bar{a}\,(xb + c) \quad \text{and} \quad \bar{x}' = \bar{a}\,(xb + c),$$

or by the similarly-defined equations:

$$x' = (\gamma + \beta x)\,\bar{\alpha} \quad \text{and} \quad \bar{x}' = (\gamma + \beta x)\,\bar{\alpha}\,.$$

We extend the manifold thus described from now on to a closed continuum in which we likewise allow systems of values that satisfy the equations $\mathfrak{X}_0 = \mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0$, and refer to them as the coordinates of a *pseudo-soma*. We shall not enter into a discussion of the geometric meaning of this analytic concept, but simply formulate the following theorem, whose proof and closely-related generalization can likewise find no place here:

Any analytic transformation that is everywhere defined, single-valued, and continuous in the continuum of the ∞^6 *actual and* ∞^3 *pseudo-somas belongs to a so-called mixed group* $\mathcal{G}_{28}$, $\mathcal{H}_{28}$ *with twenty-eight parameters whose continuous subgroup* $\mathcal{G}_{28}$ *is simple.*

These transformations will be, in fact, exhausted by the linear transformations of coordinates $\mathfrak{X}_0, \dots, \mathfrak{X}_{12}$ *that do not affect the validity of the quadratic equation* (18).

It now follows from known facts that the group $\mathcal{G}_{28}$, $\mathcal{H}_{28}$ is imaginary-similar to the group of conformal transformations of a space of six dimensions, and that it will be characterized completely by the invariance of the (additional to (18)) Monge equation:

(19) $$d\mathfrak{X}_0\, d\mathfrak{X}_{123} + d\mathfrak{X}_{01}\, d\mathfrak{X}_{23} + d\mathfrak{X}_{02}\, d\mathfrak{X}_{31} + d\mathfrak{X}_{03}\, d\mathfrak{X}_{12} = 0.$$

For this reason, we call our transformations of somas *pseudo-conformal* *) and remark that equation (19), and likewise the corresponding finite equation:

(20) $$(\mathfrak{X}\,\mathfrak{Y}) = \mathfrak{X}_0\, \mathfrak{Y}_{123} + \dots + \mathfrak{X}_{123}\, \mathfrak{Y}_0 + \dots = 0,$$

are easy to interpret geometrically:

The pseudo-conformal transformations of somas likewise have the characteristic property that consecutive actual somas that can be taken to each other by an infinitesimal rotation (or translation) are associated with somas with the same property.

As a result of this, they also have the further property of allowing one to go from any rotation chain in general to another one, or also to a translation chain, viz., a pencil of parallel somas.

The additional "in general" that was inserted here has the following *precise* sense: One will consider only such a neighborhood of an actual soma in which there are no somas that go to a pseudo-soma by the transformation.

We cannot treat the rich geometry of the group $\mathcal{G}_{28}$, $\mathcal{H}_{28}$ thoroughly here either, as we did with the group $\mathcal{G}_{31}$. We consider the group $\mathcal{G}_{28}$, $\mathcal{H}_{28}$ only on systematic grounds here. However, we would like to shed light on its meaning by presenting its connection with a

*) Incidentally, the expression "pseudo-projective" would be just as good – or just as bad – in its place.

beautiful theorem of G. Königs *). This theorem can be, in fact, formulated and extended in the following way:

If an analytic family of ∞^r actual somas has the property that any two neighboring somas of the family can be coupled by a rotation chain or a translation chain then its dimension number r can have at most the value r.

If it has the value three then any two finitely different somas of the family can also be linked by one of the chains described that lies completely in the family.

The figure then depends upon six constants and can be generated geometrically in one of the following ways:

1) *A soma will be reflected in all possible ways either in planes or points of space (or, in the second case, a soma will be subjected to all translations)* **).

2) *A soma will be subjected to the three-parameter group of all rotations around an actual or ideal point* **).

The ∞^6 families of the first of these two families will be permuted transitively amongst themselves by the transformations of the group $\mathcal{G}_{28}$. They will then define a single class of equivalent figures. Likewise, the ∞^6 families of the second family will define a class after one adds the continuum of the ∞^3 pseudo-somas. The transformations of the family $\mathcal{G}_{28}$ permute both classes.

