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1

PROJECTIVE RELATIVITY THEORY

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Foreword

This little book makes no claim to completeness. It offers little more than a representation of a personal viewpoint that I hope will be of use in the further treatment of the geometries described here and other applications. The same is true for the bibliography at the end. No attempt has been made to make it complete. On the other hand, it includes numerous treatises for which no specific mention is made in the text, but which I hope can perhaps be of use for further investigations of the topic.

First, I would like to thank three young German mathematicians very much. F. JOHN assisted me in the preparation of my lecture at Göttingen in the Summer of 1932. His *elaboration* of this lecture defined a first draft of the present manuscript. I would like to make note of Herrn NÖBELING, for the supplementary material that he contributed on the occasion of the lectures that I gave in Vienna. Finally, after my lectures in Hamburg G. HOWE helped me with a sweeping revision of the entire manuscript and thus also stimulated many various improvements.

Fynshav, in August 1932. O. VEBLEN.

Upon the conclusion of the corrections, I would like to express my thanks to my Princeton collaborators, and in particular J. L. VANDERSLICE, who has carefully maintained the unity of the manuscript.

Princeton, in November 1932. O. VEBLEN.

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I. Unified theory of gravitation and electromagnetism.

One of the many achievements of EINSTEIN's general theory of relativity is that the theory of gravitation has been geometrized. This geometrization arises from the assumption that one must regard the world of physical phenomena as a four-dimensional spacetime continuum. Such a continuum is, by definition, representable by coordinate systems. A coordinate system is merely a map of a class of world-points to a class of number-quadruples, or, as one can also say, *number-points* (x^1, x^2, x^3, x^4). Therefore, the first axiom, or the first group of axioms, of relativity theory must also represent a statement of the existence and the properties of this map. Whether or not I consider it to be important that these axioms be clearly formulated, nevertheless, I will not go into the particulars of that problem in this work, since J. H. C. WHITEHEAD and myself have thoroughly presented those axioms in a recently-appearing work (Bibliography 1932, 10).

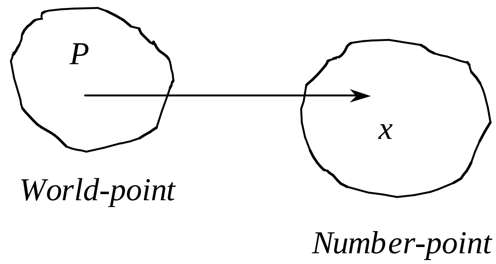

Fig. 1.

Furthermore, this geometrization implies the assumption of a definite spatial structure to the universe. In fact, this structure resides in nothing more than the existence of ten functions of position:

$$g_{ij}(x^1, x^2, x^3, x^4) \qquad (i, j = 1, \ldots, 4)$$

$(g_{ij} = g_{ji})$ in each coordinate system. Since these functions are uniquely defined in each coordinate system we denote them as the components of a *geometrical* (or physical) *object*. If these components are given in one coordinate system then they are determined in any other coordinate system by a simple linear transformation law. Due to the particular linear form of this transformation law this geometric object is called a tensor, and indeed one refers to it as the *fundamental tensor* of a RIEMANNIAN space.

It is not essential for one to use the geometrical language. Everything proceeds in a logically smooth fashion if we regard the g_{ij} as ten gravitational potentials and treat the entire theory in a purely analytical way. However, this is not as interesting and stimulating (at least for the scientists of our epoch) as when we regard the g_{ij} as the coefficients of a quadratic differential form:

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

that RIEMANN defined along a curve:

$$x^i = x^i(t)$$

by the integral:

$$\int ds,$$

and can thus geometrically express and motivate an entire series of theorems. However, one must freely observe that in the case of EINSTEIN's theory the geometrical measurements are connected with gravitational phenomena.

Each particular choice of fundamental tensor distinguishes a particular RIEMANNIAN space. Classes of RIEMANNIAN spaces may be obtained as solutions of systems of partial differential equations in which the g_{ij} represent independent variables. By a clever choice of such a system of equations, EINSTEIN succeeded in singling out particular classes of RIEMANNIAN spaces that are capable of being given a genuine physical interpretation.

In order to find this interpretation, one employs coordinate systems (normal or inertial coordinates) that have a definite geometrical meaning and allow for a particular decomposition into spacelike and timelike components. By this means, the geometrical theorems of EINSTEIN's classes of RIEMANNIAN spaces may be translated into ordinary physical theorems. One thus finds that a large part of classical physics is contained in the ten components g_{ij} , and since gravitational phenomena play a leading role in this part of the theory, the identification of the g_{ij} as gravitational potentials seems justified. The unified character of this theory finds its expression the fact that the gravitational potentials are the components of a single geometrical object.

The essential difference between this theory and the previous NEWTONIAN theory of gravitation that we would like to emphasize is the following: In the older theory, one thought of a Euclidian space as being given to begin with, and then introduced gravitational potentials into this theory. However, these potential functions have no influence on space itself. The properties of space are completely independent of those of the potentials. In EINSTEIN's theory, by contrast, the properties of space are identical with the properties of the gravitational potentials g_{ij} .

On the other hand, in the world of these gravitational potentials, electromagnetic phenomena have nothing to with the geometrical structure of space. In EINSTEIN's theory the electromagenetic potentials are – so to speak – foreign, just as the NEWTONIAN potential functions were in Euclidian space. Whether more less matter exists in the universe leaves the Euclidian geometry unchanged. Likewise, no direct effect of electricity on spacetime structure was present in the general theory of relativity.

The problem of discovering a spacetime structure that depends, not only upon the gravitational potentials, but also on the electromagnetic potentials was first attacked by H. WEYL in the year 1918. Despite the fact that WEYL's attempt was physically unsuccessful, he has produced a very beautiful geometry as a fruit of that labor. The next attempt was made by T. KALUZA in 1921. KALUZA replaced the four-dimensional continuum with a five-dimensional one, and then introduced a RIEMANNIAN metric into this continuum, and he succeeded in establishing field equations that yielded the EINSTEIN gravitational equations and the MAXWELL electromagnetic equations in the first approximation.

The KALUZA theory was simplified by O. KLEIN (Bibliography 1926, 5; 1927, 11) in such a way that the EINSTEIN-MAXWELL theory emerged, not approximately, but in its precise form. Since then, various other mathematical physicists have pursued the theory and found its formal structure very tempting. However, a fundamental question with no satisfactory answer still remains: What is the meaning of the fifth dimension? One has found no compelling basis for doubting our conviction that the physical universe is four-dimensional. Therefore, theoreticians, and above all, EINSTEIN himself, have carried out a series of investigations with the purpose of creating a four-dimensional theory. Some of these efforts are listed in the Bibliography at the end of this book.

After five or six years, the thought came to me that a possible solution to the unification problem for the spaces that many people have investigated in the last ten years might be in finding a generalization of projective geometry. Before we present the basic geometrical ideas upon which our solution of the unification problem rests, we discuss some notions of ordinary relativity theory.

We thus now work with the ordinary spacetime that relates to the coordinates x^1, x^2, x^3, x^4. We consider the differentials dx^1, dx^2, dx^3, dx^4 of the coordinates. How are they to be geometrically interpreted? Their basic property is that they are transformed linearly by a coordinate transformation:

$$\bar{x}^i = \bar{x}^i(x)$$

according to the formulas [1]:

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j ; \tag{1}$$

the dx^i may thus be interpreted as affine coordinates in a four-dimensional space. Any point (x^1, x^2, x^3, x^4) of the *base space* is therefore associated with an affine "tangent space." However, the point (0, 0, 0, 0) of the tangent space, whose coordinates remain unchanged under all transformations (1), can thus be identified with the point (x^1, x^2, x^3, x^4) of the base space and regarded as a contact point. Each coordinate transformation of the base space induces an affine transformation of each tangent space.

If our base space were – say – one-dimensional then we could represent it as a curve, and the tangent space at P would be the usual tangent to the curve at P. The variable x is a parameter that establishes the position of a point on the curvem and the parameter dx determines the position of a point on the tangent line. In the four-dimensional case, such an intuitive picture is no longer possible due to the limitations of our visual imagination. Therefore, the corresponding geometrical expressions prove to be helpful and suggestive. We must therefore start with a four-dimensional base space or universe and then introduce a set of tangent spaces, each of which is attached to a definite point of the base space.

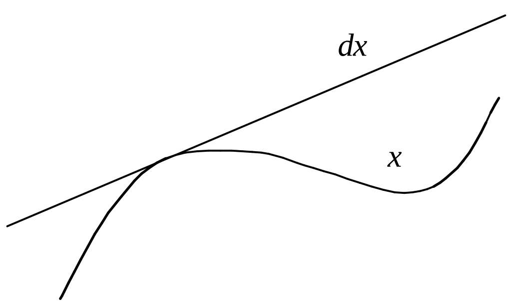

Fig. 2.

The $g_{ij}(x)$ are constant on any given tangent space. Therefore:

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

is the square of the distance between the origin and the point dx relative to a Euclidian metric on the tangent space at x. The points dx that satisfy the quadratic equation:

$$g_{ij}\, dx^i\, dx^j = 0,$$

[1] In this work on relativity theory, it is self-explanatory that we use the EINSTEIN summation convention.

define a cone through the origin: the light cone. In the case of relativity theory, this cone is real since the quadratic form g_{ij} is indefinite. According a viewpoint that was stressed by E. CARTAN in particular (Bibliography 1928, 1) the RIEMANNIAN geometry of the base universe should be regarded as the theory of these associated Euclidian tangent spaces.

The generalization that we have in mind is the following one:

Instead of a cone, such as the light cone that we would encounter in relativity theory, we would like to associate a completely general non-degenerate surface of second order in each tangent space. By means of this quadric surface, a quadratic cone $g_{ij}\,dx^i\,dx^j = 0$ is likewise distinguished in each tangent space, namely, the tangent cone through the origin, and a hyperplane: the polar plane of the origin. The polar plane includes the contact point of the tangent cone with the surface. The polar hyperplane shall represent the electromagnetic potentials, whereas the cone represents the gravitational potentials (Fig. 3).

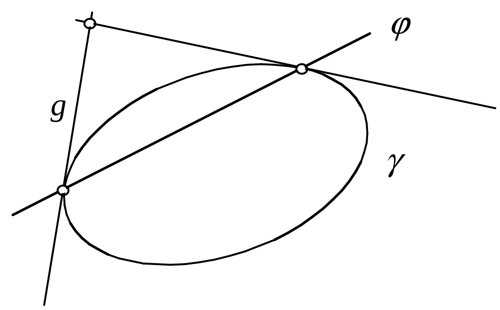

Fig. 3.

Instead of a Euclidian geometry in each tangent space, we now have a non-Euclidian geometry, in which our quadric surface is the absolute surface in the sense of CAYLEY. Our new geometry is therefore the collective theory of this set of CAYLEY spaces, just as RIEMANNIAN geometry was the theory of Euclidian spaces that were tangent to the base space.

The computational apparatus that seems to be the most suitable for our purposes was briefly presented by B. HOFFMANN and myself in the "Physical Review" (1930) at the conclusion of a previous work in the "Quarterly Journal of Mathematics" (Oxford Series, 1930). The correspondence with the formalism of the KALUZA-KLEIN theory is so complete that HOFFMANN and myself have regarded our theory to be the geometrical basis for the KALUZA-KLEIN theory. We therefore emphatically stress that our theory arises from viewpoints that are completely different from the KALUZA one. In particular, we claim no relationship between electrical charge and the fifth coordinate; our theory is thoroughly four-dimensional.

Independently of our investigations, EINSTEIN and MAYER (Bibliography, 1931, 3) have published a "unified field theory" that leads to essentially the same results as ours (cf. chap. VIII). Furthermore, various papers of J. A. SCHOUTEN and D. van DANTZIG (Bibliography, 1931, 7; 1932, 3, 4, 8, 9; 1933, 1) have recently appeared, in which the projective theory of relativity was treated in a different form.

It is noteworthy that – in a mathematical manner of speaking – all of these theories seem to converge to each other. Therefore, one might hope that one will actually arrive at a definitive solution of the unification problem in the manner that is entered into here. We will discuss the limitations of this solution in chap. VII.

II. Projective tensors.

The generalized non-Euclidian geometry that was sketched out in chapter I is the theory of a set of tangent spaces, each of which contains a quadric surface. The theory of a quadric surface finds its most satisfactory form in the spaces of ordinary projective geometry. For this reason, it is natural to look for a representation of the generalized non-Euclidian geometry in the spaces of a generalized projective geometry. In fact, such a generalized projective geometry is not hard to find now. It is a branch of the differential geometrical investigations of the last decade. The various ideas of this geometry are gradually increasing from the efforts of a great number of mathematicians. In particular, the investigations of H. WEYL, E. CARTAN, J. A. SCHOUTEN, L. P. EISENHART, and T. Y. THOMAS must be mentioned. These ideas have been presented in very many different forms. I would like to sketch them out in the form that I myself have adopted.

Affine tensors.

Now, for the sake of orientation, we make a few remarks about ordinary – or affine – tensors. There is a large set of admissible coordinate systems on the base space, which all go over to each other by means of analytic transformations:

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3, x^4). \tag{1}$$

On the other hand, in the tangent spaces there is only the relatively small set of coordinate systems that are connected with each other by linear transformations:

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j}\, dx^j\,. \tag{2}$$

Therefore, we call the tangent spaces "centrally affine" spaces. Their geometry depends on the affine group, and there is a distinguished point – the contact point (0, 0, 0, 0) – whose coordinates are unchanged by the transformations (2).

In this way, the theory of the base space may be reduced to the simultaneous affine geometry of this set of affine spaces. Tensors define a suitable device for the treatment of this simultaneous affine geometry. As a first example, we take a contravariant vector, i.e., a contravariant tensor of rank one. That is a geometrical object that possesses four components:

$$V^1(x), \ldots, V^4(x),$$

that are functions of x in any coordinate system. A particular point $(dx^1, \ldots, dx^4)$ in each tangent space is determined by the equations:

$$dx^j = V^i(x).$$

This relationship is independent of the choice of coordinates, since differentials transform precisely like contravariant vectors according to well-known tensor transformation laws. Therefore, one can associate each point of any tangent space with a contravariant vector.

Likewise, one can associate each hyperplane through the origin in any tangent space with a covariant vector. The points that satisfy an equation $A_i\, dx^i = 0$ define a flat space of dimension three. In general, the components of an n^{th} rank covariant tensor are the coefficients of the equation:

$$A_{ij\ldots k}\, dx^i\, dx^j \ldots dx^k = 0,$$

and the points of any tangent space that satisfy this equation define an n^{th} order cone. Above all, the theory of tensors is an affine-algebraic geometry of the tangent spaces with respect to their simultaneous behavior.

Introduction of homogeneous coordinates.

The question now lies close at hand of whether there are spaces of other kinds that can play the role of tangent spaces and whose totality can then be the subject of a new theory. This question can be answered in the affirmative in various ways. In our case, we extend the ordinary tangent spaces to projective spaces; another viewpoint will be discussed in chap. VIII.

We know how we can geometrically proceed with this extension of the affine tangent spaces to the projective spaces. Each bundle of parallel lines is associated with a figurative, or imaginary, point. This imaginary point represents an "infinitely distant" point for each line of the bundle. Three infinitely distant points are called collinear when and only when they are the infinitely distant points of three lines in the same plane. Four infinitely distant points are called coplanar when they are the infinitely distant points of four lines in the same three-dimensional space. However, it must be stressed that this introduction of infinitely distant points must take place in each tangent space.

It is more convenient to use homogeneous coordinates for the analytical treatment of projective tangent spaces. Instead of the four affine coordinates $dx^1, \ldots, dx^4$, we would like to introduce five coordinates $X^0, X^1, \ldots, X^4$, such that $(X^0, X^1, \ldots, X^4)$ and $(hX^0, hX^1, \ldots, hX^4)$ define one and the same point, which implies that only the ratios of the X have any meaning. A relation of the form:

$$dx^i = \frac{X^i}{\varphi_\alpha X^\alpha} \tag{3}$$

shall exist between the affine and the projective coordinates, in such a way that the infinitely distant points of our space satisfy the equation [1]:

$$\varphi_\alpha X^\alpha = 0. \tag{4}$$

[1] Greek symbols shall always take the values 0, 1, …, 4; Latin ones only take the values 1, …, 4.

We will return to the connection between this definition of the homogeneous coordinates and the one that is used in elementary textbooks $\left(dx^i = \frac{X^i}{X^0}\right)$.

The proportionality factor.

The homogeneous coordinates are therefore still not completely determined by these formulas. They lack a rule that would specify how the X^α and the φ_α behave under coordinate transformations. In order to obtain a theory of transformations that is able to respond to this question, we start with the fact that our homogeneous coordinates are determined only up to a proportionality factor k. It puts forth the fact that we can relate the choice of proportionality factor to the choice of infinitely distant plane.

The proportionality factor is arbitrary for any point of a given tangent space, and also for any point of the base space; in particular, it varies from tangent space to tangent space. Here, however, we are not seeking the most general theory that is possible, but only a generalization of the usual theory that is suitable for our purposes.

In particular, nothing changes when we multiply all homogeneous coordinates by a function of position $\sigma(x^1, \ldots, x^4)$. In order to treat the process of extension to an analytical function of position, we find it convenient to express the proportionality factor in the form:

$$k = e^{x^0}.$$

The homogeneous coordinates are then extended by a function of position $1/\rho(x)$ through the substitution [1]:

$$\bar{x}^0 = x^0 + \log \rho(x). \tag{5}$$

It is clear that geometrical or physical quantities that are described by a projective geometry must be invariant under (5). As in the usual tensor analysis, we further demand that they also must be invariant under coordinate transformations:

$$\bar{x}^i = \bar{x}^i(x). \tag{6}$$

This raises the question of finding the simplest invariants under both classes of transformations.

Projective scalars.

In order to study these invariants, we begin with scalars. An affine scalar has only one component in any coordinate system. The components in two coordinate systems $P \to x$ and $P \to \bar{x}$ are connected by the transformation law:

$$\bar{A}(\bar{x}) = A(x). \tag{6′}$$

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[1] Instead of this, one can use, e.g., $\int v_i\, dx^i$ in place of $\log \rho$, in which $v_i\, dx^i$ refers to a not-necessarily-integrable differentiable form.

For the definition of a projective scalar, we retain the transformation law:

$$\bar{A} = A.$$

Furthermore, we assume that x^0 enters into A as a parameter in the simplest way. Namely, A shall have the form:

$$A = e^{Nx^0} f(x),$$

in which N is a fixed number that obviously does not transform [1]. From (5), we have:

$$A = \bar{A} = e^{N\bar{x}^0} \frac{f(x)}{(\rho(x))^N}.$$

The number N shall be called the *index* of the scalar A. It roughly plays the role of a weight. However, we must reserve the word "weight" for another purpose since further invariants with a weight in the usual sense can arise.

The part of A that is independent of x^0 obeys the law:

$$\bar{f} = \frac{f(x)}{\rho(x)^N},$$

whereas A itself is subject to the simple law (6′). By comparison to an affine scalar, which has only one component, a projective scalar is understood to have infinitely many components. Any transformation of the parameter x^0 produces a new component from a given one.

However, if one does not transform the parameter x^0 then the components of a projective scalar behave exactly like the components of an affine scalar. To say that the parameter x^0 is not transformed is to say that we keep our space in a particular state, so to speak. If any scalar is given then each such state corresponds to a definite component of the scalar. As we will soon see, that is also true for all of our projective tensors. In particular, each state is associated with a definite coordinate system in each tangent space.

Gauges.

Due to the close connection between this notion and the one that was given the same name in WEYL's theory, I would like to call it a *gauge*. The parameter x^0, which I previously regarded as simply a factor, following a suggestion of J.H.C. WHITEHEAD, I would now like to call a *gauge variable*. We refer to a transformation of the form (5) as a *gauge transformation*.

Any projective scalar $e^{Nx^0} f(x)$ may be put into the form e^{Nx^0}. One need use only the gauge transformation:

$$\bar{x}^0 = x^0 + \log(f(x))^{1/N}.$$

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[1] A possible generalization might be to set, e.g., $N = \varphi(x^1, \ldots, x^4)$.

This gauge transformation is uniquely determined by the requirement that it take a given component of a projective scalar with an index that is different from zero to another given one. This assignment is independent of the choice of coordinate system.

