THE GREAT PROBLEMS OF SCIENCE
WORKS COLLECTED BY Mme. P. FIVRIER

V.

STRUCTURE OF MICRO-OBJECTS IN THE CAUSAL INTERPRETATION OF QUANTUM THEORY

BY

Jean-Pierre VIGIER

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Preface by **L. de BROGLIE**
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To the glorious memory of

ALBERT EINSTEIN

MATERIALISTIC AND DETERMINISTIC PHYSICIST

"There is something like the "real state" of a physical system that exists objectively, independently of any observation or measurement, and that may, in principle, be described by means of physical expressions [What means of expression is adequate, and, consequently, what concepts are to be used in this regard, are, to my knowledge, actually unknown (Material points? Field? A means to determine these things that must first be created?)]

…

Indeed, all the men who comprise the quantum theoreticians hold closely to that thesis on reality, whereas they do not exactly discuss the fundamentals of quantum theory. For example, no one doubts that the center of gravity of the moon occupies a definite position at a definite instant in the absence of an arbitrary real or potential observer. If one considers this thesis on reality in a purely logical and arbitrary way, then it is very difficult to escape the solipsism. In the sense indicated above, I do not blush at the thought of elevating the concept of "the real state of a system" to the center of my meditation itself ([1])."

"Something that does not please me in the arguments of the quantum theoreticians who feel that quantum theory gives a complete description of elementary phenomena is their positivistic way of seeing things, which, to my way of seeing things is inadmissible, and which is the same, in essence, as the principle of Berkeley: "esse est percipi ([2])."

A. EINSTEIN

([1]) In *Louis de Broglie, Physicien et penseur,* Albin Michel edition, Paris, 1953, pp. 7.
([2]) In *Physics Today,* 1950.

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PREFACE

Since the appearance of wave mechanics, which goes back to my first works in 1923-24 and, notably, since the confirmation of the ideas at the basis of that theory by the works of Schrödinger in 1926, and by the discovery of electron diffraction in 1927, the question of the interpretation that one agrees to give to the duality of waves and particles, whose general character is therefore found to be established, is posed acutely.

Since I was desiring to respect the ideas of determinism and the objectivity of the physical world that are almost always accepted in science, and more or less consciously guided by the concern of attaching my interpretation to the ideas of Einstein on the representation of corpuscles as a sort of field singularity, I developed a curious and subtle theory in 1927 that I called the "theory of the double solution;" in the spirit of that theory, it permits us to reconcile the probabilistic significance that one first attributes to the wave Ψ of wave mechanics with an objective causal representation of wave-particle duality. I have also published my ideas in a truncated form, which is less complete and profound, in my opinion, and which I call the "pilot-wave theory," which leads into the hydrodynamical interpretation of Madelung.

However, my ideas raise great difficulties that I have yet to completely resolve, and, from the series of discussion that took place at the Solvay Conference of October, 1927, the vast majority of physicists have adopted an interpretation of wave mechanics that is very different from what I proposed. This interpretation, which was first put forth by Born, Bohr, and Heisenberg, may be qualified as "purely probabilistic;" it rests on an abstract formalism and removes all concrete physical character from the waves of wave mechanics, renounces the notions of determinism and objectivity in the physical world, leads to the interesting, but imprecise, idea of "complementarity." It is uncontestable that this purely probabilistic interpretation presents considerable formal elegance and that, overall, the calculations that it allowed one to perform have generated predictions that have been quite remarkably confirmed by experiment.

Nevertheless, certain isolated, but not insignificant, physicists – Einstein and Schrödinger, for example – have always protested against the purely probabilistic interpretation and the abandonment of the ideas of objectivity and causality that it implies, and they have posed troubling objections that, in my opinion, no one has responded to in a truly satisfactory fashion. Einstein, while recognizing that the present theory is entirely exact in its statistical predictions, has always affirmed that it does not give a "complete" description of physical reality. As for myself, discouraged by the very real difficulties that I encountered in the interpretation of the double solution, for the last twenty-five years I am rallying to the viewpoint that is becoming "orthodox," but I have always had a certain difficulty in my education regarding clearly expressing myself, and I have often felt an impression of malaise in studying certain of its aspects.

It was in 1951 that David Bohm, then in the United States, published articles that tended to recall the fundamental ideas of the older attempt at a hydrodynamical interpretation of the Madelung-pilot-wave type in a slightly modified and corrected form. This publication directed attention to my old attempts in 1927, and Jean-Pierre Vigier, who worked for some time at the l'Institut Henri Poincaré, and diligently, I am sure, remarked to me that there was an analogy between the "guidance law" that I proved in 1927, in the case of the theory of the double solution, and a very important proof that was given independently in the same epoch by Einstein and Georges Darmois on the movement of material objects in general relativity when one considers the corpuscles to be field singularities. The curious remark of Vigier interested me greatly; it led me to think that it is indispensable to introduce into the theory of the "double solution," not the linear wave equations that were originally envisioned, but nonlinear equations, whose nonlinearity is, moreover, significant only in very small regions of space that define the "corpuscles." Thus, a cordial and fruitful collaboration was established almost three years ago between Vigier and myself in order to try to recall my attempts of 1927, with an eye towards obtaining a causal objective interpretation of wave mechanics, and trying to remove the difficulties that it raised in its original form.

In the course of that common effort, Vigier has obtained very important results that properly belong to him. This is why he has extended the theory of the double solution (which I had formerly developed in the framework of the Klein-Gordon wave equation, which was the only one known at the time) to the Dirac equations of the electron, and to equations that are valid for particles of spin greater than $h/4\pi$. Likewise, this is why he, together with Bohm, developed a justification that was more rigorous than the one that I sketched in 1927 for the statistical interpretation of $|\Psi|^2$ and the ideas upon which its reasoning rests seems to be clearly of interest.

It was also Vigier who was responsible for the introduction of very important hypothesis: that the wave u with singularity (or a singular nonlinear region) envisioned by the theory of the double solution must have an external part whose form coincides, in general at least, with the wave Ψ that is envisioned by the usual wave mechanics. This hypothesis, which seems to me to run into strong objections from the outset, also seems to me, upon reflection, to be the only one that is capable of justifying the success of the calculus of proper values in wave mechanics, and explaining interference phenomena, such as Young's double slit, when one assumes that photons (and, more generally, corpuscles) exist and are localized; it realizes a sort of reconciliation between the viewpoint of the double solution and that of the hydrodynamical theory upon attributing a physical reality, not to the statistical wave Ψ, but to a wave with an objective character (called φ by Vigier in his text), which very often has the same mathematical form. Today, I think that the idea of Vigier is absolutely necessary for the development of the theory of the double solution, and that in formulating it, he has allowed for important progress to be made in that direction.

Finally, Vigier has shown how one may try to introduce the concepts of the double solution in the formalism of general relativity. Naturally, this is only a first attempt, since it is almost certain that in order to give the theory of the double solution everything that it implies, one must develop it in the framework of a unitary relativistic theory in which gravitational, electromagnetic, and mesonic fields have singular regions (corpuscles) that are found in the context of waves with vorticial singularities, which seems necessary for the representation of electrons and other particles with spin. However, surely a synthesis that vast will result only from a long-winded effort.

In the doctoral thesis that defines the object of the present volume, Vigier has given a clear and detailed exposition of the collection of his attempts at a causal and objective interpretation of wave mechanics that is founded upon the idea of the double solution, and, in particular, upon numerous important results that he has personally derived along those lines. This research, in which certain points will obviously be the object of criticism or revision, is very brilliantly presented, and will not fail to interest the intended readers. The enthusiasm that Vigier has for his research and the incessant ardent and creative activity of his imagination render his efforts particularly noteworthy and worthy of encouragement.

Certainly, at this present moment one may not consider the causal objective interpretation of wave mechanics by the double solution to have triumphed over all of the obstacles that seemingly must be abandoned. If certain difficulties appear to be minimized, then others persist that seem to be very great; notably, I am thinking of the ones that relate to the spreading and division of wave trains, the conservation of energy in quantum processes, etc., questions that are particularly delicate, and whose study Vigier has not begun in his work. As long as the set of these difficulties has not been reduced, one is not sure that they are surmountable and that one might not affirm that the causal objective interpretation of wave mechanics by the concepts of the double solution must replace the current purely probabilistic interpretation. Meanwhile, if the former enjoys a day that concludes by superseding the latter then Jean-Pierre Vigier must then be considered as having contributed powerfully to this new and unforeseen evolution of theoretical physics by his research.

Louis de Broglie

GENERAL INTRODUCTION

The development of the theory of general relativity and the discovery of the wavelike properties of micro-objects have completely disrupted the classical ideas on the nature of the physical world. Concepts that seemed indestructible, such as the deterministic character of phenomena, have been challenged, and the controversies that they created have not been settled yet, even after more than thirty years. Likewise, one may confirm without paradox that the crisis generated by new theory is yet to be resolved: Witness the difficulties that one actually encounters in quantum theory in the context of nuclear phenomena.

The research that is described in this work is concerned entirely with the problems that were raised in the course of the preceding controversies. Therefore, we briefly recall them, since a discussion of them is liable to clarify the mathematical developments that must follow ([1]).

To make things precise, we first study fundamental ideas of the kind of physics that one calls "classical," i.e., both pre-quantum and pre-relativistic. This study, which is necessarily schematic, has the goal of exhibiting the difficulties of this classical theory and the solutions that are proposed by the new theory. Therefore, it does not do justice to the results obtained that constitute, without a doubt, an essential stop in the march of humanity towards the comprehension and domination of nature.

For the classical physicists, Isaac Newton and Clerk Maxwell, external physical reality existed independently of observers and it would be possible to construct a model that reproduces the objective behavior of phenomena.

This model rests essentially on the distinction that is established between the general space in which the phenomena evolve – classical Euclidian spacetime – and the material substance that it contains.

The existence of this space and the nature of the situation in which one finds the matter permit a complete spatio-temporal description of the evolution of that substance.

Having said this, in conformity with the Cartesian ideal of explaining by figures and motions, the object of the theory is to analyze this evolution. In order to do this, we assume that:

1. The matter is decomposable into material points that are endowed with mass and a negligible volume in such a way that its motions permit us to understand the aforementioned evolution. (Obviously, this amounts to a representation that is only approached by the actual behavior of the body.)

2. The accelerations of these particles may be attributed to force fields with unspecified characteristics that nevertheless allow us to describe the objective behavior of these accelerations; thus, these fields also constitute a primitive given in the theory.

([1]) This monograph does not exactly reproduce the historical development of the ideas that the reader will find remarkably summarized in the work of Louis de Broglie, *La Physique quantique restera-t-elle indéterministe?* General Introduction.

In this model, the complete description of phenomena implies, as a consequence, two classes of facts that encompass the entire set of laws that govern the evolution of the physical world.

One must first know the laws that determine the evolution of the fields. These laws, when expressed in differential form, allow us to predict their value at each point of spacetime if we take certain boundary conditions into account.

One must then discover the laws that dictate the motion of the body in the field under consideration when one gives the body a particular initial position and velocity.

Maxwell's laws belong to the first type; Newton's celebrated law $\boldsymbol{f} = m\boldsymbol{a}$ continues to be the classical example of a law of the second type.

Such is the model of the world that was proposed by classical physics, in all of its elegant simplicity. It is essentially and irreducibly deterministic, since being given the initial conditions (initial positions and velocities of the all of the particles and the initial values of the fields) allows us, in principle, to calculate the ultimate evolution of the ensemble of material processes in full rigor. It was in this form that the celebrated mechanistic determinism of Laplace was scientifically expressed, and this was a theory that rested on the differential expression that Hamilton gave to the laws of mechanics.

We shall not recall the impressive successes and experimental verifications of this theory at this point. We only note that they present two internal difficulties that are impossible to resolve.

The first one is that it is impossible to understand the nature of the classical fields in this schema. In particular, the separation of the notions of the spatio-temporal context and the field inevitably leads to the problem of action at a distance. As a result, the form that Maxwell gave to the electromagnetic field equations suggests some propagation effects that continue to be difficult to reconcile with classical ideas.

The second one is that it is impossible to understand the nature of the laws of motion, i.e., the interaction between the fields and the particles.

Therefore, up until the definitive experiments of Michelson and the discovery of the wavelike properties of micro-processes permitted us to predict almost all of the known properties of bodies, the classical model did not provide a complete explanation, which is contrary to the ideals of the theory. The great masters of classical physics, including Newton himself, were not satisfied, and the scientific history of the Twentieth Century was, in part, dominated by the verification of the theories that demanded the construction of an ether that was capable of lifting the aforementioned difficulties.

In Chapter I we shall ultimately return to the ideas of Einstein that relate to the problems that were posed. In effect, they constitute a prolongation of the deterministic realism of classical physics and largely ignore the profound upheaval that was provoked by the development of the probabilistic interpretation of quantum theory in the context of micro-phenomena.

Micro-phenomena.

As Louis de Broglie ([2]) emphasized, the great drama of contemporary microphysics has been to explain the duality of waves and particles. First established for light, it was progressively extended to all of the known particles in the course of the first third of the

([2]) *La Physique quantique restera-t-elle indéterministe?* loc. cit.

Twentieth Century. We shall insist upon this point, and it is at the basis for all of the research that follows.

In order to do this, we shall recall a very interesting discussion of Janossy ([3]) that accentuates the problems that were raised by this discovery and the nature of the solutions that were proposed.

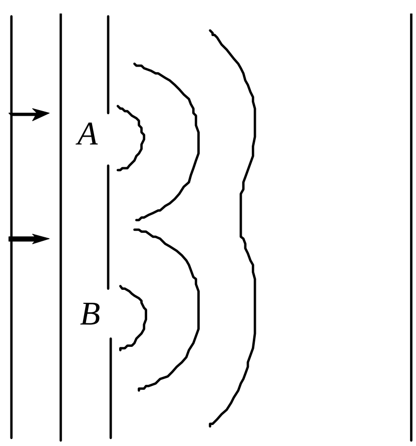
A. – Corpuscular aspects of micro-phenomena.

Numerous properties permit us to give meaning to the corpuscular aspect of micro-phenomena. For example, experience shows that the impact of light on a photoelectric plate is composed of distinct localized individual photons that carry an energy of $h\nu$. These impacts are independent of each other, and it is possible to realize experiments that involve sufficiently weak sources that one can observe the arrival of isolated photons that are separated by appreciable time intervals. Similarly, one exhibits the localized actions of the other particles – electrons, nucleons, mesons – on the various apparatuses – counters, etc. Moreover, one must emphasize that the only observable aspects of micro-objects are related to their particular properties; all known experiments finally come down to the observation of the quasi-pointlike aspects of their evolution or the interactions of matter.

B. – Wavelike aspect of micro-phenomena.

The wavelike aspect of micro-processes is clearly manifested in interference and diffraction experiments.

Consider two screens, I and II, in which the first is pierced with two holes, *A* and *B*. A plane light wave impinges on I. If only *B* is open then one obtains a uniform illumination on II. However, if *A* and *B* are open then one obtains an interference pattern on II that varies with the form and separation of *A* and *B*, and which differs from the sum of the illuminations produced by *A* and *B* alone.



I Fig. 1. II

Of course, the illumination that is observed on II is formed by the superposition of the point-like effects of the photons that comprise the ray. Similarly, one may arrange for them to appear one by one; in this case, the interference pattern is comprised of the progressive superposition of the spots on II that result from the action of the individual photons. This pattern is therefore actually composed of the independent contributions of the isolated photons.

Therefore, in this experiment, the micro-objects successively manifest themselves one by one in the form of spots that appear on either I or II. However, although the distribution of these spots is uniform on I, the same is not true on II, where they tend to

[3] *Acta Physica Academiae Scientiarum Hungaricae,* tome I., fasc. 4, pp 391.

accumulate only in certain regions (white fringes) that are systematically separated from the other regions (dark fringes).

This phenomenon is adequately explained if one assumes that the plane wave that falls on *A* and *B* is formed of continuous cylindrical waves that are centered on *A* and *B*. When just one of the waves falls on II (by closing the opening) it generates a uniform distribution. When two waves form simultaneously, they interfere and give rise to an interference distribution.

The foregoing result clearly shows the wavelike character of luminous phenomena. One may make analogous demonstrations with the aid of any known type of particle. Classical experiments ([4]) that were made with electrons show the wavelike character of electronic beams. Instead of slits, one simply uses crystals or metallic networks with angles of incidence. However, there is no difference in principle, and the reasoning that must follow is independent of the exact nature of the micro-particles or the experimental setup used.

It is clear that these experiments exhibit properties that are absolutely foreign to the classical model.

Indeed, observe our interference device under conditions that will assure that there is not more than one object in the apparatus at that time.

With regard to II, this micro-object manifests itself in a corpuscular form. With regard to I, it behaves like a wave, since the distribution of the individual isolated photons on II is influenced by the presence of two slits *A* and *B*. Therefore, each photon that acts on II as a particle individually presents a wavelike aspect. Everything happens as if it interferes with itself at I, and is finally preferentially absorbed in the regions where the diffraction pattern presents maxima.

Experiment has then allowed us to derive a fundamental property: The particle aspects of a set of micro-objects subject to an experiment of the preceding type are distributed in space with a density $|\psi|^2$, where ψ designates a continuous wave that satisfies one of the linear equations of wave mechanics. For example, in the device of interest to us the probability for a photon to be present at a point of space is proportional to the square of the amplitude of the light wave to which it is associated.

Such properties obviously pose difficult problems. They are obviously incompatible with the usual ideas about the structure of micro-objects. Even so, it amounts to understanding what happens and explaining, in particular, the type of interference that comes from things interfering with themselves.

C. – The probabilistic interpretation and complementarity.

A first attempt at treating the question consists, to quote an expression of Rosenfeld, "in solving it at the highest plane of the theory of knowledge ([5])" and is the celebrated probabilistic theory that was developed by the school called "the Copenhagen School." Indeed, its promoters – Niels Bohr, Heisenberg, Born, and Pauli – cleave to the same idea that the model is understood to be based in the possible objective knowledge of micro-objects.

([4]) See chap. IV.

([5]) Louis de Broglie, *Physicien et penseur,* ed. Albin Michel. See also chap. II of the present work.

For the Copenhagen School, the classical notion of knowledge is senseless. The object of physics is not to describe the actual behavior of things, but only to construct a mathematical symbolism that permits one to account for experimental results. The position known by philosophers under the name of positivism is clearly expressed in the declaration of N. Bohr, who was quoting Heisenberg ([6]): "…the (microscopic) phenomena are generated by repeated observations in some way."

According to Pauli, any property that is not actually accessible to measurement is devoid of real significance.

The theory goes on like this: Knowing anything that is external to the process of measurement is definitely prohibited by Bohr because micro-objects are not describable in the context of space and time. Moreover, this is the profound sense of the notion of complementarity. For the Copenhagen School, micro-objects are neither waves (as in I, for example) nor corpuscles (as in II), but both at once. Depending on how you observe them, they present one of these complementary aspects to observers and always behave "like ill-defined individuals in finite regions of space-time."

In this way of thinking, there is precisely one wave ψ and one corpuscle, but they may not be represented in the classical manner. The corpuscle has neither a precise position, nor velocity, nor trajectory; it only acquires them at the moment of the experiment. Therefore, it is simultaneously endowed with an infinitude of possible positions and velocities, in general, which are realized by measurements with certain probabilities. The wave ψ is devoid of physical reality and simply represents the set of all experimental potentialities of the corpuscle with their respective probabilities. These waves constitute "quantum fields" whose properties (interaction with the measuring apparatus) and evolution (furnished by the wave equations of Schrödinger, Dirac, etc.) embody all of what we may know about the particles that are associated to them. We return to this concept in more detail later on in chapter IV.

According to Bohr ([7]), the preceding interference experiment must therefore be interpreted in the following manner (when the micro-objects arrive at the apparatus of Fig. 1 one-by-one):

To the left of screen I, the photon has neither position nor trajectory. It is represented by a continuous plane wave ψ that permits us to calculate its probability of existing at each point $|\psi|^2$. In regard to I, this wave generates two cylindrical waves, *A* and *B*. After I, these probability waves interfere and generate an interference pattern on II that represents the distribution of the eventual impacts of the photon considered. Finally, the photon appears on II at a point that is impossible to determine in advance, in principle.

If one then repeats the experiment with a large number of photons that are associated to identical waves ψ then they will spread across II with a distribution that conforms to the experimental results.

What explains the great interest in that experiment and the interpretation that was given by Bohr is that it brings out the essential and very particular role that is played by the notion of probability.

([6]) N. Bohr, *La théorie atomique et la description des Phénomenes* (Gauthier-Villars, Paris, 1932).
([7]) Cf. P.A.M. Dirac, *Principles of Quantum Mechanics* [Introduction].

A great number of physicists, who base their claims on the analysis of fictitious experiments made by Heisenberg (of which, the best known is the microscope experiment), figure that the Copenhagen School does not contest the reality of the movement of the body apart from the observers. They simply believe that the notion of probability, which was introduced because the measurement apparatus perturbs the observed micro-object, makes it impossible to make any precise observation of their true behavior. This interpretation of the position of Bohr, Pauli, and Rosenfeld (which appears in numerous works) is not exact. It is radically opposed to the preceding explanation of interference phenomena that are associated with isolated particles.

Indeed, the explanation of Bohr assumes that one does not have the right to assume that the micro-object actually passes through one of the two slits. As Dirac remarked, if one can say that it passes through A or B then this amounts to saying that there exists one reality (the trajectory) for which the theory does not work, and one comprehends only how opening B modifies the arrival of a photon that passes through A on II.

Therefore, according to Bohr, the interference pattern results from the fact that the particle is objectively represented at I by his probability wave ψ. Since this wave describes only the results of eventual experiments, the micro-object is actually confused with these results, and it is impossible to conceive of anything else. In full rigor, the particle that moves to the left of the figure does not exist, as well as the particle before it reached the screen II. The interference pattern is essentially related to the non-existence of a trajectory of the particle and not, as some who wish to express the beliefs of the Copenhagen School would say, the practical impossibility of observing that trajectory without destroying the interference phenomena by that very act.

As was strongly emphasized by Louis de Broglie ([8]), the Copenhagen School reduces all of physics to the notion of probability and gives this notion a sense that is quite new in science. In quantum physics, probability does not result from our ignorance of the actual behavior of things or the complexity of the phenomena; it results from pure contingency.

Such an interpretation naturally entails important consequences on the orientation itself of the research that it inspires. Here are two examples:

If all knowledge is necessarily statistical then only linear wave equations can have any physical significance, since $|\psi|^2$ must satisfy well-known laws on the composition of probabilities ([9]).

If micro-objects cannot be correctly described in the context of space and time then it is vain to ask about their individual structure because such a structure introduces hidden parameters into the theory that are incompatible with the notion of complementarity.

Such prohibitions – a sort of irreducible limitation on the nature of our knowledge – result from taking a philosophical position, not from experimental necessity.

Indeed, as Blokinzef and David Bohm have shown, the beautiful results that were obtained by quantum theory do not prove that our knowledge of the isolated particles is necessarily statistical. They simply establish that the theory correctly accounts (obviously within certain limits) for the statistical behavior of sets of micro-objects. The predictions that calculated are therefore independent of the probabilistic interpretation of their individual behavior. There does not exist an experiment that proves that the exchange of energy by quanta is necessarily beyond analysis nor a theorem that allows us

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([8]) *La physique quantique restera-t-elle indéterministe,* loc. cit.
([9]) Cf. W. Pauli, *Wellenmechanik*, Introduction.

to prove this. As far as this is concerned, a detailed examination ([10]) has shown that the celebrated reasoning of Von Neumann, which seems to proscribe the use of hidden parameters, rests on the idea that these parameters may not exist in the system being observed and the apparatus used for observation simultaneously. However, as Von Neumann himself has recently recognized, this is not necessary.

Nevertheless, the statistical success that is obtained in the context of the probabilistic interpretation imposes a certain number of conditions on any possible interpretation, which we shall enumerate:

– One must first explain why a continuous wave ψ that satisfies linear equations allows us to account for the statistical behavior of sets of micro-objects placed in specific conditions.
– One must then account for the success of the Schrödinger equation in configuration space.
– Finally, one must interpret the relativistic equations and the properties that are obtained with the aid of the corresponding statistical quantities called the Bose-Einstein and Fermi-Dirac distributions.

However, let us return to interference experiments. The interpretation that was given by the Copenhagen School is not the only one possible. One may imagine at least two ways of accounting for the wave-particle character of the individual micro-objects, without, in principle, posing the impossibility of writing them in the context of space and time or renouncing determinism.

D. – The interpretation of Schrödinger and Janossy.

The first proposal for the origin of wave mechanics by Schrödinger consists in rejecting the point-like character of the particles, except in their interaction with the apparatus.

In this way of thinking, one generally assumes that only the waves actually exist, and that the function ψ actually represents an extended particle with a density $|\psi|^2$.

Therefore, ψ no longer represents a probability, but a physical phenomenon. To the left of I, the photon is distributed over its wave packet. On I, this real packet is divided into two parts that interfere afterwards. Finally, on II the wave abruptly contracts upon the impact of the photon.

One then explains the observed statistical distribution by assuming that this latter contraction operates precisely with a probability $|\psi|^2$ for an arbitrary interaction.

This interpretation (which amounts to saying that the micro-objects are actually sometimes waves and sometimes particles) had been abandoned at the outset by Schrödinger himself (who did not accept the postulates of Bohr, for that matter) because it raises complications that seem to be insurmountable. We first state them without discussion:

First difficulty: If the micro-objects are identified with the usual continuous wave packets of wave mechanics then it seems difficult to associate them with actual fields since one knows that they necessarily disperse over time; the amplitude of the associated

([10]) Cf. L. de Broglie, loc. cit., and D. Bohm (*Phys. Rev.*, 85, pp. 166-180, 1952).

wave ψ should then tend to disappear. One may likewise imagine devices that systematically reduce this amplitude to the scale of the laboratory.

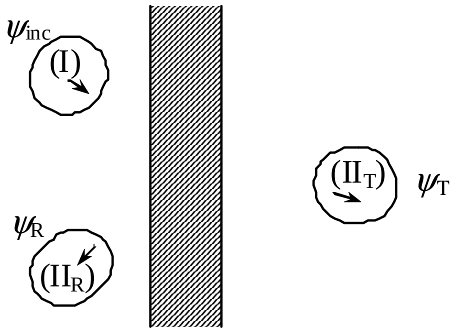

Fig. 2.

Consider a semi-transparent mirror *M* for which the power of reflection is equal to the power of transmission. An incident wave packet (I) strikes this mirror. After interaction, it is decomposed into two reflected and transmitted packets of the same form, (II_R) and (II_T). These two packets are quite real since it is possible to reduce the one to the other and obtain interference effects. However, if they have the same form and account for the probability then the amplitudes of the transmitted and reflected packets ψ_R and ψ_T are equal to the initial amplitude multiplied by $1/\sqrt{2}$. The device thus acts like an amplitude reducer. If I then consider a large number of such mirrors then I finally obtain from this sequence a transmitted packet (N_T) whose amplitude is as small as one pleases. Nevertheless, the packet that was used in the preceding interference experiment will give me results that are identical to those of the initial packet (I). Later on, we shall encounter this question again in another form as it relates to the causal interpretation.

Second difficulty: it has not been possible to account for the processes of contraction that this theory implies.

Go back to the initial device and open just one slit *A*. After passing through the slit, the wave spreads out and is found to be distributed on II with a decreasing density starting from a line *P* that is the geometric image of the slit *A*. One must then understand how to bring about the appearance of a corpuscle at an arbitrary point *Q*. As Einstein indicated, such a process of contraction must happen instantaneously (with a velocity greater than that of light), which violates the results that were obtained by the theory of relativity. Indeed, the wave contracts no matter what its extent, and a photon that appears on II collects on a photographic film, even if the slit *A* is found on Sirius.

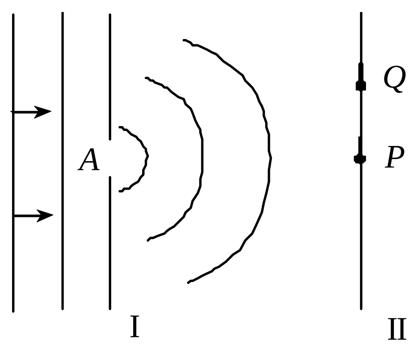

Fig. 3.