One may enumerate the manifolds that correspond to the assumption $r = 2$ in a similar way (except for the sufficiently well-known case of $r = 1$). However, there are naturally an infinite number of classes that one can divide into *five families.* The families of each family will be permuted by the group $\mathcal{G}_{28}$ only amongst themselves or (in two cases) with manifolds of pseudo-somas. Three of these families are included in one or two of the aforementioned families of ∞^3 somas; the remaining two families, however, are of great interest. In fact, of the most beautiful (in terms of basic ideas) theorems of kinematics belongs to the theory of such families, for which one has Ribaucour to thank ***). One further comes to an interesting theorem of P. Stäckel by these considerations, which has already derived a group of 28 parameters from the theory of the deformation of surfaces ****). This group, in fact, arises from ours (to which it is, in addition, real-similar) by a change of spatial element.

We must postpone a thorough examination of the aspects of this that we believe are most important to differential geometry to another occasion. Here, we remark only that one must proceed carefully as long as one also has to consider imaginary figures. We do not need to use the word "soma" at all in this case. It is only in the real domain that, at the very least, the actual somas are identical to the related figures that belong to the group $\mathcal{G}_{28}$, $\mathcal{H}_{28}$, as we have explained before. (One confers the discussion in § 28 on the

*) Cf., Königs, *Leçons de Cinématique*, art. 84, 85.

**) It thus arises especially for the figures that appear in our geometry of chains. Cf., pp. 571-573.

***) Ribaucour, *Sur la déformation des surfaces*, Comptes rendus, t. LXX, 1870, pp. 330. Cf., Darboux, *Théorie des Surfaces, I*, Paris, 1887, art. 58-61. Königs, *Cinématique*, art. 84. The theorem of Ribaucour that is reproduced by the cited authors is, moreover, incorrect, and the present version seems to stray somewhat far from reality. We shall communicate a corrected formulation in another place.

****) Comptes rendus, t. CXXI, 1895, pp. 396.

difference between the concepts of ray and line.) Certainly, any further advances into this domain will necessitate a careful construction of the terminology.

The group $\mathcal{G}_{28}$ includes a continuous subgroup with 22 parameters whose transformations have the characteristic property of consistently associating actual (real) somas with other ones, and which then subsumes all analytic transformation that associate rotation chains with other ones (and parallel somas with other parallel somas) *without exception.*

In this group, which one can consider to be a piece of the group of affine transformations in the line continuum, one finds the group of all transformations of actual somas that are simultaneously pseudo-conformal and projective, and can thus be compared to the similarity transformations of the real actual lines or rays that are simultaneously projective and radial-projective. However, while the ∞^7 similarity transformations define a continuous group (in the usual sense of the word), here, this produces a deviation from the analogy up to now (that is necessitated merely by the differing characters of the dimension numbers):

The totality of all transformations of actual somas that are both pseudo-conformal and projective defines a mixed group $\mathcal{G}_{13}$, $\mathcal{H}_{13}$ *with thirteen parameters.*

This group is therefore described by the fact that linear chains and rotation chains go to other such chains in the infinitesimal – and consequently, also in the finite.

It consists of all transformations of the group $\mathcal{G}_{31}$ *that coincide with the discordants to them, or therefore, that do not affect the coincidence of two somas on different sheets.*

The similarity transformations of the somas (pp. 560) also belong to this group in a trivial way. We consider only the synectic transformations that are contained in $\mathcal{G}_{13}$, $\mathcal{H}_{13}$ more closely, which have extremely remarkable properties. We call them *orthogonal*, because, when they are represented in dual coordinates, the change the dual-quadratic form:

(21) $$(X\,X) = X_0^2 + X_1^2 + X_2^2 + X_3^2$$

only by a (dual) factor.

The geometry of orthogonal transformations of somas.

As is clear with no further assumptions, the orthogonal transformation define a mixed group $\mathcal{G}_{12}$, $\mathcal{H}_{12}$ with twelve parameters whose transformations can be distinguished as *actual* ($\mathcal{G}_{12}$) and *ideal* ($\mathcal{H}_{12}$).