By differentiating a component of a given projective scalar A with an index of N, we obtain five functions:

$$\frac{\partial A}{\partial x^0}, \frac{\partial A}{\partial x^1}, \ldots, \frac{\partial A}{\partial x^4},$$

of which the first one corresponds to the scalar itself, up to a factor N. Under an arbitrary gauge transformation:

$$\bar{x}^0 = x^0 + \log \rho(x) \tag{5}$$

and an arbitrary coordinate transformation:

$$\bar{x}^i = \bar{x}^i(x) \tag{6}$$

these functions go to five functions:

$$\frac{\partial \bar{A}}{\partial \bar{x}^0}, \frac{\partial \bar{A}}{\partial \bar{x}^1}, \ldots, \frac{\partial \bar{A}}{\partial \bar{x}^4},$$

which are given by the formulas:

$$\frac{\partial \bar{A}}{\partial \bar{x}^\alpha} = \frac{\partial A}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^\alpha} \qquad (\alpha, \beta = 0, \ldots, 4). \tag{7}$$

We agree to call the pair consisting of a given gauge and a given coordinate system a *representation*. In what follows, one can regard the pair of transformations (5) and (6) collectively as a transformation of the representation.

Projective vectors.

Equation (7) is a special case of the following one:

$$\bar{\varphi}_\alpha = \varphi_\beta \frac{\partial x^\beta}{\partial \bar{x}^\alpha}. \tag{8}$$

A geometrical object that has, in any representation, five components of the form:

$$\varphi_\alpha = e^{Nx^0} f_\alpha(x^1, \ldots, x^4)$$

that obey the transformation law (8) is called a *projective covariant vector of index N*. Each system of five components is thus associated with a definite representation.

Its null component φ_0 is a projective scalar, since one has:

$$\bar{\varphi}_0 = \varphi_\beta \frac{\partial x^\beta}{\partial \bar{x}^0} = \varphi_0.$$

In particular, the matrix of the transformation law is:

$$\left\| \frac{\partial x^{\alpha}}{\partial \bar{x}^{\beta}} \right\| = \left(\begin{array}{c|ccc} 1 & -\dfrac{\partial \log \rho}{\partial x^1} & \cdots & -\dfrac{\partial \log \rho}{\partial x^4} \\ \hline 0 & & & \\ \vdots & & \dfrac{\partial x^i}{\partial \bar{x}^j} & \\ 0 & & & \end{array} \right).$$

A coordinate transformation:

$$x \to \bar{x}$$

induces the transformation:

$$\bar{\varphi}_i = \varphi_j \frac{\partial x^j}{\partial \bar{x}^i}$$

of any non-null component. That is to say: The non-null components of a covariant projective vector behave like the components of an affine covariant vector under coordinate transformations. A gauge transformation then induces the transformation:

$$\bar{\varphi}_i = \varphi_0 \frac{\partial x^0}{\partial \bar{x}^i} + \varphi_i \; .$$

In particular, when φ_α has index zero and obeys the invariant condition:

$$\varphi_0 = 1,$$

then the gauge transformation looks like:

$$\bar{\varphi}_i = \varphi_i - \frac{\partial \log \rho}{\partial \bar{x}^i} \; .$$

We can then say: The four non-null components are determined only up to a gradient $\dfrac{\partial \log \rho}{\partial \bar{x}^i}$. This property is already recognized for the electromagnetic potentials. In other words: The four electromagnetic potentials are the non-null components of a projective vector whose null component is one.

A contravariant vector is defined analogously to a covariant vector. Now, we must assume the representation transformation law has the form:

(9)
$$\bar{X}^{\alpha} = X^{\beta} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}}$$

instead of (8). When we separate the null component from the other components this law looks like:

(10)
$$\begin{cases} \bar{X}^0 = X^0 + X^j \dfrac{\partial \bar{x}^0}{\partial x^j} \\ \bar{X}^i = X^j \dfrac{\partial \bar{x}^i}{\partial x^j}. \end{cases}$$

In other words: The non-null components behave exactly like the components of an affine vector. By contrast, the null component behaves like a scalar under a coordinate transformation, but under a gauge transformation it takes on a linear combination of the other components.

This difference in behavior between the null component and the other ones in the case of a covariant and contravariant vector implies that the projective tensor calculus is non-trivial. Otherwise, one could think of these tensors as only arising from a mere concatentation of affine tensors. In fact, one does find a decomposition into affine tensors, but it behaves differently for covariant and contravariant tensors.

The co- and contravariant tensors of higher rank are now formally defined in exactly the same way as the corresponding affine tensors. We will discuss some particular cases when the occasion to use them arises.

Homogeneous coordinates in tangent spaces.

We are now in a position to define the homogeneous coordinates in the tangent spaces precisely. Suppose we are given a covariant vector φ_α of index 0 that is arbitrary, but determined once and for all, and has:

$$\varphi = 1.$$

In order to characterize the homogeneous coordinates of a given point dx of the tangent space, we choose an arbitrary number k and set:

(11)
$$\begin{cases} X^i = k\, dx^i \\ X^0 = k(1 - \varphi_i dx^i). \end{cases}$$

To choose another number k means only that one multiplies the X^α by a proportionality factor.

In order to invert these equations, we remark that one has:

$$\varphi_\alpha X^\alpha = k,$$

and therefore:

$$dx^i = \frac{X^i}{\varphi_\alpha X^\alpha},$$

which constitute the previously introduced equations (3).

A coordinate transformation:

$$x \to \bar{x}$$

of the base space induces the transformation:

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j ,$$

and that, in turn, induces the transformation:

$$X \to \bar{X} ,$$

in which:

$$\bar{X}^i = K\, d\bar{x}^i = \frac{K}{k} X^j \frac{\partial \bar{x}^i}{\partial x^j}$$
$$\bar{X}^0 = K(1 - \bar{\varphi}_i d\bar{x}^i) = K(1 - \varphi_i\, dx^i) = \frac{K}{k} X^0 ,$$

because the φ_i behave like the components of an affine vector. Therefore, the X^i transform like the components of a contravariant vector under coordinate transformations. Therefore, the origin and the opposite side of the reference simplex remain invariant.

A gauge transformation induces no transformation of the dx^i other than the transformation:

$$\bar{\varphi}_i = \varphi_i - \frac{\partial \log \rho}{\partial x^i}$$

of φ_α. Thus, we have the transformations:

$$\bar{X}^i = \frac{K}{k} X^i ,$$
$$\bar{X}^0 = K(1 - \bar{\varphi}_i d\bar{x}^i) = K\left(1 - \varphi_i\, dx^i + \frac{\partial \log \rho}{\partial x^i} dx^i\right) = \frac{K}{k}\left(X^0 + \frac{\partial \log \rho}{\partial x^i} X^i\right) .$$

Under a gauge transformation, the X^α behave like the components of a projective contravariant vector.

As we previously suggested and have just now proved, a gauge represents not only a particular choice of components for each projective tensor, but also a particular choice of the side of the coordinate simplex that is opposite to the origin. Each gauge therefore corresponds to a particular equation for the infinitely distant hyperplane. Only in one particular case can we introduce the projective coordinates by the simple formula $dx^i = \frac{X^i}{X^0}$, or, what amounts to the same thing, by way of $dx^i = \frac{X^i}{NX^0}$. When a projective scalar A exists for which we have:

$$\varphi_\alpha = \frac{\partial \log \rho}{\partial x^\alpha} ,$$

then we can always have that $A = e^{Nx^0}$ and $\varphi_\alpha = N\delta^0_\alpha$ by a gauge transformation. Therefore, we have arrived at the announced conclusion of an elementary definition of projective coordinates.

Since our homogeneous coordinates behave like components of a contravariant vector under coordinate transformations we can regard the equation:

$$X^\alpha = A^\alpha,$$

as the characterization of one and only one point in each tangent space. Here, the index can be arbitrary, but different from zero; i.e., the functions A^α are of the form:

$$A^\alpha = e^{Nx^0} f^\alpha(x) \qquad N \neq 0.$$

Exactly as in the affine case, we can now interpret the various projective tensors geometrically. For example, let A_α be a projective covariant vector. Then:

$$A_\alpha X^\alpha = 0$$

is the equation of a hyperplane. It is possible to choose the gauge such that the equations of all of these hyperplanes reduce to the form $X^0 = 0$ when and only when A_α is a projective gradient:

$$A_\alpha = \frac{\partial A}{\partial x^\alpha}.$$

We do indeed know that A can be brought into the form $\bar{A} = e^{Nx^0}$ by gauging.

Projective tensors of rank two.

As our next example, we take a projective covariant symmetric tensor of rank two and index $2N$. On this occasion, we can make the previously sketched figure upon which the projective theory of relativity rests somewhat more distinct.

The tensor $G_{\alpha\beta}$ obeys the transformation law:

$$\bar{G}_{\alpha\beta}(\bar{x}) = G_{\sigma\tau} \frac{\partial x^\sigma}{\partial \bar{x}^\alpha} \frac{\partial x^\tau}{\partial \bar{x}^\beta}.$$

Due to the particular form of a transformation of representation:

$$\bar{x}^0 = x^0 + \log \rho(x)$$
$$\bar{x}^i = \bar{x}^i(x),$$

we have that:

$$\frac{\partial \bar{x}^\alpha}{\partial x^0} = \delta^\alpha_0 \qquad \text{and} \qquad \frac{\partial x^\alpha}{\partial \bar{x}^0} = \delta^\alpha_0.$$

It follows that:

$$\overline{G}_{00} = G_{00}$$

and:

$$\overline{G}_{0\alpha} = G_{0\tau}\,\frac{\partial x^\tau}{\partial \overline{x}^\alpha}\,.$$

Therefore, G_{00} is a projective scalar and $G_{0\alpha}$ is a projective covariant vector.

We write:

(12) $$G_{00} = \Phi^2 = e^{2Nx^0} f(x)$$

and:

(13) $$\frac{G_{\alpha\beta}}{G_{00}} = \gamma_{\alpha\beta}\,,$$

as well as:

$$\frac{G_{0\alpha}}{G_{00}} = \varphi_\alpha\,.$$

The quantities $\gamma_{\alpha\beta}$ and φ_α represent a projective tensor and a projective vector, respectively, and both have index 0. Φ is a scalar of index N. We have the invariant conditions:

$$\varphi = 1 \qquad \text{and } \gamma_{00} = 1.$$

The equations:

$$\gamma_{\alpha\beta} - \varphi_\alpha\varphi_\beta = g_{\alpha\beta}$$

determine a projective tensor that satisfies the invariant condition:

$$g_{0\alpha} = 0.$$

As a result, we have the transformation law:

$$\overline{g}_{ij} = g_{pq}\,\frac{\partial x^p}{\partial \overline{x}^i}\,\frac{\partial x^q}{\partial \overline{x}^j}\,;$$

i.e., the g_{ij} are the components of an affine tensor of rank two.

The projective tensor $G_{\alpha\beta}$ includes – so to speak – a scalar Φ, a projective vector φ_α, and an affine tensor g_{ij} . In fact, we have:

(13) $$G_{\alpha\beta} = \Phi^2(g_{\alpha\beta} + \varphi_\alpha\varphi_\beta) = \Phi^2\gamma_{\alpha\beta}.$$

The tangent space that is associated with a point of the base space is indicated by the coordinates X^0, X^1, …, X^4. Our tensor $G_{\alpha\beta}$ determines a quadric surface in our tangent plane by the equation:

$$G_{\alpha\beta}\, X^\alpha X^\beta = 0.$$

The polar hyperplane of a point A^α of the tangent space relative to this quadric surface is:

$$A_\alpha X^\alpha = 0,$$

in which:

$$\gamma_{\alpha\beta} A^\alpha = A_\beta .$$

Exactly as in ordinary relativity theory, we will raise or lower covariant or contravariant indices by means of the tensor $\gamma_{\alpha\beta}$ and the associated tensor $\gamma^{\alpha\beta}$. The tensor
$\gamma^{\alpha\beta}$ is completely defined by the equation:

$$\gamma_{\alpha\beta}\,\gamma^{\alpha\sigma} = \delta_\beta^\sigma .$$

Raising or lowering an index corresponds to passing to the polar form relative to the quadric surface.

The homogeneous coordinates of the origin are:

$$X^\alpha = \varphi^\alpha = \delta_0^\alpha .$$

In this, we have made use of the relation:

$$\varphi_\beta\,\gamma^{\beta\alpha} = \varphi^\alpha = \delta_0^\alpha .$$

Thus, the polar hyperplane of the origin has the equation:

$$G_{0\alpha}\,X^\alpha = 0$$

or:

$$\varphi_\alpha\,X^\alpha = 0.$$

The equation of the tangent cone with its vertex at the origin is:

$$\gamma_{\alpha\beta}\,X^\alpha\,X^\beta - (\varphi_\alpha\,X^\alpha)^2 = 0$$

or:

$$g_{ij}\,X^i\,X^j = 0.$$

As is well known, the contact point of the tangent cone with the quadric surface lies on the polar hyperplane.

The decomposition of our projective tensor into the affine tensor g_{ij} and the projective vector φ_α has the simple geometrical interpretation that the origin and the quadric surface in any tangent space determine the tangent cone through the origin and the polar hyperplane relative to the surface, respectively.

As we have already remarked, it is a basic assumption of projective relative theory that the coefficients g_{ij} in the equation of the cone are gravitational potentials and the coefficients φ_α in the equation of the hyperplane are the electromagnetic potentials. In fact, as we will see, the most natural field equation for the $\gamma_{\alpha\beta}$ is a unification of the EINSTEIN gravitational equations and the MAXWELL field equations.

III. Applications to classical projective geometry.

Projective coordinates.

In order to present our viewpoint more clearly, we would like to apply it to a particular case, which is, in fact, the case of classical projective geometry. A basic projective space is characterized by being given a set of distinguished homogeneous coordinate systems that are related to each other by equations of the form:

$$\bar{Z}^\alpha = p^\alpha_\beta Z^\beta. \tag{1}$$

We will refer to these homogeneous coordinate systems as *projective*. An arbitrary allowable coordinate system x is related to an arbitrary projective coordinate system by equations of the form:

$$Z^\alpha = e^{x^0} f^\alpha(x^1, \ldots, x^4). \tag{2}$$

Here, $x^1, \ldots, x^4$ are the coordinates of an arbitrary point and x^0 is an arbitrary parameter. The functions on the right-hand side of these equations are obviously projective scalars, since the choice of another coordinate system simply means that we substitute:

$$x^i = x^i(\bar{x})$$

in (2), and also that a substitution:

$$x^0 = \bar{x}^0 - \log \rho(\bar{x})$$

changes nothing in the meaning of (2).

From these considerations, we conclude that classical projective geometry is characterized by a family of projective scalars:

$$Z = p_\alpha A^\alpha, \tag{3}$$

in which the constants p_α are arbitrary. None of these scalars is distinguished from the other ones. An arbitrary homogeneous projective coordinate system is determined by the choice of five arbitrary independent scalars of the family.

Differential equations of projective geometry.

For the purposes of differential geometry, it is useful to eliminate the constants p_α, and thus to remove the apparent exceptional character of the five scalars $A^0, A^1, \ldots, A^4$ in the representation of an arbitrary scalar:

$$Z = p_\alpha A^\alpha,$$

in the family (3).

Due to the role of the proportionality factor, we now have:

(4) $$\frac{\partial Z}{\partial x^0} = Z.$$

This equation includes the statement that the proportionality factor is e^{x^0}. It can just as well have the value:

$$e^{Nx^0},$$

but for our problem a value of N that is non-zero means the same thing as $N = 1$.

We differentiate (3) twice and obtain:

(5) $$\frac{\partial Z}{\partial x^\beta} = p_\alpha \frac{\partial A^\alpha}{\partial x^\beta},$$

(6) $$\frac{\partial^2 Z}{\partial x^\beta \partial x^\gamma} = p_\alpha \frac{\partial^2 A^\alpha}{\partial x^\beta \partial x^\gamma}.$$

We then define a_β^α by the equations:

$$a_\beta^\gamma \frac{\partial A^\alpha}{\partial x^\gamma} = \delta_\beta^\alpha,$$

and find from (5) that:

$$p_\alpha = a_\alpha^\beta \frac{\partial Z}{\partial x^\beta}.$$

We then substitute these expressions for p_α in (6) and obtain:

$$\frac{\partial^2 Z}{\partial x^\alpha \partial x^\beta} = \Pi_{\alpha\beta}^\sigma \frac{\partial Z}{\partial x^\sigma},$$

in which:

(7) $$\Pi_{\alpha\beta}^\sigma = a_\gamma^\sigma \frac{\partial^2 A^\gamma}{\partial x^\alpha \partial x^\beta}.$$

One easily sees that these functions $\Pi_{\alpha\beta}^\sigma$ are independent of the choice the five functions A^α. Another choice of these functions must be given by a linear equation:

$$\bar{A}^\alpha = p_\beta^\alpha A^\beta;$$

under such a substitution of the variables A^α, the $\frac{\partial^2 A^\alpha}{\partial x^\beta \partial x^\gamma}$, however, behave cogrediently, whereas the a_γ^α are contragredient. Therefore, the $\Pi_{\beta\gamma}^\alpha$ remain unchanged.

I would like to call the differential equations:

(8) $$\frac{\partial^2 Z}{\partial x^\alpha \partial x^\beta} - \Pi_{\alpha\beta}^\sigma \frac{\partial Z}{\partial x^\sigma} = 0, \qquad \frac{\partial Z}{\partial x^0} = Z,$$

the *differential equations of projective geometry*, since all of projective geometry can be considered to be a theory of these equations.

Projective connections.

Each coordinate system and each gauge is associated with a particular set of 5^3 functions:

$$\Pi^{\alpha}_{\beta\gamma}.$$

Under a gauge and coordinate transformation:

$$\text{(9)} \qquad \begin{cases} \bar{x}^0 = x^0 + \log \rho(x), \\ \bar{x}^i = \bar{x}^i(x), \end{cases}$$

these functions transform like the components of an affine connection:

$$\text{(10)} \qquad \bar{\Pi}^{\alpha}_{\beta\gamma} = \left(\Pi^{\sigma}_{\rho\tau} \frac{\partial x^{\rho}}{\partial \bar{x}^{\beta}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\gamma}} + \frac{\partial^2 x^{\sigma}}{\partial \bar{x}^{\beta} \partial \bar{x}^{\gamma}} \right) \frac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}}$$

in a five-dimensional space.

One can easily verify this by direct computation. In these computations we employ only formula (7), but not the particular form of the transformation (9). Obviously, we can always interpret our transformations of representation as coordinate transformations in a five-dimensional space. The computation is precisely the same as for the corresponding introduction of an affine connection into any flat affine space. (Bibliography 1932, 10, pp. 41-43.)

We will call any invariant, or any geometric object, whose components are functions of the coordinates $x^1, \ldots, x^4$, and behave like the components of a five-dimensional affine connection under a transformation of representation a *projective connection*. The functions $\Pi^{\alpha}_{\beta\gamma}$, which we defined by (7) above, are then the components of a particular projective connection. The theory of a general projective connection is a generalization of the classical projective geometry.

If we now transform only the coordinates then our transformation law (10) reduces to:

$$\text{(10a)} \qquad \bar{\Pi}^{i}_{jk} = \left(\Pi^{s}_{rt} \frac{\partial x^{r}}{\partial \bar{x}^{j}} \frac{\partial x^{t}}{\partial \bar{x}^{k}} + \frac{\partial^2 x^{s}}{\partial \bar{x}^{j} \partial \bar{x}^{k}} \right) \frac{\partial \bar{x}^{i}}{\partial x^{s}},$$

$$\text{(10b)} \qquad \bar{\Pi}^{0}_{jk} = \Pi^{0}_{rt} \frac{\partial x^{r}}{\partial \bar{x}^{j}} \frac{\partial x^{t}}{\partial \bar{x}^{k}}.$$

Under coordinate transformations, the Π^{i}_{jk} behave like the components of an affine connection and the Π^{0}_{jk} behave like the components of an affine tensor.