We remark that this phenomenon also constitutes an objection against the probabilistic interpretation. As de Broglie has remarked: "With our habitual ideas of space and time, it is impossible to comprehend the fact that a photographic effect that is produced at *Q* interferes with the production at any other point of the film, at least as far as admitting that the corpuscle is, in reality, localized, and occupies a well-defined point in the associated wave at each instant. Any other way of thinking seems to be irreconcilable with the idea that physical phenomena may be completely represented in framework of space and time or, similarly, Einsteinian spacetime."

Third difficulty: one must explain how contractions of the preceding type that are associated with a set of micro-objects reproduce the distribution $|\psi|^2 dv$, for example.

Last difficulty: one must account for the behavior of systems of interacting particles in this model and explain the reasons for the success of the Schrödinger equation in configuration space.

We immediately point out that there exists no quantitative theory of the preceding model that allows us to solve these difficulties. Janossy, who has given his account of the theory, studied them qualitatively without arriving at a mathematical formulation of the solutions that he discussed.

Like Schrödinger, he proposed to describe the electron or the photon by an extended structure. When there are no important perturbations, this structure displaces according to the linear equations of Schrödinger and Maxwell, although the exact evolution is nevertheless defined with the aid of nonlinear equations. The solution of these nonlinear equations differs in the case of strong interactions and during the process of contraction of the solution of the linear equation. However, these solutions reduce to solutions of linear equations, at least with respect to certain parameters that relate to initial conditions. If one prefers, the actual waves differ from the statistical waves ψ, which describe their mean behavior.

In this work, we shall not study the manner by which this model might allow us to surmount the difficulties raised. It has not been sufficiently developed to permit a quantitative discussion and a sound examination like our subject. We shall see, moreover, that there exists a simple model that suppresses the second difficulty, and whose quantitative development offers encouraging perspectives.

E. – The causal interpretation.

The second model, which was likewise proposed in 1927 by de Broglie (and which he subsequently abandoned when he rallied to the probabilistic interpretation) has been recently reprised and developed. As a result of a memoir of D. Bohm, which surmounted one part of the difficulties raised in 1927 by Pauli, and work done at l'Institut Henri Poincaré by the author along these lines, Louis de Broglie has reprised the question. Moreover, in our opinion, the results obtained, which already recover the better part of the results that were described by the interpretation of Bohr, justify the systematic discussion that shall make. They constitute what one may call the "causal interpretation" of quantum theory. This interpretation rests essentially on the idea that, conforming to classical concepts, it is possible to furnish a deterministic model that approaches the behavior of the individual micro-objects in the framework of space and time.

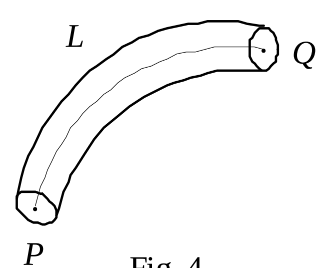

Fig. 4.

One first assumes that the micro-objects always present a particle aspect (limited to a region of space that is small in extent), even when we do not see them. This signifies that there exist intense phenomena that propagate along the interior of a tube of small cross section centered on a line L (a trajectory) that joins two points P and Q when a micro-object is emitted at P and observed at Q.

This fundamental hypothesis, which introduces elements into the theory that are unobservable at the moment, obviously breaks with the positivistic theory of knowledge because it assumes that the particle aspect of micro-objects exists independently of any observation. It presents the advantage of explaining the point-like effects of micro-objects without recourse to incomprehensible interactions.

Nevertheless, this point-like aspect does not suffice. One must endow the particles with an extended aspect if one wants to comprehend what happens. We return to our fundamental experiment again. Consider particles that come from the left and impinge upon the two slits. If one places counters behind *A* and *B* that cover them completely then one may establish for each particle whether it passes through *A* or *B*. Of course, most of the particles do not pass through any slit and will be absorbed by I; however, we are interested only in the ones that are not absorbed.

The particles thus observed in passing are obviously absorbed by the counters and do not reach screen II. Now remove the counters; conforming to the idea that the objects exist even when one does not observe them, we must admit that these particles, which were not decelerated by the counters, are effectively passing through *A* or *B*, and that any particle passes through a point *P* to the left of I and follows a trajectory *L* in order to arrive at its point of impact *Q* on II. This hypothesis on the actual existence of an unmeasured trajectory qualitatively determines the characteristics of the proposed model. Indeed, if one assumes, as we just did, that there exists a trajectory *L* then one concludes from this that each particle passed through one of the slits to reach *Q* (see figure above).

However, the necessity of associating a field, properly speaking, to each micro-object clearly appears then. Indeed, if one places oneself on II in the neighborhood of a minimum in the interference pattern and considers the particles that pass through *A* then one sees that they have a tendency to not fall in that region. By contrast, if one closes *B* then this tendency must disappear. There are thus particles passing through A whose trajectories are modified by the opening or closing of *B*. This makes it indispensable that we consider the idea that the motions of the isolated particles are perturbed by the modification of the macroscopic boundary conditions (such as the presence or absence of *B* for particles passing through *A*), therefore that they are influenced by an extended phenomenon that accounts for these conditions.

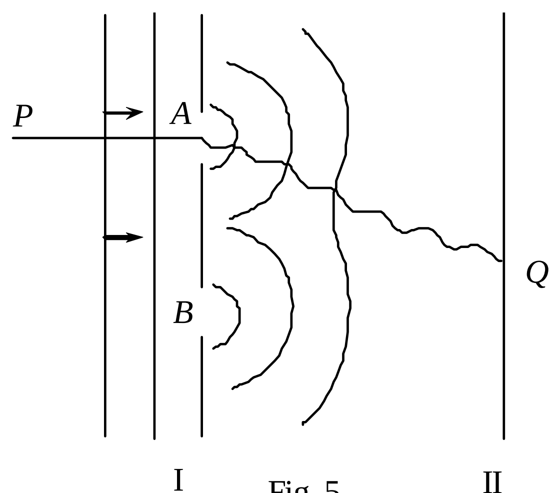

Fig. 5.

It results from the preceding analysis that if one admits the objective existence of micro-objects and the existence of movements *L* of their point-like aspect then one is necessarily led to also attribute a real extended wavelike aspect. This hypothesis thus leads almost irresistibly to the "model" for the causal interpretation in which one must consider the micro-objects to be a singular region ([11]) in an extended phenomenon.

([11]) By the term "singular" region, we mean a region that is endowed with the particular properties and characteristics that differentiate it in a unique and mathematically proper manner from the extended region

This viewpoint, which is the objective synthesis of the wavelike and particle-like aspects of micro-objects, thus amounts to considering them as both waves and particles; each micro-object is simultaneously a wave and particle.

As we have seen, in order to develop such an interpretation one must solve a certain number of problems, which we shall enumerate.

In the first place, one must furnish a deterministic model of individual micro-objects that permits us to quantitatively describe their actual behavior – notably, the (non-classical) motions of their particle aspect.

One must then show that the distribution of the motions of a given set of such objects furnishes, at least approximately, the statistical distribution that is associated to the known continuous solutions ψ of wave mechanics.

Finally, one must treat the question of particles in interaction, interpret the Schrödinger equations, and treat the questions of relativistic equations and those of quantum statistics.

These questions determine the plan of this work. We successively study them and indicate both the results that are obtained and the results that remain to be solved. We therefore hope to provoke discussion or research that could bring progress to a theory that tends to commit microphysics to an exploration of the properties of matter that are subordinate to the statistical phenomena that are described by the habitual interpretation.

In chapter I, the reader will find an analysis of the historical development of the individual "model" of the micro-object that was proposed in the first versions of the causal interpretation, as well as a discussion of the problems that it raised.

In chapter II, we discuss in more detail certain properties of the model of the theory of the double solution with singularity, by insisting on the notion that the particle singularities are "guided" by continuous waves φ that correspond to quantum fields.

Chapter III will be dedicated to the discussion of the possible relationships that that one can establish between the preceding causal theory and the essential ideas of the relativistic unitary theories. In particular, we prove that it is possible to find solutions with singularities for the relativistic equations that follow the "trajectories" of the causal interpretation.

In chapter IV, we then show that the statistical mechanics that is associated with a collection of micro-objects of the preceding type is described precisely by continuous solutions of the usual linear wave equations. This proof, which agrees with that of Bohm, rests on hypotheses and analysis that eventually exhibits the difference between the classical mechanistic determinism and the dialectical determinism that is associated with the new interpretation.

Chapter V involves a discussion of the problem of micro-objects in interaction and quantum statistics. These questions are not completely solved, but sufficient progress has been made recently to justify a systematic discussion nonetheless.

In the last chapter, we conclude with a description of the theory of measurement that is associated with this new interpretation. In our opinion, this theory, which is essentially due to the work of David Bohm, constitutes a very important first step because it begins for the first time the objective study of interactions between the measuring apparatuses

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into which it is included as a subset; to a first approximation, one may undoubtedly mathematically represent this difference by a singularity.

and the observed micro-objects. In part, it justifies certain results that were postulated by the old theory and gives a statistical interpretation of the Heisenberg uncertainties that strips away the barrier from their character that is imposed by knowing them.

This work makes no pretense of constituting a complete or definitive description of the causal interpretation of quantum theory. Qualified as a "program" by Pauli more than two years ago, this interpretation has passed that stage today; its adversaries themselves contest only its internal coherence. It is therefore useful to undertake its examination with the goal of making the results obtained precise and expanding upon the problems that are likely to orient the research towards new experimental discoveries.

In terminating this introduction, I would like to express my profound recognition of Louis de Broglie, whose work and counsel have guided the research that was undertaken at l'Institut Henri Poincaré. I would also like to thank Prof. G. Darmois, A. Lichnerowicz, and R. Fortet, as well as Mme. Tonnelat and G. Petiau for their encouragement and counsel.

Finally, I would like to express my gratitude to a certain number of personal friends: In the first place, Prof. David Bohm, F. Halbwachs, A Régnier, E. Schatzmann, and F. Fer, for numerous discussions that we have had in the course of latter years. Their criticisms and their suggestions have greatly contributed to giving this monograph the form that it presently takes; in particular, chapter IV resulted from collaboration with Bohm in Sao Paolo. The detailed results of our collaboration will be ultimately published.

CHAPTER I

§ 1. – Conforming to the plan we just described, we shall dedicate this first chapter to the study of the possible deterministic models for the individual micro-objects, which are considered to have both a point-like aspect and an extended aspect. In order to facilitate the exposition, we shall first study the causal theory in the classical approximation (in which the particle aspect is reducible to a point), such as was presented by Bohm ([1]), and we indicate how one may extend this to the relativistic equations of Dirac and Petiau-Kemmer.

§ 2. – Before discussing the models of micro-objects that are proposed by the causal interpretation, we shall recall an old version of the theory ([2]) that is interesting for the fact that it exhibits the difficulties of the enterprise.

In this version, de Broglie began with the idea that one may describe the objective behavior of micro-particles in a deterministic fashion with the aid of continuous movements that are different from the classical movements.

As a consequence, he reduced the micro-objects to material points in movement in fields of a new type, and he proposed to define real trajectories that can explain the success of the laws of quantum statistics.

In order to do this, one might make the following hypotheses:

A. One introduces a continuous field ψ, which we write in the form:

$$\psi = R \exp\left(\frac{iS}{\hbar}\right) \tag{1.1}$$

in which S and R are real functions and $\hbar$ is an arbitrary constant. This field satisfies a field law that is defined by the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi + V\psi \tag{1.2}$$

in which V denotes the classical potential. This equation, which one may also write as:

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t} + H\left(\vec{x}, \frac{\hbar}{i}\nabla, t\right)\right\}\psi = 0,$$

splits into two distinct equations (which correspond to the real and imaginary parts of (1.2)) when one writes ψ in the form (1.1). Upon setting $P = R^2$ they become:

([1]) *Physical Review,* (85, pp. 166, 1952).
([2]) Proposed in 1927 at the Solvay Congress by de Broglie.

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{1}{2m}(\nabla S)^2 + V - \dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R} = 0 & \text{(J)} \\ \dfrac{\partial S}{\partial t} + div\left(P \cdot \dfrac{\nabla S}{2m}\right) = 0 & \text{(C)} \end{cases} \tag{1.3}$$

whose physical significance we shall ultimately discuss.

B. Having said this, the particle is restricted in this field to follow the particle trajectory that is defined by:

$$\vec{v} = \frac{\nabla S}{m}, \tag{1.4}$$

in which we have denoted the velocity of the particle by $\vec{v}$. Upon taking the gradient of equation (J), one then obtains:

$$\begin{aligned} m\frac{d\vec{v}}{dt} &= -\nabla\left(V - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right) \\ &= -\nabla\left(V - \frac{\hbar^2}{4m}\left\{\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right\}\right) \end{aligned} \tag{1.5}$$

which may be interpreted by saying that the particles follow a congruence (L) of trajectories that conform to the laws of classical mechanics, provided that one adds a quantum potential:

$$Q = -\frac{\hbar^2}{4m}\frac{\Delta R}{R} = -\frac{\hbar^2}{4m}\left\{\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right\},$$

to the usual potential.

One therefore sees the appearance of an essential difference between this theory and the usual field theory.

In classical theory, the trajectory of a particle in a field is defined by (1.5) and by initial conditions (position and velocity) that may be given arbitrarily. One may thus obtain an infinitude of possible motions. This is not the case here. Relation (1.4) selects a particular family (L) from among all of these trajectories that is the only one that can be described by the particle considered. For example, one may make this selection by imposing the following initial condition on these motions:

$$m\vec{v}_{t=0} = \nabla S(\vec{x}, 0), \tag{1.7}$$

which is obviously compatible with (1.5).

This already strongly suggests the idea that the relationship between the particle and the quantum field is not of a classical nature. Everything happens as if the particle is related to the field in such a way that it cannot be displaced arbitrarily. This is the idea that will be developed later on in the model of the double solution.

C. Finally, we suppose that $P = R^2$ represents the density of a set of identical particles that are associated to the field ψ. This last hypothesis is coherent, since the continuity equation (C) shows precisely that $P(\vec{x}, t)$ behaves like the density of a set of particles that are restricted to follow the congruence (*L*).

Here again, the foregoing amounts to choosing the particular distribution:

$$P(\vec{x}, 0) = R^2(\vec{x}, 0) \tag{1.8}$$

from among all of the possible initial densities of the particle that one may associate to (*L*).

Before indicating the reasons that led us to abandon this first attempt, we must first emphasize that it permits us to formally account for all of the results of the Schrödinger equation (1.2). It therefore suggests a profound analogy, which served as the point of departure for the research of de Broglie, between the wave equation and the expression that was given by Hamilton and Jacobi for the laws of classical mechanics. Indeed, what jumps out at the eyes is that equation (J) represents a generalization of Jacobi equation, since it suffices to make $\hbar$ tend to zero in this equation to recover the classical equation.

Having said this, it is clear that this version of the theory is physically inadmissible, since it gives two incompatible meanings to the function *R*, namely:

– in (B), *R* defines a real quantum potential that influences the trajectories;
– in (C), *R* defines a probability density of the particles that are associated to these same trajectories.

One concludes from this that everything happens in this version of the causal theory as if the motion of the particle is determined by the set of possible motions, since $R(\vec{x}, t)$ corresponds to both a real field and a probability. Such hypotheses obviously contradict the objective that was pursued by de Broglie. As Pauli emphasized, this introduces an unexplained statistical hypothesis at the basis for the theory that leads back to the Bohr interpretation.

This last objection seemed decisive to L. de Broglie, and he agreed to abandon his attempts to defend determinism in the context of micro-phenomena.

Nevertheless, recent work has shown that it is possible to escape the preceding difficulties while preserving the same equations, but attributing a different physical significance to the symbols.

One thus arrives at what one may call "the classical approximation of the theory of the double solution." This version of causal interpretation, which has been defended by Bohm ([3]), in particular, obviously admits the objective reality of the micro-phenomena and proposes to furnish a deterministic description of them. It may be presented in two different forms.

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[3] *Physical Review*, loc. cit.

I. – First form.

Along with the classical field, one introduces a quantum field φ (which is represented by a continuous solution of the usual wave equations) that is composed of a wave that propagates in spacetime and one agrees that:

A_1. $\varphi = R \exp iS / \hbar$ satisfies the Schrödinger equation (1.2),

B_1. The momentum $\vec{p}$ of the associated particle is given by $\vec{p} = \nabla S$.

C_1. If we may predict or control the exact position of this particle at an exact time then one must nevertheless introduce a statistical ensemble that has a probability density $P = R^2 = |\varphi|^2$; the use of statistics is not considered to be inherent to the conceptual structure of the theory, but results from our ignorance of the precise initial conditions of the particle.

This amounts to saying:

1. That each micro-object is formed by a point-like particle and a real wave φ; the first is restricted to follow a streamline of the second.

2. That an ensemble of micro-objects of the preceding type that is associated with a wave φ identically is necessarily endowed with a density:

$$P = |\varphi|^2.$$

In this version, the wave φ represents a true field that is distinct from the usual statistical wave φ. All of the formulas from (1.1) to (1.7) are obviously applicable to it.

The last hypothesis C_1 assumes that the statistical behavior of a set of identical particles that are associated to the same field φ is furnished by a fictitious wave ψ that satisfies the equality:

$$\psi = C\varphi, \tag{1.9}$$

in which C is a normalization constant. One therefore distinguishes two waves φ and ψ that have a different physical significance. We shall see that this distinction is at the basis of the theory of the double solution.

We immediately point out that, apart from any other difficulties, the model suggested by this form of the theory does not appear to be satisfactory. In the first place, one sees that the field φ is not a field in the usual sense of the word, since it is mathematically represented by a complex quantity.

If one then accepts the correspondence principle (which seems necessary in order for the macroscopic ensembles of micro-objects to obey the laws of classical mechanics) – a principle that ensures that one recovers the classical model and behavior when $\hbar \to 0$ – then one encounters difficulties that seem hard to interpret. Indeed, one may not blindly make $\hbar$ go to zero in (1.1) and (1.2), when applied to φ, because then the field φ and its motion then lose all significance. In order to retrieve the classical result, one must make $\hbar$ go to zero only in (1.3) (with $\varphi = R \exp iS/\hbar$). Indeed, in this case R and S become independent; equation (J) tends to the Jacobi equation, and (C) defines a continuity equation that is associated to the particle density *P*.

This suggests a second form of the model considered that appears to be more physical than the previous one to me, and which was the result of private discussions between Bohm and the author of this work.

II. – Second form ([4]):

A quantum system – such as an electron – is essentially composed of:

1. A quasi-pointlike particle that is endowed with a well-defined position (which varies continuously in time) and a velocity that is defined by a potential *S*.

2. A real quantum field *Q* that is important in the atomic realm, but negligible in the macroscopic realm.

One then recovers the usual results by postulating that:

A_2. The field *Q* is calculable with the aid of an auxiliary function $\varphi = R \exp iS/\hbar$ that satisfies the Schrödinger equation (1.2) by setting:

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R}.$$

B_2. The particle moves in this field according to the classical laws with:

$$m\frac{d^2 x}{dt^2} = -\nabla\{Q+V\},$$

but this is true only on the trajectories of $\vec{v} = \nabla S/\hbar$ (which amounts to imposing the initial condition (1.7)).

C_2. The density of an ensemble of such particles that are associated with the same field *Q* is given by $P = R^2$.

(It also suffices to postulate (1.8)).

This second form seems more satisfactory than the first, since if $\hbar \to 0$ then the quantum field *Q* disappears, and one simply comes back to the classical model of a point-like particle moving in the potential $V(x, t)$; $S(x, t)$ plays the role of a Jacobi function.

One must nevertheless emphasize, as L. de Broglie has remarked, that in this form, one may not be content with the given of the potential *Q* upon eliminating *S* (which is defined only by an initial condition), because in concrete problems the boundary conditions are imposed on the wave φ, which is considered to be a solution of the wave equation. Being given such conditions defines – in the same stroke – *Q* and the possible trajectories that appear to be inseparable. This is evidently a peculiar aspect of the quantum field that distinguishes it from classical fields. In classical mechanics, one is given boundary conditions on the fields themselves when one devises equations that involve potentials in order to calculate them (as in the case of the electromagnetic field).

([4]) Discussed by D. Bohm, *Phys. Rev.*, 89, 1458 (1953).

In this second form, it is necessary to give them on φ and not on Q and S separately, to which they are indeed related. Here, one recovers precisely the non-classical relationship between the field and the trajectories that was previously pointed out. L. de Broglie considers this to be an objection to the second form. Personally, I think that it amounts to a special property, and not a difficulty. It is not surprising that the introduction of wavelike properties makes us leave behind classical procedures in the mathematical treatment of boundary conditions.

As in the first form of the theory, here one distinguishes two waves φ and ψ (ψ = const. φ) once more; however, the first one φ has a particular meaning. Instead of representing the field, it plays the role of an auxiliary function (a sort of intermediate calculation); the field itself is composed of the quantum potential Q. The two models are therefore physically different:

- In the first, the micro-object is represented by a point-like particle and a complex field φ.
- In the second, it is represented by the same particle and a real quantum potential field $Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R}$; in summary, $\hbar$ plays the role of a coupling constant between this field and particle considered.

Be that as it may, these two models lead to the same motions and the same statistical distribution; the first two postulates, A and B, describe the individual behavior of the micro-objects, and the last two, C_1 and C_2, describe their statistical behavior.

§ 3. - This manner of presenting the causal interpretation has been the object of a certain number of criticisms that one may discuss immediately in the context of the causal interpretation of the Schrödinger equation because they do not depend on the exact form of the wave equation that is used.

The first refers to postulates C_1 and C_2. In particular, Pauli ([5]) has contested their legitimacy. Indeed, according to him one has no right to introduce such a hypothesis into a theory of the preceding type. In order to comprehend its proper sense, "one seeks to justify the restriction to particular ensembles [$P = R^2$, ed. note] by the fact that the continuity equation guarantees the density distribution of the parameters if it is realized in the initial state, provided that the system remains closed. Just the same, as far as that is concerned, one must add that a hypothesis on the subject of probability is out of place in a deterministic theory. The name of the continuity equation seems to me insufficient to fix this in a general fashion. For example, if the experimenter arbitrarily divides this ensemble into two parts then the distribution of parameter values will no longer be given by the amplitude of the function φ, at all. From the fact that there exist all sorts of phenomena in which the values of these parameters must manifest themselves, likewise in an indirect fashion, these parts do not have to behave in the same fashion, despite the equality of their functions. The hypothesis of the general validity of a probability distribution for parameters that is determined by just one function is therefore not justified from the viewpoint of the deterministic scheme."

([5]) Cf., Louis de Broglie, *Physicien et Penseur*, Albin Michel.

In chapter IV of the present work, the reader will find a detailed discussion of the manner by which Bohm and myself proposed to surmount this objection.

Indeed, according to us, postulates C_1 and C_2 are not necessary because one may show that the density P of an ensemble of particles that has the same quantum field (in the sense of forms I or II) and is subject to the same very complex stochastic external conditions will necessarily tend toward a limit state that corresponds to:

$$P = |\varphi|^2.$$

In other words, quantum ensembles constitute the equilibrium distribution that is attained by the particles under the influence of external fluctuations for any initial distribution $P(\vec{x}, 0)$. The distribution $P = R^2$ thus plays a role that is analogous to the Maxwell distribution in the kinetic theory of gases, and it is found to be automatically re-established in the case (envisioned by Pauli) in which it has been destroyed by the particular conditions.

One may therefore prove postulates C_1 and C_2 of the preceding model, and we shall no longer speak of individual micro-objects in this chapter.

The second objection is due to Francis Perrin. It is attached to the difficulties that relate to the amplitude discussed in the general introduction in the context of the Schrödinger interpretation that considers the continuous wave φ to also represent an actual phenomenon. We state as follows: "consider a micro-object that is described by a wave that has been reduced to a wave packet. If I subject this packet to an arbitrary amplitude reducer (composed of the semi-transparent mirrors that were introduced in the case of the photon), or if I let it evolve for a sufficient time then one knows that the amplitude R tends to "flatten out" in the course of time. It therefore becomes very difficult to comprehend how a wave whose amplitude tends to zero may continue to govern the movement of the particle and produce macroscopic interference effects. Similarly, when the slits A and B are separated by macroscopic distances one obtains, for example, interference effects from the light of the distant stars, even though the wave that is associated with it has seen its amplitude decrease like $1/r$ (where r designates the distance traveled)."

It is possible to treat the problem thus posed in different ways.

In form I of the interpretation, in which φ directly represents the quantum field, one may first remark, with Fer, that this objection amounts to posing the following question: What amplitude must one start with in order that one may consider the wave to lose its "guidance" law and its physical significance? It is difficult to respond. Obviously, from the classical viewpoint one knows that an increasingly weak field produces effects that tend to zero, but this is not true in the context considered.

As L. de Broglie suggested, in principle, one may also suppose that φ satisfies a nonlinear wave equation such that the usual equations constitute valid approximations only when φ and its derivatives are small. The packet then tends to the form that is sketched in the figure; P denotes the position of the particle. It is then exactly true that one must neglect the nonlinear terms in the interval BC; the same may not be true on the boundary of the packet in the regions AB and CD, where the derivatives of φ may take considerable values. The supplementary terms thus introduced into the equation are likely to stop the flattening of φ. If we start with a certain extent (of a sort that is impossible to realize in interference experiments of the Young double-slit type) then the packet is displaced along with the particle (fig. 7), a little like protoplasm that accompanies the kernel-particle. For example, conforming to the very old conception of Einstein, a light ray is composed of the superposition of cells that are constituted from photon particles and their associated packets (Nadelstrahlung). In the interior of each packet one will obviously have $\psi = C\varphi$; ψ satisfies the linear equation everywhere. This conception of the structure of micro-objects is very interesting. Unfortunately, it is very difficult to mathematically formulate and analyze what happens in the particular case considered. As it has not been possible, up till now, to systematically develop it, we shall therefore not discuss it in more detail here. Nevertheless, note that it presents the advantage of making the wave fronts play a particular physical role, which is precisely the case in nature (which is systematically ignored in most of the works on wave mechanics and optics).

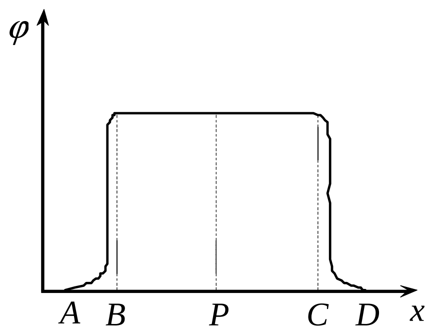

Fig. 6.

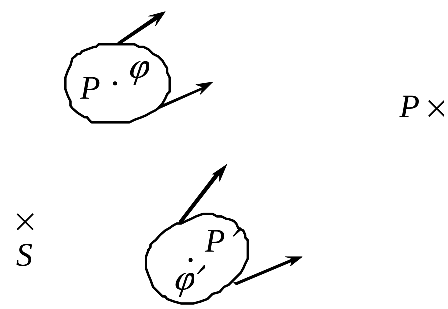

Fig. 7.

However, the objection of Perrin presents a different aspect when one adopts form II of the causal interpretation. In this case, the real field that is responsible for the movement is the quantum potential:

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R},$$

and not φ, which only serves as an intermediate calculation. Since the field Q does not vanish when $R \to 0$ because Q does not change when multiplies φ by a constant, the objection does not apply to our new version of the theory; L. de Broglie shares this opinion. He finds it difficult to admit that a real physical quantity must be determined by the ratio of two quantities ΔR and R that simultaneously tend to zero, or similarly if this ratio remains finite in time. I must say that I am not convinced that the latter argument is valid. Indeed, there is no shortage of physical examples of real magnitudes that are defined by ratios of this nature. For example, this is the case with the instantaneous velocity $(\vec{v} = dx/dt)$ of a body in classical mechanics, which is defined by the ratio of two quantities (dx and dt) that tend to zero; then again, the ratio of V/I defines the resistance of a circuit no matter how small V and I are. We shall not belabor this particular point further, and we leave it to the reader to form his own opinions, because we shall return to this question in the context of the new interpretation of the theory of relativity.

The third difficulty that has been discussed by various authors ([6]) relates to the nature of the quantum field in relation to the laws of motion.

Indeed, it is clear that the two proposed models leave the classical difficulty regarding the relationship between fields and quasi-pointlike particles intact. As we saw in sec. 1, this arises again because the proposed models restrict the eventual motion of the particles that are associated with φ to only one congruence: (L). In particular, Takabayasi has emphasized the strange character of this restriction, which shows that the quantum potential is not a potential like the others.