A glimpse at the expression (21) *now shows that the entire conceptual content of non-Euclidian geometry in spaces of positive curvature must find a new and essentially geometric interpretation in the geometry of these groups the orthogonal somatic transformations* *).

*) For to the applicable precautions, one confers pare 569.

We will prove that the basic notions of non-Euclidian geometry, in particular, the so-called elliptical spaces, are almost as simple as they were before, and thus the concepts of kinematics that we will explain can be placed beside the tools of elementary geometry, which have the same connection to each other – for the most part – as the other ones, and we will then illustrate these thoughts with some applications.

We have already seen that the concepts of *point*, *line, and plane* run parallel to the concepts of *actual soma*, *linear chain*, *and planar chain*, resp. *However, the question that we must now ask refers to the concept of "the distance between two points,"* since the further concepts of angle, surface area, and volume, and ultimately the most esoteric theorems of differential geometry, rest upon it, and indeed any possibility of a transition to kinematics. The answer leaves nothing to be desired in the name of simplicity. If 2ϑ (determined mod 2π) means the rotation angle and 2η the magnitude of translation of the screw that makes the actual soma X cover another one Y, and we then call $\vartheta + \eta\varepsilon$ *the dual distance* between the two somas (in which the sign will always remain arbitrary) then we find immediately:

$$\arccos \frac{(XY)}{\sqrt{XX}\sqrt{YY}} = \vartheta + \eta\varepsilon. \tag{22}$$

The orthogonal transformations of somas therefore have the (characteristic) property that the dual distance between two actual somas, or therefore that the rotation angle and the magnitude of translation of the motion that is determined by both of them remains unaffected.

Of the numerous consequences, which come about with little effort, moreover, we now explicitly cite only a few entirely special ones that are nonetheless especially important. We cover – say – the first sheet of the soma continuum, in turn, with two sheets by the adjunction of $\sqrt{XX}$. We thus obtain a new continuum of *oriented* somas that is analogous to the point continuum of *spherical* (Riemannian) geometry.

All synectic transformations that are everywhere defined, single-valued, and continuous in the continuum of (real) oriented actual somas define a continuous group $\mathcal{G}_{20}$ *with* 20 *parameters.*

It consists of all dual-conformal transformations of the somas, i.e., the synectic transformations that change the dual distance between consecutive actual somas X, $X + dX$ *by a (dual) proportionality factor that merely depends upon the location* X.

This group of somatic transformations can also be characterized by the fact that its transformations are synectic and always take soma spheres to other ones.

By the term "soma sphere," we naturally understand this to mean a manifold that is described by the equation $\vartheta + \eta\varepsilon$ = const., i.e., the totality of all actual somas Y that emerge from a given soma X by screws whose rotation angle 2ϑ and translation magnitude 2η are given.

The complete intersection of the group $\mathcal{G}_{20}$ with the group of projective-somatic transformations, or also the group of pseudo-conformal transformations, is the group of orthogonal transformations of the somas.

Now that we have thus characterized our group $\mathcal{G}_{12}$, $\mathcal{H}_{12}$ in no less than three different ways as the intersection of groups that are interesting in their own right, it will seem worthwhile to consider it somewhat more closely. We leave to the reader the task of making it clear just *which* concepts and theorems of non-Euclidian geometry will carry over to kinematics, moreover.

The parameter groups of motions [*]).

The group $\mathcal{G}_{12}$, $\mathcal{H}_{12}$, and likewise also a more comprehensive group that comes about by the addition of any transfer (*the somas;* cf., pp. 560), can be represented very conveniently with the help of quaternion algebra. The transformations of – e.g. – $\mathcal{G}_{12}$ are all described in the form:

(23) $$\dot{X}' = \dot{\bar{A}} \cdot \dot{X} \cdot \dot{B} \qquad (\mathfrak{a}_{00},\ \mathfrak{b}_{00} \neq 0)$$

(cf., pp. 557), and one can combine several of them according to the rule:

(24) $$\dot{\mathfrak{a}} \cdot \dot{\mathfrak{a}}' = \dot{\mathfrak{a}}'', \qquad \dot{\mathfrak{b}} \cdot \dot{\mathfrak{b}}' = \dot{\mathfrak{b}}''.$$

These formulas also yield a complete insight into the structure of our group, which cannot, however, be explained more precisely here.