Other hand, if we change only the gauge then we obtain the transformation formulae:

$$
(10c)\qquad
\begin{cases}
\overline{\Pi}^i_{jk} = \Pi^i_{jk} - \Pi^0_{jk}\dfrac{\partial\log\rho}{\partial x^j} - \Pi^i_{j0}\dfrac{\partial\log\rho}{\partial x^k} + \Pi^i_{00}\dfrac{\partial\log\rho}{\partial x^j}\dfrac{\partial\log\rho}{\partial x^k} \\
\overline{\Pi}^0_{jk} = \Pi^0_{jk} + \Pi^i_{jk}\dfrac{\partial\log\rho}{\partial x^i} + \Pi^i_{0k}\dfrac{\partial\log\rho}{\partial x^i}\dfrac{\partial\log\rho}{\partial x^j} \\
\qquad - \Pi^i_{j0}\dfrac{\partial\log\rho}{\partial x^i}\dfrac{\partial\log\rho}{\partial x^k} + \Pi^i_{00}\dfrac{\partial\log\rho}{\partial x^i}\dfrac{\partial\log\rho}{\partial x^j}\dfrac{\partial\log\rho}{\partial x^k} - \Pi^0_{0k}\dfrac{\partial\log\rho}{\partial x^j} \\
\qquad - \Pi^0_{j0}\dfrac{\partial\log\rho}{\partial x^k} + \Pi^0_{00}\dfrac{\partial\log\rho}{\partial x^j}\dfrac{\partial\log\rho}{\partial x^k} - \dfrac{\partial^2\log\rho}{\partial x^j\partial x^k}.
\end{cases}
$$

Under transformations of representation, the $\Pi^\alpha_{0\beta}$ and $\Pi^\alpha_{\beta 0}$ behave like the components of a projective tensor. Thus:

$$\Pi^\alpha_{0\beta} = \Pi^\alpha_{\beta 0} = \delta^\alpha_\beta$$

is an invariant equation. This equation is obviously satisfied for the particular projective connection (7). In this particular case, the transformation formulae (10c) reduce to:

$$
(10d)\qquad
\begin{cases}
\overline{\Pi}^i_{jk} = \Pi^i_{jk} - \delta^i_j\dfrac{\partial\log\rho}{\partial x^k} - \delta^i_k\dfrac{\partial\log\rho}{\partial x^j} \\
\overline{\Pi}^0_{jk} = \Pi^0_{jk} - \dfrac{\partial\log\rho}{\partial x^j}\dfrac{\partial\log\rho}{\partial x^k} - \dfrac{\partial^2\log\rho}{\partial x^j\partial x^k} + \Pi^i_{jk}\dfrac{\partial\log\rho}{\partial x^i}.
\end{cases}
$$

Five-dimensional representation.

At this point, we have made advantageous use of an interpretation that is essentially due to T.Y. THOMAS (Bibliography 1925, 8; 1926, 13). Our entire theory finds a representation in a five-dimensional space with coordinates $x^0, \ldots, x^4$, which are not the most general coordinates, but are subject to the transformations:

$$
\begin{aligned}
x^i &= x^i(\bar{x}), \\
x^0 &= \bar{x}^0 - \log\rho(x).
\end{aligned}
$$

We can therefore interpret our transformations of representation as the transformations of this five-dimensional space. The lines $x^1, \ldots, x^4$ = const., x^0 = arbitrary, play a distinguished role, due to the particular form of the transformations of representation.

One now obtains a precise picture of our four-dimensional projective geometry when one regards these x^0-lines in the five-dimensional space as points in a four-dimensional space. Just as a five-dimensional affine space defines a four-dimensional projective space in the elementary geometry of lines through a fixed point, so also does a four-dimensional "projective" space arise from a five-dimensional "affine" space in the theory of a general projective connection. The role of lines through the fixed point is played by the x^0-lines here. The common point of the x^0-lines is transformed to infinity here (Bibliography 1931, 16; 1929, 2).

Projective derivative.

With the help of a projective connection we can construct new projective tensors of higher rank from the components of an arbitrary projective tensor by means of the formulae of covariant differentiation. For example, if A^α is an arbitrary projective vector then:

$$\frac{\partial A^\alpha}{\partial x^\beta} + \Pi^\alpha_{\sigma\beta} A^\sigma$$

will be the components of a projective tensor A^α_β, which we refer to as the *projective derivative* of A^α. That the A^α_β are actually the components of a tensor follows directly from the five-dimensional affine interpretation of the transformations of representation.

The same theorems and formulas that were valid for the covariant derivative of a general tensor in the affine theory are likewise valid for projective differentiation. In general, there are further theorems that are not found in the affine theory that depend on the special form of the gauge transformation. We will, however, develop those theorems only when they are necessary.

Integrability conditions.

Ordinary projective geometry can be characterized as the theory of systems of differential equations (8). These differential equations are therefore not the most general differential equations of the form (8). Rather, they must satisfy a series of integrability conditions. We write the differential equations in the form:

$$\frac{\partial Z_0}{\partial x^\alpha} = Z_\alpha, \tag{11a}$$

$$\frac{\partial Z_\alpha}{\partial x^\beta} = \Pi^\sigma_{\alpha\beta} Z_\sigma, \tag{11b}$$

in which Z_0 now means Z. As is well known, the integrability conditions for these equations are:

$$\Pi^\alpha_{\beta\gamma} = \Pi^\alpha_{\gamma\beta}, \tag{12}$$

$$R^\lambda_{\alpha\beta\gamma} = \frac{\partial \Pi^\lambda_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial \Pi^\lambda_{\alpha\gamma}}{\partial x^\beta} + \Pi^\sigma_{\alpha\beta}\Pi^\lambda_{\sigma\gamma} - \Pi^\lambda_{\sigma\beta}\Pi^\sigma_{\alpha\gamma} = 0. \tag{13}$$

The computation is exactly the same as for the corresponding problem in affine geometry (Bibliography 1927, 22; 1932, 10).

$R^\lambda_{\alpha\beta\gamma}$ is a projective tensor of rank four, namely, the curvature tensor of the connection $\Pi^\alpha_{\beta\gamma}$. If it vanishes then we call the connection *flat*.

From (11a) and (11b), one also obtains the invariant relation:

$$\Pi^\alpha_{\beta 0} = \delta^\alpha_\beta. \tag{14}$$

We now must prove that the integrability conditions for the case of projective geometry are in fact satisfied. The functions Π have the form (7). From this, one immediately infers the validity of (12). One can likewise verify condition (13) by elementary computation on the basis of (7). However, the consequences are exactly the same as in the affine theory, such that we immediately see that the integrability conditions are necessary.

Conversely, we will now prove that our integrabilitiy conditions are also sufficient. With that, we will also prove that equations (8), together with conditions (12), (13), and (14), are characteristic of projective geometry.

We must therefore show that the validity of the integrability conditions implies the existence of five independent functions $A^0, \ldots, A^4$, from which the usual solutions are obtained by forming linear combinations with constant coefficients.

Next, it is clear from the five-dimensional theory that because of (12) and (13), (11b) has precisely one solution vector $A_0, \ldots, A_4$, that takes given values at a particular point. We obtain five independent solution vectors A^α_β when we start at a definite point with five independent vectors as initial values.

From the five-dimensional representation it further follows that because of the symmetry of the Π, the solution vectors are gradients of five affine scalars A^α in five dimensions. However, we must now prove that these scalars take the form:

$$A^\alpha = e^{x^0} f^\alpha(x^1, \ldots, x^4). \tag{15}$$

From (14), it follows that:

$$\frac{\partial A_\alpha}{\partial x^0} = \Pi^\alpha_{\beta 0} A_\sigma = A_\alpha,$$

or:

$$A_\alpha = e^{x^0} f_\alpha(x^1, \ldots, x^4).$$

If we write f instead of f_0 then we have that:

$$Z = e^{x^0} f(x^1, \ldots, x^4)$$

is a solution of (8), whereas, due to (11a), the solution vectors of (11b) are the gradients of the function Z.

We have thus proved that classical projective geometry is completely equivalent to the theory of differential equations (8) with the conditions (12), (13), and (14), at least for a given domain in the base space.

Homogeneous projective coordinates as functions of the boundary conditions.

We denote five independent solutions of the equations (8) by Z_α and the corresponding solution vectors by $Z^\alpha_\beta = \dfrac{\partial Z^\alpha}{\partial x^\beta}$. At the point $x = q$ we now choose $(Z^\alpha_\beta)_q = \delta^\alpha_\beta$ as initial value. With that, we determine five independent solutions $Z^\alpha = e^{x^0} f^\alpha(x^1, \ldots, x^4)$ that we regard as functions of the coordinates, as well as functions of the initial values:

$$Z^\alpha = Z^\alpha(x, q) .$$

It now follows that Z^α takes the form:

$$Z^\alpha = e^{x^0 - q^0} f^\alpha(x^1, \ldots, x^4, q^1, \ldots, q^4) . \tag{16}$$

In fact, (11) includes the equation:

$$\frac{\partial^2 f^\alpha}{\partial x^j \partial x^k} = \Pi^i_{jk} \frac{\partial f^\alpha}{\partial x^i} + \Pi^0_{jk} f^\alpha .$$

As initial values, we can take $f^\alpha(q) = \delta^\alpha_0, \left(\frac{\partial f^\alpha}{\partial x^i}\right)_q = \delta^\alpha_i$. When regarded as functions of the coordinates, as well as the initial values, the f^α then have the form $f^\alpha = f^\alpha(x^1, \ldots, x^4, q^1, \ldots, q^4)$. Since any linear combination of solutions is again a solution, we can, in fact, put Z^α into the form (16), and one confirms that the Z^α_β take on the values $(Z^\alpha_\beta)_q = \delta^\alpha_\beta$ at the point $x = q$. Due to the uniqueness of the solutions of (11), it follows that Z^α must necessarily have the form (16).

If we subject the x^i to the transformation:

$$x^i = x^i(\bar{x}),$$

then they behave like:

$$q^i = x^i(\bar{q}) .$$

In the transformed coordinate system we now seek a system of solutions with the initial values $Z^\alpha_\beta(\bar{q}) = \delta^\alpha_\beta$.

Since there can be only five independent solutions, it is clear that there must be relations of the form:

$$\bar{Z}^\alpha = p^\alpha_\beta Z^\beta$$

between the new and the old solutions, in which the p^α_β are constants and $|p^\alpha_\beta| \neq 0$. If we differentiate these equations then we obtain:

$$\frac{\partial \bar{Z}^\alpha}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^\gamma} = p^\alpha_\beta \frac{\partial Z^\beta}{\partial x^\gamma}$$

identically at x. For $x = q$, when we employ the initial values we obtain:

$$\left(\frac{\partial \bar{x}^\alpha}{\partial x^\gamma}\right)_{x=q} = p^\alpha_\gamma$$

or:

$$\bar{Z}^\alpha(\bar{x},\bar{q}) = \frac{\partial \bar{q}^\alpha}{\partial q^\gamma} Z^\gamma(x, q) .$$

Z^α, when regarded as a function of q, is therefore a contravariant projective vector that has the index –1, on account of (16).

Inhomogeneous projective coordinates.

For the sake of our further development, we employ a special coordinate system to our advantage. A coordinate transformation $y^i = y^i(x)$ is determined by the equation:

$$y^i = \frac{Z^i}{Z^0}.$$

We call the resulting coordinate system y^i an inhomogeneous projective coordinate system.

From this coordinate system, we can go to yet another coordinate system; for example, by the formulas:

(17) $$Z^i = (y^i - q^i)\, e^{y^0 - q^0}, \qquad Z^0 = e^{y^0 - q^0} .$$

From the Z^α, we further obtain the most general form for our homogeneous coordinates when we simultaneously subject the y and the q to the same transformation of representation.

We can compute the Π in this special coordinate system, and the same consequences that we demonstrated for the A^α at the start of this chapter are likewise valid for the Z^α. One thus obtains:

(18) $$\Pi^\alpha_{\beta\gamma} = \delta^\alpha_\beta \delta^0_\gamma + \delta^\alpha_\gamma \delta^0_\beta - \delta^\alpha_0 \delta^0_\beta \delta^0_\gamma .$$

This means that $\Pi^\alpha_{jk} = 0$ and $\Pi^\alpha_{\beta 0} = \delta^\alpha_\beta$. By comparison, if one makes use of the fact that the Z^α must satisfy equations of the form (8) in any case then from:

$$\frac{\partial^2 Z^\alpha}{\partial y^i \partial y^k} = 0, \qquad \frac{\partial^2 Z^\alpha}{\partial y^\beta \partial y^0} = \delta^\gamma_\beta \frac{\partial Z^\alpha}{\partial y^\gamma}$$

we again obtain $\Pi^\alpha_{jk} = 0$ and $\Pi^\alpha_{\beta 0} = \delta^\alpha_\beta$.

We further remark that under a gauge transformation:

$$\bar{x}^0 = x^0 + \log \rho(x^1, \ldots, x^4)$$

the components (18) of the projective connection take on the form:

(19) $$\Pi^\alpha_{\beta\gamma} = \delta^\alpha_\beta \varphi_\gamma + \delta^\alpha_\gamma \varphi_\beta - \varphi_\beta \varphi_\gamma \delta^\alpha_0 + \frac{\partial \varphi_\beta}{\partial x^\gamma} \delta^\alpha_0 ,$$

in which $\varphi_i = -\dfrac{\partial \log \rho}{\partial x^i}$, $\varphi_0 = 1$.

We can characterize inhomogeneous coordinates as the only coordinates in which the Π have the form (19).

Two inhomogeneous coordinate systems are related to each other by a piecewise-linear transformation:

(20) $$\bar{z}_i = \frac{p^i_j z^j + p^i_0}{p_i z^i + p_0}.$$

This follows immediately from the fact that two homogeneous systems as connected with each other by a linear substitution.

From our present viewpoint, we can say that any two coordinate systems in which the Π take on the form (19) are linked with each other by a substitution of the form (20). When we make the further demand that the form (18) of the Π shall remain invariant then we must couple each coordinate system with a certain gauge transformation, namely:

$$\rho = k\,u^{-\frac{1}{5}},$$

in which u is the functional determinant of our coordinate transformation and k is a constant. With these remarks, the relationship between our present theory and the formal apparatuses that were presented in the earlier work of T. Y. THOMAS and others (Bibliography 1926, 13; 1928, 10); 1930, 5) becomes clearer.

Homogeneous projective coordinates as functions of the boundary conditions (continued).

We previously saw that the functions:

$$Z^{\alpha} = e^{x^0 - q^0} f^{\alpha}(x^1, \ldots, x^4, q^1, \ldots, q^4)$$

with $Z^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ for $x = q$, mediate the transition between the arbitrary coordinate system x and the homogeneous coordinate system Z that is coupled to x by means of the boundary conditions. Each point q and each parameter value q^0 of the base space is associated with a definite coordinate system. How do the Z behave as functions of q?

The answer is as follows: The Z are functions of q that satisfy the following differential equations:

(21) $$\frac{\partial Z^{\alpha}}{\partial q^{\beta}} + \Pi^{\alpha}_{\sigma\beta} Z^{\sigma} = 0.$$

We know that the Z are the components of a contravariant vector of index –1. We now consider the projective derivative:

$$\frac{\partial Z^{\alpha}}{\partial q^{\beta}} + \Pi^{\alpha}_{\sigma\beta} Z^{\sigma}$$

of this vector. In our special coordinate system we can now easily calculate that the projective derivative of Z^α vanishes. Π has the values (18) in the homogeneous system, whereas the Z assume the form (17). The Z indeed satisfy our boundary conditions precisely. If we give P the value (18) then the components of a tensor of rank two:

$$\frac{\partial Z^\alpha}{\partial q^\beta} + \Pi^\alpha_{\sigma\beta} Z^\sigma$$

reduce to the form:

$$\frac{\partial Z^\alpha}{\partial q^0} + Z^\alpha, \qquad \frac{\partial Z^i}{\partial q^j} + \delta^i_j Z^0, \qquad \frac{\partial Z^0}{\partial q^j}.$$

Due to (17), however, all of these components must vanish. With that, we have established the proposed equation (21).

Since the left-hand side of the equation is independent of the representation we know in full generality that the homogeneous coordinates satisfy the equation (21). Equation (21) then characterizes projective geometry just as equations (8) did.

Its integrability conditions are:

$$R^\alpha_{\beta\gamma\delta} = 0.$$

If these integrability conditions are not satisfied then we are dealing with generalization of classical projective geometry that we will consider in later chapters.

IV. Projective translations.

In the last chapter, we regarded the solutions Z^α of the differential equations of projective geometry as functions of the boundary conditions. We saw that the Z^α are a projective vector of index –1 as functions of q when we demand that:

$$(Z^\alpha_\beta)_{x=q} = \delta^\alpha_\beta .$$

Furthermore, we found that the Z^α satisfied the differential equations:

(1) $$\frac{\partial Z^\alpha}{\partial q^\beta} + \Pi^\alpha_{\sigma\beta} Z^\sigma = 0.$$

In all of this we have always assumed the validity of the integrability conditions.

Translation along a curve.

We now put forth a generalization that is somewhat analogous to the transition from Euclidian geometry to RIEMANNIAN geometry. Namely, we no longer assume that the equations (1) are integrable; the tensors:

$$\Pi^\alpha_{\beta\gamma} - \Pi^\alpha_{\gamma\beta}$$

and:

$$R^\alpha_{\beta\gamma\delta}$$

are thus not necessarily equal to zero.

By contrast, we now retain the condition:

$$\Pi^\alpha_{\beta 0} = M\delta^\alpha_\beta .$$

Here, we have made a slight generalization by writing $M\delta^\alpha_\beta$ instead of δ^α_β. Correspondingly, x^0 and q^0 enter into Z by way of:

$$e^{M(x^0 - q^0)}.$$

From a general technique for treating partial differential equations, we now choose an arbitrary curve:

(2) $$q^i = x^i\,(t)$$

and set, with no loss of generality:

(3) $$q^0 = x^0(t).$$

If we multiply (1) by dq^β/dt then we obtain the equations:

(4) $$\frac{\partial Z^\alpha}{\partial t} + \Pi^\alpha_{\lambda\beta} Z^\lambda \frac{dq^\beta}{dt} = 0.$$

Due to the condition:

$$\Pi^{\alpha}_{\beta 0} = M\delta^{\alpha}_{\beta}$$

we then have:

(5)
$$\frac{\partial Z^{\alpha}}{\partial q^{0}} + \Pi^{\alpha}_{\sigma 0} Z^{\sigma} = 0,$$

as long as Z^{α} has the form $e^{-Mq^{0}} f(t)$, and for that reason equations (4) mean the same thing as the equations:

(6)
$$\frac{\partial Z^{\alpha}}{\partial t} + \Pi^{\alpha}_{\lambda j} \frac{dq^{j}}{dt} Z^{\lambda} = 0.$$

The five-dimensional affine theory then implies that (4) is invariant under transformations of representation, and from this, the invariance of the four-dimensional equations (6) follows as well. These equations depend only upon the functions Π and the curve in the parameter representation (2), and not on (3).

We now write equations (6) in the form:

(7)
$$\frac{\partial Z^{\alpha}}{\partial t} + \Pi^{\alpha}_{\lambda}(t) Z^{\lambda} = 0$$

in which:

$$\Pi^{\alpha}_{\lambda}(t) = \Pi^{\alpha}_{\lambda j} \frac{dq^{j}}{dt}.$$

From the well-known existence theorem for systems of linear differential equations, the solutions to our system have the form:

(8)
$$X^{\alpha} = A^{\beta} p^{\alpha}_{\beta}(t),$$

in which the A are the given initial values at the point $t = t_0$. The p^{α}_{β} must therefore reduce to the values δ^{α}_{β} at the point $t = t_0$.

In this sense, we can say that the equations:

$$dX^{\alpha} + \Pi^{\alpha}_{\lambda j} X^{\lambda} dx^{j} = 0$$

represent an infinitesimal projective transformation. Thus, when we connect two points a and b with a curve, equation (8) determines a projective transformation of the tangent space at a to the tangent space at b. This map is completely determined by the givens of a, b, and the connecting curve. In the flat case, i.e., when equations (12) and (13) of chap. III are satisfied, equations (1) are completely integrable, and map that is determined by two points and a connecting curve does not change when we deform the curve arbitrarily.

In the integrable case, we can interpret the A^{α} and X^{α} as homogeneous coordinates of the base space, and indeed interpret the A^{α} as coordinates in the coordinate system about the point $q^i = x^i(t_0)$ and the X^α as coordinates in the coordinate system about the point $q^i = x^i(t)$. Equation (8) then represents a transformation of the base space to itself.

Generalized projective geometry.

We formulate the facts once more in another form: a map of the tangent space at q is given by the equation:

$$Z^\alpha(q) = X^\alpha,$$

and one can regard it as a covering of the base space by the tangent space. Since the differential equations of projective geometry are completely integrable, all of the tangent spaces coincide with the base space, so to speak.

Therefore, if the differential equations are not integrable then we have no projective coordinate system $Z(q)$, and it then follows that we also have no such covering of the tangent space over the base space. In this case, we have only translations along arbitrary curves. The tangent spaces are related to each other by means of these translations, but this relationship is not as close as in the integrable case, in which we can regard it as a coincidence. To borrow a notion from surface theory, we can say: The tangent spaces fall apart when the differential equations (1) are not integrable.

We now see how one can geometrically regard the theory of a non-integrable projective connection as a generalization of ordinary projective geometry. Our general viewpoint is the following one:

A geometry is a theory of geometric objects. If one of these objects is a projective connection then we have generalized projective geometry. If the projective connection satisfies the previously considered integrability conditions then one obtains a classical projective geometry, at least locally.