This character becomes more striking if one remarks, with de Broglie, that everything in this model happens as if the point-like particle is influenced only by its quantum potential proper (properly speaking, this does not constitute an objection since this property accounts for experimental facts, although it resorts to the "non-classical" aspect of the quantum potential). For example, this is true when one realizes fundamental interference experiments with two families of photons that are associated with distinct waves φ and φ' that are taken in such a way that spherical waves that are produced by the slits AB and $A'B'$ that were made in the screens I and I′ are superimposed in space and time, in conformity with the figure. If the fields that are defined by φ and φ' are ordinary fields then one might not comprehend why the experiment shows that the photons that belong to the family φ' give an interference pattern at II′ that definitely does not depend on φ (and vice versa), which is explained only by assuming that these photons are influenced only by φ', and then only if they are found in a region in which the field φ is different from zero ([7]).

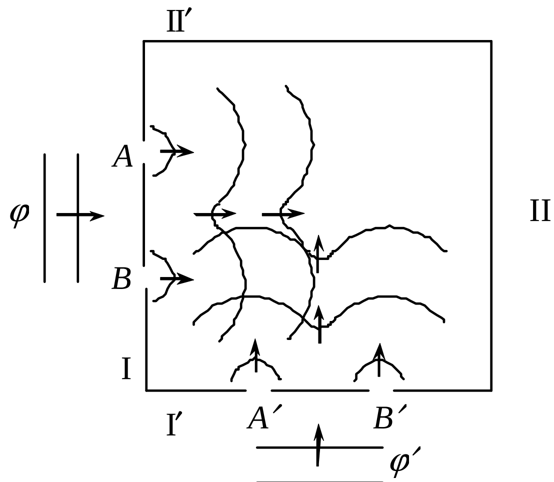

Fig. 8.

In our opinion, this last property is very important because it exhibits exactly the insufficiency of the two models that we just discussed. Obviously, one may respond to this only by perfecting them, and we shall verify later on that one may explain such behavior effectively by the condition of passing to the theory of the double solution, hence, abandoning the classical ideas on the nature of particles and, conforming to Einstein, assimilating them to singular regions (bunch-like solutions) of the potentials considered. The preceding discussion of the principal difficulties that are raised by forms I and II of the causal theory makes no pretense of exhausting the subject. The study of these difficulties – particularly the second – is underway, and may lead to perfecting the proposed models. Similarly, in chapter V we verify, in the context of systems of interacting particles, that these difficulties raise new problems, which are also being examined today. At the moment, as D. Bohm has emphasized, the models that are associated with micro-objects in the causal interpretation may not be considered to be

([6]) Cf., D. Bohm, *Progress of Th. Physics* (Japan), 9, 273 (1953).

([7]) Furthermore, we return to this difficulty in chap. III, in the context of systems of interacting particles.

definitive. The present interpretation presents just this character, and – in my opinion – this is a strength, not a weakness, and this suggests a means for developing the particular models that were studied in section 2 ([8]).

Before doing this, we shall first extend the models in question to relativistic equations for particles with spin. Indeed, we are doing this because, up till now, the perfections that we have envisioned have not called into question the class of motions of the corpuscle (L). We therefore proceed with that extension in the following paragraphs, in order to facilitate the ultimate examination of the theory of the double solution.

§ 4. – First, we briefly return to the Schrödinger equation as it applies to the wave:

$$\varphi = R \exp \frac{iS}{\hbar}.$$

The two field equations (J) and (C) may be considered as two Euler equations that are derived from the Lagrangian:

$$\mathcal{L} = -P\left\{\dot{S} + \frac{1}{2m}(\nabla S)^2 + V + \frac{\hbar^2}{8m}\frac{\nabla P}{P^2}\right\},$$

in which $\dot{S}$ designates the derivative of S with respect to time. From this, one immediately concludes that the momenta that are canonically conjugate to S and P are $\Pi_s = -P$ and $\Pi_p = 0$, respectively. This gives us the following expression for a Hamiltonian density:

$$H = -P\dot{S} - \mathcal{L} = P\left\{\frac{1}{2m}(\nabla S)^2 + V + \frac{\hbar^2}{8m}\frac{\nabla P}{P^2}\right\},$$

which, when considered as a functional of S and Π_s, gives back equations (J) and (C) as its Hamilton equations.

One then extends forms I and II of the causal interpretation to the general Schrödinger equation:

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t} + \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A}\right)^2 + e\Phi + V\right\}\varphi = 0,$$

in which we have denoted the general electromagnetic field by the quadri-vector $(\vec{A}, \Phi)$. Indeed, (J) and (C) become the relations:

$$\dot{S} + \frac{1}{2m}\left(\nabla S - \frac{e}{c}\vec{A}\right)^2 + e\Phi + V - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0 \qquad \text{(J)}$$

([8]) We shall ultimately indicate some of these proposed modifications.

$$\dot{P} + div\left\{P\left(\nabla S - \frac{e}{c}\vec{A}\right)/m\right\} = 0 \qquad \text{(C)}$$

and it suffices to attribute the velocity:

$$\vec{v}(x,t) = \left(\nabla S - \frac{e}{c}\vec{A}\right)/m$$

and the energy:

$$E(\vec{x},t) = -\dot{S} - e\Phi$$

to the particles in order to define the congruence (L) of possible trajectories (on which an ensemble of particles will end up being distributed with the density R^2, as we shall verify in chapter IV).

The following step consists of establishing a model for a micro-object that corresponds to the relativistic wave equation of Klein-Gordon. This extension, which was proposed by de Broglie in 1927 ([9]), is carried out without difficulty.

Starting with this equation, which we write in its classical form:

$$\left\{\left(\frac{\hbar}{i}\partial_\mu - \frac{e}{c}A_\mu\right)^2 + m^2c^2\right\}\varphi = 0\,, \qquad (1.10)$$

in which A_μ represents the electromagnetic potential, ∂_μ represents the operator $\left(\nabla, \frac{1}{c}\frac{\partial}{\partial t}\right)$, and μ = 0, 1, 2, 3 are the indices that denote the spacetime coordinates. If one sets, as before:

$$\varphi = R\exp\frac{iS}{\hbar} \qquad (P = R^2)$$

then, by separating into real and imaginary parts in equation (1.10) one again obtains the fundamental relations:

$$\begin{cases} \left(\partial_\mu S - \dfrac{e}{c}\vec{A}\right)^2 + m^2c^2 - \hbar^2\dfrac{\Delta R}{R} = 0 & \text{(J)} \\ \partial_\mu\{P(\partial^\mu S - eA^\mu/c\} = 0 & \text{(C)} \end{cases}$$

and it remains for us to interpret them. This may be accomplished by remarking, with L. de Broglie, that the streamlines of equation (1.10) may be obtained by attributing a variable proper mass M_0 to the particle, which is defined by the relation:

$$(\partial_\mu S - eA_\mu/c)^2 = -M_0^2c^2 = -m^2c^2 + \hbar^2\frac{\Delta R}{R}\,,$$

([9]) Cf., *La Physique Quantique restera-t-elle indéterministe?*

which amounts to setting:

$$M_0 = m + \delta m = m + \frac{\lambda Q}{c^2}$$

with:

$$\begin{cases} \lambda Q = mc^2(\gamma - 1) \\ \gamma = \left(1 - \chi^2 \dfrac{\Delta R}{R}\right) \\ \chi = \dfrac{\hbar}{\sqrt{2mc}}, \end{cases}$$

which implies that $\chi^2 \, \Delta R / R \leq 1$.

One then sees that these streamlines are extremal lines of the usual relativistic Lagrangian, in which one has replaced m with M_0. One therefore has:

$$\delta \int_{\tau_0}^{\tau_1} \mathcal{L} d\tau = 0,$$

with

$$\mathcal{L} = -M_0 c^2 \sqrt{1-\beta^2} + \frac{e}{c} u_\mu A^\mu = -\left(mc + \frac{\lambda Q}{c^2}\right)(-u_\mu u^\mu)^{1/2} + \frac{e}{c} u_\mu A^\mu,$$

in which u_μ designate the components of the world-velocity that is associated to the streamlines (L). The components obviously satisfy the relations:

$$u^\mu = \left(\partial^\mu S - \frac{e}{c} A^\mu\right) / m\gamma = \left(\partial^\mu S - \frac{e}{c} A^\mu\right) / M_0$$

and

$$u_\mu \, u^\mu = -c^2.$$

This permits us to interpret the relativistic equations (J) and (C). The first one (J) is nothing but the relativistic Hamilton-Jacobi equation that is associated with the classical motion of a body in a scalar potential Q with the coupling constant λ, namely:

$$\left(\partial_\mu S - \frac{e}{c} A_\mu\right)^2 + \left(mc + \frac{\lambda Q}{c}\right)^2 = 0.$$

The second one (C) is nothing but the continuity equation that is associated with these trajectories, namely:

$$\partial_\mu \left(P_0 \cdot \frac{\partial^\mu S - \frac{e}{c} A^\mu}{m + \lambda Q / c^2} \right) = 0,$$

in which $P_0 = P / m\gamma$ represents the density in the proper system.

The equations of motion on L are then established without difficulty. In the absence of an electromagnetic field they become:

$$\frac{d}{d\tau}(M_0 c u_\mu) = c\partial_\mu M_0$$

or, more generally $(F^{\mu\nu} = (\mathrm{rot}\,\vec{A})^{\mu\nu})$:

$$m\frac{d}{d\tau}\left\{\left(1+\frac{\lambda Q}{mc^2}\right)u^\mu\right\} = \frac{e}{c}u_\nu F^{\mu\nu} - \lambda\partial^\mu Q\,.$$

The components (P_i , W) of the energy-momentum quadri-vector are written:

$$P_i = -\frac{\partial\mathcal{L}}{\partial x_i}$$
$$W = \frac{\partial\mathcal{L}}{\partial t}$$

and one finally recovers the Hamilton equations in their classical form:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial P_i} \qquad\qquad \frac{dP^i}{dt} = -\frac{\partial H}{\partial x^i}$$

with:

$$H = P_\mu u^\mu - \mathcal{L} = \frac{1}{2}\left(\frac{\left(P_\mu - \frac{e}{c}A_\mu\right)}{m+\lambda Q/c^2} + (mc^2 + \lambda Q)\right).$$

The preceding considerations show that one may extend the previous two forms of the causal interpretation to the Klein-Gordon equation, which amounts to reducing each micro-object:

– in form I, to a point-like particle and a quantum field φ with:

$$\varphi = R\exp\left(\frac{iS}{\hbar}\right)$$

such that:

A_1. This field is governed by equation (1.10).

B_1. The particle follows the streamlines L (with the world-velocity u_μ) with which it initially coincides.

– in form II, to a point-like particle and a quantum field Q that is defined by (1.6) such that:

A_2. This field is calculable by the intermediary of φ, which satisfies (1.10).
B_2. The particle follows the preceding line L.

As in the case of the Schrödinger equation, one sees that the motions are effected classically in the aforementioned two forms under the influence of a supplementary quantum potential Q, with the condition that a particular initial condition that is analogous to (1.7) be imposed upon the motions. In order to abbreviate our discussion, we leave to the reader the task of agreeing that this extension of the causal "model" to the relativistic theory of particles of spin 0 adds nothing to the discussion of section 3, whose conclusions may be systematically applied to the relativistic micro-objects that we just defined.

§ 5. – It remains for us to extend the causal theory to the case of particles with spin 0, ½, and 1. When that extension was performed by the author – on the basis of a suggestion of David Bohm – it presented several apparent difficulties because these particles are defined by spinors that have several components:

$$\varphi_{(\alpha)} = R_{(\alpha)} \exp\left(\frac{iS_{(\alpha)}}{\hbar}\right),$$

which does not permit us to split the wave equation into real and imaginary parts that are easily interpreted.

We must therefore proceed differently. In order to do this, first recall a classical property of tensor analysis:

Lemma. If one lets $f_{\mu\nu}$ denote the components of a second-order antisymmetric tensor that satisfies the following relations:

$$\partial_\sigma f_{\mu\nu} + \partial_\mu f_{\nu\sigma} + \partial_\nu f_{\sigma\mu} = 0 \tag{1.11}$$

then one may always calculate the components k_μ of a world-vector $\vec{k}$, such that one has:

$$f_{\mu\nu} = \partial_\nu k_\mu - \partial_\mu k_\nu,$$

in which the k_μ are determined up to the gradient of an arbitrary scalar function S (because one sees that if one sets $k'_\mu = k_\mu + \partial_\mu S$ then the rotation does not change).

It results from this that if I am given the components J_μ of a vector then I may always uniquely determine a function S and a vector $\vec{k}$ that does not contain a gradient, so that:

$$\begin{cases} J_\mu = \partial_\mu S + k_\mu & (1.12) \\ \partial_\mu J_\nu - \partial_\nu J_\mu = \partial_\mu k_\nu - \partial_\nu k_\mu = f_{\mu\nu} & (1.13) \end{cases}$$

Indeed, the given of J_μ determines the $f_{\mu\nu}$ that satisfy (1.11). One may therefore calculate the k_μ because of (1.13) and then deduce S from (1.12).

Having said this, we return to particles with spin.

The spinors to which they are associated satisfy the classical relations:

$$\begin{cases} \hbar\alpha^\nu D_\nu\varphi + \mu\varphi = 0 \\ \hbar D_\nu^*\varphi^+\alpha^\nu - \mu\varphi^+ = 0 \end{cases} \tag{1.14}$$

in which the α_ν denote the Dirac – or Petiau-Kemmer – matrices, the D_ν denote the operators $\partial_\nu - i\varepsilon A_\nu$, and μ and ε are the constants m_0c and $e/\hbar c$, which we leave undetermined.

Now introduce the magnitude G that represents the expression $-2i\varphi^+\varphi\,/\,\mu$ in the Dirac representation, for which:

$$\alpha^\sigma\alpha^\nu + \alpha^\nu\alpha^\sigma = 2\delta^{\sigma\nu},$$

or $-i\varphi^+\varphi/\mu$, in the Petiau-Kemmer representation, for which:

$$\alpha^\sigma\alpha^\nu\alpha^\rho + \alpha^\rho\alpha^\nu\alpha^\sigma = \alpha^\sigma\delta^{\nu\rho} + \alpha^\rho\delta^{\nu\sigma},$$

and the quadri-vector:

$$s^\nu = \varphi^+\alpha^\nu\varphi\,.$$

By applying the preceding lemma, I may always define a function S and a vector $\vec{k}$, such that one has:

$$\varphi^+\alpha^\nu\varphi = G(\partial^\nu S + k^\nu) \tag{1.15}$$

because it suffices to replace J^ν with Gs^ν in relations (1.12) and (1.13). Having done this, one introduces a new spinor Φ with the aid of the relations:

$$\begin{cases} \varphi_{(\beta)} = \exp(-iS/\hbar)\Phi_{(\beta)} \\ \varphi^+_{(\beta)} = \exp(-iS/\hbar)\Phi^+_{(\beta)} \end{cases} \tag{1.16}$$

and one transforms the expression $s^\nu = \varphi^+\alpha^\nu\varphi$ with the aid of relations (1.14). If one extracts φ and φ^+ from (1.14) then one first obtains the identities:

$$\varphi^+\alpha^\rho\varphi = \frac{\hbar}{2\mu}(\mathrm{D}_\nu^*\varphi^+\alpha^\nu\alpha^\rho\varphi - \varphi^+\alpha^\rho\alpha^\nu\mathrm{D}_\nu\varphi)$$

that we shall elaborate upon with the aid of the classical commutation relations.

Upon setting $I_{\nu\rho} = \alpha_\nu \alpha_\rho - \alpha_\rho \alpha_\nu$ one obtains:

1. For the Dirac representation ($\alpha^\sigma \alpha^\nu + \alpha^\nu \alpha^\sigma = 2\delta^{\sigma\nu}$):

$$\varphi^+ \alpha_\rho \varphi = \frac{\hbar}{2\mu}\left[2(D^*_\rho \varphi^+ \varphi - \varphi^+ D_\rho \varphi) + \frac{1}{2}\partial_\nu(\varphi^+ I^\nu_\rho \varphi)\right],$$

or again, upon introducing the function S that was previously calculated:

$$\varphi^+ \alpha_\rho \varphi = \frac{2i\varphi^+\varphi}{\mu}\left[\partial_\rho S + \left(\frac{e}{c}A_\rho + \left\{\frac{\hbar}{2i\varphi^+\varphi}(\partial_\rho \Phi^* \Phi - \Phi^+ \partial_\rho \Phi) + \frac{\hbar}{8i\varphi^+\varphi}\partial_\nu \mathrm{I}^\nu_\rho \varphi\right\}\right)\right]. \tag{1.17}$$

2. For the Petiau-Kemmer representation:

$$\alpha^\sigma \alpha^\nu \alpha^\rho + \alpha^\rho \alpha^\nu \alpha^\sigma = \alpha^\sigma \delta^{\nu\rho} + \alpha^\rho \delta^{\nu\sigma},$$

$$\varphi^+ \alpha_\rho \varphi = \frac{\hbar}{2\mu}\left[2(D^*_\rho \varphi^+ \varphi - \varphi^+ D_\rho \varphi) + \partial_\nu(\varphi^+ I^\nu_\rho \varphi) - \frac{i\varepsilon\hbar}{\mu c} F_{\nu\gamma}\varphi^+ \alpha^\nu \alpha_\rho \alpha^\gamma \varphi\right]$$

or again, upon introducing S:

$$\varphi^+ \alpha_\rho \varphi = \frac{2i\varphi^+\varphi}{\mu}\left[\partial_\rho S + \frac{e}{c}A_\rho + \left\{\frac{\hbar}{2i\varphi^+\varphi}(\partial_\rho \Phi^* \Phi - \Phi^+ \partial_\rho \Phi) + \frac{\hbar}{2i\varphi^+\varphi}\partial_\nu(\varphi^+ I^\nu_\rho \varphi) - \frac{\hbar}{2i\varphi^+\varphi} F_{\sigma\gamma}\varphi^+ \alpha^\sigma \alpha_\rho \alpha^\gamma\right\}\right]. \tag{1.18}$$

Formulas (1.17) and (1.18) are interesting for the fact that they permit us to calculate the components k_μ in (1.15); we write these components in the form:

$$k_\mu = \frac{e}{c}A_\mu + P_\mu,$$

in which P_μ denotes the terms in the {} in the right-hand side of the preceding expression.

This result permits us to define the desired model by supposing that the point-like aspect of the micro-object that is associated with a function φ is restricted to follow one of the streamlines L that are tangent to each point of $\vec{s}$.

This hypothesis, which constitutes a generalization of the preceding hypotheses to quantum trajectory and seems necessary in order to account for the experimental results (in particular, the ones that relate to the stationary states of the hydrogen atom), defines, in the same stroke, the quantum forces.

Indeed, set:

$$M_0 = \frac{[(\varphi^+\alpha^\nu\varphi)(\varphi^+\alpha_\nu\varphi)]^{1/2}\,\mu'}{2i\varphi^+\varphi},$$

in which μ' designates the quantity μ in the Dirac representation and 2μ in the Petiau-Kemmer representation.

Upon introducing the unitary world-vector $\vec{u}(u_\nu)$ that is collinear with $\vec{s}(s_\nu)$, one thus obtains the equality:

$$M_0 u_\nu = \partial_\nu S + k_\nu,$$

which permits us to write the Lagrange equations without difficulty.

They become:

$$\begin{aligned}
\frac{d}{d\tau}(cM_0u_\nu) &= cu^\beta\partial_\beta(u_\nu M_0) \\
&= cu^\beta\partial_\beta(\partial_\nu S + k_\nu) \\
&= cu^\beta\partial_\nu(\partial_\beta S + k_\beta) + cu^\beta(\partial_\beta k_\nu - \partial_\nu k_\beta) \\
&= cu^\beta\partial_\nu(M_0u_\beta) + cu^\beta(\partial_\beta k_\nu - \partial_\nu k_\beta) \\
&= c\partial_\nu M_0 + cu^\beta(\partial_\beta k_\nu - \partial_\nu k_\beta),
\end{aligned}$$

from which one finally deduces that the point-like part of the micro-object behaves on L like a classical particle that is subject to a supplementary quantum potential that may be decomposed into:

1. An invariant potential M_0.
2. A potential quadri-vector $\vec{P}(P_\nu)$.

One then painlessly generalizes the preceding two forms of the model that are associated with the individual micro-objects.

A particle with spin may thus be considered to be:

1. A point-like particle that is associated with a wave such that:

   A_1. φ satisfies equations (1.14).
   B_1. This particle follows the streamlines with which it initially coincides with a world-velocity $\vec{u}$.

2. A point-like particle that is accompanied by the extended quantum potentials M_0 and $\vec{P}$ that were previously defined in such a way that:

   A. These fields are calculable by starting with a wave φ that satisfies (1.14).
   B. The particle again follows a streamline L with the world-velocity $\vec{u}$.

Here again, the conclusions of the discussion in section 3 may be applied without modification. In our opinion, the models of micro-objects that are endowed with spin must therefore be developed in the sense of the double solution if one wants to surmount the difficulties that were previously analyzed.

§ 6. – We complete the preceding considerations upon briefly examining several properties of the streamlines that are associated with the functions φ.

In order to accomplish this we generalize the formalism of the canonical equations of motion in the space of special relativity.

Suppose that we write them in the invariant form:

$$\begin{cases} \dfrac{dx^\nu}{d\tau} = \dfrac{\partial W}{\partial p_\nu} \\ \dfrac{dp_\nu}{d\tau} = -\dfrac{\partial W}{\partial x^\nu} \end{cases} \tag{1.19}$$

in which $d\tau$ denotes proper time, W is a scalar function that does not explicitly depend on the interval, and the dx^ν are the components dx, dy, dz, $icdt$ (we have introduced i in order to simplify the $g_{\mu\nu}$ that pointlessly complicate the calculations), which satisfy the classical relation:

$$c^2 d\tau^2 = -\sum_\nu (dx^\nu)^2 \ .$$

One then sees that W is a constant of the movement. Indeed, if W depends on the x^ν *and the* p_ν *, and not on explicitly on proper time* $d\tau$ *then we obtain:*

$$\frac{dW}{d\tau} = \frac{\partial W}{\partial x^\nu}\frac{dx^\nu}{d\tau} + \frac{\partial W}{\partial p_\nu}\frac{dp_\nu}{d\tau} = \frac{\partial W}{\partial x^\nu}\frac{\partial W}{\partial p_\nu} - \frac{\partial W}{\partial p_\nu}\frac{\partial W}{\partial x^\nu} = 0$$

on account of the preceding relations. From this, one deduces that the function $W(x^\nu, p_\mu)$ is a constant of the motion with respect to any system; hence, it is an absolute constant since it acts as a scalar function.

Formula (1.19) permits us to simplify the study of the motions.

For example, in the classical case of a charged particle in the electromagnetic field, upon setting:

$$W = -m_0 c^2 = -c\sqrt{-\sum\left(p_\mu - \frac{e}{c}A_\mu\right)^2} \ ,$$

one obtains:

$$\frac{dx^\mu}{d\tau} = \frac{1}{m_0}\left(p_\mu - \frac{e}{c}A_\mu\right)$$

namely:

$$p_\mu = m_0 \frac{dx^\mu}{d\tau} + \frac{e}{c} A_\mu$$

as well as the usual equations of motion:

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}} \frac{dx_i}{dt} \right) = e(\vec{E} + \frac{1}{c}\vec{v}\vec{H})_i \,, \qquad i = 1, 2, 3$$

$$\frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}} \right) = e(\vec{E} \cdot \vec{V})$$

with:

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}} \frac{dx_\mu}{dt} \right) = \frac{e}{c} F_{\mu\nu} \frac{dx_\mu}{dt} \,.$$

One may use the operator notation in the case of particles with spin, and notably in the case of the Dirac equation.

If one takes into account the classical relations of non-commutative algebra:

$$\begin{cases} \dfrac{i}{\hbar}(x^\mu f - f x^\mu) = -\dfrac{\partial f}{\partial p_\mu} \\ \dfrac{i}{\hbar}(p_\mu f - f p_\mu) = \dfrac{\partial f}{\partial x^\mu} \end{cases}$$

which are valid for the symbol f when f is a function of the x^μ and p_μ, then one obtains the following expressions for the equations of motion:

$$\frac{dx^\mu}{d\tau} = \frac{i}{\hbar}(W x^\mu - x^\mu W) = [W, x^\mu]$$

$$\frac{dp_\mu}{d\tau} = \frac{i}{\hbar}(W p_\mu - p_\mu W) = [W, p_\mu]$$

and, by analogy with the classical case, one further sets:

$$W = -ic \sum_\mu \alpha^\nu \left(p_\mu - \frac{e}{c} A_\mu \right),$$

which is the proper mass operator that corresponds to the wave equation:

$$\alpha^\nu \left(\frac{\hbar}{i} \partial_\nu - eA_\nu \right) \varphi + i m_0 c \varphi = 0 \,.$$

This gives the following equality for the velocity operator:

$$\frac{dx_\nu}{d\tau} = [W, x_\nu] = ic\alpha_\nu ,$$

and for the velocity itself, we have the expression:

$$-ic\varphi^+\alpha_\nu\varphi = <\varphi, ic\alpha_\nu\varphi> = \left(\overline{\frac{dx_\nu}{d\tau}}\right),$$

which furnishes the following values for the tri-dimensional components:

$$v_i = \frac{dx_i}{dt} = \frac{\dfrac{dx^i}{d\tau}}{\dfrac{dt}{d\tau}} = -c\frac{\varphi^+\alpha_i\varphi}{\varphi^+\varphi}.$$

The acceleration operator will then be:

$$\frac{d^2x_\nu}{d\tau^2} = -ic\frac{d\alpha_\nu}{d\tau}.$$

On account of the relations:

$$i\hbar\frac{d\alpha_\nu}{d\tau} = \alpha_\nu W - W\alpha_\nu$$

and

$$\alpha_\nu W - W\alpha_\nu = -2icp_\nu ,$$

one may then write:

$$i\hbar\frac{d\alpha_\nu}{d\tau} = 2\alpha_\nu W + 2icp_\nu ,$$

namely:

$$i\hbar\frac{d^2\alpha_\nu}{d\tau^2} = 2\frac{d\alpha_\nu}{d\tau}W ,$$

from which one derives:

$$\frac{d\alpha_\nu}{d\tau} = \left(\frac{d\alpha_\nu}{d\tau}\right)_0 e^{\frac{-4\pi i}{\hbar}W\tau}.$$

Therefore, one finally has:

$$\alpha_\nu = -icp_\nu W^{-1} + \frac{i\hbar}{2}\left(\frac{d\alpha_\nu}{d\tau}\right)_0 e^{-\frac{2i}{\hbar}W\tau}\cdot W^{-1}$$

and

$$\frac{dx_\nu}{d\tau} = -ic\alpha_\nu = -c^2p_\nu W^{-1} + \frac{\hbar c}{2}\left(\frac{d\alpha_\nu}{d\tau}\right)_0 e^{-\frac{2i}{\hbar}W\tau}\cdot W^{-1},$$

whose first term is the ordinary velocity, since:

$$-c^2 p_\nu W^{-1} = -c^2 p_\nu \frac{1}{-m_0 c^2} = p_\nu / m_0 ,$$

and the second term represents an oscillating movement whose frequency is given by:

$$\nu = \left(\frac{2W}{\hbar} \sqrt{1-\beta^2} \right) = \frac{2m_0 c^2}{\hbar} \sqrt{1-\beta^2} \, ,$$

with:

$$2m_0 c^2 = \hbar \nu / \sqrt{1-\beta^2} \, .$$

One then sees that the trajectories (L) of the Klein-Gordon equation that correspond to the relations of L. de Broglie:

$$\frac{dx_j}{\partial_i S - \frac{e}{c} A_i} = \frac{dt}{\partial_t S - \frac{e}{c} A_4} \qquad i = 1, 2, 3$$

give us the mean trajectories of the trajectories of particles with spin; the latter orbit around the former in a helicoidal motion that is defined by the preceding relations.

This precession effect, which furnishes an intuitive representation of the Schrödinger "zitterbewegung," corresponds to the spin effects that are associated with the relativistic trajectories of the causal interpretation.