We next emphasize:

The group $\mathcal{G}_{12}$ of actual-orthogonal transformations of the somas can be decomposed into two mutually-commuting groups $\mathcal{G}_6$, $\mathcal{G}_6'$. They are identical with (or similar to) the two parameter groups of the group of Euclidian motions, and the one of them $\mathcal{G}_6$ coincides with this group itself, moreover, when one chooses the actual (real) soma to be the spatial element.

The motions of somas will be obtained, in fact, when one chooses the combinations that are represented by the upper sign in the equations:

[*]) One may treat all of the parameter groups with a bilinear combination of the parameters in a manner that is similar to these parameter groups, and to a certain degree, also any arbitrary parameter groups. Cf., Leipz. Ber. 1889, pp. 177, *et seq.*

$$(25)\qquad \left\{\begin{array}{c} \mathcal{G}_6 \\ \dot{X}' = \dot{X}\cdot\dot{A} \\ \hline \dot{X}' = \dot{A}\cdot\dot{X} \\ \mathcal{G}'_6 \end{array}\right\} \qquad \begin{aligned} X'_0 &= \mathfrak{a}_0 X_0 \mp \mathfrak{a}_1 X_1 \mp \mathfrak{a}_2 X_2 \mp \mathfrak{a}_3 X_3, \\ X'_1 &= \pm\mathfrak{a}_1 X_0 + \mathfrak{a}_0 X_1 + \mathfrak{a}_3 X_2 - \mathfrak{a}_2 X_3, \\ X'_2 &= \pm\mathfrak{a}_2 X_0 - \mathfrak{a}_3 X_1 + \mathfrak{a}_0 X_2 + \mathfrak{a}_1 X_3, \\ X'_3 &= \pm\mathfrak{a}_3 X_0 + \mathfrak{a}_2 X_1 - \mathfrak{a}_1 X_2 + \mathfrak{a}_0 X_3. \end{aligned}$$

This now immediately raises the further question: How does one characterize the transformations of the group $\mathcal{G}'_6$ that is (in Lie's terminology) reciprocal to the group $\mathcal{G}_6$? Furthermore, what do the transformations of the family $\mathcal{H}_{12}$ mean? An answer to this – among others – is contained in the following theorems, the second of which completely represents one of the surprising special results of our investigation:

The group $\mathcal{G}_{12}$ of actual-orthogonal transformations consists of all synectic transformations of somas that take congruent linear chains – or also coaxial linear chains – to others of that kind *).

The ideal-orthogonal transformations (viz., those of the family $\mathcal{H}_{12}$) likewise have the characteristic property that amongst all synectic transformations they take congruent (coaxial, resp.) linear chains to coaxial (congruent, resp.) ones.

There really are such transformations!

The group $\mathcal{G}'_6$ that is reciprocal to the group of motions of somas consists of all analytic transformations that permute the individual members of any arbitrary family of coaxial linear chains (at most) amongst themselves.

The group $\mathcal{G}_{12}$ thus permutes the ∞^4 axes of the various families of coaxial chains only in a six-parameter way, and indeed the same is true for its subgroup:

$$(26)\qquad \dot{X}' = \dot{A}\cdot\dot{X}\cdot\dot{A} \qquad (\mathfrak{a}_{00} \neq 0),$$

which is similar to the so-called *adjoint group of motions*, and since its place is entirely representative, we may refer to them in precisely the same way here. This "adjoint group" then consists of all transformations of $\mathcal{G}_{12}$ that leave the protosoma, and thus, also the reciprocal plane C_4, at rest. The rays will be permuted amongst themselves by the motions in precisely the same way as the axes in the theorem on page 568 under the twists that take the protosoma to the somas of the reciprocal planar chain $X_1 = 0$.