If the relation $\Pi^\alpha_{\beta 0} = M\delta^\alpha_\beta$ is satisfied then a general projective geometry includes a theory of projective translations of the tangent spaces along an arbitrary curve, as we suggested above. We would now like to pursue this a little further.

One next sees that, just as one usually does in affine theory, the projective translation of a hyperplane $B_\alpha X^\alpha = 0$ is defined through the differential equations:

$$\frac{dB_\alpha}{dt} - \Pi^\sigma_{\alpha j} B_\sigma \frac{dx^j}{dt} = 0. \tag{9}$$

Likewise, we can describe the projective translation of algebraic structures of higher order, i.e., projective tensors of higher rank. In particular, we obtain the following equations for the projective translation of a quadric surface:

$$\frac{dG_{\alpha\beta}}{dt} - \Pi^\sigma_{\alpha j} G_{\beta\sigma} \frac{dx^j}{dt} - \Pi^\sigma_{\beta j} G_{\alpha\sigma} \frac{dx^j}{dt} = 0. \tag{10}$$

Translations in inhomogeneous coordinates.

In order to describe translations precisely, we must employ the introduction of inhomogeneous coordinates that was given in chap. II, i.e., we must make use of the relation:

$$\frac{X^i}{\varphi_\alpha X^\alpha} = dx^i . \tag{11}$$

The precise form of transformations of tangent spaces must depend upon the projective derivative of φ; we will now calculate this. From (14), we obtain:

$$X^i = \varphi_\alpha X^\alpha V^i, \tag{12}$$

in which we have set $V^i = dx^i$.

We differentiate (12):

$$\frac{\partial X^i}{\partial x^j} = \varphi_\alpha X^\alpha \frac{\partial V^i}{\partial x^j} + \varphi_\alpha \frac{\partial X^\alpha}{\partial x^j} V^i + \frac{\partial \varphi_\alpha}{\partial x^j} X^\alpha V^i .$$

We now multiply this by dx^j/dt. By the use of (1), this yields:

$$-\Pi^i_{\lambda j} X^\lambda \frac{dx^i}{dt} = \varphi_\alpha X^\alpha \frac{dV^i}{dt} + \varphi_\alpha \Pi^\alpha_{\lambda j} \frac{dX^\alpha}{dt} V^i + \frac{d\varphi_\alpha}{dt} X^\alpha V^i ,$$

or, if we divide this by $\varphi_\alpha X^\alpha$:

$$\frac{dV^i}{dt} + \varphi_{\alpha;j} \frac{X^\alpha}{\varphi_\sigma X^\sigma} V^i \frac{dx^j}{dt} + \Pi^i_{\lambda j} \frac{X^\lambda}{\varphi_\sigma X^\sigma} \frac{dx^j}{dt} = 0. \tag{13}$$

With the help of (12) and:

$$X^0 = X^\alpha \varphi_\alpha (1 - \varphi_i V^i)$$

we ultimately obtain:

$$\left\{ \begin{aligned} &\frac{dV^i}{dt} + \Pi^i_{kj} V^k \frac{dx^j}{dt} + (1 - \varphi_i V^j) \Pi^i_{0j} \frac{dx^j}{dt} + \varphi_{k;j} V^k V^i \frac{dx^j}{dt} \\ &\qquad + \varphi_{0;j} (1 - \varphi_i V^j) \frac{dx^j}{dt} V^i = 0. \end{aligned} \right. \tag{14}$$

In this expression, $\varphi_{\alpha,\beta}$ means the projective derivative of φ_α. With this, we have derived the inhomogeneous form of a projective translation.

Paths.

Let a curve be given by the parameter representation:

$$x^i = x^i (t).$$

Its "velocity vector:"

(15) $$V^i = \frac{dx^i}{dt}$$

determines a point in each tangent space to the points of the curve.

We now ask whether curves exist for which these points go back to themselves under translation along a curve. For this, we must replace V^i with dx^i/dx in (14). In the case where:

$$\Pi^{\alpha}_{\beta 0} = M\delta^{\alpha}_{\beta}$$

we find that:

$$\frac{d^2x^i}{dt^2} + \Pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \frac{dx^i}{dt}\left[\varphi_{k;j}\frac{dx^k}{dt}\frac{dx^j}{dt} + M\left(1-\varphi_k\frac{dx^k}{dt}\right)^2\right] = 0.$$

Since the expression in the square brackets is independent of i, we can put this equation into the following form:

(16) $$\frac{\dfrac{d^2x^i}{dt^2} + \Pi^i_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}}{\dfrac{dx^i}{dt}} = \frac{\dfrac{d^2x^l}{dt^2} + \Pi^l_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt}}{\dfrac{dx^l}{dt}}.$$

It is noteworthy that these equations are completely independent of φ. (16) is a well-known expression for a system of paths. By the phrase "a system of paths," we understand that we are dealing with a system of curves that has the property that inside of a sufficiently small neighborhood one and only one curve of the system goes through any two given points. Paths are a generalization of the geodetic lines of RIEMANNIAN geometry.

As one easily verifies, the form of equations (16) does not depend upon the parameterization of the paths. Likewise, one shows, on the basis of equations (10a) of chap. III, that equation (16) is also invariant under coordinate transformations. Finally, it follows from (10c) that they are also invariant under gauge transformations.

Namely, under a gauge transformation the expression:

$$\frac{\partial \log\rho}{\partial x^k}\frac{dx^k}{dt} - \frac{\partial \log\rho}{\partial x^j}\frac{dx^j}{dt}$$

gets added to the left-hand side of (16). Since this expression is, however, independent of i, it cancels out the corresponding expression on the right-hand side of (16), by which the invariance under gauge transformations is proved.

With this, we have obtained the theorem that in the case where:

$$\Pi^{\alpha}_{0\beta} = \Pi^{\alpha}_{\beta 0} = M\delta^{\alpha}_{\beta}$$

our connection gives rise to a uniquely determined system of paths. In the case of classical projective geometry these paths are straight lines.

General projective connections.

Up till now, we have considered only the particular case of a projective connection, for which we have:

$$\Pi^{\alpha}_{\beta 0} = M\delta^{\alpha}_{\beta}.$$

We would now like to look for a translation that does not satisfy this condition.

We cannot use the equation:

$$\overline{(17)} \qquad \frac{\partial X^{\alpha}}{\partial x^{\beta}} + \Pi^{\alpha}_{\sigma\beta} X^{\sigma} = 0$$

since in the case of $\beta = 0$ it will contradict the assumption that x^0 enters into X in the form e^{-Mx^0}. However, we must retain this assumption since translation must depend only on the curve and its parametric representation:

$$x^i = x^i\,(t),$$

not on the particular choice of the parameter:

$$x^0 = x^0\,(t).$$

In order to arrive at a suitable definition for a translation, we remark that the choice of the tensor 0 that appears in the right-hand side of $\overline{(17)}$ is likewise basically as arbitrary as the choice of any other tensor that is invariantly associated with Π.

We now define a translation by the equations:

$$(17) \qquad X^{\alpha}{}_{;\,\beta} = X^{\alpha}{}_{;\,0} C_{\beta}\,,$$

in which C_β is a covariant vector that must satisfy the condition $C_0 = 1$. Equation (17) is satisfied identically for $\beta = 0$.

Instead of equation $\overline{(17)}$, we now have a set of equations (17), which one obtains when one sets C_β equal to all possible systems of four functions. The totality of these equations is invariantly linked with the connection.

We can now write equation (17) in the form:

$$\frac{\partial X^{\alpha}}{\partial x^{\sigma}} + \Lambda^{\alpha}_{\beta\sigma} X^{\beta} = 0\,,$$

in which we have:

$$(18) \qquad \Lambda^{\alpha}_{\beta\sigma} = \Pi^{\alpha}_{\beta\sigma} - \Pi^{\alpha}_{\beta 0} C_{\sigma} + M\delta^{\alpha}_{\beta} C_{\sigma}\,,$$

where M is an arbitrary index.

The $\Lambda^{\alpha}_{\beta\sigma}$ are the components of a projective connection that satisfies the conditions:

$$\Lambda^{\alpha}_{\beta\sigma} = M\delta^{\alpha}_{\beta}.$$

We thus obtain a set of projective translations for any projective connection.

The associated projective connection for the tensor $G_{\alpha\beta}$.

Two symmetric projective connections are associated with $G_{\alpha\beta}$, which are both characterized by the vanishing of the projective derivatives of $G_{\alpha\beta}$ and $\gamma_{\alpha\beta}$. These two connections, which we shall call Π and Γ, are therefore defined by the equations:

(19)
$$G_{\alpha\beta\,|\,\gamma} = \frac{\partial G_{\alpha\beta}}{\partial x^{\gamma}} - G_{\alpha\sigma}\Pi^{\sigma}_{\beta\gamma} - G_{\sigma\beta}\Pi^{\sigma}_{\alpha\gamma} = 0$$

and:

(20)
$$\gamma_{\alpha\beta\,;\,\gamma} = \frac{\partial \gamma_{\alpha\beta}}{\partial x^{\gamma}} - \gamma_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma} - \gamma_{\sigma\beta}\Gamma^{\sigma}_{\alpha\gamma} = 0.$$

We denote projective differentiation with respect to Π or Γ by a " |" or a ";", respectively. If one solves equation (20) in the usual way then one obtains the CHRISTOFFEL formulae:

(21)
$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2}\gamma^{\alpha\beta}\left(\frac{\partial \gamma_{\beta\sigma}}{\partial x^{\gamma}} + \frac{\partial \gamma_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial \gamma_{\beta\gamma}}{\partial x^{\sigma}}\right).$$

One obtains equations for Π in which one replaces Γ with Π and γ with G in (21).

Obviously, the Π must be calculated from the Γ with the help of the scalar Φ. By a simple application of the product rule for differentiation, one obtains:

$$\Pi^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \left(\delta^{\alpha}_{\beta}\frac{\partial \log \Phi}{\partial x^{\gamma}} + \delta^{\alpha}_{\gamma}\frac{\partial \log \Phi}{\partial x^{\beta}} - \gamma_{\beta\gamma}\gamma^{\alpha\sigma}\frac{\partial \log \Phi}{\partial x^{\sigma}}\right).$$

If we set:

$$\Phi_{\alpha} = \frac{\partial \log \Phi}{\partial x^{\alpha}}$$

then we obtain:

(22)
$$\Pi^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \left(\delta^{\alpha}_{\beta}\Phi_{\gamma} + \delta^{\alpha}_{\gamma}\Phi_{\beta} - \gamma_{\beta\gamma}\gamma^{\alpha\sigma}\Phi^{\alpha}\right).$$

The associated translation for $G_{\alpha\beta}$.

The projective connection Γ satisfies the conditions:

$$\Gamma^{\alpha}_{\beta 0} = \gamma^{\alpha\sigma}\varphi_{\alpha\beta},$$

as one can immediately read off from (21). In this, we have set:

$$\varphi_{\alpha\beta} = \tfrac{1}{2}\left(\frac{\partial \varphi_{\alpha}}{\partial x^{\beta}} - \frac{\partial \varphi_{\beta}}{\partial x^{\alpha}}\right).$$

The following important formula is valid for $\varphi_{\alpha\beta}$:

$$\varphi_{\alpha;\,\beta} = \frac{\partial \varphi_\alpha}{\partial x^\beta} - \varphi_\sigma \Gamma^\sigma_{\alpha\beta} = \varphi_{\alpha\beta}\,.$$

Namely, if we multiply (21) by:

$$\varphi_\sigma = \gamma_{\sigma 0}$$

then we obtain:

$$\varphi_\sigma \Gamma^\sigma_{\alpha\beta} = \tfrac{1}{2}\delta^\tau_0 \left(\frac{\partial \gamma_{\beta\tau}}{\partial x^\alpha} + \frac{\partial \gamma_{\alpha\tau}}{\partial x^\beta} - \frac{\partial \gamma_{\alpha\beta}}{\partial x^\tau} \right) = \tfrac{1}{2}\left(\frac{\partial \varphi_\alpha}{\partial x^\beta} + \frac{\partial \varphi_\beta}{\partial x^\alpha} \right).$$

From this, one immediately obtains the stated formula:

(23) $$\varphi_{\alpha;\,\beta} = \varphi_{\alpha\beta}\,.$$

In general, we now have:

$$\Gamma^\alpha_{\beta 0} = \varphi^\alpha_\beta \neq \delta^\alpha_\beta\,.$$

We require formula (17) for this purpose, in order to arrive at a translation. In this case, this suggests that we use, not the arbitrary vector C, but the vector φ that is invariantly related to the $G_{\alpha\beta}$. Thus, the resulting translation is also invariantly related to Γ itself, whereas in the general case this is only true for the set of all translations collectively.

We define translations in our case by way of:

(24) $$X^\alpha_{\ ;\,\beta} = X^\alpha_{\ ;\,0}\ \varphi_\beta\,.$$

For Λ, we then obtain:

(25) $$\Lambda^\alpha_{\beta\sigma} = \Gamma^\alpha_{\beta\sigma} + M\delta^\alpha_\beta \varphi_\sigma - \Gamma^\alpha_{\beta 0}\varphi_\sigma\,,$$

under the assumption that X^α is of index – M.

Furthermore, Λ depends upon M. For $\sigma = 0$, one has:

$$\Lambda^\alpha_{\beta\sigma} = M\delta^\alpha_\beta\,,$$

such that one again has that for $\beta = 0$ the equation:

(26) $$\frac{\partial X^\alpha}{\partial x^\beta} + \Lambda^\alpha_{\sigma\beta} X^\sigma = 0\,,$$

is satisfied identically.

We will pursue the theory of the translations Λ in greater detail in chap. VI.

V. Non-Euclidian geometry.

In chapter III we applied our general theory to the special case of classical projective geometry. We would now like to take the specialization one step further and consider a quadric surface in ordinary projective geometry.

Equation of a quadric surface.

We assume that that a system of functions Π is given in our base space, which has the coordinates $x^1, \ldots, x^4$, such that the differential equations:

$$\frac{\partial^2 Z}{\partial x^\beta \partial x^\gamma} = \Pi^\sigma_{\beta\gamma} \frac{\partial Z}{\partial x^\sigma} \tag{1}$$

or:

$$\frac{\partial Z^\alpha}{\partial q^\beta} + \Pi^\alpha_{\lambda\beta} Z^\lambda = 0 \,, \tag{2}$$

are soluble. Here, q denotes the point at which the Z^α possess the initial values:

$$\frac{\partial Z^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \,.$$

If we choose a coordinate system Z that is arbitrary, but fixed by q, then we obtain a quadric surface by way of the equation:

$$G_{\alpha\beta} Z^\alpha Z^\beta = 0, \tag{3}$$

in which G is only determined up to a common factor. Two arbitrary homogeneous coordinate systems go over to each other by a linear homogeneous substitution. Due to (3), the G then transform like the coefficients of a quadratic form under a linear substitution.

The $G_{\alpha\beta}$ as functions of q.

We now consider the Z to be functions of q. Therefore, the ratios of the G must also depend upon $q^0, \ldots, q^4$, whereas they are naturally constant in a particular projective coordinate system.

The Z^α are contravariant vectors of index −1 when regarded as functions of q. Therefore, it follows from the considerations above that the G transform according to the formula:

$$\overline{G}_{\alpha\beta}(\overline{q}) = \lambda \, G_{\sigma\tau}(q) \frac{\partial q^\sigma}{\partial \overline{q}^\alpha} \frac{\partial q^\tau}{\partial \overline{q}^\beta} \,, \tag{4}$$

in which λ can be an arbitrary function of the q.

In order to establish the independence of the G from q we differentiate (3) and replace the $\partial Z^\alpha/\partial q^\beta$ with the values in (2). Equation (2) certainly shows how the Z^α behave as functions of the origin q.

Thus, we find that the system of equations:

$$\left(\frac{\partial G_{\alpha\beta}}{\partial q^\gamma} - G_{\alpha\sigma}\Pi^\alpha_{\beta\gamma} - G_{\sigma\beta}\Pi^\sigma_{\alpha\gamma}\right) Z^\alpha Z^\beta = 0 ,$$

must be satisfied for all values of Z^α that satisfy equation (3). Therefore, the expression:

$$\frac{\partial G_{\alpha\beta}}{\partial q^\gamma} - G_{\alpha\sigma}\Pi^\alpha_{\beta\gamma} - G_{\sigma\beta}\Pi^\sigma_{\alpha\gamma}$$

must be proportional to the $G_{\alpha\beta}$, in which the proportionality factor will naturally be different for the various γ, in general. We thus obtain the system of equations:

$$\frac{\partial G_{\alpha\beta}}{\partial q^\gamma} - G_{\alpha\sigma}\Pi^\alpha_{\beta\gamma} - G_{\sigma\beta}\Pi^\sigma_{\alpha\gamma} = G_{\alpha\beta} A_\gamma . \tag{5}$$

Equations (5) describe the change in G under a change in the origin q.

Normalization of the $G_{\alpha\beta}$.

It is easy to show now that there is no loss in generality geometrically if we set:

$$A_\gamma = 0.$$

Namely, if we multiply (5) by $G^{\alpha\beta}$ then since:

$$G^{\alpha\beta}\frac{\partial G_{\alpha\beta}}{\partial q^\gamma} = \frac{\partial \log G}{\partial q^\gamma}$$

we obtain the equations:

$$\frac{\partial \log G}{\partial q^\gamma} - 2\Pi^\sigma_{\sigma\gamma} = 5A_\gamma , \tag{6}$$

in which:

$$G = |\, G_{\alpha\beta} \,| .$$

Now, it follows immediately from chap. III (13) that $\Pi^\sigma_{\sigma\gamma}$ satisfies the equation:

$$\frac{\partial \Pi^\sigma_{\sigma\gamma}}{\partial q^\delta} - \frac{\partial \Pi^\sigma_{\sigma\delta}}{\partial q^\gamma} = 0.$$

We can therefore put $\Pi^\sigma_{\sigma\gamma}$ into the form:

$$\Pi^{\sigma}_{\sigma\gamma} = \frac{\partial \log f}{\partial q^{\gamma}},$$

in which we do not need to make any more specific statements about how f behaves under changes of representation. f is a function of only $q^1, \ldots, q^4$ since q^0 does not enter into Π anywhere.

On the basis of (6) we therefore obtain:

(7) $$A_{\gamma} = \frac{1}{5}\frac{\partial \log A}{\partial q^{\gamma}},$$

in which:

$$A = \frac{G}{f^2}.$$

If we now make the replacement:

$$G_{\alpha\beta} = G^{*}_{\alpha\beta} A^{\frac{1}{5}}$$

in (5) then it follows that:

$$\frac{\partial G^{*}_{\alpha\beta}}{\partial q^{\gamma}} A^{\frac{1}{5}} + G^{*}_{\alpha\beta}\frac{\partial A^{\frac{1}{5}}}{\partial q^{\gamma}} - A^{\frac{1}{5}} G^{*}_{\alpha\sigma}\Pi^{\sigma}_{\beta\gamma} - A^{\frac{1}{5}} G^{*}_{\beta\sigma}\Pi^{\sigma}_{\alpha\gamma} = A^{\frac{1}{5}} G^{*}_{\alpha\beta} A_{\gamma},$$

or:

(8) $$\frac{\partial G^{*}_{\alpha\beta}}{\partial q^{\gamma}} - G^{*}_{\alpha\sigma}\Pi^{\sigma}_{\beta\gamma} - G^{*}_{\beta\sigma}\Pi^{\sigma}_{\alpha\gamma} = 0 .$$

Obviously, we can replace $G_{\alpha\beta}$ with $G^{*}_{\alpha\beta}$ in (3):

(3*) $$G^{*}_{\alpha\beta} Z^{\alpha} Z^{\beta} = 0 .$$

It is self-evident that $G^{*}_{\alpha\beta}$ also obeys the transformation law (4).

Since the $G^{*}_{\alpha\beta}$ are chosen in such a way that (8) is valid in every coordinate system it then follows that the quantity λ in (4) must be constant. If we set $q = \bar{q}$ in (4) then we find that λ has the value 1. From now on, we omit the asterisks.

We thus obtain the transformation law:

(4′) $$\bar{G}_{\alpha\beta}(\bar{q}) = G_{\sigma\tau}\,\frac{\partial q^{\sigma}}{\partial \bar{q}^{\alpha}}\frac{\partial q^{\tau}}{\partial \bar{q}^{\beta}}$$

for the $G_{\alpha\beta}$.

Computation of the $G_{\alpha\beta}$ from Φ.