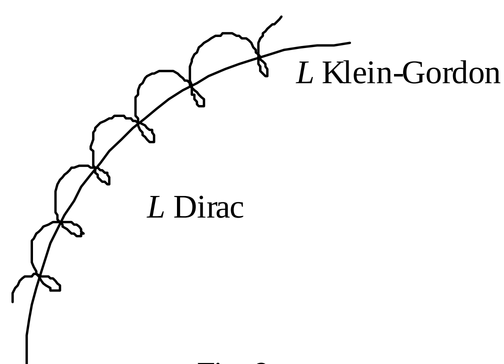

Fig. 9

§ 7. – We end this first part with several considerations that relate to what we may call the hydrodynamical representation of quantum fields.

Numerous authors have studied this representation in the context of the probabilistic interpretation.

Indeed, one knows that one may make a fictitious conservative fluid correspond to any wave function φ that satisfies linear equations, and that this fluid represents the evolution of the probability of existence $|\varphi|^2$ for the corpuscle in space. This type of "probability fluid" permits us to intuitively represent the behavior of the cloud of possible points that represent the corpuscle in the interpretation of Bohr. One may study them directly by starting with the wave equations of Schrödinger, Klein-Gordon, or Dirac.

The results that one obtains may be transposed to the causal theory (with a different interpretation) because one sees that the preceding fluid permits us to describe both the behavior of the quantum field that is associated with the wave φ and its effects on the corpuscular aspect of micro-objects. For example, its streamlines (L) constitute possible trajectories for that point-like aspect, and the forces that act on the streamlines correspond to the quantum field in the second form of the causal interpretation.

As Takabayasi ([10]) has pointed out, this representation presents the additional interest of suggesting possible generalizations of the field equations.

A. In what follows, in order to simplify matters, we have systematically reprised the notations that were used in the first part of the chapter.

The hydrodynamical representation of the Schrödinger equation has been studied since the beginning of wave mechanics by Madelung and L. de Broglie.

The Schrödinger fluid is reducible to an irrotational fluid (in the absence of exterior potentials) that is endowed with a velocity potential S / m, a density R^2, and a stress potential $-\frac{\hbar^2}{2m}\frac{\Delta R}{R}$ that is equivalent to a stress tensor.

This tensor is furnished by the relation:

$$\sigma_{ik} = \frac{\hbar^2}{2m}\left(R\frac{\partial^2 R}{\partial x^i \partial x^k} - \frac{\partial R}{\partial x^i}\frac{\partial R}{\partial x^k} \right) = \frac{\hbar^2}{2m} P \frac{\partial^2 (\log R)}{\partial x^i \partial x^k}$$

because the equation of motion in the presence of an exterior potential $V \left(K_i = -\frac{\partial V}{\partial x^i} \right)$, may be written:

$$m\frac{\partial}{\partial t}(P \cdot V_i) + m\frac{\partial}{\partial x^k}(P \cdot V_i V_k) = P \cdot K_i + \frac{\partial}{\partial x^k}\sigma_{ik} ,$$

with:

$$\frac{\partial}{\partial x^k}\sigma_{ik} = P\frac{\partial}{\partial x^i}\left(\frac{\hbar^2}{2m}\frac{\Delta R}{R} \right).$$

The continuity equation (C) is always satisfied, and in the absence of external fields the total energy-momentum tensor of the fluid may be written:

$$T_{ki} = mPV_k V_i + \frac{\hbar^2}{4m}(P^{-1}\partial_k P \cdot \partial_i P - \delta_{ki}\Delta P) ,$$

in which mPV_kV_i represents the components of the energy-momentum tensor for the motion of the fluid molecules and $\frac{\hbar^2}{4m}(P^{-1}\partial_k P \cdot \partial_i P - \delta_{ki}\Delta P)$ represents the components of an internal tension tensor.

The reader is referred to the original literature for more details.

([10]) *Progress of Th. Physics,* 8, pp. 143 (1952).

B. The generalization of the preceding considerations to the case of the Klein-Gordon equation was carried out in 1927 by L. de Broglie ([11]), and ultimately was independently recovered by Takabayasi ([12]) and the author ([13]).

As before, one is concerned with a quasi-irrotational fluid moving under the influence of a quantum stress potential that is equivalent to the stress tensor $\sigma^{\mu\nu}$ that is furnished by the relations:

$$\sigma^{\mu\nu} = \left(\frac{\hbar^2}{2m}\right)(R\partial^\mu\partial^\nu R - \partial^\mu R\partial^\nu R) = \left(\frac{\hbar^2}{4m}\right)P\partial^\mu\partial^\nu(\log P)\,.$$

Once more, the fictitious molecules of the fluid obviously follow the trajectories (L).

If one neglects the effects of the gravitational field then, upon denoting the Galilean values of $g_{\mu\nu}$ by $\varepsilon_{\mu\nu}$, one also obtains the equations of motion in the form given by L. de Broglie ([14]):

$$\frac{d}{d\tau}(M_0 c u_\mu) = c u^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) + c\partial_\mu M_0\,,$$

which is interpreted by saying that the corpuscles are endowed with a variable mass M_0, and they displace under the influence of a supplementary quantum potential M_0 .

These equations are immediately deduced from the expression $\partial_\mu T_\nu^\mu = 0$, in which the T_ν^μ represent the components of a total energy-momentum tensor of the fluid, which one may write in the form:

$$T_\nu^\mu = T_{(H)\nu}^\mu + T_{(\mathrm{I})\nu}^\mu + \tau_\nu^\mu\,,$$

in which:

$$T_{(H)\nu}^\mu = mP_0\gamma u^\mu u_\nu$$

represents the energy-momentum tensor of the molecules:

$$T_{(I)\nu}^\mu = \varepsilon^{\mu\sigma}[2\partial_\sigma R\partial_\nu R - \varepsilon_{\mu\nu}(\partial_\lambda R\partial^\lambda R - R\Delta R)]$$

represents the internal stresses of the fluid, and:

$$\tau_\nu^\mu = F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\varepsilon_\nu^\mu F_{\lambda\sigma}F^{\lambda\sigma}$$

represents the Maxwell electromagnetic tensor.

We shall now make an important remark:

([11]) L. de Broglie, *C. R.* 185 (1927) pp. 1118.
([12]) *Progress of Th. Physics*, loc. cit.
([13]) In an unpublished work.
([14]) *C. R.*, 185-380 (1927).

The foregoing theory of the charged Klein-Gordon fluid may be obtained by introducing just two variables: an electromagnetic potential A_μ and a scalar R, provided that one imposes a gauge condition on this potential that is distinct from the classical relation $\partial_\mu A^\mu = 0$.

Indeed, set, by definition:

$$\partial_\mu A_\nu - \partial_\nu A_\mu = -\frac{e}{mc} F_{\mu\nu} \tag{1.20}$$

with the gauge condition:

$$A_\mu A^\mu = -c^2 + \frac{\hbar^2}{m^2}\frac{\Delta R}{R} \tag{1.21}$$

and the classical field equations:

$$\partial_\nu F^{\mu\nu} = \frac{e}{c} P A^\mu . \tag{1.22}$$

One derives the field equations (J) and (C) without difficulty; case (J) results from (1.21) upon setting:

$$A_\mu = \frac{1}{m}\left(\partial_\mu S - \frac{e}{c} A_\mu\right),$$

and (C) is deduced from (1.22).

This signifies that the classical approximation of the Klein-Gordon theory agrees with the new classical theory of the electrons that was proposed by Dirac ([15]), because if one makes $\hbar$ go to zero in equation (1.21) then one is left with the Dirac relations:

$$A_\mu A^\mu = -c^2$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = -\frac{e}{mc} F_{\mu\nu}$$

$$\partial_\nu F^{\mu\nu} = \frac{e}{c} P A^\mu .$$

We return to this property later on.

Takabayasi ([16]) sought to introduce rotational movements into the preceding fluid by using the Clebsch parameters ξ and η from classical hydrodynamics.

One thus obtains a very interesting generalization of the attempts made by Dirac to introduce effects that are analogous to the preceding classical theory.

For example, if one sets:

$$\vec{V} = \nabla S + \xi \nabla \eta$$

then one obtains a generalization of the Schrödinger and Klein-Gordon fluids that approaches the hydrodynamical representation of the Pauli fluid (which corresponds to

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([15]) *Proc. Roy. Soc.*, A. 209 (1951), pp. 291.
([16]) *Progress of Th. Physics,* loc. cit.

the two-component equations for φ) that were recently established by Tiomno, D. Bohm, and Schiller ([17]). We shall not deal with questions of this sort in our discussion.

C. The study of the hydrodynamical representation of the Dirac equation is obviously more difficult because the corresponding fluids are endowed with spin. The study of such fluids poses a large number of problems because one must introduce a supplementary quadri-vector $\vec{\sigma}$ that represents the proper kinetic moment. This was notably undertaken by Lyon ([18]) and, more recently, by L. de Broglie ([19]) on the basis of a classical theory of particles with spin that was developed by Weyssenhoff ([20]).

In order to define a fluid endowed with spin one introduces, along with the components u_μ of the world-velocity, an energy-momentum quadri-vector g^β that is not collinear with this velocity.

If one then defines two types of derivation with respect to time, namely:

1. The classical Lagrangian derivative that follows the particle:

$$d_{t_0} f = \dot{f} = u^\nu \partial_\nu f$$

2. The derivation for densities:

$$D_{t_0} f = d_{t_0} f + f \partial_\nu u^\nu = \partial_\nu (f u^\nu),$$

then one may define an energy-momentum tensor $T^{\alpha\beta}$ with the aid of the relations:

$$T^{\alpha\beta} = g^\alpha u^\beta,$$

which is represented (in the absence of stresses) by the asymmetric matrix:

$$\begin{vmatrix} g^j u^i & c\vec{g} \\ \dfrac{W}{c}\vec{u} & W \end{vmatrix}.$$

The equations of motion then become:

$$\partial_\beta T^{\alpha\beta} = D_{t_0} g^\alpha = 0,$$

because of the preceding definitions, relations to which one may add the supplementary condition:

$$D_{t_0} s^{\alpha\beta} = g^\alpha u^\beta - g^\beta u^\alpha = T^{\alpha\beta} - T^{\beta\alpha},$$

([17]) *Nuovo Cimento.* Supplement, no. 1, 1955.
([18]) J. YVON, *Journal de Physique et le Radium*, 1940.
([19]) L. de BROGLIE, *La théorie des particules de spin* $\frac{1}{2}$.
([20]) J. WEYSSENHOFF, *Acta Physica Polonica*, 1947.

which expresses the conservation of total moment of momentum (orbital + spin) in the proper system. This introduces a spin tensor $s^{\alpha\beta}$ that is defined by the relation:

$$\mu_0 = -\frac{1}{c^2} u_\beta g^\beta ,$$

which gives, on account of the identity $u^\beta u^\beta = -c^2$:

$$g^\alpha = \mu_0 u^\alpha - \frac{1}{c^2} u_\beta D_{t_0} s^{\alpha\beta} = \mu_0 u^\alpha + \frac{1}{c^2} s^{\alpha\beta} u_\beta$$

and also:

$$s^{\alpha\beta} u_\beta = 0.$$

If one then introduces the following integral quantities by summing in the proper system:

$$G^\alpha = \int g^\alpha d\omega_0$$
$$S^{\alpha\beta} = \int s^{\alpha\beta} d\omega_0$$
$$m_0 = \int \mu_0 d\omega_0$$

then one extracts the following relations from the preceding equations:

$$\dot{m} = 0$$

and

$$S^{\alpha\beta} \cdot \dot{S}_{\alpha\beta} = 0 ,$$

which express that the total mass and total proper momentum are constants.

Weyssenhoff has integrated the preceding equations in different cases, and has established that in their proper system particles with spin are animated with circular motions that are perpendicular to the spin vector, which furnishes a very interesting image of the their behavior.

The preceding theory has been applied to the case of the Dirac equation by L. de Broglie in the case of plane waves by reducing the tensor $T^{\alpha\beta}$ of Weyssenhoff to the canonical energy-momentum tensor of Dirac:

$$T^{\mu\nu} = \frac{hc}{2i} (\psi^+ \alpha^\mu \partial^\nu \psi - \partial^\nu \psi^+ \alpha^\mu \psi) .$$

Indeed, (on account of the fact that the u_μ must be collinear with the components S_μ of the current) these relations define components g^α and $s^{\alpha\beta}$, and permit us to apply the Weyssenhoff theory to the fluid thus specialized.

One may generalize this theory without much difficulty to the case in which the fluid is placed in an external electromagnetic field, but we shall not develop this aspect of the theory here, and we shall return to it in a later work.

CHAPTER II

§ 1. – Following our program, we shall now begin to examine some more complicated models of micro-objects that are related to what one calls the "theory of the double solution.

This theory, which was introduced in 1927 by L. de Broglie ([1]), is quite interesting in the way that it introduces concepts into wave mechanics that were proposed for the first time by Einstein and Darmois in order to surmount the classical difficulties that relate to the nature of the laws of motion.

We shall develop this point. As we saw in the general introduction, the classical model does not permit us to understand the character of these laws in a satisfactory fashion, and presents genuine difficulties as well. Indeed, if one reduces the particles to singular points in the fields then one confirms that:

1. They present infinite proper energies whether gravitational or electromagnetic.

2. The presence of such points signifies that the field equations are not valid everywhere, and may not account for the global evolution of the fields that are generated by bodies in motion. This signifies that one must introduce equations of motion for the singularities in order to obtain a complete description for the behavior of the fields.

There is more. The use of these equations obviously constitutes only a phenomenological description of the interaction between the fields and the particles. In particular, the experimenters – who are scarcely inclined to agree on a mathematical formalism – raise objections that Faraday expressed in these terms:

"I feel it very difficult to conceive of the atoms of matter that are assumed to be in solids, fluids, and vapors, which are more or less separated from each other and swimming in a space that is not occupied by atoms; I also perceive great contradictions ensuing from such a viewpoint. I can hardly imagine the difference between a small rigid particle and the forces that surround it. The matter of an atom touches that of its neighborhood. Matter is continuous from one to the other. Matter fills all of space, or at least, everywhere gravitation extends."

This criticism introduced a very important idea into the history of physics, which may be formulated as: There is no difference in nature between fields and matter.

Or furthermore: material particles are nothing but very small regions in which fields take on values that are large with respect to their normal values.

Such a conception constitutes an important simplification of the classical model since it reduces material substance to the notion of a field by reducing particles to localized condensations of fields (called "bunched solutions" by Einstein) that must obviously behave like the point-like aspect of micro-objects. Their importance has been emphasized by Einstein, who sought to surmount the classical difficulties relating to the

[1] Cf. *La Physique Quantique restera-t-elle indéterministe?*

nature of the laws of motion while developing them in the framework of general relativity. We shall reproduce his analysis because it does not depend on the nature of the field considered.

Einstein first remarked that if one reduces particles to singular regions of fields then only one system of laws – the field laws – suffices, in principle, to describe the behavior of physical reality since there no longer exists the means to separate fields and particles.

In the second place, one confirms – and this is a fundamental property – that it is impossible to develop such a theory in the framework of the linear equations that habitually used. This is proved without difficulty because in the linear theory the sum of two solutions of the field equations is again a solution. I may therefore arbitrarily superpose an arbitrary continuous field and a given particular solution that represents a particle without necessarily demanding a relationship between the solutions; this amounts to saying that one may arbitrarily arrange the trajectories of a particle in an exterior field.

For example, if one consider Maxwell's equations, which are deduced from the linear Lagrangian $\mathcal{L} = -1/4F^{\mu\nu}F_{\mu\nu}$, then one sees that one may always add a solution that is symmetric in $1/r$ (which represents an electron) and is aligned along an arbitrary trajectory for a given potential.

This analysis leads to the following conclusion: one must resort to nonlinear field equations if one wants to deduce the behavior of the particular solutions that are associated with the particles from the general properties of the field.

Indeed, in the nonlinear theory, it is possible to add two particular solutions to obtain a third one only on the condition that they satisfy supplementary relations. In general, a given solution of particle type may be superposed with an exterior field only if its trajectory satisfies certain conditions that are comparable to laws of motion.

As Einstein said: "In a theory of this type, the task of the physicist consists of discovering particular solutions to the field equations (which are associated with particles) such that their agreement with the exterior fields leads naturally to the physical laws of motion."

In developing this idea, Einstein and Darmois have shown that if one starts with the theory of general relativity, in which the gravitational field is represented by fundamental metric tensor $g_{\mu\nu}$, which satisfies nonlinear relations $R_{\mu\nu} = 0$, then one may add a Schwarzschild solution that is singular in $1/r$ (which describes a particle) to a given continuous exterior field only if the center of that singularity describes a geodesic of that exterior field.

Therefore, in relativity the preceding supplementary conditions, which express, in summary, the compatibility between the singular field of the particles and the exterior field, are precisely equivalent to the laws of motion. This is a remarkable qualitative result that constitutes – in our opinion – one of the more important contributions of the theory of relativity to the history of ideas in physics. Indeed, it suppresses the classical duality between the laws of the field and the laws of movement that seemed irreducible since the second automatically results from the first, provided that one adopts a convenient definition for the particles.

This discussion applies point-by-point to the causal theory of micro-objects.

Indeed, the individual models that one studies are distinguished in the classical manner:

– quantum fields that satisfy the field laws A_1 and A_2.
– Point-like particles that obey the laws of motion B_1 and B_2.

One may therefore seek to generalize:

1. by associating the point-like aspect to the particular singular regions of the quantum fields,
2. by introducing nonlinear wave equations such that the trajectories (L) of these regions, which are defined by B_1 and B_2, automatically result from the field equations.

This generalization constitutes precisely what one may call the theory of the double solution because it distinguishes:

1. waves u that occupy a singular region, which permit a physical characterization of the individual micro-objects and satisfy nonlinear wave equations.
2. continuous waves ψ that obey classical linear equations, which describe the statistical behavior of micro-objects placed in the particular conditions.

Before we develop these notions, we must emphasize three important points:

a) As we have seen, it is impossible to interpret nonlinear wave equations in the framework of the ideas of Bohr. This results from the fact that their solutions cannot be superposed according to the laws of the composition of probabilities. Therefore, they may not represent statistical phenomena in the usual sense of the word.

b) The causal theory that we studied in the first part quantizes the wave equations with the aid of trajectories (L) that account for the behavior of the corpuscular aspect of micro-objects. This process of quantization is interesting because it may be extended to nonlinear equations for which the usual processes of the probabilistic interpretation are inapplicable, in general. For this, it suffices to associate the nonlinear regions of the field with point-like aspects of the micro-objects by choosing them in such a way that they follow families (L) of the particular trajectory.

c) In the theory of the double solution this quantization is a consequence of the internal structure of the micro-objects since the motions (L) result from the form of the particular solutions that represent the particles considered.

This suggests a physical idea:

The phenomena of quantization are related to the internal structure of the individual micro-objects, which is subordinate to the experimentally observed statistical phenomena. One therefore generalizes the conceptions of Einstein that deduce the relativistic extension of classical mechanics from the structure of the singularities that are associated with the particles.

From the viewpoint that has been suggested by Fock, quantization seems to be a property that is associated with the individual micro-objects (and not to statistical ensembles) that have been prepared under particular physical conditions, since the statistical mechanics of the quanta that are associated to the objects considered is derived in our model from the individual behavior of the isolated micro-objects.

§ 2. – The examination of the theory of the double solution may be undertaken in various ways. We shall choose an approach that permits us generalize the results that were obtained in the course of chapter I. To simplify the discussion, we begin with the scalar case.

1. In the theory of the double solution each micro-object is represented by a wave $u = f\exp(i\omega/\hbar)$ that satisfies a nonlinear equation.

This wave generalizes the function $\varphi = R\exp(iS/\hbar)$ that was previously introduced in the sense that it involves a singular region that represents the corpuscular aspect of the micro-object.

2. As before, we thus have two possible interpretations:

I. – The micro-object is represented by a particular solution u such that:

A_1. u satisfies the nonlinear wave equation.

B_1. The center of the singular region automatically follows one of the previously defined trajectories (L) due to the nature of the chosen solution.

II. – The micro-object is represented by the quantum field Q that is calculated from f (instead of R), which presents a singular region (the singularity of the potential corresponds to the singularity of u such that:

A_2. Q may be calculated from the preceding particular solution u, which satisfies the nonlinear equation.

B_2. The singular region also follows a trajectory (L).

The interpretation that we choose to represent the micro-object will obviously depend on the exact form of the nonlinear wave equation.

Here, we are presented with a first difficulty: One sees that in any sort of linear theory there is an infinitude of possible wave equations, and it is hard to choose between them.

In the absence of physical criteria, one reduced to postulating one directly or deducing it from more general considerations, which we now review.

For example, one may, with Rosen ([2]) and Finkelstein ([3]), start with the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu u)(D^{*\mu} u^*) + \sigma^2 u u^*,$$

in which $D_\mu = \partial_\mu - i\varepsilon A_\mu$ represents an operator that contains the total electromagnetic potential (and not just the exterior potential), and look for spherically symmetric solutions of the field *u* and the potential *A* that behave like particles.

More generally, one may start with a nonlinear electromagnetic Lagrangian L_E (of Born-Infeld type), and add a generalized scalar Lagrangian L_M:

$$L_M = L_M\,(\text{Klein - Gordon}) + \frac{\gamma}{2}(uu^*)^2 + \cdots$$

in which L_M (Klein-Gordon) denotes the expression $D_\mu\, u\, D^{*\mu}\, u^* + \sigma^2 uu^*$ (ε, σ, γ, etc., represent arbitrary constants), and seek to solve the same problem.

The study of these equations (as well as the equations that generalize the Dirac equation) is quite interesting from the viewpoint of the causal interpretation. A number of the results that were obtained without interpretation by the aforementioned authors are indeed susceptible to being transposed into the causal theory; we shall discuss them qualitatively later on. However, we must emphasize the difficulty in the mathematical problems that are raised by such equations, whose analysis is still in its infancy. This is why de Broglie ([4]) and myself have taken a different approach from the beginning, namely, the approach of seeking to deal with the general character independently of the exact form of the equations used.

In the absence of selection criteria regarding the choice of wave equation, one may indeed ponder the difficulty just pointed out by posing the following question:

What conditions must the solutions of nonlinear equations satisfy in order for the singular region to behave like the point-like particles that were introduced in the causal interpretation that was defined in the first part of this chapter?

The response to that question obviously rests on a certain number of general hypotheses that relate to the properties of equations and the nature of the solutions used – hypotheses that one may not directly justify *a priori*.

We postulate them provisionally in the form of conditions that the completely nonlinear theory must satisfy. We therefore suppose that:

1. The function *f* is governed everywhere by a nonlinear equation that reduces to the usual linear wave equation when *u* is small.
2. The singularity of *u* is contained in a small region of radius r_0 (that has dimensions of order the classical radius of particles, namely 10^{-13} cm.), so that the nonlinear terms play a role in the *interior* of the region and *u* satisfies the preceding linear equation in its *exterior* (see figure).

([2]) Rosen, *Phys. Review*, 85 (1952).
([3]) Finkelstein, *Phys. Review*, 83-326 (1951).
([4]) *La physique quantique restera-t-elle indéterministe?*

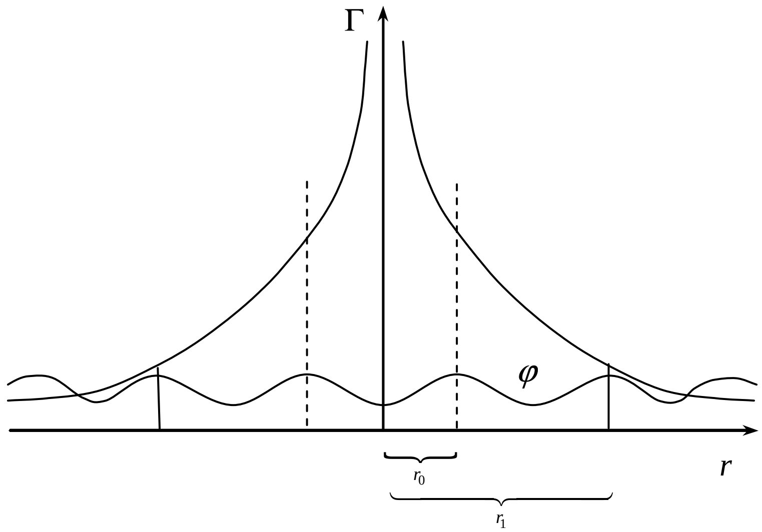


Fig. 10.

3. This function u may be written in the form:

$$u = f\,\exp\left(i\frac{\omega}{\hbar}\right) \tag{2.1}$$
$$= u_0 + \varphi,$$

in which φ denotes the previously-introduced physical wave and u_0 is a wave function such that:

a) $u_0 + \varphi$ satisfies the nonlinear equation everywhere.
b) u_0 satisfies the linear equation in the exterior of the surface S' of radius r_0.
c) One has $u \approx \varphi$, so $\varphi \gg u_0$ at a distance from the center of that singular region (for $r > r_1$).
d) $u \approx u_0$ in the proximity of r_0, so $u_0 \gg \varphi$.

One immediately sees that these hypotheses permit us to describe a wave with singularity that behaves like the desired model provided that one imposes certain matching conditions – called guidance conditions – on u_0 and φ in a neighborhood of r_0; these conditions must be ultimately deduced from the nonlinear theory one uses.

Let us look at these conditions.

To establish them, we shall generalize what L. de Broglie has called the "guidance theorem" with the aid of the following lemma.

Lemma. Consider a conservative world-fluid of scalar density ρ and world-velocity v_μ, i.e., such that one has:

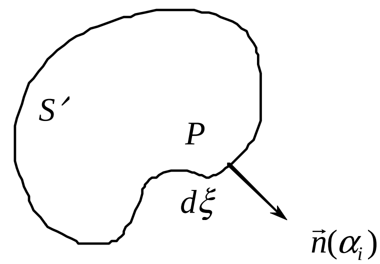
$$s^\mu = \rho v^\mu$$

at each point, with:

$$\partial_\mu s^\mu = 0 ,$$

in which s^μ denotes the world-current, and study the motion of a bump that displaces in this fluid without changing form.

By "bump," we mean a zone such that:



ρ = const.

Fig. 11.

1. It is very small and enclosed by a surface S', in which one has ρ = const., and displaces as a unit in the course of its motion.

2. ρ takes on values there that are very much larger than its external values, so that one has:

$$\frac{\rho}{\partial\rho/\partial x_\mu}\varphi \approx 0,$$

in a neighborhood of S', which amounts to saying that the boundaries of this bump are of pole type.

3. At a small distance from S' one has $\rho = \rho_r$, where ρ_r and v_r denote that values that ρ and v take on when there is no bump; these values correspond to the regular part of the fluid being considered.

Having said this, we let α^i (i = 1, 2, 3) denote the direction cosines of the unit vector $\vec{n}$ that is a normal at an arbitrary point P of S', and let $d\xi$ denote the normal displacement of a point of the bump in the course of a very small time interval, where the τ symbol refers to an ordinary space vector.

Since ρ = const. on S', we get:

$$d\xi \frac{\partial\rho}{\partial x^i}\alpha^i + \frac{\partial\rho}{\partial x^i} dt = 0 , \qquad i = 1, 2, 3,$$

which gives a normal velocity of displacement equal to:

$$v_n = \frac{d\xi}{dt} = \frac{-\partial\rho/\partial t}{\sqrt{\sum_i \left(\frac{\partial\rho}{\partial x^i}\right)^2}} .$$

We then write the equation of continuity on the boundary of S' in the following form:

$$\frac{\partial\rho}{\partial t}+\frac{v_i}{v_4}\frac{\partial\rho}{\partial x_i}+\rho\frac{\partial}{\partial x_i}\left(\frac{v_i}{v_4}\right)=0\,, \qquad \text{(with } v_\mu v^\mu = 1\text{)},$$

and divide by:

$$\sqrt{\sum_i\left(\frac{\partial\rho}{\partial x^i}\right)^2}\;.$$

If we take into account the hypothesis that was made in 2) then the second term disappears, and one obtains the relation:

$$\frac{v^\mu}{\sqrt{\sum_i\left(\frac{\partial\rho}{\partial x^i}\right)^2}}\cdot\frac{\partial\rho}{\partial x^\mu}=0\,,$$

which may also be written:

$$v^i\alpha_i - v_n\,v^4 = 0.$$

If one then remarks that one has $v_n = \vec{w}\cdot\vec{n}$ at every point of S, in which $\vec{w}$ denotes the velocity of the singularity, then one finally obtains, by substituting in the preceding equality:

$$w^i = \frac{v^i}{v^4}\,,$$

which defines the velocity of the bump in the fluid.