Naturally, one can, with no further assumptions, construct the completely-determined transformation of $\mathcal{G}'_6$ that takes a given actual soma X to another arbitrary one X': One needs only to subject the soma-pair X, X' to all motions. However, it is likewise easy to find the transformations of $\mathcal{H}_{12}$. We say that two actual somas X, X' correspond to each

*) All analytical transformations with the same property define the aforementioned group $\mathcal{G}_{12}$.

other under a *reflection through an actual soma O* (or through the reciprocal planar chain) if they emerge from this soma by the opposite motions. These transformations, to which, we add the reflection through the protosoma:

$$X_0' = X_0, \qquad X_1' = -X_1, \qquad X_2' = -X_2, \qquad X_3' = -X_3, \tag{27}$$

exhaust all of the involutory transformations of $\mathcal{H}_{12}$. One can compose all ideal orthogonal transformations of somas from three of them and all actual ones from four of them. *Two distinct actual somas are symmetric when the associated reflections commute with each other, and conversely.*

We defer the further investigations that go into this to a greater degree to the reader, but we add a (very specialized) theorem to illuminate them:

There might be given two sequential, mutually-perpendicularly intersecting motion pointers (Laufstangen) (pp. 537, 552). An actual soma can be first screwed around the second motion pointer arbitrarily, and then the entire linear chain that results from this around the first motion pointer. This mechanism then allows the soma to take on ∞^4 *positions, and these define a synectic manifold that can obviously be described by* ∞^4 *congruent linear chains and* ∞^2 *coaxial linear chains.*

There are now $2 \cdot \infty^2$ *somatic reflections that leave the manifold considered completely at rest, but permute the* ∞^2 *chains that are congruent to the* ∞^2 *coaxial chains.*

The soma that belongs to such a reflection is an arbitrary one of the two well-defined linear chains that are congruent to the chains of the first family and are simultaneously coaxial to those of the second family.

The reflection through a soma is illuminated in a special case in Figure 46. In it, somas that can be permuted by motions of the indicator plane will be represented by arrows of equal length. Two coaxial rotation chains will be reflected through a certain soma *O* and will thus be taken to be congruent (and therefore, also parallel) rotation chains. The rotation, e.g., that makes the arrow 3 coincide with the arrow 0 of the first chain (the one with no number) will be the same one that makes it coincide with the arrow 3 of the last two chains.

The natural notion of equivalence between kinematics and the so-called inverse of a "motion."

If we consider all of the motions that perhaps make the protosoma *O*, which is thought of as being at rest, cover the somas *X* of an *r*-dimensional manifold of somas then we will have a family of ∞^r motions before us whose totality we, in turn, prefer to call – by a less-fortunate choice of word, it seems – an [*r*-dimensional] "motion." Let $X(t_1, \ldots, t_r)$ be the family of somas and let $S(t_1, \ldots, t_r)$ be the associated so-called motion, so one obtains a new family of somas and a new "motion," which can likewise be known, firstly, when one performs an arbitrarily-chosen motion B on the entire figure (with the exception of the protosoma), secondly, when one lets a soma that is rigidly linked to the protosoma enter in place of it that arises from O by any well-defined motion A. These r-dimensional motions, which are all representable in the form:

$$S'(t_1, \ldots, t_r) = A^{-1} \cdot S(t_1, \ldots, t_r) \cdot B, \tag{27}$$

will generally be deemed to be *equivalent* in kinematics, although generally – as it seems – it has not been deemed necessary to expressly define this notion of equivalence. We would like to say perhaps:

The notion of equivalence that belongs to the group $\mathcal{G}_{12}$ *of actual-orthogonal transformations of somas is the natural notion of equivalence in kinematics. That is, any class of r-dimensional manifolds of somas that are equivalent relative to* $\mathcal{G}_{12}$ *corresponds, according to the usual terminology, to a certain "type of motion," and conversely.*

The following remark represents an especially important extension of this:

If one subjects any of the stated classes to an arbitrary ideal-orthogonal transformation of its somas – thus, e.g., to a reflection through a soma – then the so-called inverse type of motion emerges from the associated type of motion.