Equation (8) then yields the conditions:

$$\Pi^{\sigma}_{\alpha 0} = \delta^{\sigma}_{\alpha}$$

As a result, we have, in particular, the invariant equations:

(9) $$\frac{\partial G_{\alpha\beta}}{\partial q^0} = 2\, G_{\alpha\beta}$$

and:

(10) $$\frac{\partial G_{00}}{\partial q^\gamma} = 2\, G_{0\gamma}$$

and:

(11) $$\frac{\partial G_{\alpha 0}}{\partial q^\gamma} - G_{\sigma 0}\Pi^\sigma_{\alpha\gamma} = G_{\alpha\gamma}\,.$$

From the first equation, it follows that $G_{\alpha\beta}$ is of the form:

$$G_{\alpha\beta} = e^{2q^0} f_{\alpha\beta}(q)\,.$$

If we then take (4) into account then it follows that $G_{\alpha\beta}$ is a second-rank projective tensor of index 2.

The theory of a quadric surface is therefore included in the theory of a second-rank tensor that satisfies equations (8). Π is therefore an integrable projective connection. The quantities Π are given by the existence of a projective space, in the usual sense, from the outset. One can then ascertain the $G_{\alpha\beta}$ by integrating equations (8).

From equations (10) and (11), one sees that the entire theory of our tensor depends upon the scalar:

$$G_{00} = \Phi^2.$$

From equation (10), we have, in fact:

(12) $$\frac{G_{0\alpha}}{G_{00}} = \varphi_\alpha = \frac{\partial \log\Phi}{\partial q^\gamma}\,.$$

Equation (11) means the same thing as:

$$\frac{\partial(\Phi^2\varphi_\alpha)}{\partial q^\beta} - \Phi^2\varphi_\sigma\Pi^\sigma_{\alpha\beta} = \Phi^2\gamma_{\alpha\beta}$$

or:

$$\frac{\partial\varphi_\alpha}{\partial q^\beta} - \varphi_\sigma\Pi^\sigma_{\alpha\beta} = \gamma_{\alpha\beta} - \varphi_\alpha\,\varphi_\beta\,.$$

Therefore, we have:

$$g_{ij} = \frac{\partial\varphi_i}{\partial q^j} - \varphi_\sigma\Pi^\sigma_{ij} + \varphi_{\square}\,\varphi_{\square}\,,$$

or, from (12):

(13) $$g_{ij} = \frac{\dfrac{\partial^2\Phi}{\partial q^i\partial q^j} - \dfrac{\partial\Phi}{\partial q^\sigma}\Pi^\sigma_{ij}}{\Phi}\,.$$

One thus obtains φ_α and g_{ij} from Φ with the help of equations (12) and (13), and thus, since:

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + \varphi_\alpha \varphi_\beta$$

and:

$$G_{\alpha\beta} = \Phi^2 \gamma_{\alpha\beta},$$

one also obtains $\gamma_{\alpha\beta}$ and $G_{\alpha\beta}$.

A homogeneous projective coordinate system Z is completely determined for a given choice of point of origin, coordinate system, and gauge. In this coordinate system, the surface has the equation:

$$G_{\alpha\beta} Z^\alpha Z^\beta = 0,$$

in which the coefficients $G_{\alpha\beta}$ are the values of the solutions of (9), (10), and (11) at the point q.

The equation of the surface in inhomogeneous projective coordinates.

We now take an inhomogeneous projective coordinate system and a gauge such that $\Pi^\alpha_{\beta\gamma}$ takes on the values:

$$\Pi^\alpha_{\beta\gamma} = \delta^\alpha_\beta \delta^0_\gamma + \delta^\alpha_\gamma \delta^0_\beta - \delta^\alpha_0 \delta^0_\beta \delta^0_\gamma .$$

It then follows from (8), (10), (11) that we have:

$$\frac{\partial G_{ij}}{\partial q^k} = 0 ,$$

$$\frac{\partial G_{0i}}{\partial q^j} = G_{ij} ,$$

$$\frac{\partial G_{00}}{\partial q^i} = 2G_{0i} .$$

By integration, we then find that:

(14a) $$G_{ij} = e^{2q^0} a_{ij} ,$$

(14b) $$G_{0j} = e^{2q^0} (a_{ij} q^j + a_{j0}) ,$$

(14c) $$G_{00} = e^{2q^0} (a_{ij} q^i q^j + 2a_{i0} q^j + a_{00}) ,$$

in which the $a_{\alpha\beta}$ are constants.

The equation of the quadric surface is:

$$0 = G_{\alpha\beta} Z^\alpha Z^\beta,$$

or, by making use of (14) and chap. III (17):

$$0 = e^{2(x^0 - q^0)} (G_{ij} (x^i - q^j)(x^j - q^j) + 2G_{i0} (x^i - q^j) + G_{00}) ,$$

or, finally:

$$0 = e^{2x^0}(a_{ij}\,x^i\,x^j + 2a_{i\,0}\,x^j + a_{00})\;.$$

With this, we have established that the surface is associated with a quadric surface with constant coefficients in any inhomogeneous coordinate system.

The distinguished gauge.

As we saw in chap. II, a particular gauge is always determined by a choice of projective scalar. In our case, we can assume that the scalar Φ has the form:

(15) $$\Phi = e^{x^0}\;.$$

Namely, if Φ has the form:

$$\Phi = e^{x^0}\rho(x)\;,$$

then we need to apply only the gauge transformation:

$$\bar{x}^0 = x^0 + \log\rho = \log\Phi\;.$$

Due to (12) the vector φ then satisfies:

$$\varphi_i = 0\;.$$

Since no further gauge transformations exist that leave the form (15) of Φ invariant, we no longer have a projective geometry, only an affine one. Due to the existence of the tensor g_{ij}, this affine geometry is a metric one.

Relative to our distinguished gauge, we have:

(16a) $$G_{ij} = e^{2q^0}g_{ij}$$

and:

(16b) $$G_{0\alpha} = e^{2q^0}\delta_0^\alpha\;,$$

and the equation of our quadric surface is then:

(17) $$e^{2q^0}(g_{ij}\,Z^i\,Z^j + Z^0\,Z^0) = 0\;.$$

Furthermore, from (8) and (16), we have the following equation:

$$\frac{\partial g_{ij}}{\partial x^k} - g_{is}\Pi^s_{jk} - g_{js}\Pi^s_{ik} = 0\;.$$

If we solve these equations in the usual way then we obtain:

(18) $$\Pi^i_{jk} = \left\{\begin{matrix} i \\ jk \end{matrix}\right\} = \frac{1}{2}g^{is}\left(\frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right)\;.$$

From (13), if we make use of (15), it follows that:

(19) $$\Pi^0_{jk} = -\, g_{jk}\ .$$

Due to (18) and (19), it is clear that Π^i_{jk} is an affine connection and Π^0_{jk} is an affine tensor; this is a simple application of formulas (10a) and (10b) in chap. III.

CAYLEYIAN geometry.

We are given a RIEMANNIAN metric in our base space by way of the tensor g_{ij} . Since this metric is invariantly related to our quadric surface, we surmise that the RIEMANNIAN metric is precisely the only non-Euclidian metric that our quadric surface possesses as an intrinsic structure.

A non-Euclidian – or CAYLEYIAN – metric may be easily defined with the help of a "tangential" Euclidian one. The general notion of a tangential metric that is due to E. CARTAN (Bibliography 1928, 1, chap. IV) is the following one: Two metrics that possess the same g_{ij} at some point are called *tangential* at the point in question.

A Euclidian metric exists at every point q of our space that makes the quadric surface precisely a ball of radius 1 and midpoint q. The infinitely distant hyperplane of this Euclidian space is the polar hyperplane of q relative to the quadric surface. The infinitesimal non-Euclidian distance at q shall now correspond with the infinitesimal distance of the Euclidian metric at q.

We now seek the analytical expression for our Euclidian metric.

In the non-homogenous coordinate system that is defined by:

$$\frac{Z^i}{Z^0} = z^i$$

the equation of our quadric surface is:

(20) $$g_{ij}\,(x)\, z^i\, z^j + 1 = 0\ .$$

Equation (20) is the equation of a ball of radius 1 and midpoint q relative to a Euclidian metric with the line element:

(21) $$ds^2 = -\, g_{ij}\,(x)\, dz^i\, dz^j\ .$$

Therefore, (21) is also the CAYLEYIAN metric at the point q that is given by the quadric surface.

In order to reconcile this definition with the one that CAYLEY himself gave, we calculate the CAYLEYIAN distance of the point Z from O. The line through O and Z intersects the ball at A and B. The double ratio of the four points $O, Z, A,$ and B is:

$$\alpha = \frac{OA}{OB} : \frac{ZA}{ZB} = \frac{1+\sqrt{-g_{ij}z^i z^j}}{1-\sqrt{-g_{ij}z^i z^j}}\ .$$

The CAYLEYIAN distance from O to Z is then m log α, in which m is constant. From this, it follows by taking the limit that:

$$ds = 2m \sqrt{-g_{ij} dz^i dz^j} \; .$$

Up to a constant, the CAYLEYIAN distance then corresponds with:

$$ds^2 = - g_{ij}\, dz^i\, dz^j \; .$$

Naturally, all of the formulas for the points on the absolute surface break down when:

$$\Phi = 0 \; .$$

On this basis, WHITEHEAD (Bibliography 1931, 15), who has thoroughly investigated these matters, has proposed the name "missing (n–1)-space" for this surface. Obviously, the transition from a projective geometry to an affine one arises from a choice of gauge when the missing (n-1)-space is not a quadric. This idea is developed further in the work of WHITEHEAD. It is quite possible that one might find other interesting geometries in this direction.

VI. Generalized theory of conic sections.

The metric part of geometry.

The theory of a general second-rank projective tensor of index $2N$ may be considered to be a generalization of the non-Euclidian geometry that was discussed in the previous chapter. A second-rank tensor gives rise to a metric geometry, namely, by way of the RIEMANNIAN geometry that is given by the affine tensor g_{ij} . g_{ij} is defined by the equation:

(1) $$G_{\alpha\beta} = \Phi^2(g_{\alpha\beta} + \varphi_\alpha\varphi_\beta) .$$

Our geometry is, however, not only a metric geometry; rather, it contains other elements that are not metric.

We have a CAYLEY metric in this tangent space that is related to the quadratic form:

$$G_{\alpha\beta}\, X^\alpha\, X^\beta.$$

The well-known theorems and formulas of CAYLEY geometry are thus valid in a tangent space. For instance, we have the following formula for the distance between two points of the tangent space:

(2) $$\cos id = \frac{G_{\alpha\beta}X^\alpha X^\beta}{\sqrt{G_{\alpha\beta}X^\alpha X^\beta G_{\sigma\tau}Y^\sigma Y^\tau}} .$$

The affine tensor g_{ij} is uniquely determined by the projective tensor $G_{\alpha\beta}$, and the g_{ij} then define a RIEMANNIAN metric with the line element:

(3) $$ds = \sqrt{-g_{ij}dx^i dx^j} .$$

As is well known, a RIEMANNIAN metric defines a Euclidian metric in every tangent space. Relative to the measure (3), the surface:

$$G_{\alpha\beta}\, X^\alpha\, X^\beta = 0$$

seems to be precisely a sphere of radius 1 in the tangent space, with its midpoint at the contact point. As we saw in the last chapter, the infinitesimal CAYLEY distance agrees with the one given by (3) at the contact point.

Obviously, the CAYLEY metric can be applied to the base space only in an infinitesimal neighborhood of the origin. Therefore, the CAYLEY metric in the tangent space has precisely the same influence on, say, the formula:

$$\int ds$$

for the arc-length as the as the tangent metric at the various points of the curve. From an expression that was established by HOWE, because of (3), we can regard the Euclidian metric as the metric on the tangent space to the CAYLEY tangent space at a given point when both tangent spaces have the same contact point with the base space.

If we restrict ourselves then to the metric viewpoint then we find only a RIEMANNIAN space with the line element:

$$ds^2 = -\, g_{ij}\, dx^i\, dx^j\,. \tag{3}$$

However, this line element does not account for the entire effect of the CAYLEY space on the base space since φ and Φ do not appear in (3). The non-metric properties of $G_{\alpha\beta}$ first appear when one considers the associated projective translations Γ and Π, as we began to do in chap. IV.

In particular, we will find systems of curves that are invariantly related to these translations. However, in order to describe these curves concisely, we must first develop our formal apparatus somewhat further.

Invariants of g_{ij} .

Any enumeration of the invariants of $G_{\alpha\beta}$ must include the invariants of the affine tensor g_{ij} and the projective vector φ_α, in particular. The tensor g_{ij} possesses a series of well-known invariants or associated geometric objects.

Now, the determinant:

$$g = |\, g_{ij}\,|$$

is a relative scalar[1] of weight 2. Its transformation law is:

$$\bar{g} = g\left|\frac{\partial x}{\partial \bar{x}}\right|^2,$$

in which $|\,\partial x / \partial \bar{x}\,|$ is the functional determinant of the coordinate transformation $x \to \bar{x}$.

Furthermore, we have to name the contravariant tensor g_{ij} , which is determined by the relation:

$$g_{ij}\, g^{ik} = \delta_j^k\,. \tag{4}$$

Furthermore, we obtain the components of an affine connection from the CHRISTOFFEL formula:

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2} g^{ia}\left(\frac{\partial g_{aj}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a}\right)\,. \tag{5}$$

We can covariantly differentiate affine tensors relative to this affine connection. For example, the covariant derivative of a mixed tensor φ_j^i is:

[1] Often, invariants that I am calling "relative scalars" are referred to as "scalar densities." I would therefore like to reserve this name for relative scalars of weight 1, such as $\sqrt{g}$, since physical densities are always of weight 1.

$$\varphi^i_{j,k} = \frac{\partial \varphi^i_j}{\partial x^k} + \varphi^s_j \begin{Bmatrix} i \\ sk \end{Bmatrix} - \varphi^i_s \begin{Bmatrix} s \\ jk \end{Bmatrix} .$$

We denote the affine covariant derivative by a comma (,). From (5), it is well known that that the tensor g_{ij} satisfies:

$$g_{ij,k} = 0 .$$

One obtains the fundamental RIEMANNIAN curvature tensor:

(6) $$R^i_{jkl} = \frac{\partial}{\partial x^l} \begin{Bmatrix} i \\ jk \end{Bmatrix} - \frac{\partial}{\partial x^k} \begin{Bmatrix} i \\ jl \end{Bmatrix} - \begin{Bmatrix} i \\ rk \end{Bmatrix} \begin{Bmatrix} r \\ jl \end{Bmatrix} + \begin{Bmatrix} i \\ rl \end{Bmatrix} \begin{Bmatrix} r \\ jk \end{Bmatrix}$$

from the CHRISTOFFEL symbols in the theory of the tensor g_{ij} in the usual way.

One derives the RICCI tensor from the curvature tensor:

(7) $$R_{ij} = R^k_{ikj}$$

by contraction, and, upon multiplying this by the RIEMANN tensor g_{ij} and contracting, one obtains the scalar curvature:

$$R = g^{ij} R_{ij} .$$

One derive an infinite sequence of new invariants from these tensors by the process of covariant differentiation (cf., e.g., Bibliography 1927, 22).

Affine invariants of φ_α .

A further affine invariant is determined by the projective vector φ_α :

(8) $$\frac{1}{2}\left(\frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}\right) = \varphi_{ij}$$

must be an affine tensor, since:

$$\frac{\partial \varphi_\alpha}{\partial x^\beta} - \frac{\partial \varphi_\beta}{\partial x^\alpha} ,$$

vanishes, as long as α or β assumes the value 0. The important tensor φ_{ij} plays a central role in electromagnetic theory.

We agree that Latin indices shall be raised or lowered by means of g^{ij} and g_{ij} , whereas the same processes are carried out on Greek indices by means of $\gamma^{\alpha\beta}$ and $\gamma_{\alpha\beta}$, as we already mentioned. Therefore, one has, e.g.:

(9) $$\varphi^i_j = g^{ij} \varphi_{kj} .$$

In order to define this situation uniquely we assume that when more than one lower indices are present it is always the first index that shall be raised. This assumption determines the sign of φ^i_j.

Since φ_i and $\varphi_i\, g^{ij}$ are not affine tensors, we cannot raise the Latin index of φ_i by means of g^{ij}. Rather, as we agreed, one has:

(10) $$\varphi^\alpha = \gamma^{\alpha\beta}\varphi_\beta = \delta^\alpha_0 .$$

Thus, for just the φ^j, one has:

$$\varphi^j = 0.$$

Invariants of $\gamma_{\alpha\beta}$.

We can derive a sequence of invariants from $\gamma_{\alpha\beta}$ in a way that formally agrees with the route that we took with the g_{ij}. We obtain its determinant as:

(11) $$g = |\gamma_{\alpha\beta}| = \begin{vmatrix} 1 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \varphi_1 & & & & \\ \varphi_2 & & (g_{ij} + \varphi_i\varphi_j) & & \\ \varphi_3 & & & & \\ \varphi_4 & & & & \end{vmatrix} = g .$$

We have already given the defining equation for $\gamma^{\alpha\sigma}$:

(12) $$\gamma^{\alpha\sigma}\gamma_{\sigma\beta} = \delta^\alpha_\beta .$$

The tensor $\gamma^{\alpha\sigma}$ has a very simple relationship to g^{ij} and φ_i. Namely, from (12), one has:

$$\delta^i_j = \gamma^{\alpha i}\gamma_{\alpha k} = \gamma^{\alpha i}\,(g_{\alpha k} + \varphi_\alpha\,\varphi_k)$$

or, due to (10) and $g_{0k} = 0$:

$$\delta^i_j = \gamma^{ik}\, g_{kj} ,$$

or finally, due to (4):

(13) $$\gamma^{ij} = g^{ij}.$$

Likewise, we can calculate γ^{0i} from g^{ij} and φ_j:

(14) $$\gamma^{0i} = -g^{ij}\,\varphi_j .$$

Equation (14) follows immediately from:

$$\delta^i_0 = 0 = \gamma^{\alpha i}\gamma_{\alpha 0} = g^{ij}\varphi_j + \gamma^{00},$$

or:

(15) $$\gamma^{00} = 1 + g^{ij}\varphi_i\,\varphi_j .$$

The connection Γ.

We have already introduced the projective connection Γ by the formulae:

(16) $$\gamma_{\sigma\beta\,;\,\gamma} = 0$$
and:

(17) $$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\gamma^{\alpha\sigma}\left(\frac{\partial\gamma_{\beta\sigma}}{\partial x^{\gamma}} + \frac{\partial\gamma_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial\gamma_{\beta\gamma}}{\partial x^{\sigma}}\right) .$$

Let us write (17) a bit more explicitly:

(18) $$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\gamma^{\alpha i}\left(\frac{\partial g_{i\beta}}{\partial x^{\gamma}} + \frac{\partial g_{i\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{i}}\right) + \frac{1}{2}\delta^{\alpha}_{0}\left(\frac{\partial\varphi_{\beta}}{\partial x^{\gamma}} + \frac{\partial\varphi_{\gamma}}{\partial x^{\beta}}\right) + \gamma^{\alpha i}(\varphi_{i\gamma}\varphi_{\beta} + \varphi_{i\beta}\varphi_{\gamma}) .$$

This means that:

(19) $$\Gamma^{\alpha}_{00} = 0 ,$$

(20) $$\Gamma^{\alpha}_{\beta 0} = \gamma^{i\alpha}\varphi_{i\beta} ,$$

(21) $$\Gamma^{i}_{jk} = \Gamma\left\{{i \atop jk}\right\} + \varphi^{i}_{j}\varphi_{k} + \varphi^{i}_{k}\varphi_{j} ,$$

(22) $$\Gamma^{0}_{jk} = -\varphi_{i}\Gamma^{i}_{jk} + \frac{1}{2}\left(\frac{\partial\varphi_{j}}{\partial x^{k}} + \frac{\partial\varphi_{k}}{\partial x^{j}}\right).$$

Due to (14), equation (20) may be decomposed into:

$$\Gamma^{i}_{j0} = \varphi^{i}_{j} , \qquad \Gamma^{0}_{j0} = -\varphi_{i}\varphi^{i}_{j} ,$$

Finally, we mention once more the formula that we proved in chap. IV:

(23) $$\varphi_{\alpha;\,\beta} = \varphi_{\alpha\beta} .$$

For $\beta = 0$, this means that:

(23a) $$\varphi_{\sigma}\Gamma^{\sigma}_{\alpha\,0} = 0 .$$

The curvature tensor for Γ .