One then sees that the bump behaves like a particle that is restricted to follow one of the streamlines of the regular fluid that corresponds to ρ_r and v_r if one has the fundamental equality on S':

$$v^\mu = v_r^\mu\,, \tag{2.2}$$

which generalizes the guidance theorem of L. de Broglie.

If the dimensions of the bump are weak with respect to the variations of v_r then it obviously suffices that the equalities (2.2) are satisfied in the center of the singular region.

The application of this lemma to the case of the Klein-Gordon equation and to equations with spin immediately defines the guidance conditions.

A. If we first start with the scalar equation:

$$(D_\nu D^\nu - \mu)\,u = 0,$$

which is valid outside the singular region, then, by our hypothesis, one may write:

$$u = f\,\exp\left(\frac{i\omega}{\hbar}\right) = u_0 + \varphi$$

$$= u_0 + R\exp\left(\frac{iS}{\hbar}\right),$$

in which $R\exp(iS/\hbar)$ also satisfies the wave equation with the external potential.

In the hydrodynamical representation, the function will define a current with a bump, and the function φ will define what we called the regular current in the lemma.

One will therefore have:

$$s^\mu = f^2\left(\partial^\mu\omega - \frac{\varepsilon}{c}A^\mu\right)$$
$$s_r^\mu = R^2\left(\partial^\mu S - \frac{\varepsilon}{c}A^\mu\right),$$

from which one deduces that the guidance condition may be written:

$$\partial^\mu\omega - \frac{\varepsilon}{c}A^\mu = \partial^\mu S - \frac{\varepsilon}{c}A^\mu \tag{2.3}$$

so that:

$$\omega = S + \text{const.}, \tag{2.4}$$

which signifies that the singular wave must have the same phase on S' as the regular part, up to a constant.

One therefore recovers precisely the condition that one calls "phase matching," which was introduced by L. de Broglie in the causal interpretation.

As he himself has emphasized: This condition generalizes the fundamental idea that guided the earliest research in wave mechanics because it amounts to considering a corpuscle as a small clock that must remain in phase with the wave that accompanies it.

B. If one then starts with the equation for particles with spin:

$$(\alpha^\nu D_\nu - \mu)\, u = 0,$$

and one further sets $u = u_0 + \varphi$, in which the functions u, u_0, and φ represent spinors, then one further obtains the following relations with the aid of the hydrodynamical interpretation:

$$\begin{cases} s^\nu = u^+\alpha_\nu u = u^+u\,\dfrac{u^+\alpha_\nu u}{u^+u} \\ \qquad = (u^+u)v^\nu \end{cases}$$

and

$$\begin{cases} s_r^\nu = \varphi^+\varphi\,\dfrac{\varphi^+\alpha_\nu\varphi}{\varphi^+\varphi} \\ \qquad = (\varphi^+\varphi)v_r^\nu \end{cases}$$

which permits us to write the guidance conditions as:

$$\frac{u^+\alpha^\nu u}{u^+u} = \frac{\varphi^+\alpha^\nu\varphi}{\varphi^+\varphi} \ . \tag{2.5}$$

With the aid of relations (1.15) one painlessly verifies that these relations generalize formula (2.3), which corresponds to the scalar case.

Equations (2.3) and (2.5) constitute the sufficient conditions imposed on the solutions u of the nonlinear equations in order for them to represent micro-objects.

Later on, we shall go further and show that they are also necessary.

In a work that will be ultimately published, Fer has succeeded in showing that a singularity of the pole type that is a solution to the Klein-Gordon equation necessarily follows a trajectory that is defined by $\partial^\nu\omega + \varepsilon/c\, A^\nu$. One concludes from this that condition (1.3) is indispensable in order to recover the motion (L) that was previously associated with the causal interpretation.

The calculations that we must do may be performed as follows:

If one represents micro-objects by particular solutions u of nonlinear equations that satisfy the guidance conditions then the laws of motion result naturally from the field equations.

As D. Bohm remarked, in principle, this permits us to suppress the initial conditions on the velocities that were criticized by Takabayasi (see D, first part, last section).

§ 3. – The theory of the double solution presents another advantage: it permits us to comprehend why the particles are influenced only by their proper waves, as we pointed out in the first chapter (see D of section 3).

We reason with just two particles because one painlessly extends the results to the case of N objects.

Consider two micro-objects of the same nature that are defined by two waves u_1 and u_2. Outside the two singular regions these waves separately satisfy linear equations.

To associate waves with real fields amounts to saying that the total field will be defined at an arbitrary point P by the sum of the preceding waves, namely:

$$u = u_1 + u_2.$$

I say that the two singularities continue to be displaced as if they were being guided by their particular proper fields uniquely.

Indeed, start with the equations for particles with spin in the hydrodynamical representation of the fields. The total current will be described by the expression:

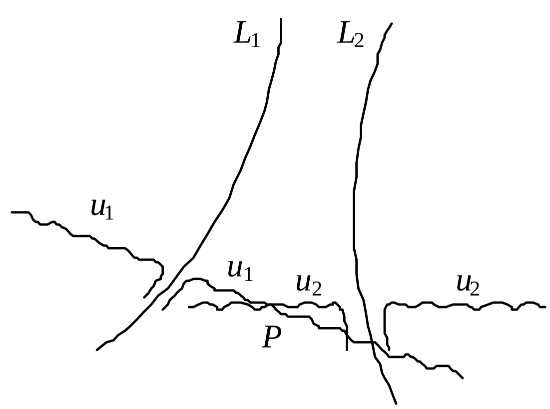


Fig. 12.

$$\begin{cases} S^{\nu} = u^{+}\alpha^{\nu}u \\ \quad = (u_1 + u_2)^{+}\alpha^{\nu}(u_1 + u_2) \\ \quad = \rho v^{\mu} \end{cases}$$

with:

$$\begin{cases} \rho = u^{+}u \\ v^{\mu} = \dfrac{u^{+}\alpha^{\nu}u}{u^{+}u} \cdot \end{cases}$$

From the guidance theorem, the motions L_1 and L_2 of the two singularities will be determined by the values taken by v_ν on S_1' and S_2'.

Now, from the hypotheses that were made in the preceding section one sees that as long as the singularities are separated by distances $\pi\, r_{o(1)} + r_{o(2)}$, one has, for example, the relation:

$$\frac{(u_1 + u_2)^{+}\alpha^{\nu}(u_1 + u_2)}{(u_1 + u_2)^{+}(u_1 + u_2)} \approx \frac{u_1{}^{+}\alpha^{\nu}u_1}{u_1{}^{+}u_1} = \frac{\varphi_1{}^{+}\alpha^{\nu}\varphi_1}{\varphi_1{}^{+}\varphi_1},$$

which signifies that bump 1 follows the trajectory L_1 that is associated with the continuous part φ_2 of its proper wave in the total field.

Therefore, the same is true for singularity 2 (Q.E.D.)

If one wants to use a physical depiction then we say that everything happens as if the waves that are associated with the particles were superposed in space without influencing the particles (except by the intermediary of the classical fields that are related to the particles) with each one piloting its proper singularity.

This result is interesting because it suggests three ideas that are capable of being developed further.

a) The first one, which is attached to relativistic unitary concepts, amounts to considering all of the micro-objects to be singularities in unique quantum electromagnetic gravitational fields that are defined by $g_{\mu\nu}$, A^{ν}, and u.

Naturally, this is possible only if one may account for the quantum structure of all of the micro-objects with the aid of just one field u. One then sees that if one wants to account for the effects of spin then it is natural to suppose that the fundamental quantum field is the Dirac field that is defined by the four-components spinors. However, conforming to the ideas of L. de Broglie, one may reconstruct any micro-object with the aid of particles of spin 1/2.

In this schema, the only elementary micro-objects will be charged micro-objects of spin 1/2, and all of the other ones will necessarily have a complex structure and might possibly be decomposed into particles of spin 1/2.

The set of micro-objects will then be represented by a unique field u, which may be decomposed at a distance into sum of waves that satisfy the linear Dirac equation, namely, $u = u_1 + u_2 + \ldots$

We shall not develop this concept here since it goes beyond the scope of our subject.

b) The second idea is that from the process of developing the preceding calculations it is possible to build a foundation for a causal theory of N micro-objects in interaction that is based on the idea that one may account for the actual motion of the singular regions by associating a proper wave u_μ that propagates under the influence of classical potentials that are associated with the other particles. This theory has already led to a certain number of results that will be discussed in chapter V.

c) The third idea is that the wave function u_0 describes the structure itself of the particle, at least outside the surface S'. The study of these singular solutions of the linear equations is therefore likely to furnish physical information about the behavior of the particles in the neighborhood of the singularities.

From this perspective, a certain number of results have been obtained at l'Institut Henri-Poincaré by L. De Broglie, Fer, and Petiau, in particular, results that succeed in giving possible solutions for u_0 in particular cases. These results raise the possibility of proving the existence of some solutions whose possible existence has been in doubt ([5]) since the beginning of the work by the author on the theory of the double solution. We summarize them briefly:

– Petiau has calculated possible values for u_0 in the case of the particle at rest in a proper reference frame and in the absence of an external field; we shall give his results. For the Klein-Gordon equation and the Dirac equation:

1. We start with the Klein-Gordon equation, written in the form:

$$(p_0^2 - p^2 - m_0^2 c^2) u_0(x, y, z, t) = 0\,, \tag{2.6}$$

and seek to determine the form of u_0 in a proper reference frame in which the singularity is centered at the point x_0, y_0, z_0 .

If one supposes that u_0 is of the form:

$$u_0 = g(x - x_0, y - y_0, z - z_0) e^{\frac{i}{\hbar} m_0 c^2 (t - t_0)}\,,$$

in which g is independent of t, then one finds, by substituting in the wave equation:

$$\Delta g = 0.$$

One then has various possibilities:

– If one considers a corpuscle as possessing spherical symmetry in the proper system, and one sets, as usual:

([5]) In particular, by Rosenfeld in the work: de Louis de Broglie, *Physicien et Penseur*, pp. 57.

$$\begin{cases} x = x_0 + r \sin\theta \cos\varphi \\ y = y_0 + r \sin\theta \sin\varphi \\ z = z_0 + r \cos\theta \end{cases}$$

in which φ refers to an angle, and no longer to the regular part of u, as well as:

$$Y_l^m(\theta, \varphi) = P_l^m(\cos\theta) e^{im\varphi},$$

then the general solution of equation (2.6), when given at a distance and presenting local singularities, may be written:

$$u(x, y, z, t) = \left[C_0 + \sum_{lm} \frac{A^{(l,m)}}{r^{l+1}} Y_l^m(\theta, \varphi) \right] e^{\frac{i}{\hbar} m_0 c^2 (t - t_0)},$$

in which the $A^{(l,m)}$ form a series of structure constants that characterize the nature of the corpuscle.

By the Lorentz transformation:

$$\begin{cases} m_0 c^2 t = \omega t' - p z' \\ m_0 c^2 z = \omega z' - p c^2 t' \\ r'^2 = (x' - x'_0)^2 + (y' - y'_0)^2 + \dfrac{[z' - z'_0 - v(t' - t'_0)]^2}{1 - \beta^2} \\ \cos\theta' = \dfrac{z' - z'_0 - v(t' - t'_0)}{\sqrt{1 - \beta^2}} \\ \varphi' = \varphi \end{cases}$$

one obtains the general solution:

$$u_0(x', y', z', t', x'_0, y'_0, z'_0, t'_0) = \left[C_0 + \sum_{l,m} \frac{A^{(l,m)}}{z'^{l+1}} Y_l^m(\theta', \varphi') \right] e^{\frac{i}{\hbar}[(\omega(t' - t'_0) - p(z' - z'_0)]}$$

that describes the motion that is associated with a classical plane wave.

– By contrast, if one considers a singularity that possesses the symmetry of an elongated ellipsoid in its proper system, and one takes an axis Ox that is parallel to the symmetry axis of the localized corpuscle at $x_0\, y_0\, z_0$, then one may write:

$$x = x_0 + a \operatorname{sh} \eta \sin\theta \cos\varphi$$
$$y = y_0 + a \operatorname{sh} \eta \sin\theta \sin\varphi$$
$$z = z_0 + a \operatorname{sh} \eta \cos\theta$$

and the equation $\Delta g = 0$ becomes:

$$\left[\frac{\partial}{\partial\eta'} + \coth\eta\frac{\partial}{\partial\eta} + \frac{\partial^2}{\partial\theta^2} + \cot g\frac{\partial}{\partial\theta} + \left(\frac{1}{\sin^2\theta} + \frac{1}{sh^2\eta}\right)\frac{\partial^2}{\partial\varphi^2}\right]g = 0,$$

which admits the solution:

$$g(\theta, \eta, \varphi) = T(\theta)\, H(\eta)\, e^{im\varphi}$$

in which:

$$T(\theta) = A_l^m \cdot P_l^m(\cos\theta) + B_l^m Q_l^m(\cos\theta)$$

and

$$H(\eta) = C_l^m P_l^m(\mathrm{ch}\,\eta) + D_l^m Q_l^m(\mathrm{ch}\,\eta),$$

and P_l^m and Q_l^m denote Legendre functions of the first and second types.

The general solution u_0 , which is bounded and has a localized singularity, may therefore be written, in the proper system:

$$u_0(x, y, z, t) = \left[\mathrm{C}_0 + \sum \mathrm{A}^{(lm)}\mathrm{P}_l^m(\cos\theta)\mathrm{Q}_l^m(\mathrm{ch}\,\eta)e^{im\varphi}\right]e^{\frac{i}{\hbar}m_0c^2(t-t_0)}.$$

It presents a logarithmic singularity along the segment $(-a, +a)$ that corresponds to $\eta = 0$.

If one passes from the *OXYZT* system to the system of the observer by a spatial rotation that makes *OZ* point in the direction of motion and a Lorentz transformation then one sees that the singular segment will give a spacetime segment that is characterized by the invariant length $2a$, which amounts to introducing a fundamental length that is associated with the dimension of the singular region.

2. As far as the Dirac equation is concerned, an equation that we classically write:

$$[p_0 + (p - \alpha) + m_0c\alpha]\, v_{0j} = 0 \qquad j = 1, 2, 3, 4, \tag{2.7}$$

one may look for solutions of the form:

$$u_{0j}(x, y, z, t) = g_j(x - x_0, y - y_0, z - z_0)e^{\frac{i}{\hbar}m_0c^2(t-t_0)},$$

in which the g_j do not depend on t.

If one sets, as usual, $g_1, g_2 = g_i^{(1)}$, $g_3, g_4 = g_i^{(2)}$, (i =1, 2) then the wave equation will give:

$$\begin{cases} 2m_0cg^{(1)} + (p\sigma)g^{(2)} = 0 \\ (p\sigma)g^{(1)} = 0 \end{cases}$$

as well as the relations:

$$\Delta g^{(1)} = 0$$
$$g^{(1)} = -\frac{1}{2} m_0 c (p\sigma) g^{(1)},$$

which admit the general solutions for a spherical corpuscle:

$$g_1 = \frac{i\hbar}{2m_0 c} \frac{1}{r^{l+2}} Y_{l+1}^m (l-m+1)[(l-m)a_2^{(lm)} - a_1^{(lm)}]$$
$$g_2 = \frac{i\hbar}{2m_0 c} \frac{1}{r^{l+2}} Y_{l+1}^{m+1} [(l-m)a_2^{(lm)} - a_1^{(lm)}]$$
$$g_3 = \frac{a_1^{(lm)}}{r^{l+1}} Y_l^m(\theta, \varphi) + c$$
$$g_4 = \frac{a_2^{(lm)}}{r^{l+1}} Y_l^{m+1}(\theta, \varphi) + c_2$$

which leads to the expression:

$$u_{oj}(x', y', z', t', x_0', y_0', z_0', t_0') = \frac{1}{\left[2m_0 c\left(\frac{w}{c} + m_0 c\right)\right]^{\frac{1}{2}}}$$
$$\times \left[\left(\frac{W}{c} + m_0 c\right) g_j(r', \theta', \varphi') + p_{jh}^{(\alpha\xi)} g^h(r', \theta', \varphi')\right] e^{\frac{i}{\hbar} w(t'-t_0') - p(z'-z_0')}$$

that is associated with ordinary plane waves.

Similarly, Petiau has found the corresponding solutions for the cases of spin 0 and 1.

The solutions are obviously valid only in particular cases, but there do not seem to be any difficulties, in principle, associated with constructing them in the general case, in which the lines *L* are associated with variable fields.

3. For example, Fer ([6]), with the goal of constructing a theory of light with the author of this work ([7]), has studied the scalar equation:

$$\Box u = 0.$$

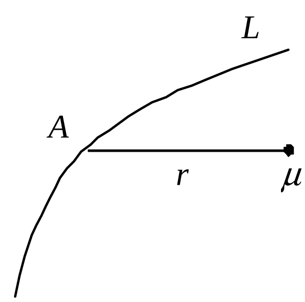


Fig. 13.

If one looks for solutions u_0 that are constrained to follow a trajectory *L* that is described by a point *A* as a function of time θ with a velocity *v* that is taken between $\alpha > 0$ and $\varpi < 1$ then one finds:

([6]) *C. Rendus*, t. 238, no. 5, pp. 567.
([7]) Which will ultimately be published.

$$u_0(M,t) = \int_{-\infty}^{\tau} \omega(\theta) \frac{\mu[c(t-\theta)-r]}{r} d\theta$$

in which the functions $\mu(\xi)$ and $\omega(\theta)$ are regular complex functions that are subject to the conditions:

$$\mu(\xi) = 0 \qquad \text{outside of the interval } (0, \chi > 0),$$

$$\omega(\theta) = \left(\frac{d\mu}{d\xi}\right)_0, \qquad \omega \text{ bounded,}$$

in which τ designates the unique value of θ for which:

$$c(t-\tau) - r(\tau) = 0.$$

The expression u_0 is interpreted by considering μ to be a signal that is emitted in a recurring fashion from the point A, and ω is a property that is propagated by that signal. For example, if one sets $\omega = e$ and $\mu = \delta$ then u_0 reduces to the Lienard-Wiechart potential of the moving electron.

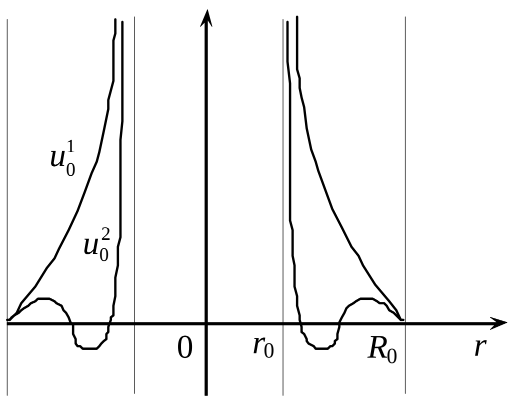


Fig. 14.

4. We conclude this section by reproducing a calculation of Petiau that shows precisely that the consideration of singular functions is apt to lead to physical consequences.

This calculation, which is based on the preceding calculations and a suggestion of the author, amounts to supposing (as is physically reasonable) that u_0 must be annulled at a distance R_0 from the center of the singularity.

Therefore, if one starts with the Gordon equation:

$$[P_0^2 - p^2 - m_0 c^2] u_0(x,y,z,t) = 0,$$

with:

$$\begin{cases} P_0 = +\dfrac{i\hbar}{c}\dfrac{\partial}{\partial t} \\ P_x = -i\hbar\dfrac{\partial}{\partial x} \end{cases}$$

and one sets $u_0 = g(x,y,z)e^{\frac{i}{\hbar}\mu_0 ct}$, in which μ_0 may be different from m_0, conforming to the guidance formula (one may interpret this as a sort of proper inertial mass), then one finds, for $\mu_0 > m_0$, that:

$$\Delta g+\frac{\mu_0^2-m_0^2c^2}{\hbar^2}g=0,$$

or:

$$\Delta g + \lambda^2 g = 0,$$

with:

$$\lambda^2=\frac{\mu_0^2-m_0^2c^2}{\hbar^2}.$$

As before, we then set:

$$g(x,y,z)=g'(r)P_l^n(\cos\theta)e^{im\varphi}$$

so that:

$$\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\lambda^2-\frac{l(l+1)}{r^2}\right]g'(r)=0\,;$$

hence:

$$g'(r)=\frac{1}{\sqrt{r}}\left[A_l^mJ_{l+\frac{1}{2}}(\lambda r)+B_l^mJ_{-\left(l+\frac{1}{2}\right)}(\lambda r)\right].$$

The functions $\frac{1}{\sqrt{r}}J_{l+\frac{1}{2}}(\lambda r)$ are regular at r = 0, whereas the functions $\frac{1}{\sqrt{r}}J_{-\left(l+\frac{1}{2}\right)}(\lambda r)$ are singular.

For example, we have:

$$\frac{1}{\sqrt{r}}J_{\frac{1}{2}}(r)=\sqrt{\frac{2}{\pi}}\frac{1}{r}\sin r \qquad \frac{1}{\sqrt{r}}J_{\frac{3}{2}}(r)=\sqrt{\frac{\pi}{2}}\left(\frac{\sin r}{r^2}-\frac{\cos r}{r}\right)$$
$$\frac{1}{\sqrt{r}}J_{-\frac{1}{2}}(r)=\sqrt{\frac{2}{\pi}}\frac{\cos r}{r} \qquad \frac{1}{\sqrt{r}}J_{-\frac{3}{2}}(r)=\sqrt{\frac{\pi}{2}}\left(\frac{\cos r}{r^2}-\frac{\sin r}{r}\right)$$

If one then annuls u_0 on the boundary of a sphere of radius $R_0\approx\eta\frac{e^2}{mc^2}$, in which η is a number of order 1, which amounts to saying that the dimensions of the singularity do not exceed the dimensions of the classical radius of particles, in conformity with experimental results, then we shall see a relation between μ_0 and m_0 appear that is analogous to a quantization of mass.

Indeed, this hypothesis amounts to writing the relations:

$$J_{-\left(l+\frac{1}{2}\right)}(\lambda R_0)=0$$

or

$$J_{-\left(l+\frac{1}{2}\right)}(\lambda\eta\frac{e^2}{m_0c^2})=0,$$

which leads to the equalities:

$$\begin{cases} \lambda\eta \dfrac{e^2}{m_0 c^2} = \beta_l^{(s)} \\ \lambda^2\eta^2 = \dfrac{m_0^2 c^4}{e^4}[\beta_l^{(s)}]^2 \end{cases}$$

when one considers the singular solutions, for example (one will obtain analogous results by studying the regular parts). One therefore has:

$$\mu_0^2 c^2 - m_0^2 c^2 = m_0^2 c^2 \left[\frac{c^2}{\eta^2} \frac{\hbar^2}{e^4} \right] (\beta_l^{(s)})^2$$

or

$$\mu_0^2 = m_0^2 \left[1 + \frac{\alpha^2}{\eta^2} (\beta_l^{(s)})^2 \right]$$

with

$$\alpha = \frac{\hbar c}{e^2},$$

so we have, approximately:

$$\mu_0 \approx m_0 \frac{\alpha}{\eta} \beta^{(s)}.$$

If one attaches the radius to the first 0 of $J_{-(l+\frac{1}{2})}$ then one makes a possible mass correspond to each value of *l*. For example, for $l = 0$ one has:

$$\beta_0^{(s)} = (2\mathrm{S}+1)\frac{\pi}{2},$$

namely:

$$\mu_0 = m_0 \frac{\alpha}{\eta} (2\mathrm{S}+1) \frac{\pi}{2}$$

with $S = 0, 1$.

Physically, this calculation, which is gross and phenomenological, may be interpreted by saying that if one restricts the singular part of the wave *u* to be annulled on a wall of dimension $2R_0$ then one sees conditions appear for supplementary quanta that might correspond to the mass spectrum of the elementary particles.

CHAPTER III

§ 1. – Before looking into the statistical problems, we shall develop one last aspect of the causal theory of micro-objects:

It is obviously not possible to pretend that the "model" of the double solution that we just summarized presently constitutes a complete theoretical edifice that is capable of solving all of the questions that were raised by quantum phenomena. Such as that is, it nevertheless presents a remarkable characteristic: As we will verify in chapter IV, it permits us explain quantum statistics in the framework of a field theory with the aid of deterministic motions that we have analyzed (which naturally prolong the classical ideas).

Now, if one compares this viewpoint with the ideas that were advanced by Einstein to surmount the classical difficulties that were indicated in the general introduction then one sees surprising analogies appear.

It is clear that the theory of the double solution, which was put forth by L. de Broglie in 1927, rests on a physical idea that is identical to the ideas that inspired the work of Darmois, Einstein, and Grommer in the same epoch. Indeed, in order to develop the theory of General Relativity they considered the field and the particle to be different manifestations of the same physical reality. They also associated material particles to the singularities of fields, which were constrained to follow world-lines that correspond to the dynamics of general relativity.

As a consequence, the two theories rest on identical concepts that relate to the nature and deterministic behavior of micro-phenomena.

This agreement suggests a new path, which has been unexplored up till now, that might effect a synthesis between Quantum theory and the theory of General Relativity.

In what follows, we shall try to examine it, without making any pretense of arriving at a complete or definitive solution of a very difficult problem.

For this, we start with the theory of General Relativity, and we analyze two successive versions of it and summarize their essential elements.

I. In order to resolve the difficulties ([1]) that relate to action at a distance, A. Einstein first disrupted the framework of the classical schema. While still preserving the concept of a real external world that is independent from the observer, he boldly abolished the classical distinction between spacetime and fields with the benefit of a non-Euclidean spacetime that is described by a Riemannian geometry.

According to Einstein, fields do not constitute real phenomena that are localized in external spacetime, but they are a part of it, and correspond, in summary, to the objective properties that define the natural geometry. For example, in his theory the $g_{\mu\nu}$ ([2]) must play the usual role that they inherit as gravitational potentials.

Nevertheless, in an early version Einstein maintained the classical duality between fields and matter, which he further defined as an assemblage of particles that were embedded in the preceding "medium."

([1]) Which were presented in the general introduction.

([2]) Starting now and up till the end of the chapter, we refer the reader to Appendix I to find the meaning of the symbols used. We therefore simplify the presentation by reducing the argument to its strict minimum.

Such a model obviously suppresses the notion of action at a distance. It reduces the universe to a three-dimensional substance (of spatial type) in perpetual evolution in which the material bodies swim. The bodies and this substance continually interact. Its successive forms may be described with the aid of a four-dimensional spacetime in which the particles follow well-defined world-lines that generalize the classical trajectories.

Such a model obviously simplifies the search for the two classes of natural laws that we described in our General Introduction.

The initial choice of a particular geometry that corresponds to experiment determines the value of the fields that represent it. For example, the law $R_{\mu\nu} = 0$ defines the nature of the gravitational field.

In a presentation of relativity that is very widespread, one then postulates that the bodies follow geodesics of the external spacetime, which thus defines a relativistic dynamic that permits one to correctly describe the behavior of the bodies in the gravitational field.

The relativistic physicists have even complicated the geometry of the medium in various ways in order to introduce the electromagnetic field. For example, one may start with an affine geometry and, with Cartan, associate the electromagnetic potential to the torsion of spacetime; we shall return to this particular point later.

This model is obviously deterministic since, as a study of the Cauchy problem shows, the givens of initial conditions on a space-like surface suffices to determine the later evolution. Nevertheless, it does not suppress the classical duality between the laws of fields and the laws of motion since the fields that describe the behavior of the type of ether that the matter lives in do not account for the behavior of the particles (which leaves the essential conceptual difficulties of the classical theory intact).

II. The relativists attacked these difficulties in a second version of the theory, a version that one may call the general theory of relativity. It amounts to abandoning the preceding definition of matter and substituting the idea that particles are singular regions of spacetime that continually agree with the external field.

In a series of remarkable memoirs, Darmois ([3]), Einstein ([4]), Grommer ([5]), and Infeld ([6]) have showed that the matching conditions lead to the relativistic laws of motion (by reason of the nonlinear character of the field equations).

The schema is found to be simple in the extreme: Nature is reduced to a unique spacelike substance that is geometrically describable and contains particle-singularities. This substance constitutes what one may call matter; its continuous part forms the material field and its singularities represent particles. In this framework, the field and the particles are different aspects, or, if you prefer, distinct modes of existence of matter in motion.