In fact, from the definition of reflection through the rest protosoma, the moving soma X' relates to it in precisely the same way that O itself relates to the moving soma X.

In our scheme of things, the notion of equivalence in kinematics seems to be an individual term, at a definite place, in a whole series (if not, several series) of notions of equivalence. However, the last term – at least, the last term of any general interest – defines another notion of equivalence in this series that we would, in fact, like to refer to as the *equivalence notion of mechanics.* Namely, in mechanics it is well-known that the position of the rigid body under scrutiny is not indifferent to its mobility-restricted mechanism, and one can then – for a certain, not-too-special choice of the body – deem to be equivalent only such families of somas that can be represented in terms of one of them by means of some transformation of the group of motions $\mathcal{G}_6$ in the form:

$$S'(t_1, \ldots, t_r) = S(t_1, \ldots, t_r) \cdot B. \tag{28}$$

Therefore, only congruent manifolds of somas are mechanically equivalent. *)

The aforementioned facts show that the relatively specialized theorems on the kinematic extension of non-Euclidian geometry and the theory of groups of motions $\mathcal{G}_6$

*) The concept of symmetry breaks down here since, from the definition of soma, symmetric rigid bodies must all be regarded as completely different figures. The fact that they obey the same laws for a simultaneous symmetry in the applied forces is so self-explanatory that no deficiency can be found when this situation does not find its expression in the formulation of the notion of equivalence above.

that is included in it give them a special place in our system. We are then immediately curious to know about the mathematical concepts that nature itself seems to present us with as occupants of Euclidian space, or at least one that carries us. However, these are the two types of equivalence that we just spoke of. Either of them points to a certain way of dealing with kinematics, and almost all of the investigations that have been carried out in this realm fall into one of these two broad categories. If such a connection actually exists between this undoubtedly important field of research and non-Euclidian geometry, which we believe we have shown, such that in the latter concatenation of concepts only other things need to be interpreted in order to yield kinematic theorems, then it may certainly be considered to be one of the most noteworthy and important of the wonderful phenomena that the presence of logically-equivalent inferences affords in different mathematical disciplines. The meaning of non-Euclidian geometry is thus cast in a new light.

For our way of dealing with kinematical problems, it is obviously characteristic that the moving rigid body is (at first) introduced in the calculations and constructions as an *atom*, so to speak, and not as a whole composed of simpler components. However, in reality, the rigid body of theoretical kinematics is the totality of all the points, curves, surfaces, etc., that are rigidly bound with it, and thus it is already an entire world unto itself, which, when placed in motion, affords a wealth of new and interesting phenomena in comparison to the geometry of figures at rest. Once the theory that was sketched out here is developed further, one will have the right to demand an extension of it in the same preferred direction in which the development of kinematics has been moving up to now. Something that is thus perhaps left for us to do as an – albeit modest – contribution is to demonstrate that there is also at least the possibility of an organic further development in this direction. For this purpose, we choose entirely simple figures – viz., the aplanar one and two-dimensional chains of somas – whose geometric manner of generation we once more recall: They come about by twisting a soma around all rays of a chain of rays or an aplanar congruence of chains with the help of figures for which we have already given numerous constructions before, which are also based in elementary geometry.

Further properties of some chains.

We next recall a theorem of Darboux, who has determined all non-trivial motions (see pp. 589) under which any arbitrary point describes a plane curve [*]). The theorem on the base point curves of chains of rays (pp. 345) now brings to light the fact that the type of motion that is determined by an aplanar C_1 belongs to this family of motions. However, the same thing is true for the inverse motion, and this property is obviously characteristic. We express this as follows:

The aplanar one-dimensional chains of somas are the Darboux families of somas that are taken to other such families under the reflection through any, and consequently each (actual), soma.

[*]) Darboux, "Sur les mouvements algébriques." (Note III in the kinematics of Königs).