Just as in the affine theory, one can also define a curvature tensor:

$$B^{\alpha}_{\beta\gamma\delta}$$

for the connection Γ. By contraction, one obtains an analogue of the Ricci tensor from it:

$$
(24) \qquad \begin{cases} B^{\sigma}_{\alpha\sigma\beta} = \dfrac{\partial \Gamma^{\sigma}_{\alpha\sigma}}{\partial x^{\beta}} - \dfrac{\partial \Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\sigma}} + \Gamma^{\varepsilon}_{\alpha\sigma}\Gamma^{\sigma}_{\varepsilon\sigma} - \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\sigma}_{\varepsilon\sigma} \\ \quad = R_{ij}\delta^{i}_{\alpha}\delta^{j}_{\beta} - \varphi^{s}_{i,s}(\delta^{i}_{\alpha}\varphi_{\beta} + \delta^{i}_{\beta}\varphi_{\alpha}) + \varphi^{s}_{\beta}\varphi_{s\alpha} + \varphi^{s}_{t}\varphi^{t}_{s}\varphi_{\alpha}\varphi_{\beta} \\ \quad = B_{\alpha\beta}. \end{cases}
$$

We can then derive a scalar from $B_{\alpha\beta}$ that corresponds to the scalar curvature:

$$
(25) \qquad B = \gamma^{\alpha\beta} B_{\alpha\beta} = R - \varphi^{s}_{t}\varphi^{t}_{s}.
$$

The projective translation Λ.

A projective translation is associated with the connection Γ for any index by way of the differential equation:

$$
X^{\alpha}_{;\,\beta} = X^{\alpha}_{;\,0}\ \varphi_{\beta},
$$

as we showed in chap. IV, pp. 35, 36. Along a curve:

$$
x^{i} = x^{i}(t)
$$

the tangent spaces are displaced according to the equation:

$$
\frac{dX^{\alpha}}{dt} + \Lambda^{\alpha}_{\sigma j} X^{\sigma} \frac{dx^{j}}{dt} = 0 .
$$

Thus, one has:

$$
(26) \qquad \Lambda^{\alpha}_{\sigma\tau} = \Gamma^{\alpha}_{\sigma\tau} - \Gamma^{\alpha}_{\sigma 0}\varphi_{\tau} + M\delta^{\alpha}_{\sigma}\varphi_{\tau}.
$$

Invariance of the non-Euclidian distance.

The translation Λ satisfies the equations:

$$
(27) \qquad \frac{\partial \gamma_{\alpha\beta}}{\partial x^{\gamma}} - \gamma_{\alpha\sigma}\Lambda^{\sigma}_{\beta\gamma} - \gamma_{\beta\sigma}\Lambda^{\sigma}_{\alpha\gamma} = -2M\ \gamma_{\alpha\beta}\ \varphi_{\gamma}
$$

and:

$$
(28) \qquad \frac{\partial \varphi_{\alpha}}{\partial x^{\gamma}} - \varphi_{\sigma}\Lambda^{\sigma}_{\beta\gamma} = \varphi_{\alpha\beta} - 2M\ \gamma_{\alpha\beta}\ \varphi_{\gamma},
$$

as one easily confirms.

An application of (27) is the theorem that the non-Euclidian distance between two points X^{α} and Y^{β} in a tangent space is preserved under the translation Λ.

The non-Euclidian distance between the points is determined by way of:

$$D = \frac{(G_{\alpha\beta} X^\alpha Y^\beta)^2}{(G_{\alpha\beta} X^\alpha X^\beta)(G_{\alpha\beta} Y^\alpha Y^\beta)} = \frac{(\gamma_{\alpha\beta} X^\alpha Y^\beta)^2}{(\gamma_{\alpha\beta} X^\alpha X^\beta)(\gamma_{\alpha\beta} Y^\alpha Y^\beta)} .$$

Differentiating $\gamma_{\alpha\beta} X^\alpha Y^\beta$ with respect to x^γ yields:

$$\frac{\partial}{\partial x^\gamma}\ \gamma_{\alpha\beta} X^\alpha Y^\beta = \left(\frac{\partial \gamma_{\alpha\beta}}{\partial x^\gamma} - \gamma_{\alpha\sigma}\Lambda^\sigma_{\beta\gamma} - \gamma_{\beta\sigma}\Lambda^\sigma_{\alpha\gamma}\right) X^\alpha Y^\beta .$$

From (27), we then obtain:

$$\frac{\partial}{\partial x^\gamma}\ \gamma_{\alpha\beta} X^\alpha Y^\beta = -\,2M\ \gamma_{\alpha\beta} X^\alpha Y^\beta\ \varphi_\gamma ,$$

or:

$$\frac{\partial \log \gamma_{\alpha\beta} X^\alpha Y^\beta}{\partial x^\gamma} = -\,2M\ \varphi_\gamma .$$

Since the right-hand side of this equation is independent of X and Y, we actually obtain:

$$\frac{\partial \log D}{\partial x^\gamma} = 0 .$$

Therefore, under translation by Λ any figure in a tangent space goes to the same figure in another tangent space.

Translation in inhomogeneous coordinates.

In chap. IV, pp. 32, we derived the equation:

$$(29)\qquad \begin{cases} \dfrac{dV^i}{dt} + \Lambda^i_{kj} V^k \dfrac{dx^j}{dt} + \Lambda^i_{0j} \dfrac{dx^j}{dt}(1 - \varphi_i V^j) + \varphi_{k;\,j} V^k V^i \dfrac{dx^j}{dt} \\ \qquad + \varphi_{0;\,j}(1 - \varphi_k V^k)\dfrac{dx^j}{dt} V^i = 0. \end{cases}$$

In this equation, the Π of chap. IV (14) has been replaced with Λ, and the projective derivative of φ_α relative to Λ is denoted by $\varphi_{\alpha;\beta}$. An application of (26) then yields:

$$(30)\qquad \begin{cases} \dfrac{dV^i}{dt} + \Gamma^i_{kj} V^k \dfrac{dx^j}{dt} - \Gamma^i_{k0} V^k \varphi_j \dfrac{dx^j}{dt} + M V^k \varphi_j \dfrac{dx^j}{dt} + (1 - \varphi_i V^j)\Gamma^i_{0j} \dfrac{dx^j}{dt} \\ \qquad + \varphi_{k;\,j} V^k V^i \dfrac{dx^j}{dt} + \varphi_{0;\,j}(1 - \varphi_k V^k)\dfrac{dx^j}{dt} V^i = 0. \end{cases}$$

If we replace $\varphi_{k;j}$ and $\varphi_{0;j}$ in (30) with their values in (28) then we arrive at the equation for translation along a curve in inhomogeneous notation in the form:

(31) $$\frac{dV^i}{dt} + \begin{Bmatrix} i \\ kj \end{Bmatrix} V^k \frac{dx^j}{dt} + \varphi_{kj} V^k \frac{dx^j}{dt} V^i + \varphi^i_j \frac{dx^j}{dt} = 0 .$$

The world-lines of an electric particle.

On the basis of (31), we can now define a system of curves that may be regarded as a generalization of the geodetic lines of a RIEMANNIAN space.

A given curve in a definite parameter representation:

$$x^i = x^i\,(t)$$

distinguishes a point in the associated tangent space to each of its points through the equation:

(32) $$k\frac{dx^i}{dt} = V^i .$$

Here, k is an arbitrary constant.

We now look at curves along which the point V is translated to itself according to (32). For this to happen, we must substitute (32) into (31). Due to the skew-symmetry of φ_{ij} , (31) then becomes:

(33) $$\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ kj \end{Bmatrix} \frac{dx^k}{dt}\frac{dx^j}{dt} + \frac{1}{k}\varphi^i_j \frac{dx^j}{dt} = 0 .$$

Equations (33) yield conditions for the parameter representation of the curve as well as for the curve itself.

When:

(34) $$\frac{1}{k} = 0 ,$$

the curves (33) are precisely the geodetic lines that are characteristic of the metric g_{ij} .

When (34) is not satisfied then the differential equations are not homogeneous in t. If we fix k then through any point, a curve with a given velocity vector is completely determined. Curves through the initial point with the initial direction are determined only for geodetic lines.

If we replace $1/k$ with e/m in (33) then we obtain:

(33a) $$\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ kj \end{Bmatrix} \frac{dx^k}{dt}\frac{dx^j}{dt} + \frac{e}{m}\varphi^i_j \frac{dx^j}{dt} = 0 .$$

In the context of general relativity theory, we must interpret these equations as the equations of motion of an electric particle. e and m are the charge and mass of the particle, whereas the g_{ij} and the φ_i are the gravitational and electromagnetic potentials of the field in which the particle moves.

The connection Π.

We would now like to consider the connection Π that we also defined in chap. IV. We have seen that Π may be computed from Γ and Φ. Just as we expressed the $\Gamma^{\alpha}_{\beta\gamma}$ in terms of $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$, φ_k , φ_{ij} , etc., by means of equations (19), (20), (21), and (22), likewise we would like to now express $\Lambda^{\alpha}_{\beta\gamma}$ in terms of $\Gamma^{\alpha}_{\beta\gamma}$, φ, and an undetermined quantity Θ.

We set:

$$\Phi_\alpha = \frac{\partial \log \Phi}{\partial x^\alpha}$$

and:

(35) $$\Theta_\alpha = N\, \varphi_\alpha - \Phi_\alpha .$$

Since $\Phi_0 = N$, we have $\Theta_0 = 0$. Θ_4 is therefore an affine tensor. We now replace Φ^α with the quantities $\Theta^i = g^{ij}\, \Theta_j$.

Next, we have:

$$\Phi^\sigma = \gamma^{0\alpha}\, \Phi_\alpha = N\, \gamma^{0\alpha} + \gamma^{0j}\, \Phi_j .$$

From this, it follows, with the help of (13), (14), and (15) that:

(36) $$\Phi^i = N\, \gamma^{\,i\,0} + \gamma^{ij}\, \Phi_j = -\, g^{ij}\, (N\varphi_j - \Phi_j\,) = -\, \Theta^i$$

and:

$$\Phi^0 = N\, \gamma^{\,00} + \gamma^{\,0j}\, \Phi_j = N - g^{ij}\, \varphi_i\, \Phi_j + N\, g^{ij}\, \varphi_i\, \varphi_j ,$$

or:

(37) $$\Phi^0 = N + \varphi_i\, \Theta^i .$$

$\Phi^\alpha \Phi_\alpha$ can be calculated from Θ in a particularly simple way:

$$\Phi^\alpha \Phi_\alpha = (N + \varphi_i\, \Theta^i\,)N + \Theta^i\, (\Theta_i - N\varphi_i\,) = N^2 + \Theta_i\, \Theta^i.$$

For $\Pi^{\alpha}_{\beta\gamma}$ we now obtain the following formulas by using (35), (36), (37), and (22) in chap. IV (pp. 35):

(38a) $$\Pi^0_{00} = \Gamma^0_{00} + N - \varphi_i \Theta^i$$

(38b) $$\Pi^i_{00} = \Gamma^i_{00} + \Theta^i$$

(38c) $$\Pi^0_{j0} = \Gamma^0_{j0} + \Theta_j - \varphi_i \varphi_j \Theta^i$$

(38d) $$\Pi^i_{j0} = \Gamma^i_{j0} + N\delta^i_j + \Theta^i \varphi_j$$

(38e) $$\Pi^0_{jk} = \Gamma^0_{jk} - (g_{jk} + \varphi_j \varphi_k)(N + \varphi_i \Theta^i)$$

(38f) $$\Pi^i_{jk} = \Gamma^i_{jk} + \delta^i_j \Phi_k + \delta^i_k \Phi_j + \Theta^i (g_{jk} + \varphi_j \varphi_k) \ .$$

The covariant derivative of Φ relative to Γ.

With the help of the quantities Θ that we just defined, we can also arrive at a sequence of simple formulas for the covariant derivatives of Φ relative to Γ. We will not, however, carry out all of the elementary intermediate calculations in detail.

For the covariant derivative of Φ, we obtain:

$$\Phi_{\alpha;\,\beta} = \frac{\partial \Phi_\alpha}{\partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\Phi_\sigma .$$

In particular, due to (19), we have:

$$\Phi_{0;\,0} = 0.$$

From (20), (21), (36), and (37), we obtain:

$$\Phi^\alpha_{\;;\,\alpha} = \frac{\partial \Phi^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\sigma\alpha}\Phi^\alpha = -\frac{\partial \Theta^i}{\partial x^i} - \begin{Bmatrix} j \\ ji \end{Bmatrix}\Theta^i, \tag{39}$$

or:

$$\Phi^\alpha_{\;;\,\alpha} = \Phi^i_{\;;\,i} = -\frac{1}{\sqrt{g}}\frac{\partial(\Theta^i\sqrt{g})}{\partial x^i}.$$

In this, we have denoted the (affine) covariant derivative relative to $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ by a comma (,).

Ultimately, we find:

$$\Phi^i_{\;;\,\tilde{0}} = -\varphi^i_j\Theta^j \tag{40}$$

and:

$$\gamma^{\sigma j}\,\Phi^i_{\;;\,\sigma} = g^{ia}\,(N\,\varphi^i_j - \Phi^i_{\;;\,a}) = -\tfrac{1}{2}g^{ia}\,g^{jb}\,(\Theta_{a,b} + \Theta_{a,a}) . \tag{41}$$

With this, we have reduced the covariant derivative of Φ relative to Γ to the covariant derivative of Θ relative to $\begin{Bmatrix} i \\ jk \end{Bmatrix}$, to a certain degree.

The curvature tensor of Π.

We now return to the connection Π and calculate the curvature tensor $P^\alpha_{\beta\gamma\delta}$ that is constructed from it. In order to express $P^\alpha_{\beta\gamma\delta}$ in terms of Γ and Φ in a convenient way, we define a new quantity $T^\alpha_{\beta\gamma}$ by the formula:

$$\Pi^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} . \tag{42}$$

From chap. IV, T has the value:

$$T^\alpha_{\beta\gamma} = (\delta^\alpha_\beta\Phi_\gamma + \delta^\alpha_\gamma\Phi_\beta - \gamma_{\beta\gamma}\Phi^\alpha) . \tag{43}$$

By the use of (42), we now obtain the following equation for $P^{\alpha}_{\beta\gamma\delta}$:

(44) $$P^{\alpha}_{\beta\gamma\delta} = B^{\alpha}_{\beta\gamma\delta} + T^{\alpha}_{\beta\gamma;\delta} - T^{\alpha}_{\beta\delta;\gamma} + T^{\sigma}_{\beta\gamma}T^{\alpha}_{\sigma\gamma} - T^{\sigma}_{\beta\delta}T^{\alpha}_{\sigma\gamma}$$

or:

(45) $$\begin{cases} P^{\alpha}_{\beta\gamma\delta} = B^{\alpha}_{\beta\gamma\delta} + \delta^{\alpha}_{\gamma}(\Phi_{\beta;\delta} - \Phi_{\beta}\Phi_{\delta} + \Phi^{\sigma}\Phi_{\sigma}\gamma_{\beta\delta}) \\ \qquad - \delta^{\alpha}_{\delta}(\Phi_{\beta;\gamma} - \Phi_{\beta}\Phi_{\gamma} + \Phi^{\sigma}\Phi_{\sigma}\gamma_{\beta\gamma}) \\ \qquad - \gamma_{\beta\gamma}(\Phi^{\alpha}{}_{;\delta} - \Phi^{\alpha}\Phi_{\delta}) \\ \qquad + \gamma_{\beta\delta}(\Phi^{\alpha}{}_{;\gamma} - \Phi^{\alpha}\Phi_{\gamma}). \end{cases}$$

By contraction, we further obtain a tensor $P_{\beta\delta}$ and a scalar P that correspond to the quantities $B_{\alpha\beta}$ and B in formulas (24) and (25):

(46) $$P_{\beta\delta} = B_{\beta\delta} + (n-1)(\Phi_{\beta;\alpha} - \Phi_{\beta}\Phi_{\delta} + \Phi^{\sigma}\Phi_{\sigma}\ \gamma_{\beta\delta}) + \gamma_{\beta\delta}\ \ \Phi^{\sigma}{}_{;\sigma},$$

and:

(47) $$P = B - 2n\Theta^{i}{}_{,i} + n(n-1)(N^2 + \Theta^{i}\,\Theta^{j}).$$

From (46), we finally obtain the affine invariants:

$$\begin{aligned} P^{ij} &= \gamma^{i\alpha}\gamma^{j\beta}P_{\alpha\beta} = B^{ij} + (n-1)[g^{ij}(N^2 + \Theta^{p}\,\Theta_{p}) - \Theta^{i}\,\Theta^{j}], \\ P^{i}_{0} &= \gamma^{j\alpha}P_{\alpha 0} = B^{i}_{0} + (n-1)[N\Theta^{i} - \varphi^{i}_{j}\Theta^{j}] \\ P_{00} &= B_{00} + (n-1)\,\Theta_{i}\Theta^{i} - \Theta^{i}{}_{,i}\,. \end{aligned}$$

The translations associated with Π.

A translation is associated with Π in a manner that is completely analogous to the invariant translation Λ that is associated with Γ, by means of the equation:

(48) $$X^{\alpha}_{|j} = X^{\alpha}_{|0}\,\varphi_j,$$

or:

(49) $$\frac{\partial X^{\alpha}}{\partial x^{j}} + \Pi^{\alpha}_{\sigma j}X^{\sigma} = (-MX^{\alpha} + \Pi^{\alpha}_{\sigma 0}X^{\sigma})\ \varphi_j\,.$$

Equations (48) correspond to equations (24) in chap. IV.

If we now write:

$$\begin{aligned} \Sigma^{\alpha}_{\beta\gamma} &= \Pi^{\alpha}_{\beta\gamma} - \Pi^{\alpha}_{\beta 0}\varphi_{\gamma} + M\delta^{\alpha}_{\beta}\varphi_{\gamma} \\ &= \Lambda^{\alpha}_{\beta\gamma} + \delta^{\alpha}_{\beta}\Phi_{\gamma} + \delta^{\alpha}_{\gamma}\Phi_{\beta} - g_{\beta\gamma}\Phi^{\alpha} - N\delta^{\alpha}_{\beta}\varphi_{\gamma} - \delta^{\alpha}_{0}\Phi_{\beta}\varphi_{\gamma}\,, \end{aligned}$$

then we obtain the analogues of (27) and (28):

(50) $$\frac{\partial G_{\alpha\beta}}{\partial x^{\gamma}} - G_{\alpha\sigma}\Sigma^{\sigma}_{\beta\gamma} - G_{\sigma\beta}\Sigma^{\sigma}_{\alpha\gamma} = 2(N-M)\,G_{\alpha\beta}\,\varphi_{\gamma},$$

(51) $$\begin{cases} \dfrac{\partial\varphi_\alpha}{\partial x^\beta} - \varphi_\sigma\Sigma^\sigma_{\alpha\beta} = \varphi_{\alpha\beta} - M\varphi_\alpha\varphi_\beta + \varphi_\alpha\Theta_\beta + Ng_{\alpha\beta} \\ \qquad\qquad\qquad = \varphi_{\alpha;\beta} + \varphi_\alpha\Theta_\beta + Ng_{\alpha\beta} \end{cases}$$

and the analogue of (31):

(52) $$\begin{cases} \dfrac{dV^i}{dt} + \left\{ {i \atop kj} \right\} V^k \dfrac{dx^j}{dt} + \varphi_{kj} V^k \dfrac{dx^j}{dt} V^i + \varphi^i_j \dfrac{dx^j}{dt} \\ \quad = (\delta^i_j\Theta_k - g_{kj}\Theta^i) V^k \dfrac{dx^j}{dt} - Ng_{kj}V^kV^i \dfrac{dx^j}{dt} - N\delta^i_j \dfrac{dx^j}{dt}\,. \end{cases}$$

We will make no use of the formulas for Π in the following chapters. However, it is not unlikely that they might be of use in some later version of the theory (cf. chap. VII).

VII. Field equations.

In the previous chapters, we have seen that a projective tensor of rank two includes the formal apparatus for a theory of gravitation and electromagnetism. We regard the quantities g_{ij} as gravitational potentials, and the φ_α as electromagnetic potentials. Furthermore, we know that the motion of an electric particle can be described by the projective translations that depend upon the $\Gamma^\alpha_{\beta\gamma}$.

The projective scalar Φ played no role in any of these considerations. We therefore set $\Phi = 1$ in this chapter, so that we only need to deal with the theory of tensors with index 0.

The field equations in projective form.