The general theory of relativity is therefore based on a unique substance whose continuous evolution may be represented by a four-dimensional spacetime. This evolution is calculable with the aid of the laws of the particular field that permit us to describe its behavior by starting with well-defined initial conditions. The corresponding

([3]) G. Darmois, *Mémorial des Sciences Mathématiques* (1926).
([4]) EINSTEIN and GROMMER, *Cit. Preuss. Akad. Wiss.*, I (1927).
([5]) EINSTEIN and INFELD, *Avoir. Math.*, 41-455 (1940).
([6]) INFELD and WALLACE, *Phys. Rev.*, 57-797 (1940).

model is therefore essentially deterministic and provides a simultaneous description of fields and particles.

Here, we recover precisely the essential ideas of the double solution. It thus seems natural to seek to introduce micro-processes by looking for a new definition of micro-particles *that furnishes the continuous motions of the causal interpretation instead of the classical relativistic motions.*

More precisely, one knows that a simple definition of the singular regions that corresponds to a static symmetric solution of the field equations leads to classical mechanics. It therefore remains to find out whether it is possible to discover more complicated solutions of these same equations that lead to complex classes (L) of motion that are necessary to account for quantum phenomena. This amounts to representing the corpuscles as the singularities of the metric of spacetime that will be accompanied by a particular gravitational wavelike field of which it is a part (in which the wave u and Planck's constant intervene).

According to this idea, the desired synthesis comes down to solving the following mathematical problem: give elementary particles a particular singular definition that satisfies relativistic unitary equations and furnishes the trajectories that are suggested by the theory of the double solution as their laws of motion.

§ 2. – The application of geometric theories to physics raises two essential questions:

1. One must first choose from the infinitude of possible spacetime metrics with affine connections that particular spacetime that we associate with actual spacetime.

2. One must then determine the geometric tensors that correspond to physical fields (gravitational, electromagnetic, etc.) in this framework and to the real phenomena that we observe in nature (particles, etc.).

The first question is solved mathematically by giving a procedure for calculating the geometric entities that characterize the spacetime envisioned. In the unitary theory, and in the case of affine geometry this amounts to determining the coefficients of the connection Γ^i_{kl}. This may be done in various ways. One may start, as Einstein did in his final attempts, with a variational principle that involves an invariant Lagrangian that is constructed by means of these quantities. One may also give the field equations that permit one to calculate these coefficients directly.

The second question consists of choosing geometric definitions of the physical quantities that one studies experimentally that behave like these quantities in the chosen spacetime ([7]).

Obviously, one may do this only by appealing to experience. In particular, it is evident that the geometric possibilities are sufficiently vast that it is not possible to determine the natural geometry and the entities that represent physical quantities *a priori.*

([7]) For example, this is why Einstein, by starting with a Riemann spacetime subject to the conditions $R_{\mu\nu} - 1/2 g_{\mu\nu} R = 0$ has shown that the $g_{\mu\nu}$, when considered as gravitational potentials, permit us to explain the law of universal gravitation.

In a schema of this type, one must proceed by generalizing the existing theories and seeking to deduce the consequences of experiment at each step. In this context, it seems that the introduction of trajectories (L) and the model of the double solution into the theory is likely to furnish a supplementary control mode. If one accepts our viewpoint then the theory must contain solutions that behave like the micro-objects of the causal interpretation.

§ 3. – In order to mathematically develop the problem raised, we shall put ourselves in the framework of the "naïve" theory of general relativity that was developed by Einstein before the recent extension to the relativistic unitary theory. This step obviously presents the logical inconvenience of introducing physical tensors, *a priori*, without specifying their geometrical significance. They nevertheless have the advantage of applying to the very general field equations that mostly correspond to the Galilean approximation of the equations that were obtained in the context of the unitary theories that have been envisioned up till now.

On the one hand, and in the absence of a universally recognized unitary theory whose validity has been experimentally demonstrated, we therefore avoid a number of discussions that are foreign to ours (for example, the precise physical significance of the geometrical entities that are derived from the affine connection), which, in our opinion, reinforces the importance and the significance of the results that follow. By reason of the analogies that we just pointed out, one may, in principle, integrate them in the context of a unitary theory of the type that was recently considered by Einstein and various authors.

The simplification thus obtained facilitates the research and will not harm the limited objective we have proposed because, at the moment, it serves only to establish the possible existence of particular solutions that are capable of reproducing the classes of motion (L) that were introduced by L. de Broglie, D. Bohm, and the author in order to furnish a causal interpretation of quantum theory, but in the context of the relativistic theory this time.

According to Einstein ([8]), physical spacetime constitutes a four-dimensional Riemann manifold (V_4) that is defined by a fundamental metric tensor $g_{\mu\nu}$.

The determination of physical space is then obviously effected by choosing a tensorial system of partial differential equations that limits the generality of that tensor and relates to the energy distribution of spacetime that is generated by the motions of matter.

As one knows, Einstein was led to these equations by looking for generalizations of the Laplace-Poisson equation ([9]) that were compatible with the usual conservation conditions. We shall write them in the classical form:

$$S_{\alpha\beta} = \chi T_{\alpha\beta} \qquad (\chi = \text{const.} = -8\pi\gamma)$$

in which $S_{\alpha\beta}$ and $T_{\alpha\beta}$ are two second-rank symmetric tensors.

([8]) We shall now follow the "naïve" presentation of the theory that was carried out by Lichnerowicz in his course at the Collège de France. The reader is referred to that work for more details.

([9]) Which determines the Newtonian potential.

For Einstein, the tensor $S_{\alpha\beta}$, which generalizes the left-hand side of the Laplace-Poisson equation, must have a purely geometrical significance that is characteristic of the structure of the Riemannian manifold considered. It therefore depends on $g_{\mu\nu}$ and its first and second derivatives, and must satisfy the conservation equations:

$$S^{\alpha}_{\beta;\alpha} = 0.$$

One may then show ([10]) that the only tensors $S_{\alpha\beta}$ that satisfy relations (3.1) may be written in the form:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(R + k),$$

in which k designates a cosmological constant that plays a role only in macroscopic problems. If one neglects it in the case of interest to us, then one may write the field equations in the classical form:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi T_{\alpha\beta}$$

that we shall use in what follows.

The second tensor $T_{\alpha\beta}$ has a mechanical significance and generalizes the right-hand side of the Poisson equation; in this theory, it corresponds to the presence of energy and momentum in the world-region considered ([11]).

Quite a number of studies ([12]) have been made of the preceding equations by adopting particular forms for $T_{\alpha\beta}$ in the right-hand side that are associated to various energetic distributions that appear in nature. One is therefore led to conceive of the $T_{\alpha\beta}$ as generally composed of a sum of terms that correspond to distributions of this type and their mutual interactions.

Depending on the particular form chosen one will thus be concerned with "different schemas" such as:

- the pure matter schema, in which $T_{\alpha\beta} = \rho\, u_\alpha u_\beta$ (u_α represents the component of the world-velocity, and ρ represents the scalar density),
- the holonomic fluid schema, in which $T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \Theta_{\alpha\beta}$ ($\Theta_{\alpha\beta}$ represents a pressure tensor such that the vector K that is defined by $\rho K_\beta = \Theta^{\alpha}_{\beta;\alpha}$ is the gradient of a function M_0); we shall use this schema in the sequel for the case of Klein-Gordon particles,
- the pure electromagnetic schema, in which $T_{\alpha\beta} = \tau_{\alpha\beta}$ ($\tau_{\alpha\beta}$ represents the Maxwell energy-momentum tensor, etc.),

([10]) Cf., E. Cartan, *J. Math. Pures et appliquées*, t. I, pp. 141-203 (1922).

([11]) By contrast, in the unitary theory it likewise takes on a geometric significance. For example, if one uses asymmetric affine connections, then it appears as a natural consequence of the field equations and (just like $S_{\alpha\beta}$) depends only on the geometric structure of the spacetime considered.

([12]) Cf., LICHNEROWICZ, Cours au Collège de France. There, one will find an analysis of a number of the cases envisioned above.

and the general case corresponds to a superposition of the above.

In any event, if one wants to account for effects of the electromagnetic type then it is necessary to introduce a world-vector k_μ into the theory along with $g_{\mu\nu}$. This vector generalizes the usual vector potential ([13]), and its components k_μ are determined by the particular field equations.

In order to simplify the presentation, we start with this case. Thus, one introduces a quadri-potential k_μ into the theory (which is determined by certain proper equations) that permits us to calculate the tensor $T_{\alpha\beta}$ that appears in the right-hand side of the Einstein field equations (which fix $g_{\mu\nu}$, in turn).

Following Einstein, we assume, in addition, that this set of equations may be derived from a variational principle. This assumption amounts to postulating that one may calculate $g_{\mu\nu}$ and k_μ by introducing an invariant function A (which depends on $g_{\mu\nu}$ and its derivatives, as well as k_μ and $f_{\mu\nu} = \partial_\mu k_\nu - \partial_\nu k_\mu$), such that one has:

$$\delta \int A \, d\omega = 0 ,$$

for all independent variations of $g_{\mu\nu}$ and k_μ..

In particular, if one writes:

$$A = \frac{1}{2}\sqrt{-g}\,R + 8\pi\gamma\mathcal{L}(k_\mu, \partial_\nu k_\mu, g_{\mu\nu}) , \tag{3.2}$$

in which γ is a constant, then one obtains fields equations for $g_{\mu\nu}$ in the form of:

$$\begin{cases} R_{\mu\nu} - \dfrac{1}{2} g_{\mu\nu} R = -8\pi\gamma T_{\mu\nu} \\ \text{with } \sqrt{-g}\,T_{\mu\nu} = -\dfrac{1}{16\pi\gamma}\dfrac{\partial\mathcal{L}}{\partial g^{\mu\nu}} \end{cases} \tag{3.3a}$$

to which one must append the equations for k_μ:

$$\partial_\nu \frac{\partial\mathcal{L}}{\dfrac{\partial k_\nu}{\partial x^\mu}} - \frac{\partial\mathcal{L}}{\partial k_\mu} = 0 . \tag{3.3b}$$

These two groups of field equations must be compatible. In principle, they suffice to determine the k_μ and the geometry of spacetime while taking boundary conditions into account.

([13]) In the affine unitary theory (k_μ) is related to the $\Gamma^\alpha_{\beta\rho}$ and geometrically corresponds to a type of spacetime torsion. For example, if one uses a conformal projective affine connection, then:

$$\Gamma^i_{kl} = \begin{Bmatrix} i \\ kl \end{Bmatrix} + \delta^i_k k_l$$

in which k defines the spacetime torsion.

Depending on the exact form of the chosen Lagrangian $\mathcal{L}$ one will obtain various possible electromagnetic theories (Born-Infeld, etc.) by associating k_μ with the electromagnetic potential. For the moment, we will not be more specific about this because it still remains for us to establish several properties that are useful in the sequel and do not depend on the choice of expression.

§ 4. – The first property relates to the solutions of the field equations, as written in the form (3.3). One may formulate it as follows: How many independent functions does it take in order to define a particular solution of (3.3)?

On the surface of things, it seems that one needs 10, corresponding to the 10 components of $g_{\mu\nu}$. In reality, this is not the case. As one knows, only 6 functions suffice to define a solution.

In order to see this, it suffices to refer to the works on the Cauchy problem in the space of general relativity; we shall briefly summarize this work along the lines of a presentation by Lichnerowicz ([14]).

The Cauchy problem – or initial value problem – may be stated in the following manner:

If one is given a gravitational field and the (electromagnetic) field $f_{\mu\nu}$ on a hypersurface S_0 then determine the corresponding metric and the (electromagnetic) field f over their entire domains of existence when these fields satisfy equations of the preceding type.

In order to treat this problem, one commences with the aid of a change of coordinates that brings us to the simplified case in which the spacetime is swept out by a family of surfaces S (x_4 = const.). The initial surface S_0 corresponds to $x_4 = 0$ (one may take S_0 to be spacelike, but this is not necessary for satisfying the condition that S is not tangent to a characteristic hypersurface).

The Cauchy data are then the twenty functions $g_{\alpha\beta}$, $\partial_4 g_{\alpha\beta}$, and the six functions $f_{\alpha\beta}$. As Lichnerowicz then proceeded to do, one may:

1. Prove the physical uniqueness ([15]) of the solution that corresponds to the preceding data.

2. Establish that if $g^{44} \neq 0$ everywhere then equations (3.3*a*) and (3.3*b*) are subdivided into two distinct groups D) and C) of equations in involution, namely, the ones that contain only the space indices $i, j = 1, 2, 3$ and the ones that do not. In particular, the metric equations furnish two groups:

$$\begin{cases} \text{(D)} \quad R_{ij} - \dfrac{1}{2} g_{ij} R + 8\pi\gamma T_{ij} = 0 \qquad i, j = 1, 2, 3 \\ \text{(C)} \quad R_\alpha^4 - \dfrac{1}{2} g_\alpha^4 R + 8\pi\gamma T_\alpha^4 = 0 \qquad \alpha = 1, 2, 3, 4. \end{cases}$$

([14]) Course given at the Collège de France, 1953-1954.
([15]) The data are preserved, up to a coordinate change.

Now, if one accounts for the continuity equations:

$$\left(R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R+8\pi\gamma T^{\alpha\beta}\right)_{;\beta}=0 \tag{3.4}$$

namely:

$$T^{\alpha\beta}{}_{;\alpha}=0,$$

which automatically results from the form itself of the chosen field equations, then one may show that the solutions of equation (3.3) satisfy the following lemma:

Lemma I. Any solution of D) that satisfies C) on S_0 is a solution to (3.3), or again: Any solution of D that satisfies C) on S_0 satisfies it everywhere.

This implies that if one is given convenient initial-value solutions that satisfy C) then the general metric solution, which depends on 10 arbitrary functions, must satisfy only 6 equations (equations D). One may therefore constrain them to satisfy four arbitrary supplementary equations that completely characterize the nature of the solution.

§ 5. – This property of the solutions is related to a well-known result that we state as follows:

Lemma II. If we are given a timelike congruence of world trajectories (L) then it is possible to determine at least one particular metric solution of equations (3.3) such that (L) constitutes a geodesic congruence that is associated with the preceding problem.

The proof is obvious: Let V^4 denote a manifold that contains (L), and let V'^4 denote a manifold that satisfies the field equations (3.3), in which we arbitrarily choose a geodesic congruence (L'). As one knows, it is always possible to define a map from V^4 to V'^4 that makes (L) and (L') coincide. This map obviously transforms V'^4 into a new manifold V''^4, which is likewise a solution of (3.3) that admits (L) as a geodesic congruence (since such an operation preserves the invariant properties of space). This manifold V''^4 therefore constitutes a particular solution that satisfies the desired properties.

This result prolongs to a well-known property of the relativistic dynamics of holonomic fluids, as was considered by Lichnerowicz. Indeed, consider a domain of V^4 that is occupied by a material distribution whose tensor $T_{\alpha\beta}$ may be written:

$$T_{\alpha\beta}=r\,u_\alpha u_\beta-\Theta_{\alpha\beta},$$

(in which r is the pseudo-density of the medium and u is the unitary velocity vector), and where one has:

$$rK_\beta=\Theta^\alpha{}_{\beta;\alpha}$$
$$K_\beta=\partial_\beta\log M_0\,.$$

Classically, one knows that everything happens in this case as if the streamlines were geodesics of the Riemannian metric:

$$\eta_{\alpha\beta} = M_0^2 g_{\alpha\beta}^0$$

that is conformal to the world metric $g_{\alpha\beta}^0$ that was used in the preceding equations.

It is clear that this conformal metric does not satisfy the field equations $R_{\alpha\beta} - 1/2 g_{\alpha\beta} R = -8\pi T_{\alpha\beta}$, in general. Nevertheless, one may transform it in such a way that it satisfies the preceding lemma conformally. The $g_{\mu\nu}$ of this solution may then be defined by adding symmetric terms $\xi_{\alpha\beta}$ to the terms $M_0^2 g_{\alpha\beta}^0$ such that the tensor $\eta_{\alpha\beta}$ + $\xi_{\alpha\beta}$, satisfies equations (C) and (D).

In the same way, as L. de Broglie has remarked, the geodesic congruence of the possible relativistic trajectories of a mass particle, which is classically determined by the relation:

$$\delta \int_{\gamma_0}^{\gamma_1} m_0 c^2 ds = 0,$$

is transformed into the relativistic trajectories (L) of the causal interpretation of the Klein-Gordon equation (subject to the action of a scalar potential M_0), provided that one replaces m_0 with the variable mass M_0. Indeed, they are furnished by the condition:

$$\delta \int_{\gamma_0}^{\gamma_1} m_0 c^2 ds = 0.$$

Here again, one sees that everything happens as if these trajectories were geodesics of the metric:

$$\eta_{\mu\nu} = \frac{M_0^2}{m_0^2} \varepsilon_{\mu\nu},$$

which is conformal to the Galilean metric. As before, one may deform this metric (which no longer satisfies the field equations) into a solution of (3.3) by setting:

$$g_{\mu\nu} = \eta_{\mu\nu} + \xi_{\mu\nu},$$

in which the $\xi_{\mu\nu}$ define a tensor that is chosen in such a way that the $g_{\mu\nu}$ satisfy the relations D) and admit (L) as a geodesic congruence.

§ 6. – The third property concerns what we call the conservation condition in relativity ([16]). One defines the Hamiltonian derivative of a world-invariant $\mathcal{L} = L\sqrt{-g}$ with respect to a tensor $m_{\mu\nu}$ to be the expression $\frac{\eta \mathcal{L}}{\eta m_{\mu\nu}}$ that is defined by the equalities:

([16]) Cf., Eddington, *Mathematical Theory of Relativity*, sec. 100.

$$\delta\int L\sqrt{-g}\,d\omega=\int\frac{\eta L}{\eta m_{\mu\nu}}\delta m_{\mu\nu}\sqrt{-g}\,d\omega\,,$$

which one may always write when the variations $\delta m_{\mu\nu}$ are annulled on the boundary of the region considered.

Therefore, let $\mathcal{L}$ be a function that depends on the $g_{\mu\nu}$, the k_{μ}, and the $f_{\mu\nu}$, and their derivatives up to no special order, such that:

$$\int\mathcal{L}\,d\omega$$

is an invariant in the given region.

Upon integrating by parts one obtains the relations:

$$\delta\int\mathcal{L}\,d\omega=\int(\mathcal{L}^{\mu\nu}\delta g_{\mu\nu}-\mathcal{L}^{\mu\nu}\delta f_{\mu\nu}+\mathcal{L}^{\mu}\delta k_{\mu})d\omega=0\,,$$

in which, by definition:

$$\mathcal{T}^{\mu\nu}=\frac{\eta L}{\eta g_{\mu\nu}}\,,\qquad \mathcal{H}^{\mu\nu}=-\frac{\eta L}{\eta f_{\mu\nu}}\,,\qquad \mathcal{J}^{\mu}=\frac{\eta L}{\eta k_{\mu}}\,,$$

and we suppose that Gothic letters denote tensor densities.

By taking the relation:

$$\int\mathcal{L}^{\mu\nu}\delta f_{\mu\nu}=\int-2\frac{\partial\mathcal{L}^{\mu\nu}}{\partial x_{\nu}}\delta k_{\mu}\,,$$

into account, and by neglecting a complete differential that transforms into a surface integral, one thus obtains the relation:

$$\delta\int\mathcal{L}\,d\omega=\int\left\{\mathcal{L}^{\mu\nu}\delta g_{\mu\nu}+(2\mathcal{L}^{\mu\nu}{}_{;\nu}+\mathcal{L}^{\mu})\delta k_{\mu}\right\}d\omega$$

(in which the symbol ";" denotes the covariant derivative), which must be annulled identically for arbitrary variations of the coordinates since $\mathcal{L}$ is an invariant.

By comparing the values of the tensors for the same values of x_{μ} in the old and new systems (which permits us to keep the same $d\omega$), these variations may be written:

$$-\delta k_{\mu}=k_{\alpha}\frac{\partial\delta x^{\alpha}}{\partial x^{\mu}}+\frac{\partial k_{\mu}}{\partial x_{\alpha}}\cdot\delta x_{\alpha}$$

$$-\delta g_{\mu\nu}=g_{\mu\nu}\frac{\partial(\delta x_{\beta})}{\partial x_{\beta}}+g_{\alpha\nu}\frac{\partial(\delta x_{\alpha})}{\partial x_{\mu}}+\frac{\partial g_{\mu\nu}}{\partial x_{\beta}}\delta x_{\alpha}\,,$$

and upon substituting them in the preceding equality (while neglecting a complete differential and accounting for the antisymmetric character of $\mathcal{H}^{\mu\nu}$):

$$\int \left\{ 2\mathcal{T}^{\nu}_{\alpha;\nu} - f_{\mu\alpha}(\mathcal{J}^{\mu} + 2\mathcal{H}^{\mu\nu}{}_{;\nu}) + k_{\alpha}\mathcal{J}^{\mu}{}_{;\mu} \right\} \delta x^{\alpha} d\omega = 0$$

namely:

$$\begin{cases} \mathcal{T}^{\nu}_{\alpha;\nu} = f_{\mu\alpha}\mathcal{H}^{\mu\nu}{}_{;\nu} + \dfrac{1}{2} f_{\mu\alpha}\mathcal{J}^{\mu} - \dfrac{1}{2} k_{\alpha}\mathcal{J}^{\mu}{}_{;\mu} \\ \mathcal{P}^{\nu}_{\mu;\nu} = -f_{\mu\alpha}\mathcal{H}^{\alpha\sigma}{}_{;\sigma} - \dfrac{1}{2}(f_{\mu\nu}\mathcal{J}^{\nu} + k_{\mu}\mathcal{J}^{\nu}{}_{;\nu}) \end{cases} \tag{3.6}$$

These relations are independent of the field equations and constitute what Schrödinger calls the "conservation equations." Each world-invariant therefore furnishes four relations that one may transform with the aid of the field equations.

We apply these considerations to the preceding theory. We start with the hypothesis that the Lagrangian $\mathcal{L}$, which depends on the components $g_{\mu\nu}$ and k_{μ}, which are necessary in order to determine the natural geometry, is composed of the sum of a term that corresponds to Einstein's theory and a term that defines the potential vector, but does not contain the derivatives of $g_{\mu\nu}$, namely:

$$\int \mathfrak{A}\, d\omega = \int \frac{1}{2} \mathfrak{R}\sqrt{-g}\; d\omega + k \int \mathcal{L}\, d\omega,$$

in which $k = 8\pi\gamma$ is the Einstein constant. Upon varying the $g_{\mu\nu}$ one obtains the field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k T_{\mu\nu},$$

with

$$\begin{cases} \sqrt{-g}\, T_{\mu\nu} = \tau_{\mu\nu} = \tau^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu} \\ \tau^{\mu\nu} = \dfrac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + \dfrac{\partial \mathcal{L}}{\partial g_{\nu\mu}}. \end{cases}$$

If one writes, as before:

$$\mathcal{L} = \mathcal{L}_{(E)}(f_{\mu\nu}, g_{\mu\nu}) + \mathcal{L}_{(M)}(k_{\mu} \ldots)$$

then the tensor $T^{\mu\nu}$ is decomposed into a sum of two terms, namely:

$$T^{\mu\nu} = T^{\mu\nu}_{E} + T^{\mu\nu}_{M},$$

with:

$$\begin{cases} \tau_{(E)}^{\mu\nu} = \dfrac{\partial \mathcal{L}_{(E)}}{\partial g_{\mu\nu}} + \dfrac{\partial \mathcal{L}_{(E)}}{\partial g_{\nu\mu}} \\ \tau_{(M)}^{\mu\nu} = \dfrac{\partial \mathcal{L}_{(M)}}{\partial g_{\mu\nu}} + \dfrac{\partial \mathcal{L}_{(M)}}{\partial g_{\nu\mu}} \end{cases}$$

or further, upon setting $\mathcal{L} = \sqrt{-g}\,L = \sqrt{-g}\,(L_{(E)} + L_{(M)})$:

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} L$$
$$T_{\mu\nu} = T_{(E)\mu\nu} + T_{(M)\mu\nu}\,,$$

with

$$T_{(E)\mu\nu} = \frac{\partial \mathcal{L}_{(E)}}{\partial g_{\mu\nu}} - \frac{1}{2} g_{\mu\nu} L_{(E)}$$
$$T_{(M)\mu\nu} = \frac{\partial \mathcal{L}_{(M)}}{\partial g_{\mu\nu}} - \frac{1}{2} g_{\mu\nu} L_{(M)}\,.$$

Upon introducing the notation:

$$\mathcal{J}^{\mu} = \frac{\partial \mathcal{L}}{\partial k_{\mu}}, \qquad \tau^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial f_{\mu\nu}} - \frac{\partial \mathcal{L}}{\partial f_{\nu\mu}}\,,$$

and writing, as before, that $\mathcal{L}$ is an invariant for any coordinate change, one obtains the relations ([17]):

$$d\mathcal{L} = \mathcal{J}^{\mu} dk_{\mu} + \frac{1}{2} T^{\mu\nu} df_{\mu\nu} + \frac{1}{2} \tau^{\mu\nu} dg_{\mu\nu}$$
$$\mathcal{L}\delta_{\nu}^{\mu} = \mathcal{J}^{\mu} k_{\mu} + T^{\mu\alpha} f_{\nu\alpha} + \tau^{\mu\alpha} g_{\nu\alpha}\,.$$

With this notation, the field equations for k_{μ} may be written:

$$\partial_{\mu} T^{\mu\nu} = \mathcal{J}^{\nu}\,, \qquad (3.3b_1)$$

which gives:

$$\partial_{\mu} \mathcal{J}^{\mu} = 0\,. \qquad (3.3b_2)$$

One then verifies painlessly that these equations annul the right-hand side of (3.6); this gives back the equality (3.4).

([17]) Proved by Born, *Théorie non linear du champ électromagnetique,* ("Annales de l'Institut H. Poincaré," 1937), pp. 172.

§ 7. – Tensors are not the only things that one can possibly introduce in affine geometry. In particular, in order to interpret the case of particles with spin, it is necessary to define the geometrical significance of the spinors that are used in equations of the sort that we just studied.

For this, one may recall an old idea of Einstein and Mayer that was ultimately extended to affine spaces by Fock and Ivanenko. It consists of associating spinors with sub-tensors (half-vectors) that permit one to carry out a new type of decomposition for the classical tensorial expressions.

In particular, one sees that one may associate spinors φ and matrices α^μ at each point of space in such a way that one must define a vector k_μ, for example, by the intermediary of the relations:

$$k_\mu = \overset{+}{\varphi}\alpha^\mu\varphi. \tag{3.7}$$

These quantities behave like the components of a vector, provided that one defines a parallel transport of φ that agrees with the corresponding geometrical transport that relates to the tensors thus defined.

We have summarized these considerations in Appendix II because the corresponding calculations are too well known to make it worth repeating in the course of the argument.

§ 8. – We conclude the study of these properties with a summary of the works of Einstein and his collaborators that relate to the motion of singular regions in general relativity.

They constitute what one may call the relativistic "theory of guidance" and, as we have said, permit us to deduce the laws of motion of the field equations by giving convenient definitions of the particle singularities.

This theory ([18]) essentially rests on the idea that, with the exception of the points that are situated on certain singular lines, the potentials and their first derivatives are everywhere continuous. In particular, this must be true when one crosses hypersurfaces S that bound time-oriented world-tubes that encircle the trajectories that embody the point-like aspect of the material particles.

Having said this, we distinguish two cases of the field equations (3.3):

The first one – which is called the "exterior case" – corresponds to the solutions of the equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0$$

(in the case for which the potential vector does not exist), or:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi\gamma\tau_{\alpha\beta}\,,$$

([18]) Here again, the reader may refer to the previously cited presentation of Lichnerowicz for the detailed proof. We shall content ourselves by recalling the essential results.

in the presence of a potential that represents the Maxwell energy-momentum tensor (or its generalization to the case of a theory of the Born-Infeld type), a tensor that corresponds to the case in which *A* does not contain "matter" terms, and depends on the k_μ only by the intermediary of the $f_{\mu\nu}$.

The second case – which is called the "interior case" – corresponds to the solutions of the equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi\gamma T_{\alpha\beta} ,$$

in the case where $T_{\alpha\beta}$ contains terms that one calls "matter terms" or depends on k_μ explicitly.

What makes this distinction so interesting is that it permits us to form an idea of the structure of physical spacetime that corresponds to the motions of matter.