Any arbitrary one of these somas (inter alia) will be found when one lets a soma slide along a direction of motion, and thus any point of the soma that does not lie on the direction of motion is confined to a plane that is not parallel to the direction of motion. All of the points that do not lie on the direction of motion then describe an ellipse, and all of the planes that are not perpendicular to the direction of motion envelop a cone of rotation whose axes are parallel to the direction of motion [**]).

We now consider two-dimensional manifolds.

Darboux has likewise made us aware of a family of motions under which any point describes a *Steiner surface*, or a degeneration of this figure. One comes to these motions (which are not, in fact, the only ones of this kind) when one represents the parameters α_i in the formulas on page 176 as homogeneous linear functions of three *essential* parameters $\sigma_1 : \sigma_2 : \sigma_3$, and simultaneously expresses the quantities a_{10} , a_{20} , a_{30} as quadratic functions (forms) of just these parameters. We would like to say that the corresponding manifolds of somas belong to *Darboux families.* One now finds (with the help of a minor computation) similarly to above:

Any family of somas of the Darboux family whose mirror image relative to a soma likewise belongs to a Darboux family is a two-dimensional aplanar chain of somas, and conversely.

From what we just stated, there are precisely as many kinematically-different classes of these special Darboux "motions" as there are different classes of aplanar congruences of chains under motions. For the sake of brevity, we consider only the interesting case, which corresponds to the *first* type of our chain congruences, and then immediately obtain from the properties of the basepoint surface that were discussed on pages 346, 347 the following further kinematic way of generating our manifolds:

The figure of two points o', o'' *and two planes* ω', ω'' *that was described on page 464 can be doubled, and one then lets the two congruent figures that thus arise be distinguished by the indices* 0, 1. *Points of both figures shall next be identified in such a way that:*

$$o'_1,\ o''_1;\ \omega'_1,\ \omega''_1 \qquad \textit{coincides with} \qquad o'_0, o''_0;\omega'_0,\ \omega''_0,$$

resp.

An arbitrary soma O shall now be rigidly linked with the figure (0), *and likewise the soma X that emerges from O by a twist around the axes* o' o'' *or* ω', ω'' *is rigidly linked with the figure* (1).

One now lets the second figure, and the soma X along with it, move in such a way that the points and planes:

$$o'_1,\ o''_1;\ \omega'_1,\ \omega''_1,$$

resp., are united with the planes and points:

[**]) Cf., further, Schoenflies, Math. Ann., Bd. **40** (1891), pp. 317, *et seq.* By the way, one can also derive a further broadly-encompassing classification principle form the ideas of Darboux.

$$\omega_0', \ \omega_0''; \ o_0', o_0'',$$

resp. The soma X then runs through a two-dimensional aplanar chain of somas. Namely, it remains symmetric to the soma O at rest in each of its possible positions, and the associated twist axes fill up the aplanar congruence of chains that belong to the two planar pencils (o_0', ω_0') *and* (o_0'', ω_0'').

Any two-dimensional chain of somas that comes about by reflection of a soma through the rays of an aplanar congruence of chains of the first type will be generated this way, and indeed in a single – real – way.

However, the two-dimensional aplanar chains of somas may be characterized in yet another way. Namely, one comes to the following theorem (cf., page 385, 413) very easily:

If a two-dimensional family of somas X arises when one subjects a soma O to all twists around the (real) rays of an analytic congruence whose rays cannot be distributed on the cylinder then in the neighborhood of a point X in general position the axes of all screws that link consecutive somas of the family will generally define a ray complex that can be described by ∞^2 *chains of rays. If one reflects the entire figure through the soma O then it will either remain at rest or one will obtain the same figure twice.*

If one now takes the second of these two congruent figures away from their original position in such a way that one makes O coincide with X by a twist then this will produce a new coincidence of two mutually corresponding chains of rays of both complexes. The common principal axis of these two chains or this double chain will be the twist axis. Any infinitesimal screw that takes X to a neighboring position will take the form of a ray in the associated double chain.

The only exception to this rule is defined by the aplanar two-dimensional chains of somas.

In fact, in this case, and only in this case, does one obtain only a congruence instead of the complex. The present congruence is then an aplanar congruence of chains, and the one that is derived from it is its reciprocal.