We now seek a class of $\gamma_{\alpha\beta}$ that is a limiting class, in a certain sense, and indeed one that will hopefully occur. Just as in ordinary relativity theory, we look for differential equations here that are not as reduced as:

$$B^\alpha_{\beta\gamma\delta} = 0 .$$

The next level of complexity might be to use field equations of the form:

$$\Gamma_{\alpha\beta} = \mathrm{B}_{\alpha\beta} - \tfrac{1}{2}\, \gamma_{\alpha\beta}\, B = 0 .$$

However, this would not work, since we would obtain fifteen, instead of fourteen, equations. The Ansatz:

(1) $$\Gamma_{\alpha\beta} - \varphi_\alpha \varphi_\beta \, \Gamma = 0 ,$$

seems more promising, in which Γ is defined by the equation:

$$\Gamma = \gamma^{\alpha\beta}\, \Gamma_{\alpha\beta} = -\tfrac{3}{2} B .$$

The affine expression for B is:

$$B = R - \varphi^s_t \varphi^t_s .$$

Incidentally, the tensor $\Gamma_{\alpha\beta}$ satisfies a series of equations that correspond to the conservation law in EINSTEIN's theory:

$$\Gamma^\alpha_{\beta;\alpha} = 0 .$$

These theorems are obtained from the five-dimensional affine interpretation of our theory with no further assumptions.

Equations (1) are the differential equations of a four-dimensional variational principle. Namely, we demand that the integral:

(2) $$\int B \; g^{\frac{1}{2}} dx^1\, dx^2\, dx^3\, dx^4 ,$$

should be stationary under variations of the $\gamma_{\alpha\beta}$ with the extra condition that $\gamma_{00} = 1$, then one obtains the EULER-LAGRANGE equations (1) precisely. We will explore this further at the end of this chapter.

It is not out of the question that this property might give physical meaning to our equations. They are also of interest due to the fact they make the consistency of our equations apparent.

In the sequel, we will therefore assume that equations (1) are the differential equations of empty space; here, "empty" means that neither mass nor charge density is present. Only in this case will the field equations be valid, along with the path equations of electric particles.

Decomposition of the field equations.

We would now like to decompose equations (1) into their affine parts. The left-hand side of (1) represents a projective tensor $T_{\alpha\beta}$, such that we can write the field equations in the abbreviated form:

(4) $$T_{\alpha\beta} = 0 .$$

By raising the indices with the help of $\gamma^{\alpha\beta}$, we obtain two more systems of equations that are equivalent to (4):

(5) $$T^{\alpha}_{\beta} = 0$$

and:

(6) $$T^{\alpha\beta} = 0 .$$

Next, one can conclude from the form of the transformation of representation that T^{ij} is an affine tensor. Likewise, we know that T_{00} and T_0^i are affine invariants. The following equations then follow from (4):

$$T^{ij} = 0, \qquad T_0^i = 0, \qquad T_{00} = 0 .$$

Or, more specifically:

(7) $$R^{ij} - \tfrac{1}{2} g^{ij} R + 2S^{ij} = 0 ,$$

(8) $$\varphi^{js}{}_{,s} = 0 ,$$

(9) $$R = 0 .$$

Thus:

(10) $$S^{ij} = g^{st} \varphi^i_s \varphi^j_l + \tfrac{1}{4} g^{ij} \varphi^s_t \varphi^t_s$$

is the MAXWELL stress tensor, whereas:

$$\varphi^{js}{}_{,s} = J^i$$

represents the electric current vector. Equations (7), (8), (9) agree with the ones that are derived in relativity theory.

The limitations of the solution.

We now have a purely formal solution to the unification problem. Many physicists have hoped that some suitable solution to the unification problem might lead to knowledge of new physical phenomena; unfortunately, that is not the case here. The solution that we just described leads to only the field equations of classical relativity and precisely the same equation of motion for an electric particle that one obtains from relativity theory and MAXWELL's theory. On the contrary, the theory contains no foreign elements.

Geometrical restrictions of the theory.

We now ask about the extent to which we have actually used the geometrical apparatus of our theory.

By the decomposition into the theory of gravitation and electromagnetism, we have made use of the position of the contact point of the tangent space with the base space relative to the quadric surface.

On the contrary, we have subjected φ to restricted assumptions. First, through the introduction of inhomogeneous coordinates with the help of the formula:

$$dx^i = \frac{X_i}{\varphi_\alpha X^\alpha}$$

we demanded that the polar hyperplane to the origin agrees precisely with the hyperplane at infinity relative to the quadric surface. This implies a restriction on the position of the quadric surface in the tangent space; this assumption has no influence on the field equations. Physically, it first comes to light in the differential equations of electric particles.

Furthermore, we used the vector φ instead of an arbitrary vector of index zero in the definition of the translation that Γ defines; this is likewise meaningful for the motion of electric particles. Perhaps it might be possible to replace this assumption with other convenient assumptions. The field equations therefore remain unchanged.

Generalizations of the theory.

Furthermore, we must emphasize that we have made no use of Φ. Φ determines another projective vector of index 0, namely:

$$\frac{\partial \log \Phi}{\partial x^\gamma} = \Phi_\gamma,$$

which is amenable to a geometrical discussion.

In order to have an apparatus that is capable of formulating far-reaching physical theories, it is necessary to introduce new geometrical ideas. One can expect that one might be able to find physical applications that go beyond projective relativity theory for the extension of our geometrical apparatus that follows from the introduction of Φ.

The $\Gamma^{\alpha}_{\beta\gamma}$ no longer represent the generalization of the connection Π that is introduced in ordinary projective geometry. Rather, instead of the Γ, we must consider the connection Π that is defined by the $G_{\alpha\beta}$ in the same way that Γ is defined by the $\gamma_{\alpha\beta}$. In this way, one might succeed in constructing the field equations, which are perhaps of the type:

$$\Gamma^{*}_{\beta\gamma} = 0 . \tag{11}$$

The $\Gamma^{*}_{\beta\gamma}$ are thus constructed from the $\Pi^{\alpha}_{\beta\gamma}$ in the same way that the $\Gamma_{\alpha\beta}$ are constructed from the $\Gamma^{\alpha}_{\beta\gamma}$ (Bibliography 1930, 9). One equation that is so obtained is identical with the SCHRÖDINGER equation in quantum theory. For that reason, it seems possible that a unification of quantum theory with projective field theory might exist in this direction. Admittedly, it is disappointing that the consistency of equations (11) can no longer be proved in this case. Furthermore, up till now no one has succeeded in deriving the equations from a four-dimensional variational principle. Finally, the physical interpretation of the equations raises difficulties.

It is not entirely miraculous that an equation of the SCHRÖDINGER form appears at this point. It is merely due to the fact that one can transform a second-order differential equation of the general type in such a way that the theory of its invariants is precisely a projective theory (COTTON, WIENER, and STRUIK, Bibliography 1900, 1; 1927, 19, 20).

The generalized non-Euclidian geometry is only one of an entire sequence of geometries that are mathematically very interesting. For example, it is possible to shape a generalized conformal geometry with tools that are completely similar to our projective tensors. It is not impossible that these geometries might be amenable to physical applications in extensions of relativity theory. Making a choice of one or the other such geometry will, however, require a new physical insight, and not merely the concatenation of two theories. New physical facts can be obtained from the theory only by introducing new quantities.

Derivation of the field equations from a variational principle.

We will now show, as we previously suggested, that the field equations (1) are the EULER-LAGRANGE equations of the variational principle:

$$\int B\, g^{\frac{1}{2}}\, dx^1\, dx^2\, dx^3\, dx^4 \qquad \text{is stationary.}$$

Thus, we must vary the $\gamma_{\alpha\beta}$ under the subsidiary condition:

$$\gamma_{00} = 1.$$

For the calculation of the variation, we employ an elegant method that goes back to PALATINI (Bibliography 1919, 2).

We have:

$$B = B_{\alpha\beta}\gamma^{\alpha\beta} . \tag{12}$$

We next show that one has:

(13) $$\int \gamma^{\alpha\beta}\, g^{\frac{1}{2}}\;\delta B\, dx^1\, dx^2\, dx^3\, dx^4 = 0\,.$$

The functions:

$$\delta\Gamma^{\alpha}_{\beta\gamma}$$

are the components of a projective tensor, since they are the differences of the components of two projective connections. We now take the covariant derivative of this tensor:

$$(\delta\Gamma^{\alpha}_{\beta\gamma})_{;\delta} = \frac{\partial(\delta\Gamma^{\alpha}_{\beta\gamma})}{\partial x^{\delta}} - (\delta\Gamma^{\alpha}_{\beta\varepsilon})\Gamma^{\varepsilon}_{\gamma\delta} - (\delta\Gamma^{\alpha}_{\varepsilon\gamma})\Gamma^{\varepsilon}_{\beta\delta} + (\delta\Gamma^{\varepsilon}_{\beta\gamma})\Gamma^{\alpha}_{\varepsilon\delta}\ .$$

From this, it follows that:

$$(\delta\Gamma^{\alpha}_{\beta\gamma})_{;\delta} - (\delta\Gamma^{\alpha}_{\beta\delta})_{;\gamma} = \delta B^{\alpha}_{\beta\gamma\delta}\,,$$

or, by contracting α and γ:

$$(\delta\Gamma^{\alpha}_{\beta\alpha})_{;\delta} - (\delta\Gamma^{\alpha}_{\beta\delta})_{;\alpha} = \delta B_{\beta\delta}\,.$$

If we multiply this by $\gamma^{\beta\delta}$ then we obtain:

$$\gamma^{\beta\delta}\,\delta B_{\beta\delta} = \{\gamma^{\beta\delta}\,(\delta\Gamma^{\alpha}_{\beta\alpha}) - \gamma^{\beta\alpha}\,(\delta\Gamma^{\delta}_{\beta\alpha})\}_{;\,\delta}\,.$$

By using the fact that $\gamma = g$ we then have:

$$\begin{aligned}&\int \gamma^{\alpha\beta}\, g^{\frac{1}{2}}\;\delta B\, dx^1\, dx^2\, dx^3\, dx^4\\ &= \int \frac{\partial}{\partial x^{\delta}}\Big[\gamma^{\beta\delta}\,(\delta\Gamma^{\alpha}_{\beta\alpha})g^{\frac{1}{2}} - \gamma^{\beta\alpha}\,(\delta\Gamma^{\delta}_{\beta\alpha})g^{\frac{1}{2}}\Big]\; dx^1\, dx^2\, dx^3\, dx^4\,.\end{aligned}$$

We can omit the term with $\delta = 0$, since the expression in the square brackets does not depend upon x^0. From the generalized GREEN theorem, it now follows that:

$$\int \frac{\partial}{\partial x^{i}}\Big[\gamma^{\beta i}(\delta\Gamma^{\alpha}_{\beta\alpha})g^{\frac{1}{2}} - \gamma^{\beta\alpha}\,(\delta\Gamma^{i}_{\beta\alpha})g^{\frac{1}{2}}\Big]\; dx^1\, dx^2\, dx^3\, dx^4 = 0\,.$$

The expressions $\delta\Gamma^{\alpha}_{\beta\gamma}$ indeed vanish on the boundary of the integration domain. The validity of (13) is thus proved.

The extremum condition then yields the equation:

$$\delta\int B\; g^{\frac{1}{2}}\; dx^1\, dx^2\, dx^3\, dx^4 = \int B_{\alpha\beta}\;\delta(\gamma^{\alpha\beta} g^{\frac{1}{2}})\; dx^1\, dx^2\, dx^3\, dx^4 = 0\,.$$

Due to the fact that $\delta\gamma = -\,\gamma\,\gamma_{\alpha\beta}\;\delta\gamma^{\alpha\beta}$, it then follows that:

$$\int \Big(B_{\alpha\beta}\, g^{\frac{1}{2}}\,\delta\gamma^{\alpha\beta} - B_{\sigma\tau}\,\gamma^{\sigma\tau}\,(\tfrac{1}{2}\, g^{\frac{1}{2}})\,\gamma\,\gamma_{\alpha\beta}\;\delta\gamma^{\alpha\beta}\Big)\; dx^1\, dx^2\, dx^3\, dx^4 = 0\,,$$

or:

$$
(14) \qquad \begin{cases} \int (B_{\alpha\beta} - \frac{1}{2} B\gamma_{\alpha\beta}) g^{\frac{1}{2}} \delta\gamma^{\alpha\beta} dx^1 dx^2 dx^3 dx^4 \\ \quad = \int \Gamma_{\alpha\beta} g^{\frac{1}{2}} \delta\gamma^{\alpha\beta} dx^1 dx^2 dx^3 dx^4 = -\int \Gamma^{\alpha\beta} g^{\frac{1}{2}} \delta\gamma_{\alpha\beta} dx^1 dx^2 dx^3 dx^4 = 0. \end{cases}
$$

Due to the fact that $\gamma_{00} = 1$, we have $\delta\gamma_{00} = 0$. The remaining $\delta\gamma_{\alpha\beta}$ are thus arbitrary. From this, we can derive that:

$$\Gamma^{\alpha\beta} = \delta_0^\alpha \delta_0^\beta K,$$

in which K is a function of x^1, x^2, x^3, x^4. If we multiply this by $\gamma_{\alpha\beta}$ then we have:

$$\Gamma = K .$$

We thus obtain the necessary condition for an extremum:

$$\Gamma^{\alpha\beta} - \delta_0^\alpha \delta_0^\beta \Gamma = 0 .$$

By lowering the indices, we thus obtain the desired equation (1), in fact.

We can also derive equation (1) by demanding that:

$$\int B g^{\frac{1}{2}}\, dx^1\, dx^2\, dx^3\, dx^4$$

be stationary, except that now the functions $G^{\alpha\beta}$ are what we vary.

Since $B g^{\frac{1}{2}}$ depends only upon $\gamma_{\alpha\beta}$ and Φ, we now have:

$$
(15) \qquad \delta \int B g^{\frac{1}{2}}\, dx^1\, dx^2\, dx^3\, dx^4 = \int \Gamma_{\alpha\beta} g^{\frac{1}{2}}\, \delta\gamma^{\alpha\beta}\, dx^1\, dx^2\, dx^3\, dx^4 = 0 .
$$

However, we have:

$$\gamma^{\alpha\beta} = \Phi^2 G^{\alpha\beta} = G_{00} G^{\alpha\beta},$$

and:

$$
\begin{aligned}
\delta G_{00} &= \delta(G_{0\alpha} G_{0\beta} G^{\alpha\beta}) \\
&= G_{0\alpha} G_{0\beta} \delta G^{\alpha\beta} + 2\, G_{0\alpha} G^{\alpha\beta}\, \delta G_{0\beta} \\
&= G_{0\alpha} G_{0\beta} \delta G^{\alpha\beta} - 2\, G_{0\alpha} G_{0\beta}\, \delta G^{\alpha\beta} \\
&= -\, G_{0\alpha} G_{0\beta}\, \delta G^{\alpha\beta} \quad .
\end{aligned}
$$

We thus obtain the following for the variation of $\gamma^{\alpha\beta}$:

$$\delta\gamma^{\alpha\beta} = G^{\alpha\beta} \delta G_{00} + G_{00} \delta G^{\alpha\beta} = G_{00} \delta G^{\alpha\beta} - G_{0\sigma} G_{0\tau} \delta G^{\sigma\tau} G^{\alpha\beta}.$$

From (15), it then follows that:

$$\int (\Gamma_{\alpha\beta}\, G_{00} - G_{0\alpha} G_{0\beta} \Gamma. g^{\frac{1}{2}}\, \delta\Gamma^{\alpha\beta}\, dx^1\, dx^2\, dx^3\, dx^4 = 0,$$

or, since $\Phi \neq 0$:

$$\Gamma_{\alpha\beta} - \varphi_\alpha \varphi_\beta \Gamma = 0 .$$

VIII. Five-dimensional associated spaces.

Homogeneous coordinates in the tangent spaces.

In chap. II, we established the connection between homogeneous projective coordinates and inhomogeneous coordinates in the tangent space by way of the formula:

$$dx^i = \frac{X^i}{\varphi_\alpha X^\alpha}\,. \tag{1}$$

Furthermore, we saw that an arbitrary projective vector X^α determines a point dx^i of the tangent space by way of (1). We can also represent the relationship between projective vectors and tangent spaces in another form.

Associated spaces.

Suppose one is given a point x, a choice of representation, and five arbitrary numbers $X^0, X^1, \ldots, X^4$. *We now collect the totality of all contravariant projective vectors of a given index whose components can be assumed to have the values $X^0, X^1, \ldots, X^4$ in the given representation into a single geometrical object* [1]. The vectors of a given geometric object can take on completely arbitrary forms at points different from x even though they all assume the values $X^0, X^1, \ldots, X^4$ at x.

We will call the totality of these objects for a given point and a given index a space, and indeed, we will call it *the associated space of index N at x*.

This space is understood by way of coordinate systems, and indeed $X^0, X^1, \ldots, X^4$ are coordinates of the point that is defined with the help of these five numbers. These coordinate systems on the associated space are associated with a particular representation of the base space. Thus, every point of base space is associated with an associated space of index N, and every representation of the base space is associated with a certain coordinate system in each associated space.

It follows with no further assumptions from the basic properties of a projective vector that the definition of the associated space is independent of the coordinates. Thus, we can actually regard the associated spaces as geometric objects.

Each transformation of representation of the base space defines a transformation of the coordinates X^α in the associated space:

$$\begin{cases} \bar{X}^0 = X^0 + v_i^0 X^i \\ \bar{X}^i = v_j^i X^j. \end{cases} \tag{2}$$

The larger set of transformations of representations is thus connected with the smaller set of linear transformations of X^α.

---

[1] At the point x, the components of a vector are functions of x^0 of the form e^{Nx^0}. Two different vectors at the point x that are associated with the same geometric object can thus assume the same values $X^0, X^1, \ldots, X^4$ when we choose two suitable x^0-values; however, this remark is valid only when $N \neq 0$.

For $N \neq 0$, the associated spaces are projective spaces since all vectors that can assume the values X^0, X^1, …, X^4 at a point x will be multiplied by a factor k under a change of x^0. The point:

$$X^\alpha = \delta_0^\alpha$$

plays a special role in any associated space since its coordinates remain unchanged under all transformations (2).

Correspondence between projective associated spaces and tangent spaces.

First, we observe that our associated spaces have nothing to do with the tangent spaces. However, we can relate them to the tangent spaces by way of (1):

(3) $$X \rightarrow dx\,.$$

Thus, the point (1, 0, 0, 0, 0) of the projective space corresponds to the origin of the tangent space. If we then assume that the φ_α are the components of the polar hyperplane relative to the previously considered quadric surface then, from (1), this hyperplane is precisely the hyperplane at infinity.

Thus, we have reached the conclusion of our previous development. Our position is now essentially that of chap. II, in which we regarded the X^α as the homogeneous coordinates in the tangent spaces.

The associated projective spaces serve as aids for the introduction of the homogeneous coordinates. If we do not assume the existence of the map (1), or (3), resp., then we obtain only one theory of associated spaces. Furthermore, the form of our field equations does not depend upon the validity of (1). However, we next obtain a relationship between the associated spaces and special curves, surfaces, etc., in the base space by way of the map (3) or other suitable assumptions. Thus, e.g., a correspondence between a curve:

$$x^i = x^i\,(t)$$

and the tangent spaces that are associated with its points will be defined by the equation:

$$dx^i = \frac{dx^i}{dt}\,dt.$$

However, one possibly requires the map (1) in order to define a relationship between the dx^i and the X^α, and thus, a relationship between the points of the curve and the associated spaces. Furthermore, we have made use of (1) in the derivation of the differential equations of massive electric particles.

Five-dimensional associated spaces.

Our definition of an associated space is also valid in the case of $N = 0$. However, the associated spaces are no longer four-dimensional since the components of a vector of index 0 are independent of x^0, and because their absolute values are also thus determined.

From our definition, the associated spaces of index 0 are thus five-dimensional affine spaces. A coordinate transformation of the form (2) for the five-dimensional associated affine space is defined by the transformation of representation of the four-dimensional base space:

$$\text{(4)} \qquad \begin{cases} \bar{x}^0 = x^0 + \log \rho \\ \bar{x}^i = \bar{x}^i(t). \end{cases}$$

Thus, we have:

$$\text{(5)} \qquad \begin{cases} v_i^0 = \dfrac{\partial \log \rho}{\partial x^i} \\ v_j^i = \dfrac{\partial \bar{x}^i}{\partial x^i}. \end{cases}$$

From the form of the transformation (2) it follows that the line:

$$\text{(6)} \qquad X^i = 0, \qquad X^0 \text{ arbitrary,}$$

has the same equation in any of these associated spaces. We call this line the distinguished line.

First map of the five-dimensional associated space onto the tangent space.