Indeed, the first case contains the case of the physical "vacuum" because one may prove that any exterior ds^2 that satisfies the axioms of general relativity and is everywhere regular must be locally Euclidian ([19]).

This signifies that the presence of matter is necessarily associated with the existence of either gravitational or electromagnetic singularities.

This notion of agreement permits us to prove a large number of propositions, so we enumerate only the ones that refer to our problem. In the first place, one confirms that any solution of the interior case that is bounded by a hypersurface *S* that is generated by time lines may agree with a stationary solution of the exterior case on *S* only if this solution is singular in the interior of *S*. The particles are then necessarily associated with singularities (of the Schwarzschild type) of the classical exterior field.

One then establishes that the spatio-temporal trajectory of any singularity of the exterior case (which is associated with a particle) that is placed in a regular (interior or exterior) field necessarily follows a trajectory that is oriented in time and corresponds to the equation:

$$\eta^\alpha \eta_{\beta;\alpha} = k f_{\beta\alpha} \eta^\alpha \qquad \left(k = \frac{e}{m} \right),$$

if η is the unitary vector that is tangent to this trajectory (the $g_{\mu\nu}$ and $f_{\mu\nu}$ that figure in the preceding expression correspond to the regular solution considered).

We prove this proposition because it is important in what follows. In order to do this, we shall follow the presentation that was given by Infeld and Wallace ([20]).

We start with the field equations (which are valid everywhere except on a set of measure zero) and which we write, in a convenient system of units, as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 8\pi\gamma T_{\mu\nu} = 0 .$$

([19]) Similarly, one proves in the stationary case that if a gravitational field and an electromagnetic field are regular on V_4 and satisfy equations of the purely electromagnetic schema (exterior unitary case), as well as the axioms of general relativity, then the electromagnetic field is null and ds^2 is locally Euclidian.

([20]) *Phys. Review*, v. 57 (1940), pp. 797.

By contraction, one thus obtains $R = -8\pi\gamma T$, which gives, upon setting $T'_{\mu\nu} = 8\pi\gamma\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$:

$$R_{\mu\nu} + T'_{\mu\nu} = 0 .$$

One then sets:

$$g_{\mu\nu} = \varepsilon_{\mu\nu} + h_{\mu\nu}.$$

The field equations are then equivalent to 6 independent equations. I will therefore introduce the quantities:

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} h_{\rho\sigma},$$

and give myself 4 supplementary conditions on the coordinates, which I write:

$$\varepsilon^{\nu\rho}\gamma_{\mu\nu;\rho} = 0 \qquad (;\rho = \partial_\rho) .$$

They lead to the equations:

$$R_{\mu\nu} = \frac{1}{2}\Delta\gamma_{\mu\nu} \tag{3.8}$$

with:

$$-\Delta\gamma_{\mu\nu} = 2T'_{\mu\nu} .$$

If we use Latin indices that vary from 1 to 3 then this says:

$$\begin{cases} \gamma_{mn,ss} - \gamma_{mn,00} = 2T'_{mn} \\ \gamma_{0\alpha,ss} - \gamma_{0\alpha,00} = 2T'_{0\alpha} \end{cases} \tag{3.9a}$$

The conditions on the coordinates are written:

$$\begin{cases} \gamma_{mn,n} - \gamma_{m0,0} = 0 \\ \gamma_{0n,n} - \gamma_{00,0} = 0 \end{cases} \tag{3.9b}$$

Relations (3.9) are obviously equivalent to the following system:

$$\begin{cases} K_{mns,s} = (\gamma_{mn,s} - \gamma_{ms,n})_{,s} = 2T'_{mn} + \gamma_{mn,00} - \gamma_{m0,n0} \\ K_{0ns,s} = (\gamma_{0n,s} - \gamma_{0s,n})_{,s} = 2T'_{0n} + \gamma_{0n,00} - \gamma_{00,n0}. \end{cases}$$

We denote the direction cosine of normal to a surface S that surrounds a singular line of the field.

One then proves (on account of the fact that K_{mns} and K_{0ns} are anti-symmetric with respect to the indices n and s) that:

$$\begin{cases} \int_S K_{mns,s} \lambda^n dS = 0 \\ \int_S K_{0ns,s} \lambda^n dS = 0. \end{cases}$$

As a consequence, the field equations (3.9) imply the following relations, which are valid no matter how $r \to 0$:

$$\begin{cases} \int_S (\gamma_{mn,00} - \gamma_{m0,n0} + 2T'_{mn}) \lambda^n dS = 0 & (3.10a) \\ \int_S (\gamma_{0n,00} - \gamma_{00,n0} + 2T'_{0n}) \lambda^n dS = 0 & (3.10b) \end{cases}$$

which express the compatibility of equations (3.9*a*) and (3.9*b*), and which we – with Einstein – call "the equations of motion" of the body.

Indeed, choose a hypertube of radius r around L (we ultimately make r go to zero) and let $\eta^r(t)$ designate the coordinates of a point of L in a reference system in which the singularity is at rest at the time origin, and where one has:

$$h_{\mu\nu} = \varepsilon_{\mu\nu}\left(1 - \frac{2m}{r}\right).$$

The calculation shows that one may subdivide the solutions $\gamma_{\mu\nu}$ of (3.9*a*) into two terms, namely:

$$\gamma_{\mu\nu} = \Gamma_{\mu\nu} + \bar{\gamma}_{\mu\nu},$$

in which the $\bar{\gamma}_{\mu\nu}$ are solutions of the homogenous equation, $\Delta\bar{\gamma}_{\mu\nu} = 0$, and the $\Gamma_{\mu\nu}$ are particular solutions of equations (3.3).

The equations of motion then subdivide into a sum of integrals that contain:

1) the $\bar{\gamma}_{\mu\nu}$
2) the $\Gamma_{\mu\nu}$
3) the $T'_{\alpha\beta}$.

The only things that naturally persist are the terms that contain $1/r^2$ in the denominator because the surface element dS may be written $dS = r^2 d\Omega$ (Ω = solid angle).

One then immediately shows that:

a) the contribution of the term $\bar{\gamma}_{\mu\nu}$ is $m\ddot{\eta}_m$ in (3.10) and 0 in (3.10*b*).
b) the contribution of the terms is null in the case considered (it is the same in the Maxwellian case, in which their development starts with a term in e^2/r^2 instead of a term in $1/r$) because they can be written:

$$\lim_{r \to 0} m(r)\ddot{\eta}_m = 0$$

c) the only thing left is therefore the contribution of the terms $T'_{\mu\nu}$, or, more precisely, the fraction of $T'_{\mu\nu}$ that contains r^2 in the denominator.

One concludes from this that the equations of motion, which may be written:

$$\ddot{\eta}_m = \int_r T'_{mn} \lambda^n ds = h_m$$

in the chosen system, can be finally expressed in an arbitrary system in the form:

$$\frac{d^2\eta^\mu}{ds^2} + \begin{Bmatrix} \mu \\ \rho\sigma \end{Bmatrix} \frac{d\eta^\rho}{ds} \frac{d\eta^\sigma}{ds} = h^\mu .$$

As is well known, when $T'_{\mu\nu} = 0$ these agree with the geodesic equations. When $T'_{\mu\nu} \neq 0$, they signify that the trajectories are subject to gravitational forces and supplementary forces that depend on the $1/r^2$ part of $T'_{\mu\nu}$.

This is important because one may thus obtain an infinitude of possible trajectories according to the nature of the singular solutions that define the particle. The problem of sec. 2 thus comes down to the search for a Lagrangian L that furnishes singularities that correspond to the classes (L) of motions that were defined in the preceding chapter.

For example, if one starts with the classical Maxwell Lagrangian:

$$L = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} = -\frac{1}{2} F$$

then Infeld has proved that when one associates the particle with a gravitational singularity of Schwarzschild type that is joined to a stationary singularity (half the sum of advanced and retarded potentials) of the electromagnetic field ([21]) then one obtains as the equations of motion in preceding particular system:

$$m\dot{v}^\mu = m\ddot{\eta}^\mu = h^\mu_{(e)} = eF^{\mu}_{\nu(e)} v^\nu \tag{3.11}$$

in which $h^\mu_{(e)}$ denotes the Lorentz force that corresponds to the (exterior) continuous part of the field, and $f_{\mu\nu(e)}$ and v^μ are the components of the world-velocity of the particle, which corresponds to a classical motion.

By contrast, in the case of an electromagnetic singularity that corresponds to a retarded potential, one obtains terms that correspond to radiation, namely:

$$m\dot{v}^\mu = ef^{\mu}_{\nu(e)} v^\nu + \frac{2}{3} e^2 \dot{v}^\mu + \frac{2}{3} e^2 \dot{v}^2 v^\mu .$$

([21]) *Loc. cit.*

One arrives at similar results by using nonlinear Lagrangians, such as, for example:

$$L_{(E)} = \frac{1}{2}\log(1+F).$$

Moreover, in the latter case, as was proposed by Infeld ([22]), one shows without difficulty ([23]) that the association of the particle with a spherically symmetric stationary singularity is equivalent to the association of the particle with a classical motion (3.11).

§ 9. – These general properties will serve to begin the general problem that was posed concerning the existence of solutions of (3.3) that behave like micro-objects.

As we have seen, we are necessarily tempted to define the $g_{\mu\nu}$ and k_μ to be functions of the wave u in the double solution, which presents both an extended character and a singular one that might account for the properties that are attributed to micro-objects by the causal interpretation.

In the first place, we systematically leave aside the latter character in order to concentrate on the extended aspect. This amounts to provisionally reducing the particle to a point ([24]) and preserving only the continuous part φ of the actual wave $u = u_0 + \varphi$ that defines the particle in this conception.

We shall therefore try to determine the solutions of (3.3) that depend on φ and give them a geometric significance that is as simple as possible.

A first remark must be made: The unitary theory that we use involves, at the very least, a congruence of curves that have a particular geometric significance besides being geodesics. They amount to curves that are tangent to the potential vector k at each point, a vector that obviously defines a privileged direction in spacetime. It is therefore tempting to attribute a physical significance to these "streamlines" as well. In fact, we shall see that the proposed solutions associate these trajectories with the trajectories (L) of the causal interpretation.

§ 10. – Having said this, it is reasonable in the second place to first attack the quantum "models" in the classical approximation before trying to interpret the more complicated solutions.

As one knows, if one makes h tend to zero and one neglects the effects of spin in the previously described models then one obtains the classical theory of the ether that was recently proposed by Dirac. We shall thus try to interpret this in the context of the unitary theory that we are using.

We start with the $g_{\mu\nu}$ and k_μ that define the metric and the torsion potential vector. One may do the calculations only if one is given the precise form of the Lagrangian L. For example, one may write:

$$L = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu}$$

([22]) *Proceedings of the Cambridge Philosophical Society*, v. 33 (1937), pp. 70.
([23]) INFELD and HOFFMANN, *Phys. Rev.*, v. 51, 1937, pp. 766.
([24]) As in the first version of general relativity.

(with $f_{\mu\nu} = \partial_\mu k_\nu - \partial_\nu k_\mu$), which gives us the Maxwell theory, or use more complicated Lagrangians that correspond to the nonlinear theories. As one is concerned only with the continuous part of k, which is assumed small, it is reasonable to suppose that the preceding value constitutes a good approximation.

As in the example of Dirac, we then seek particular solutions such that the potential vector:

1. is timelike and
2. has a constant length.

They may be obtained by using the method of Lagrange multipliers: i.e., by adding a term:

$$L_{(M)} = -\frac{1}{2} m\lambda^2 (u_\nu u^\nu + c^2),$$

to the preceding Lagrangian, with, by definition:

$$\begin{aligned} k_\nu &= -\frac{mc}{e} u_\nu \\ &= A_\nu - \frac{c}{e}\partial_\nu S, \end{aligned}$$

in which A_ν represents the part of k_ν that is not a gradient (calculated by effecting the decomposition that was defined in the course of chapter I).

By varying the k_ν and the λ, one obtains the Dirac equations:

$$\begin{cases} u_\nu u^\nu = -c^2 \\ \partial_\mu u_\nu - \partial_\nu u_\mu = -\dfrac{e}{mc} f_{\mu\nu} \\ \partial_\nu f^{\mu\nu} = \dfrac{e}{c}\lambda^2 u^\mu \end{cases}$$

which shows that λ^2 behaves like a world-density.

By varying the $g_{\mu\nu}$, one arrives at the field equations (3.3) with $T_{\mu\nu} = T_{(E)\mu\nu} + T_{(M)\mu\nu}$, in which:

$$T_{(E)\mu\nu} = -2E_{\mu\nu}, \qquad \text{(Maxwell tensor)}$$

$$= 2\left(f_\nu^\alpha f_{\mu\alpha} - \frac{1}{4} g_{\mu\nu} f^{\alpha\beta} f_{\alpha\beta} \right)$$

and

$$T_{(M)\mu\nu} = m\lambda^2 u_\mu u_\nu,$$

equations that one may always solve, in principle.

The conservation relations $T^{\mu}_{\nu;\mu} = 0$ then give us the equations of the classical Dirac trajectories – which agree with the lines of the potential here – quite simply because they may also be written:

$$u^{\nu} u_{\nu;\mu} = \frac{e}{mc} f_{\nu\mu} \cdot u^{\mu} \qquad (3.11\textit{cont.})$$

The preceding calculations suggest two important remarks:

a) The Dirac gauge condition is obtained precisely by varying λ, but it will be simpler – hence preferable – to obtain it directly by starting with the field equations and adding a term $L_{(M)}$, which is a function of the k_μ, and gauge terms to $L_{(E)}$.

b) One may physically interpret this classical theory by saying that the particles are constrained to follow the lines of torsion that correspond to a timelike vector potential with constant length. This is a very restrictive condition, and the idea immediately becomes one of seeing whether it is not possible to skip this second condition and see if one does not naturally arrive that the trajectories of the causal interpretation by supposing simply that this vector has variable length. From what we have seen, if this is the case then one must recover, in particular, the Klein-Gordon trajectories by replacing m_0 with the function M_0 of L. de Broglie, which was defined in chapter I.

§ 11. – We now see what one may accomplish by using remark a) of the preceding section.

We must first define the gauge term $L_{(M)}$.

The function $\varphi = R\, e^{iS/\hbar}$, which we introduced previously, appears in it.

We further set:

$$k_\mu = A_\mu - \frac{c}{e} \partial_\mu S \qquad (3.12a)$$

and give the Lagrangian $L_{(M)}$ the following expression:

$$L_{(M)} = \frac{1}{2m}\left(\frac{\hbar}{i}\partial_\nu + \frac{e}{c}A_\nu\right)\varphi^*\left(\frac{\hbar}{i}\partial^\nu - \frac{e}{c}A^\nu\right)\varphi - \frac{1}{2}mc^2\varphi^*\varphi, \qquad (3.12)$$

which is nothing but the Klein-Gordon Lagrangian. It is clear that this must make the length of k_μ depend on a variable function R.

We then arrive at the field equations that are deduced from k_μ (with $L = -1/4\, f^{\mu\nu} f_{\mu\nu} + L_{(M)}$):

$$\partial_\mu f^{\mu\nu} = j^\nu$$

with

$$j_\mu = \frac{e}{m} R^2 \left(\partial_\mu S - \frac{e}{c} A_\mu\right)$$

and

$$\partial_\mu j^\mu = 0\,. \qquad (3.13)$$

The equations that correspond to the variation of the $g_{\mu\nu}$ may be further written in the form (3.3) with:

$$T_{\mu\nu} = T_{(E)\mu\nu} + T_{(M)\mu\nu}, \qquad \text{or} \qquad T_{(E)\mu\nu} = -2E_{\mu\nu},$$

and

$$T_{(M)\mu\nu} = -\frac{1}{2m}\left\{\left(\frac{\hbar}{i}\partial_\mu + \frac{e}{c}A_\mu\right)\varphi^*\left(\frac{\hbar}{i}\partial_\nu - \frac{e}{c}A_\nu\right)\varphi + conj.\right\} + g_{\mu\nu}L_{(M)}$$

One therefore has:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -k(T_{(E)\mu\nu} + T_{(M)\mu\nu}) \tag{3.14}$$

with the conservation equations:

$$T^{\mu\nu}{}_{;\nu} = 0,$$

which, on account of the field equations (3.3) one may write:

$$\partial_\nu T^{\mu\nu}_{(\mathrm{M})} = j_\nu f^{\mu\nu}. \tag{3.14}$$

One then sees without difficulty:

1. Equations (3.13) and (3.14) automatically lead to the usual wave equation for φ if one supposes that φ satisfies a linear equations ([25]). We do not indicate the proof here in order to abbreviate the presentation. The reader will find it in Appendix III.

Conforming to the preceding remark a), one sees that the introduction of $L_{(M)}$ leads to the generalized gauge condition:

$$k_\mu k^\mu = e^2 M_0^2. \tag{3.15}$$

2. The potential lines once more agree with the streamlines of the wave equation because:

$$k_\mu = \frac{mc}{e^2}\frac{j_\mu}{\varphi^*\varphi}. \tag{3.16}$$

3. One returns to the classical Dirac approximation by letting $\hbar$ to zero. Indeed, one then finds:

$$\begin{cases} u_\nu = \dfrac{1}{m}\left(\partial_\nu S - \dfrac{e}{c}A_\nu\right) \\ \lambda = mR^2 \\ L \to -\dfrac{1}{4}f_{\mu\nu}f^{\mu\nu} - \dfrac{1}{2}\lambda(u_\nu u^\nu + c^2). \end{cases}$$

([25]) Which amounts to assuming that this field is weak, and corresponds to the linear approximation of the solutions of the field equations.

4. It is possible find solutions of equations (3.13) such that the trajectories (L) may be written in the form:

$$m_0 \left\{ \frac{d^2 x^\mu}{ds^2} + \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right\} = f_\nu^\mu \, j^\nu ,$$

in which the term $\begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix}$ denotes the usual Christoffel symbol, which is defined in Appendix I, and $\frac{dx^\mu}{ds}$ is the unit vector that is collinear with j^μ.

In order to see this, it suffices to refer to the study in sec. 4.

One further seeks solutions of the form:

$$g_{\mu\nu}^{(0)} = \frac{M_0^2}{m_0^2} \varepsilon_{\mu\nu} + S_{\mu\nu} , \tag{3.17}$$

which depend on 10 arbitrary functions. Six functions suffice to define a solution of (3.3), and, from lemma II, one has the right to impose four more arbitrary supplementary conditions on the $g_{\mu\nu}$ that transform the congruence of trajectories subject to only the potential M_0 into a geodesic congruence.

These solutions, whose explicit form we will not discuss here, obviously depend on φ, hence on the k_μ. They define a proper extended field $g_{\mu\nu}^{(0)}$ that defines the wave-like aspect of the micro-object without spin in this simplified model (in which the particle is reduced to a point). Moreover, one must remark on the manner by which these $g_{\mu\nu}^{(0)}$ depend on φ (by the intermediary of M_0). This seems to favor the second form of the causal interpretation, in which the physical quantum field is related to a potential of a new type.

§ 12. – We shall introduce the singular aspect of micro-objects by substituting the waves of the theory of the double solution for the waves u. As far as the classical Dirac theory is concerned, this introduction presents no difficulty.

In order to see, by applying the relativistic guidance formula, that the nonlinear character of the field equations (3.3) restricts a particle to describe those trajectories (3.11*cont.*) that it initially coincides with, it suffices to define the particle aspect of the micro-object by means of a stationary singularity of the metric and a vector potential with spherical symmetry (called Schwarzschild and Maxwell singularities, resp.).

This is a well-known property whose proof we will not reproduce ([26]).

We simply note two points:

([26]) In particular, it is developed in the previously cited memoir of Infeld and Wallace.

a) As it amounts to a classical approximation, there is obviously a degree of independence between the trajectories and the field, the former being determined by the position and velocity of the center of the singularity.

b) Nothing changes in the preceding result if, instead of the Maxwell Lagrangian:

$$L_{(E)} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} = -\frac{1}{2} F\,,$$

one uses a more complicated Lagrangian that corresponds to a nonlinear theory of the electromagnetic field, for example:

$$L_{(E)} = \sqrt{-\mathrm{Det}\,|\,g_{\mu\nu} + f_{\mu\nu}\,|} \qquad \text{Born-Infeld theory}$$

or (3.24*b*)

$$L_{(E)} = \frac{1}{2}\log(1+F) \qquad \text{Infeld-Hoffmann theory}$$

The latter theory is particularly interesting because the spherically symmetric solutions whose explicit form is given in Appendix IV furnish fields that are annulled at the center of the singularity. This permits us to add the proper field of the particles to the exterior field in the wave equations without perturbing the trajectory of the singularity. One may therefore write:

$$D_\mu = \frac{\hbar}{i}\partial_\mu - \varepsilon(A_\mu^{ext.} + A_\mu^{pr.})$$

in the formulas:

$$(\alpha^\nu D_\nu - \mu)\varphi = 0$$

(in which $A^{ext.}$ and $A^{pr.}$ denote the exterior electromagnetic potential and the proper potential of the object, respectively), because it clear that this supplementary potential $A^{pr.}$ perturbs all of the streamlines (by the intermediary of the corresponding Lorentz force) except for the one that is effectively followed by the center of the electromagnetic singularity (since $f_{\mu\nu}^{(0)} = 0$). The same is true for the Klein-Gordon equation.

In order to not overburden the presentation, we refer the reader to Appendix IV for more details. There, one will find the calculations and references that are necessary for the comprehension of that particular nonlinear theory.

In particular, as in the note of Infeld, one sees that the use of the tensor $T_{\mu\nu}^{(E)}$ that is associated with the Lagrangian $L_{(E)}$ gives back the Lorentz force on (*L*) because the expression $p^{ij} f_{ij}$ that appears in it is the only one that contains a term in $1/r^2$ when one substitutes it in (3.9*c* and *d*).

d) In the theory of relativity, it is not rigorously legitimate to isolate singular regions in the total field. The field that results from the presence of several particles in space is not the sum of the fields of the isolated individual particles. The deformation of spacetime that is induced by one particle is profoundly integrated into the deformation that is produced by the other particles, which cannot be separated from it. The determination of the field of a body may be carried out only by studying the motion of a sufficiently weak particle that makes a negligible perturbation on the motion (test particles). Since an actual experiment necessarily perturbs the observed system in quantum mechanics, such "test particles" do not exist for a given field, and it becomes impossible to directly study them in the experimental context, given the actual state of our knowledge.

This raises a certain number of difficulties that relate to the wave equations that are used.

In all of the preceding calculations, we have systematically introduced the electromagnetic potential A that is produced by the *other* particles and not the total A_μ that is the sum of the exterior potential and the proper potential $A_\mu^{(0)}$ of the particle considered into the wave equations of micro-objects. From the foregoing, this is not natural and requires a particular theoretical justification.

For example, one confirms that when one uses the Maxwell electromagnetic singularity in $1/r$ the introduction of the potential A_μ into the Klein-Gordon equation leads to the appearance of infinite terms in the wave at the center of the singularity. This suggests the following idea: *If one postulates that the elementary particles are charged* ([27]) *then it is natural to suppose that the appearance of nonlinear terms in the wave equations corresponds to the small region in which the proper potential* $A_\mu^{(0)}$ *of the particle takes values that are sufficiently strong that one leaves the linear approximation.*

Moreover, the same is true in the classical approximation. Indeed, equations (3.3) show that the use of the Lagrangian $L_{(E)}$ strongly implies the appearance of singular terms in the right-hand side when one uses solutions with spatial spherical symmetry (similarly, when the potential $A_\mu^{(0)}$ remains finite, as is the case in the nonlinear theories of Born-Infeld and Infeld-Hoffmann). One must therefore necessarily associate a singularity of the metric to a singularity of the vector potential in a unitary of the preceding type.

From this, one concludes that if one associates the micro-object with certain values of the vector potential and the metric that involve a singular region with spherical symmetry of the potential then that region likewise corresponds to the singularities of the metric and the waves that we used to define the gravitational gauge and wavelet that characterize this micro-object.

In other words, we say that the point-like aspect of the micro-object necessarily corresponds to a triple singularity of the vector potential of the metric and the wave u that serves to define them. The first singularity generates the other two by virtue of the field equations, which explains the observed relation between the metric and electromagnetic singularities that translates into the classical Lorentz law of translation ([28]).

§ 13. – We will now see what happens in the case of the Klein-Gordon equation when we adopt this postulate.

([27]) Neutral particles result from a "fusion" of the charged particles, in the sense of L. de Broglie.

([28]) Cf., Lichnerowicz, *Sur les équations relativistes de l'électromagnetisme,* Annales de l'E. N. S., pp. 269.

Suppose that the geometry is again defined by the field equations (3.3), in which $L_{(E)}$ and $L_{(M)}$ take the values (3.12) and (3.24), but in which one has replaced the function:

$$\varphi = R \exp\left(i\frac{S}{\hbar}\right), \qquad \text{with} \qquad u = f \exp\left(i\frac{\theta}{\hbar}\right)$$

(in which $u = u_0 + \varphi$), and the electromagnetic potential with the total potential:

$$A_\mu^{(0)} + A_\mu^{(e)},$$

which is the sum of the proper potential and the exterior potential. Therefore, one has:

$$\begin{cases} L_{(E)} = \dfrac{1}{2}\log(1+F) & \text{in the Infeld-Hoffmann theory} \\ L_{(E)} = \dfrac{1}{2}F & \text{in the Maxwell theory} \end{cases}$$

$$L_{(M)} = \frac{1}{2m}\left(\frac{\hbar}{i}\partial_\nu + \frac{e}{c}A_\nu\right)u^*\left(\frac{\hbar}{i}\partial^\nu - \frac{e}{c}A^\nu\right)u - \frac{1}{2}mc^2 u^* u \tag{3.26}$$

with

$$F = \frac{1}{2}f^{\mu\nu}f_{\mu\nu}, \qquad f_{\mu\nu} = \operatorname{rot} k_\mu = \operatorname{rot}\left(\frac{e}{c}j_\mu\begin{pmatrix}\text{Klein}\\ \text{Gordon}\end{pmatrix}\right).$$

From our hypotheses, we shall look for solutions u that satisfy linear equations everywhere except for a small region in which the nonlinear terms are involved.

In the exterior of that region the field equations obviously give back the conservation equations (3.20) and (3.21), in which φ is always replaced by u. By applying the theorem of Appendix III, one thus obtains the Klein-Gordon equation for u again.

In the interior of that region one may establish only whether these conservations are sufficient (combined with the hypothesis of relativistic invariance) to fix the nonlinear form of the wave equation in u or whether it is necessary to start with a nonlinear gauge condition $L_{(\mu)}$ (which reduces to the preceding condition (3.26) in the linear approximation) *a priori*. We leave aside this point because it does not pertain to the object of this study directly. Moreover, it is not indispensable in what follows if one assumes the "guidance conditions" between u and φ that were furnished in the preceding chapter.

Be that as it may, it is now possible to solutions of (3.3) that account for the extended and pointlike aspects of the micro-object by using u instead of φ.

They will be defined like the combination of:

1. A wavelet of the vector field k_μ accompanied by a gravitational wavelet $g_{\mu\nu}^{(0)}$ that is defined by formulas (3.12) and (3.17), in which one replaces φ with u. *We impose the condition that we shall use only those solutions u that satisfy the guidance conditions:*

$$\frac{1}{uu^*}(u^* D_\mu u - D_\mu^* u^* u) = \frac{1}{\varphi\varphi^*}(\varphi^* D_\mu \varphi - D_\mu^* \varphi^* \varphi) \tag{3.27}$$

in which φ denotes a continuous solution that was used in sec. 11. Indeed, one sees painlessly that the preceding guidance conditions entail the equality of both the phases and the quantum potentials, so:

$$\frac{\hbar^2}{2m}\frac{\Box R}{R} = \frac{\hbar^2}{2m}\frac{\Box f}{f},$$

on the boundary of the singular region. In that regard, they also thus entail the equality of the wavelets that were constructed with u and φ.

2. Spherically symmetric stationary gravitational singularities and the associated electromagnetic singularities that necessarily follow the center of u_0, from what we saw in the preceding paragraph.

They thus follow the trajectories (L) because of (3.27) and also because of the relativistic gauge conditions ([29]) that correspond to the field equations (3.3). Since these trajectories satisfy equation (3.11), if we take into account equations (3.18) and the equality of the expressions then:

$$M_0(u, u^*) = M_0(\varphi, \varphi^*)$$

on the singular region, which is deduced from relations (3.27).