Naturally, no ray of the reciprocal congruence belongs to a single soma then, but to a singly-infinite number of them, and these somas will also again define an (aplanar one-dimensional) chain.

Without a doubt, the three-dimensional chains of somas will yield a much richer bounty of geometric properties. However, this situation, which far exceeds – in scope and, for that matter, in difficulty – the bounds of the investigation of congruences of chains that was carried out in §§ 37-39 allows us to use it as a basis entirely, just as it immediately affords us completely different extensions of the problems treated.

Concluding remarks.

If we now apply the extension principle that was just set down to the special constructions of non-Euclidian geometry that were spoken of in § 14 then we will recognize that all of the theorems that we were concerned with in Part I are capable of a meaningful generalization (under which the connection with the composition of forces is lost). However, a similar statement will be true of the constructions of quaternion theory, in which the geometric addition of vectors is intertwined with the composition of finite rotations around a fixed point into a greater whole. We can say no more here about the many new questions that this raises; however, in the following statement we would like to point out – at least, superficially – a further direction of possible development:

The geometry of dynames, or at least a large part of the constructions that have been summarized under this terminology, and the study of the constructive composition of finite motions define different pieces of a more comprehensive theory of certain geometrical constructions that is analogous to Hamilton's geometric theory of quaternions and subsumes it [*]), *and in which the geometry of dynames itself occupies a position that is similar to that of the geometric addition of vectors in the theory of quaternions.*

The elementary theory that we have in mind here will exceed the possibility of construction in terms of quaternions to the same extent that the theory of the geometry of dynames exceeds the addition of vectors. The associated analytical apparatus will consist of a system of complex numbers with eight – or, better yet – *sixteen* units.

We thus ultimately return to the ideas that defined the starting point for the author's kinematic investigations, and thus for all of the results that were contained in the present volume. These are the same ideas that were found in a vague and incomplete form in many of the papers of Clifford, and which were further developed in another direction by Lipschitz (in his investigations of sums of squares), but were scarcely noticed, however.

We believe that even with the examples that are presently known this does not exhaust the cases in which systems of complex quantities can be employed in geometric investigation for the purpose of discovering new truths. E. g., the theory of certain (finite and infinite) groups of contact transformations affords a broad field for further investigations of this type.

Some English mathematicians have sought to clarify and apply the aforementioned ideas of Clifford. However, it seems to this author – to the extent that the author knows of his work – that it lacks clearly-posed problems and, above all, the necessary skills. It and others might be the lamentable casualty of an earlier and, in part, ostensibly also still present, customary system of education in England.

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[*]) On this subject, we do not desire to say that these things should also be represented in the form that is the most circuitous and most distant from the classical paradigm that Hamilton himself and his followers chose for the representation of quaternion algebra, and whose conservation – we remark in passing – seems to be the actual goal of the recently-formed "Society for the advancement of quaternion theory." The fact that these deviations from the forms of expression that are customary for mathematical ideas are unnecessary can, as is clear with no further assumptions, be seen from the author's elementary treatment of the theory of quaternions. (Communications of the Society for Natural Science in Greifswald, v. 31, 1899 [Berlin, 1900, pp. 1-49].)

In other countries, the development of such ideas seems to stand in the path of the opinion (which has also been expressly formulated by some) that investigations into the wide variety of complex quantities must be "unfruitful." One must therefore give an interpretation of a well-known observation of Gauss that, in our opinion, must be regarded as incorrect. The fact that other systems of complex quantities cannot be employed "in general arithmetic" in the usual manner has certainly been established by the investigations of Weierstrass and others connected with him. However, at the very least, nothing can be ruled out concerning its usefulness in limited realms and for specific purposes, insofar as one knows nothing at all about these purposes. No type of judgment requires more care than the assessment of the future benefits of any direction of research. Useful ideas do not always appear in an equally suitable form, and indeed at any time facts can come to light that open up a new domain of applications for a line of reasoning that was previously in the background. Gauss himself also seems to have taken one such possibility into account, so we may therefore hardly assume that the restriction that lies, in his own words, "in general arithmetic" was added by him without some deliberation.