We can also use equation (1) to map the five-dimensional associated space at the spacetime point x onto the tangent space that is belongs with x. This map:

$$X \to dx$$

is obviously not uniquely invertible. Namely, if a point X of the five-dimensional space goes to a certain point dx of the tangent space then all points (kX^0, kX^1, …, kX^4) with k arbitrary will also go to the same point dx. (1) thus represents a map of the points of the tangent spaces onto the lines through the origin (0, 0, 0, 0, 0) of the associated space. Thus, the origin of a tangent space corresponds to the distinguished line of the five-dimensional space.

The difference between these considerations and the previous ones is admittedly not very significant. Indeed, it is known that the lines through a fixed point of a five-dimensional affine space define a four-dimensional projective space.

A projective contravariant vector $A^\alpha(x)$ of index 0 singles out a certain point:

$$X^\alpha = A^\alpha(x)$$

in any five-dimensional associated space. By contrast, a projective contravariant vector $B^\alpha(x)$ with a non-zero index determines a line through the origin in any five-dimensional associated space. The equations of these lines are:

$$\text{(7)} \qquad X^\alpha = kA^\alpha .$$

in which k is arbitrary. Furthermore, a line through the origin of each five-dimensional associated space is defined by an affine contravariant vector $V^i(x)$. Its points satisfy the equations:

$$\frac{X^i}{\varphi_\alpha X^\alpha} = V^i . \tag{8}$$

One can realize the connection between the lines through a fixed point of a five-dimensional affine space and the points of a four-dimensional affine space more intuitively by intersecting the lines with hyperplanes that do not pass through the fixed point. The hyperplane:

$$\varphi_\alpha X^\alpha = p$$

is particularly suitable for this purpose. If we assume that a certain point X that is determined by (8) shall lie in this hyperplane for a given V then equations (8) (equations (1), resp.) give us a one-to-one map between the points of the hyperplane and the points of the tangent spaces. Homogeneous coordinates are defined by (7) in this hyperplane.

For various p we obtain an entire band of hyperplanes. Our map thus breaks down for $p = 0$.

Differentials as coordinates of the five-dimensional space.

The differentials of the coordinates $x^1, \ldots, x^4$ and the differential of the gauge variable x^0 transform precisely like the components of a contravariant vector:

$$\begin{cases} d\bar{x}^0 = dx^0 + \dfrac{\partial \log \rho}{\partial x^j}\, dx^i, \\ d\bar{x}^i = \dfrac{\partial \bar{x}^i}{\partial x^j}\, dx^i. \end{cases} \tag{9}$$

One can thus consider the differentials dx^α to be the intrinsic coordinates of the five-dimensional associated spaces. The equation:

$$dx^\alpha = A^\alpha(x) ,$$

in which A^α is a projective vector of index 0, determines a point in each five-dimensional associated space.

One might reach the conclusion that the hyperplane $dx^0 = 0$ can simply be regarded as the tangent space with the coordinates $dx^1, \ldots, dx^4$. That is, however, impossible, since $dx^0 = 0$ is not an invariant condition. One cannot extend the tangent spaces to the five-dimensional associated spaces in this way. In fact, there is a one-to-one map between the points of the tangent spaces and the lines of the associated spaces that are parallel to the distinguished line; one sees this immediately from (9). One can also regard this relationship as a sort of map from the points of the five-dimensional associated spaces to the points of the tangent space. We represent this map by the equations:

(10) $$dx^i = \delta^i_\alpha \, \partial x^\alpha,$$

in which we now denote the five-dimensional coordinates by ∂x^α and the four-dimensional ones by dx^i. In the sequel, $dx^1, \ldots, dx^4$ will always mean coordinates in the tangent spaces and $\partial x^0, \ldots, \partial x^4$ will always mean coordinates in the five-dimensional associated spaces.

Euclidian metric of a five-dimensional associated space.

In the five-dimensional associated spaces, one can interpret:

$$\gamma_{\alpha\beta} \, \partial x^\alpha \, \partial x^\beta$$

as the square of a Euclidian distance. In each of these spaces:

$$\gamma_{\alpha\beta} \, \partial x^\alpha \, \delta^\beta_0 = \varphi_\alpha \, \partial x^\alpha$$

is the orthogonal projection of the vector ∂x^α onto the distinguished line since:

$$\delta^\beta_0 = \varphi^\beta$$

is the unit vector of the distinguished line.

The formula:

(11) $$\gamma_{\alpha\beta} \, \partial x^\alpha \, \partial x^\beta = g_{ij} \, \partial x^i \, \partial x^j + (\varphi_\alpha \, \partial x^\alpha)^2$$

represents the square of the length of the vector ∂x^α as the quadratic sum of the components that are orthogonal and parallel to the distinguished line.

The hyperplane:

(12) $$\varphi_\alpha \, \partial x^\alpha = p$$

is orthogonal to the distinguished line. Thus, from (11):

$$g_{ij} \, \partial x^i \, \partial x^j$$

is the Euclidian distance in such a hyperplane.

The map (8) of the five-dimensional space onto the tangent space thus represents a sort of orthogonal projection of the Euclidian space with the metric $\gamma_{\alpha\beta}$ onto the Euclidian space with the metric g_{ij}.

Translations of the five-dimensional associated spaces.

The projective connection Γ was defined (pp. 48) by the equation:

(13) $$\gamma_{\alpha\beta;\,\gamma} = 0,$$

and the associated translations were defined by the equations:

$$X^{\alpha}{}_{;\beta} = X^{\alpha}{}_{;0}\ \varphi_{\beta}\ .$$

We have also described these equations (pp. 50) in the form:

(14) $$\frac{\partial X^{\alpha}}{\partial x^{\gamma}} + X^{\sigma}\Lambda^{\alpha}_{\sigma\gamma} = 0\ ,$$

in which Λ takes on the value:

(15) $$\Lambda^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\beta 0}\varphi_{\gamma}$$

in the case that is of interest to us here, namely $M = 0$.

On the basis of the formula that we derived in chap. IV (pp. 49):

$$\Gamma^{\alpha}_{\beta 0} = \gamma^{i\alpha}\ \varphi_{i\beta}\,,$$

one recognizes that an equation of the form (13) is also valid for Λ. Namely, we have:

(16) $$\frac{\partial \gamma_{\alpha\beta}}{\partial x^{\gamma}} - \gamma_{\sigma\beta}\,\Lambda^{\sigma}_{\alpha\gamma} - \gamma_{\alpha\sigma}\,\Lambda^{\sigma}_{\beta\gamma} = 0\ .$$

This means that the length:

$$\gamma_{\sigma\beta}\ X^{\alpha}\,X^{\beta}$$

of a vector X and also the angle between two vectors in the five-dimensional space remain unchanged under translation.

The distinguished line does not go to itself under this translation. In particular, we find that the projective derivative of φ^{α} relative to Λ is:

(17) $$\frac{\partial \varphi^{\alpha}}{\partial x^{j}} + \varphi^{\sigma}\Lambda^{\alpha}_{\sigma j} = \Gamma^{\alpha}_{0j} = \gamma^{i\alpha}\,\varphi_{ij}\ .$$

Since the process of projective differentiation with Λ is interchangeable with raising and lowering of indices, (17) yields:

(17a) $$\frac{\partial \varphi_{\alpha}}{\partial x^{j}} - \varphi_{\sigma}\Lambda^{\sigma}_{\alpha j} = \varphi_{\alpha\beta}\ .$$

We now calculate the covariant derivative of the parameter:

$$p = \varphi_{\alpha}\,X^{\alpha}$$

of the previously considered hyperplane. It is:

$$\frac{\partial \varphi_{\alpha}X^{\alpha}}{\partial x^{j}} = \left(\frac{\partial \varphi_{\alpha}}{\partial x^{j}} - \varphi_{\sigma}\Lambda^{\sigma}_{\alpha j}\right)X^{\alpha} + \varphi_{\alpha}\left(\frac{\partial X^{\alpha}}{\partial x^{j}} + X^{\sigma}\Lambda^{\alpha}_{\sigma j}\right)\ .$$

On the basis of (17a) and (14) we thus obtain:

$$\frac{d(\varphi_\alpha X^\alpha)}{dt} = \varphi_{ij} X^i \frac{dx^i}{dt} . \tag{18}$$

Equation (18) is valid for the translation of p relative to Λ along a curve:

$$x^i = x^i (t) .$$

We now ask when the left-hand side of (18) vanishes, i.e., when a point remains in the same hyperplane under translation relative to Λ. Due to the skew-symmetry of φ_{ij} this is obviously the can when one has:

$$X^i = k\frac{dx^i}{dt}. \tag{18a}$$

From (11), it also follows that the distance of a point from the distinguished line is invariant in the case (18a).

We now write formulas (14) in somewhat more detail. If we single out the case α = 0 then, due to chap. VI, (20) and (21), we obtain:

$$\frac{dX^i}{dt} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} X^k \frac{dx^j}{dt} + \varphi^i_j \varphi_\sigma X^\sigma = 0 .$$

A similar differential equation is also valid for X^0.

The point in the five-dimensional space:

$$X^i = k\frac{dx^i}{dt}, \qquad \varphi_\alpha X^\alpha = \text{const.},$$

is associated with the velocity vector of the curve. If we demand that this point goes to such a point under translation along the curve then the following differential equations for the curve and its parameter are true:

$$\frac{d^2x^i}{dt^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt} + \frac{e}{m}\varphi^i_j \frac{dx^j}{dt} = 0 . \tag{19}$$

We have thus set the constant $\varphi_\alpha X^\alpha$ equal to e/m.

These equations for the distinguished curves in the base space agree precisely with the equations of chap. VI (33a), which we obtained for the case of $M \neq 0$, hence, for a four-dimensional associated space. Thus, we have also obtained a geometric interpretation for the world-lines of an electric particle in terms of the five-dimensional associated spaces.

Introduction of general coordinates in the associated spaces.

We can introduce completely general affine coordinates into the tangent spaces through the non-degenerate transformation:

(20) $$W^i = M^i_j \, dx^j \, .$$

If we assume, e.g., that $M^1_j, \ldots, M^4_j$ are four covariant vectors then the W^i are scalars. We refer to coordinates that are obtained in this way as *scalar* coordinates. However, we are still free to choose other transformation laws for the M^i_j.

Likewise, we can introduce completely general affine coordinates in the five-dimensional associated spaces. The equation:

(20a) $$\partial x^\beta = N^\beta_\alpha U^\alpha \, .$$

All of our formulas assume a general form in the generalized coordinates. However, their geometrical and physical meaning naturally remains unchanged. For example, in general coordinates, the non-degenerate map (10) takes on the form:

(21) $$W^i = t^i_\alpha U^\alpha \, ,$$

in which:

$$t^i_\alpha = N^\beta_\alpha \delta^j_\beta M^i_j \, .$$

EINSTEIN and MAYER (Bibliography 1931, 3) always use this general coordinate system. In their work, the map (21) plays an essential role, as we hope to clarify in the sequel. EINSTEIN and MAYER use the notation γ^j_α instead of our t^i_α. Furthermore, they write $g_{\alpha\beta}$ instead of our $\gamma_{\alpha\beta}$ and A_α instead of φ_α.

Second map of the five-dimensional space onto the tangent space.

For our present purpose it suffices for us to work with the coordinates dx^i and ∂x^α. Nevertheless, we shall write (10) in the form:

(21) $$dx^i = t^i_\alpha \partial x^\alpha \, ,$$

in order to emphasize the fact that this transformation is a geometric object. In our coordinates, one thus has:

(22) $$t^i_\alpha = \delta^i_\alpha \, .$$

We now seek to present the relationship that was given by (21) between the coordinates of the associated spaces and tangent spaces in another geometrical form. In order to do this, we must invert (21). The inversion of (21) is not uniquely determined since (21) is not a unique transformation. In any case, the formula:

$$\partial x^i = dx^i$$

must be true, whereas the choice of the ∂x^0 that goes with $(dx^1, \ldots, dx^4)$ is unrestricted. We determine the inverse of t^i_α while making the least possible demands upon the relationship between the metrics in both spaces. We thus define:

(23) $$t^{\alpha i} = \gamma^{\alpha\beta}\, t^i_\beta, \qquad t^\alpha_i = g_{ij}\, t^{\alpha j}, \qquad t_{\alpha i} = g_{ij}\, t^j_\alpha .$$

If we use (22) then we obtain:

(24) $$t^{\alpha i} = \gamma^{\alpha i},$$
(25) $$t^\alpha_i = (\gamma_{i\beta} - \varphi_i\, \varphi_\beta)\, \gamma^{\alpha\beta} = \delta^\alpha_i - \delta^\alpha_0\, \varphi_i ,$$
(26) $$t_{\alpha i} = \delta^i_\alpha\, g_{ij} .$$

We thus have the identities:

(27) $$t^\alpha_i\, t^j_\alpha = \delta^j_i, \qquad t^\alpha_i\, t^i_\beta = \delta^\alpha_\beta - \delta^\alpha_0\, \varphi_\beta ,$$
(28) $$t^{\alpha i}\, t_{\alpha j} = \delta^i_j, \qquad t^{\alpha i}\, t_{\beta i} = \delta^\alpha_\beta - \delta^\alpha_0\, \varphi_\beta ,$$
(29) $$t^\alpha_i\, t_{\alpha j} = g_{ij}, \qquad t^{\alpha i}\, t^\beta_i = \gamma^{\alpha\beta} - \delta^\alpha_0\, \delta^\beta_0 .$$

In harmony with our definition (3), we consider the transformation:

(30) $$\partial x^\alpha = t^\alpha_i\, dx^i$$

to be the inverse of (21). Under the assumption (22), we then have:

$$\partial x^\alpha = \delta^\alpha_i\, dx^i - \delta^\alpha_0\, \varphi_i\, dx^i ,$$

or:

(31) $$\begin{cases} \partial x^0 = -\varphi_i dx^i \\ \partial x^i = dx^i . \end{cases}$$

From (31), one immediately sees that all of the points dx in the tangent space will be mapped uniquely onto the points of the distinguished hyperplane:

(32) $$\varphi_\alpha\, \partial x^\alpha = 0$$

in the five-dimensional space. If one restricts oneself in (21) to the points of this hyperplane then (21) is the unique inverse of (30). It is noteworthy that (32) is the only hyperplane in the band:

$$\varphi_\alpha\, \partial x^\alpha = p$$

in which the previously given transformation (8) fails.

With the help of the quantities t_α^i and t_i^α, we can now present the desired correspondence between the structures in the five-dimensional space and those in the four-dimensional one.

Thus, e.g., the affine vector:

$$B_i = t_i^\alpha A_\alpha = A_i - A_0 \varphi_i \tag{33}$$

corresponds to the projective covariant vector A_α. In order to interpret this formula geometrically, we consider the hyperplane:

$$B_i\, dx^i = k$$

in the tangent space. We have:

$$B_i\, dx^i = B_i\; t_\alpha^i \partial x^\alpha = (A_i\; t_\alpha^i - A_0 \varphi_i\; t_\alpha^i)\; \partial x^\alpha,$$

or:

$$B_i\, dx^i = A_\alpha\, \partial x^\alpha - A_0 \varphi_\alpha\, \partial x^\alpha.$$

If we further restrict ourselves to the points of the hyperplane (32) then we obtain a one-to-one map of the hyperplane:

$$B_i\, dx^i = k$$

in the four-dimensional space onto the intersection manifold of both hyperplanes:

$$A_\alpha\, \partial x^\alpha = k$$

and:

$$\varphi_\alpha\, \partial x^\alpha = 0$$

in the five-dimensional space.

In particular, if $A_\alpha = \varphi_\alpha$ then the transformation (33) reduces to:

$$0 = t_i^\alpha\, \varphi_\alpha,$$

as we would expect geometrically.

Conversely, an affine covariant vector corresponds to the projective vector:

$$t_\alpha^i B_i = t_\alpha^i (A_i - A_0\, \varphi_i) = A_\alpha - A_0\, \varphi_\alpha .$$

The intersection manifold of $(A_\alpha - A_0 \varphi_\alpha)\partial x^\alpha = k$ with $\varphi_\alpha\, \partial x^\alpha = 0$ agrees with the intersection manifold of $A_\alpha\, \partial x^\alpha = k$ and $\varphi_\alpha\, \partial x^\alpha = 0$.

The correspondence between the fundamental tensors is mediated by the formulas:

$$\gamma_{\alpha\beta} t_i^\alpha\, t_j^\beta = g_{ij}$$

and:

$$g_{ij} t_\alpha^i t_\beta^j = \gamma_{\alpha\beta} - \varphi_\alpha \varphi_\beta .$$

Relations between the translations.

We now concern ourselves with the translations of the tangent spaces that are induced by the translation of the five-dimensional space. Conversely, we can also ask what sort of translation of the five-dimensional spaces is defined by an ordinary LEVI-CIVITA translation of the tangent space.

We start with the unique map:

$$V^i = t^i_\alpha X^\alpha \qquad \text{and} \qquad X^\alpha = t^\alpha_i V^i , \tag{34}$$

of the point of the tangent space to the point of the hyperplane:

$$\varphi_\alpha \, \partial x^\alpha = 0 \tag{32}$$

in the associated five-dimensional space. By differentiating (34), we obtain, after making use of (14):

$$\frac{\partial V^i}{\partial x^j} = -\Gamma^i_{\sigma j} X^\sigma + \Gamma^i_{\sigma 0} X^\sigma \varphi_j . \tag{35}$$

Due to (15), we thus obtain:

$$\frac{\partial V^i}{\partial x^j} = -\begin{Bmatrix} i \\ kj \end{Bmatrix} V^k - \varphi^i_j \varphi_\sigma X^\sigma .$$

If we now use the fact that X is restricted to the hyperplane (32) then we obtain:

$$\frac{\partial V^i}{\partial x^j} + \begin{Bmatrix} i \\ kj \end{Bmatrix} V^k = 0 . \tag{37}$$

Thus, if the map (34) breaks down then the affine translation (37) corresponds to the translation (14) of the five-dimensional space.

If we substitute (36) in (35) then this yields:

$$\frac{\partial t^i_\alpha}{\partial x^j} - t^i_\sigma \Lambda^\sigma_{\alpha j} + t^k_\alpha \begin{Bmatrix} i \\ kj \end{Bmatrix} = -\varphi^i_j \varphi_\alpha . \tag{38}$$

In this, we must use the fact that X^α can take on any arbitrary value in five-dimensional space.

Next, equation (38) is only one of the identities that one can construct out of equations (15), chap. VIII, and (20) and (21), chap. VI, as one can naturally verify immediately on the basis of these equations. However, we can regard (38) as the equation for the covariant derivative of the mixed quantities t^i_α, whose indices relate to the tangent space (five-dimensional space, resp.).

Previously, we have stressed that the map of the tangent space onto the five-dimensional space is a geometric object that takes on the components $t^i_\alpha = \delta^i_\alpha$ in our special coordinate system. Equations (38) represent the translation of this geometric object. If the quantities t_α^i are given then (38) gives us the desired relation between the translations in both spaces.

One has the equations:

$$\frac{\partial t_\alpha^i}{\partial x^j} + \Lambda_{\sigma j}^\alpha t_i^\sigma - \begin{Bmatrix} k \\ ij \end{Bmatrix} t_k^\alpha = \varphi^\alpha \varphi_{ij} . \tag{39}$$

One immediately derives equation (39) from (38) by raising and lowering α and i with the help of $\gamma^{\alpha\beta}$ and g_{ij} . If one does not want to use (38) then one can also verify (39) immediately on the basis of (25).

EINSTEIN and MAYER define the left-hand side of (37) as the "absolute derivative" of V^i and the left-hand side of (39) as the "absolute derivative" of t_i^α. In this way, equations (39) are introduced by geometric assumptions. From (37) and (39), one can then derive the properties of the translations of the associated five-dimensional spaces. The resulting geometrical structure admittedly agrees with the one that we gave above. The field equations are also equivalent to the ones that we gave.

One can also replace equations (39) with other ones. For example, EINSTEIN and MAYER, in a later work (Bibliography 1932, 5), have proposed the equation:

$$\frac{\partial t_\alpha^i}{\partial x^j} + \Lambda_{\sigma j}^\alpha t_i^\sigma - \begin{Bmatrix} k \\ ij \end{Bmatrix} t_k^\alpha = \varphi^\alpha \square_{ij} + \gamma^{\alpha r} V_{rij} . \tag{40}$$

Thus, we have:

$$F_{ij} = - F_{ji}$$

and:

$$V_{rij} = - V_{irj} = - V_{rji} .$$

Obviously, the quantities Λ in these equations must now be different from the Λ that we used before. The introduction of new equations, such as (40), therefore implies the choice of a new translation and with it, the possibility of arriving at new field equations.

The physical meaning of equations (40) is still not clear. We will not go into this matter any further, either.

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