This solution obviously corresponds to a possible solution of the problem that was posed in sec. 2 (the existence of convenient solutions) in the case that interests us.

The preceding considerations permit us to make a simple geometric representation of the micro-objects in the context of the "naïve" theory. They are conceived to be a combination of a solution of the exterior case (which represents their classical point-like aspect) and an extended solution of the interior case (which represents their extended wave-like aspect), in which one has used a tensor $T_{\alpha\beta}$ in the right-hand side that is a sum of the Maxwellian tensor $\tau_{\alpha\beta}$ and the tensor $T_{(M)\alpha\beta}$ that one calculates by starting with the hydrodynamic representation of the field u, with the condition that we choose the latter to be the solution that admits (L) as a geodesic congruence (which is possible because of Lemma II). Together, this amounts to our making an energy-momentum tensor figure in $T_{\alpha\beta}$ that corresponds to the probability fluid that is associated with the probabilistic interpretation.

This proper gravitational field presents the remarkable property of corresponding to one of the classical schemas that were studied by Lichnerowicz: the charged holonomic fluid. Indeed, it is obtained by using the usual tensor $T_{(M)\mu\nu}$ of the Klein-Gordon theory as the tensor $T_{\alpha\beta}$ in the right-hand side of equations (3.1), which is therefore interpreted without difficulty by using the hydrodynamical representation of the wave equation.

Indeed, as usual, set:

([29]) On the condition that we assume that the expression $T_{(M)\mu\nu}(u^+u)$ does not contain terms in $1/r^2$, conforming to the idea that the singular regions of u do not become infinite (cf., the solutions that were envisioned by Rosen and Finkelstein).

$$u = f\,\exp\left(i\frac{S}{\hbar}\right) \qquad u^* = f\,\exp\left(-i\frac{S}{\hbar}\right),$$

and upon using the wave equation, one immediately obtains:

$$T_{(M)\mu\nu} = T_{(M)}(\text{Klein-Gordon}) = MP_0\gamma u_\mu u_\nu + \varepsilon_\mu{}^\sigma\{2\partial_\sigma f\partial_\nu f - \varepsilon_{\sigma\nu}(\partial_\lambda f\partial^\lambda f - f\Box f)\},$$

which conforms to the result that was presented in chapter I, and one painlessly verifies that the second term of the right-hand side corresponds to the stress tensor of a fluid that is holonomic in the sense of Lichnerowicz.

§ 14. – The extension of these results to the case of particles with spin presents no mathematical difficulty. It obviously raises problems of interpretation that we intend to come back to in a later work.

We shall confine ourselves to treating the case of Dirac particles because since the work of L. de Broglie one knows that it is possible to constitute any arbitrary particle with spin with the aid of micro-objects of this type.

We again introduce the continuous wave φ with 4 components ($u = u_0 + \varphi$), which must serve to define the vector potential k_μ for us with the aid of considerations that were presented in D.

We further define the quadri-vector:

$$k_\mu = \frac{c}{\varepsilon}\frac{\varphi^+\alpha_\mu\varphi}{\varphi^+\varphi},$$

in which the spinor φ and the α_μ are defined as in Appendix II, and we introduce the gauge Lagrangian $L_{(M)}$, in which we use the notations that we defined in the beginning of this chapter:

$$L_{(M)} = -\frac{\hbar c}{2i}\varphi^+(\alpha^\nu D_\nu + \mu)\varphi + \text{conj.} \tag{3.19}$$

with:

$$L_{(E)} = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu},$$

($f_{\mu\nu} = \text{rot}\,A_\mu$). One then finds that the field equations for k_μ are once more:

$$\begin{cases}\partial^\mu f_{\mu\nu} = \varepsilon\hbar c\varphi^+\alpha_\nu\varphi = j_\nu \\ \partial_\mu j^\mu = 0.\end{cases} \tag{3.20}$$

The relations that define the $g_{\mu\nu}$ may again be written in the form (3.3), with $T_{\mu\nu} = T_{(E)\mu\nu} + T_{(M)\mu\nu}$:

$$T_{(M)\mu\nu} = \frac{\hbar c}{i}\left\{\varphi^+\alpha_\mu D_\nu\varphi + \varphi^+\alpha_\nu D_\mu\varphi - D_\nu\varphi^+\alpha_\mu\varphi - D_\mu\varphi^+\alpha_\nu\varphi\right\}.$$

The conservation equations $T_{\mu\nu;}{}^{\nu} = 0$ also give:

$$\partial_\mu T_{(M)\mu\nu} = -\frac{1}{c} j_\mu f^{\mu\nu} . \tag{3.21}$$

This permits us to generalize all of the considerations of the preceding paragraph:

1. The conservation equations (3.20) and (3.21) again lead to the Dirac equation if one supposes that φ satisfies a linear equation because the proof in Appendix III does not depend on the exact form of the Lagrangian used.

This furnishes a new gauge condition, for which the length of the vector k_μ will again be variable, and will depend on the function M_0 that was introduced in the analysis that we made for the Dirac equation.

2. The lines of the vector potential agree with the streamlines (L).

3. If one neglects spin then one returns to the equations of the preceding paragraph.

4. One may further define a "proper" metric field:

$$g_{\mu\nu}^{(0)} = \frac{M_0^2}{m_0^2} \varepsilon_{\mu\nu} + S_{\mu\nu}^{(0)} , \tag{3.22}$$

which satisfies equations (3.3) and admits the relations:

$$m_0 \left\{ \frac{d^2 x^\mu}{ds^2} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right\} = F^\mu{}_\nu \, j^\nu \tag{3.23}$$

as equations of (L), in which $F^\mu{}_\nu$ denotes the rotation of k_μ ($f_{\mu\nu}$ is reserved for the rotation of A_μ), by imposing the 4 relations (3.18) on $g_{\mu\nu}$, in which one replaces M_0 by its corresponding expression in the Dirac theory. As before, one has therefore established the existence of solutions $g_{\mu\nu}^{(0)}$ that constitute a sort of field that accompanies the micro-object.

As in the Klein-Gordon case, this proper field constitutes a particular solution of the field equations (3.3) in the interior case that we obtain by adopting a tensor $T_{\alpha\beta}$ that is the sum of a Maxwell tensor $\tau_{\alpha\beta}$ and a continuous matter tensor $T_{(M)\mu\nu}$ that corresponds to a fictitious fluid endowed with spin in such as one might introduce when starting with the hydrodynamical interpretation of the Dirac equation.

We will not develop this viewpoint further here, and it will be the object of a later publication that is in the course of preparation.

§ 15. – Here again, one may extend the theory of sec. 11 to the case of particles with spin by substituting u for φ.

1. One begins by defining the proper vector field and the gravitational field with the aid of the expressions:

$$\begin{cases} k_\mu = \dfrac{c}{e}\dfrac{u^+\alpha_\mu u}{u^+u} \\ g_{\mu\nu}^{(0)} = M_0(u^+u)\varepsilon_{\mu\nu} + S_{\mu\nu}^{(0)}, \end{cases} \tag{3.28}$$

by further supposing that the singularity of u corresponds to the proper electromagnetic potential $A_\mu^{(0)}$ of the micro-object (which is also associated with a corresponding singularity of the metric).

2. One further deduces the linear part of the wave equations on u from the conservations equations by means of the condition that one add a supplementary Lagrangian to the gauge Lagrangian $L_{(M)}(u^+u)$ that was given by formula (3.19), which has the property that the divergence of the corresponding term $T'_{(M)\mu\nu}$ is equal to:

$$f'_{\mu\nu}\, j^\nu ,$$

in which $f'_{\mu\nu}$ designates the rotation of the vector:

$$P_\mu = \frac{\hbar}{2}\partial_\nu (u^+ I_\mu^\nu u) .$$

Indeed, in this case one recovers equations (3.20) and (3.21) as conservation equations.

3. If one then uses a solution u that satisfies the guidance condition:

$$\frac{u^+\alpha_\mu u}{u^+u} = \frac{\varphi^+\alpha_\mu \varphi}{\varphi^+\varphi} , \tag{3.29}$$

in which the φ denote the usual continuous solutions, then one sees that this singular region follows a trajectory L that corresponds to this continuous solution.

On the other hand, the relativistic guidance equations coincide with the preceding motion provided that the solutions $g_{\mu\nu}^{(0)}$ that are defined by starting with equations (3.3), in which one has used the fact that $f_{\mu\nu}$ = rot k_μ in $L_{(E)\mu\nu}$ also satisfies four supplementary equations:

$$\frac{dv^\alpha}{ds} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} v^\mu v^\nu = \partial^\alpha M_0 \tag{3.30}$$

because:

$$M_0(u^+u) = M_0(\varphi^+\varphi) \qquad \text{and} \qquad k_\mu(u^+u) = k_\mu(\varphi^+\varphi)$$

at the singularity.

As in the case of the Klein-Gordon equation, the preceding argument shows that there exist solutions of (3.3) that constrain the singularities to follow the trajectories (L) that were introduced by the causal interpretation of quantum theory.

§ 16. – We conclude this chapter with several considerations concerning systems of particles in a theory of this type.

In the previous chapter we saw how the introduction of the notion of wave singularity combined with the theory of guidance permits us to understand why the particles are only "piloted" by their proper waves. This suggests a simple model that might illuminate a possible deterministic theory of micro-objects in interaction.

We shall try to use this fact while incorporating the preceding theory.

From what we just presented, the point-like aspects of an ensemble of charged particles will be represented by a set of singular regions in the vector potential. On account of the field equations (3.3) these regions generate singularities in the metric and in the unique wave that determines this potential. For example, one may use the expression:

$$k_\mu = \frac{c}{e}\frac{u^+\alpha_\mu u}{u^+u} \tag{3.31a}$$

for the latter, in which the α_μ correspond to the Dirac representation, namely:

$$k_\mu = \frac{c}{e}(u^+D_\mu u - D_\mu^* u^+ u)/u^+u\,, \tag{3.31b}$$

when one neglects the effects of spin and confines oneself to using a function u with one component ($u = f \cdot \exp i\theta/\hbar$).

As before, one then introduces wavelets for the vector potential and gravitation, which are unique for all fields and are constructed from u with the aid of the preceding formulas, (3.31a) or (3.31b), and the expression:

$$g_{\mu\nu}^{(0)} = \varepsilon_{\mu\nu} M_0(u^+, u) + S_{\mu\nu}^{(0)}(u^+, u)\,.$$

In addition, one obviously assumes that the $g_{\mu\nu}^{(0)}$ satisfy the field equations (3.3) (in which the total electromagnetic field naturally appears) and four supplementary conditions that transform the congruence of trajectories that are subject only to M_0 into a geodesic congruence.

Finally, one decomposes this wave u into a sum of functions $u_{(\mathrm{I})}$ that correspond to individual micro-objects by supposing that the individual guidance conditions are valid.

Therefore, by hypothesis, one has the identity:

$$u = \sum u_{\mathrm{I}}\,,$$

which is valid at each point of spacetime.

Since one has $u \approx u_{\mathrm{I}}$ in each singular region, in the case of (3.31*a*) one may further write ([30]):

$$\frac{u^{+}\alpha_{\nu}u}{u^{+}u} \approx \frac{u_{\mathrm{I}}^{+}\alpha_{\nu}u_{\mathrm{I}}}{u_{\mathrm{I}}^{+}u_{\mathrm{I}}} \approx \frac{\varphi_{\mathrm{I}}^{+}\alpha_{\nu}\varphi_{\mathrm{I}}}{\varphi_{\mathrm{I}}^{+}\varphi_{\mathrm{I}}}$$

for each singularity, since one has used functions φ that individually satisfy the guidance conditions.

Geometrically, this signifies that the vector potential and the total field $g_{\nu\mu}$ take values that correspond to particle I, when considered in each singular region S_{I}. It then results that this particle follows the trajectory L that is defined by its proper wave φ_{I} *in the total field.* This wave φ_{I} is naturally constrained to satisfy a wave equation in which only the classical fields (gravitational and electromagnetic) that are generated by the other particles appear.

Later, we shall develop certain consequences of this model.

We note only that it is obviously presented in the same spirit as the theory of general relativity and the celebrated mathematical treatment of the *n*-body problem that was performed for the first time by Einstein and Infeld. Indeed, by definition, this treatment associates these *n* bodies with *n* singular regions of a unique field that is collectively constrained to satisfy certain nonlinear equations. From this, one deduces that, by reason of this latter character itself, the *n* singular regions are displaced in a particular fashion, which amounts to saying, as we have already emphasized, that the laws of motion are a natural consequence of the field equations.

Properly speaking, there is thus no interaction at a distance or potential energy between micro-objects in a theory of this type since the singularity-particles are displaced according to objective laws that govern the matter fields collectively.

Since the theory that we shall develop in this work is only a particular case of Einstein's theory, one may apply the preceding considerations to it in such a way that the particles are associated with singular regions of a unique field.

One must nevertheless note that the mathematical solution of the problem thus posed encounters difficulties in the general case that are so considerable that they have not been resolved up to the present.

Meanwhile, one may show – and this is not the least interesting result of Einstein and Infeld – that if one assumes the Newtonian approximation, i.e., if the world-lines that are followed by the singularities do not involve velocities and accelerations that are too big, then it is possible to treat the problem of the individual motion of these bodies in the spirit of classical mechanics. Indeed, under these conditions, everything happens as if the trajectory of each body is approximately calculable by starting with the classical laws and action-at-a-distance that are attached to the $n-1$ other bodies.

Therefore, on account of the quadratic character of the equations that govern them, it is not legitimate, in principle, to separate the singular regions that are associated with the corpuscular aspect of the micro-objects of the field ensemble. The study of an ensemble of *n* micro-objects must be globally undertaken without introducing interaction forces

([30]) One obtains analogous formulas by starting with (3.16) since the field and its singularities form a whole that must satisfy nonlinear equations collectively.

Nevertheless, from the example of Einstein and Infeld, one may admit that because of the theory of guidance one may look for an approximate solution to the *n*-body problem in the Newtonian approximation by superposing *n* isolated micro-objects that are subject to fictitious potentials that represent the action of the other micro-objects.

Indeed, as we will verify later, this approach permits us to use the results that were developed in the preceding chapter and to build a simple theory of the *n*-body problem in the context of the causal interpretation.

CHAPTER IV

We now propose to apply the preceding results to the theory of stochastic ensembles of elementary particles. Such a theory is indeed indispensable if one wishes to reinterpret the experimental statistical results that were obtained in the context of the usual quantum mechanics in the context of the unitary "model" of the double solution.

Two principal approaches to the question of the significance that one agrees to attribute to quantum statistics are gradually extracted:

The first approach, which is almost universally adopted today, was developed by Niels Bohr and Heisenberg. It consists of what one may generally call the approach of the Copenhagen School; this is a modern variety of the positivist school of Mach.

According to the Copenhagen School, the ensemble of all possible information that one may obtain about *one* micro-object is furnished by a "state function" of a statistical nature.

However, this is not the case, as Niels Bohr has strongly emphasized in a celebrated article on the "Dialectica" of statistics in the usual sense of the word, such as what one may encounter in classical statistical mechanics, for example. The wave ψ does not describe the micro-object; it only accounts for the probability that an observer will obtain a given value for a given physical magnitude after an interaction between this micro-object and the apparatus used to measure this magnitude, an interaction that is, in principle, uncontrollable.

More precisely, if one denotes the operator that is associated with this magnitude by A, and lets φ_i and λ_i denote the functions and their corresponding proper values ($A\varphi_i = \lambda_i\varphi_i$), then one may write:

$$\psi = \sum_i c_i \varphi_i \ ,$$

in which we have denoted the components of the development of φ by c_i. One then admits that $|\, c_i\,|^2$ furnishes the probability of obtaining the value λ_i for the magnitude after measuring it after this uncontrollable interaction.

It follows from these results that the experimentally-observed statistical character of quantum phenomena results from the interactions between the micro-objects and the macroscopic apparatus, which is unpredictable in principle. This is why the partisans of the Copenhagen School defend the positivistic idea that it is impossible to know and describe the structure and behavior of micro-phenomena independently of the observer. According to Pauli, for example, the object of physics is simply that of defining a mathematical apparatus that is capable of predicting the numerical results that are furnished by particular experiments. In a discussion with the author along these lines, Rosenfeld estimated that the question of the real motion of the electrons independently of our existence is devoid of meaning and interest; he judged it to be purely metaphysical.

The second approach, which is expressed in various forms, was introduced that the onset of quantum mechanics by de Broglie and Langevin. Although it was abandoned for a score of years, it has been reprised and systematically developed in 1947-1948, first by the Academy of Sciences in the U.S.S.R. by Blokinzef and Terletski, and then more recently by Bohm, de Broglie, and the author.

For example, Terletski ([1]) summarized this viewpoint as follows:

1) Micro-objects exist independently of any observation.

2) It is possible to forge a deterministic representation from this that accounts for both the real individual behavior of these micro-objects and the objective properties of statistical ensembles of such objects. This "model" must account for the corpuscular and wavelike aspects that are presented by micro-processes.

3) At least in the eyes of the complementarity principle, quantum mechanics [as we know it, ed. note] is not a theory of individual micro-objects. Quantum mechanics is a statistical theory, i.e., a theory that is applicable to only statistical ensembles of micro-objects. Quantum mechanics may not completely represent the motion of an individual micro-object (electron, photon, etc.), but only the behavior of an ensemble of identical micro-objects that appear either simultaneously or in a series of consecutive experiments. This is due to the fact that the existing apparatus of quantum mechanics permits us to calculate only the possible values for different physical magnitudes (problem of proper values) and the probabilities of this or that physical state, or the transition probability from one state to another. The knowledge of the probability of a given state for a micro-object does not, moreover, give complete information about its true state, and, as a consequence, the description that is given by quantum mechanics with the aid of a wave function does not represent the state of the object entirely.

One may illustrate this viewpoint experimentally.

The celebrated experiments of Vavilov ([2]) on the microstructure of light, which developed the experiments of Taylor, Dempster, and Bathos, indeed showed that the interference figures that are obtained with the aid of a light flux of weak intensity present corresponding fluctuations in the arrival of the individual photons that comprise them. One then observes that if the dark fringes are maintained without change then the bright fringes present independent incoherent fluctuations. This may be interpreted by saying that the photons are subject to the usual wavelike laws, despite their apparent chaos.

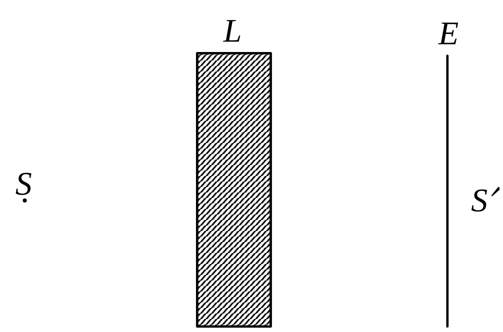


Fig. 15.

One may redo these experiments with electrons. For example, this is why the

([1]) *Questions de Physique*, ed. Réunis, Paris (1953).
([2]) S.I. Vavilov, *Progrés des Sciences Physiques*, t. XVI (1936), pp. 892-897.

experiments in the diffraction of electrons emitted one-by-one have been carried out by Bibermanm, Souchkine, and Fabrikant ([3]). Each electron that happens to traverse the diffraction system that is being used makes a small spot (electron impact) on a receiving screen. If one then prolongs the experiments sufficiently, these spots, which are dispersed without order unless they not very numerous, eventually form a diffraction figure that one may predict with the aid of the state functions of quantum mechanics.

In particular, the image S' of a point-like source S (that emits electrons one-by-one) that is produced on a screen E by a magnetic lens L is composed of a distribution of spots (with a density $\psi\psi^*$) in the diffraction ring that is predicted by the Schrödinger equation.

In summation, in the context of this second viewpoint, the essential problem that is posed to physicists is summarized as follows:

> "To find one and only one deterministic micro-mechanical 'model' of the individual micro-objects that admits the usual quantum mechanics as its statistical mechanical objective."

It is in this spirit that de Broglie, for example, developed his theory of the double solution in 1927, which distinguishes between:

- a real wave u with a singularity that represents both a particle and its field,
- a wave ψ of the same phase, which is charged with describing the statistical evolution of an ensemble of particles of the preceding type.

It is clear that only the second viewpoint is compatible with the foundations of the relativistic theory that we started with. We thus adopt it, and shall now seek to establish that the statistical laws that are associated with the definition of elementary particles that was given in chapter II give back the usual quantum mechanics (when interpreted in the context of the preceding viewpoint).

By statistical laws, we obviously mean laws that describe the real behavior of ensembles of the objects considered, laws that have (more or less) nothing to do with the knowledge that a possible observer has about this ensemble. For us, the calculation of probabilities has, in effect, the objective of correctly describing the manner by which certain events are actually produced in a very large ensemble of events that are subject to very complex subordinate causal laws. The proof that we shall give constitutes an illustration of this viewpoint, and will permit us to conclude this chapter with several aspects of the physical significance of the notion of chance in the theories of the type that was developed in these researches. Like all of the general considerations just followed, they result from the work performed in collaboration with D. Bohm (the results that were developed and detailed will be published later), and constitute the expression of our viewpoint on the nature of the statistical considerations that we agree to introduce in the context of the causal interpretation of quantum theory.

§ 2. – Before studying the statistical ensembles of particles, we must devote more attention to the physical plane and propose a more complete model of the real behavior of

([3]) *Comptes rendus l'Académie des Sciences de l'U.R.S.S.*, t. LXVI, no. 2 (1949), pp. 185-186.

the micro-objects that correspond to the "model" that was proposed in the preceding chapter. The mathematical representation that we gave in it obviously provides us with an approximation for the true properties and one must take this fact into account if one is to state laws that valid for real ensembles of objects.

In our "model" chapter, the elementary particle corresponded to a singularity (gravitational or electromagnetic) with spherical spatial symmetry that is associated with an elementary gravitational wave. It is represented mathematically by a wave u with a singularity (that may have several components u_α), whose regular part we designate by φ.

As we have seen, φ must satisfy the equation:

$$\alpha^\nu \partial_\nu \varphi - \mu\varphi = 0 \tag{4.1}$$

(if we neglect the action of the ordinary gravitational field), and the center of the singularity of u will follow the streamlines:

$$s_\mu = \varphi^+ \alpha_\mu \varphi\,, \tag{4.2}$$

with which it initially coincides.

On such a trajectory a classical particle of mass m_0 will be subject to the combined actions of:

- an invariant potential M_0 and
- a potential quadri-vector P_μ,

and the current s_μ satisfies the continuity equation:

$$\partial_\mu s^\mu = 0\,.$$

Note that from the viewpoint that is developed in this work, the given of u permits us calculate all of the properties of the micro-object that appears to be both localized (concentrated around the center of u) and extended (since it is accompanied by the "quantum field" that is defined by φ). u and φ are, in fact, indissolubly linked; how the trajectory of the center of the singularity appears will be determined by only the function φ.

If we consider the Newtonian approximation and neglect the effects of *spin* then we have seen that φ satisfies the Schrödinger equation. If we then set $\varphi = R\exp(iS/\hbar)$, this decomposes into two parts (real and pure imaginary) that furnish the two classical equations:

$$\begin{cases} \dfrac{\partial|\varphi|^2}{\partial t} + div(|\varphi|^2\, \nabla S/m) = 0 \\ \dfrac{\partial S}{\partial t} + \dfrac{(\nabla S)^2}{2m} + V - \dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R} = 0, \end{cases} \tag{4.4}$$

the first of which corresponds to the continuity equation, and the second describes the streamlines.

In order to use the language of hydrodynamics, one may also say that everything happens as if the center of the singularity is constrained to follow one of the streamlines of a pilot-fluid whose streamlines are collinear with s_μ , and the whose density is given by $\rho = \varphi^+ \alpha_4 \varphi$.

In the previously considered Newtonian approximation, this fluid therefore had a density $\rho = R^2$, and its streamlines had the velocity:

$$\vec{v} = S / m.$$

The practical application of the preceding model to the real cases rests on the following essential theorem, which is an immediate consequence of the definitions we adopted and the field equations:

Theorem:

The given of:

– the initial values of φ,
– the values of the fields that appear in (4.1),
– the initial position of the center of the singularity,

suffices, in principle, to completely calculate u if we also use the field equations of relativity ([4]).

Physically, it is clear that this description may only pretend to be an approximation of everything that actually happens in nature ([5]). Indeed, the field φ may be defined and determined only on the condition that we satisfy hypotheses (on the initial conditions at the limits, etc.) that are never realized in practice.

We clarify this point by analogy with a classical case. For example, consider a stationary macroscopic electromagnetic field in the interior of an enclosure. It is defined by macroscopic stationary conditions on the walls; it is therefore given by the corresponding stationary solution of the Maxwell equations. Theoretically, it is perfectly defined. Physically, it is obvious that this solution may represent only a sort of mean state for the field, and that this is true for two principle reasons:

In the first place, the boundary conditions that were introduced into the field equations are *a priori* unrealizable in full rigor. Other than the fact that it is not possible to absolutely isolate the system considered from the external universe, real walls are

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([4]) Here, we must emphasize that the statistical proofs that must follow do not rest on the particular model used or the form of the quadratic equations that are employed for u. They suppose only that:

a) The point-like aspect of each micro-object is guided along the streamlines by a continuous field φ that represents its wavelike aspect and satisfies (4.1).
b) The preceding theorem applies if we replace the words "suffices, in principle,…" with "suffices, in principle, to completely determine the evolution of φ, and the trajectory of the point-like aspects of the micro-objects.

([5]) To recall a remark that was made by Darmois, the level of schematization is always less than the complexity of the reality that it represents; however, it may be sufficient. In truth, it is becoming insufficient, and the search for a better level must naturally be guided by the concrete notions that result from a better knowledge – at least, a mental one – of the complexity in reality.

necessarily formed, in effect, from very complicated physical systems (molecules) that are in perpetual oscillation around certain equilibrium positions that correspond to the conditions that were used in the calculations.

As a result, Maxwell's equations may not pretend to completely describe the nature of the electromagnetic field. In fact, we know that they do not account for the corpuscular structure of radiation and that they represent only the macroscopic effect of the ensembles of photons that they define at the "quantum level."

In a similar fashion, as de Broglie has remarked, the macroscopic laws of hydrodynamics give a continuous approximation of the very complex and very rapidly varying motions of the fluid molecules.

Therefore, if one accounts for both the necessarily imperfect character of the field equations and the numerical values that one introduces into these same equations then one sees that the results obtained may represent only a sort of mean state of the real fluid. The latter always oscillates in time and at each point around the values that are obtained by calculations that correspond, in summation, to *one* mean value of the physical conditions that are actually realized in nature, at the location considered.

This analysis also applies, *mutatis mutandis*, to the field φ.

Consider a field u. Everything happens mathematically as if the center of the singularity follows a streamline of the fictitious pilot fluid that corresponds to the function φ (which is itself defined by the preceding theorem). Therefore, if one wants to rigorously give the initial values of φ on a spacelike surface and the evolution of the fields that appear in (4.4) then the singularity will quite simply follow the streamline l (the bold line in the figure), which belongs to the congruence (L) of the streamlines that are defined by s_μ, with which they initially coincide.

Physically, this might not be the case exactly, because the external fields and the real field necessarily fluctuate around the calculated values of φ, which do not take into account the complexity of the real systems that interact with u, or the subordinate processes that were neglected in this representation.

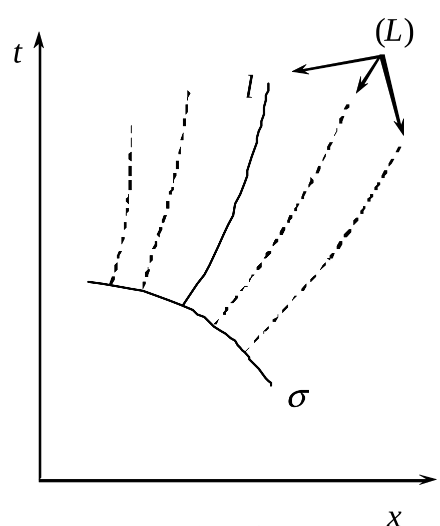

Fig. 16.

In order to complete our description, it is therefore necessary to make certain hypotheses on the properties of these fluctuations. These hypotheses constitute the statistical hypotheses at the basis for statistical mechanics that we propose to associate with the micro-mechanics of the causal interpretation. In our theory, they play a role that is equivalent to the hypothesis of molecular chaos in kinetic theory, and may be justified only *a priori*. They are nevertheless sufficiently plausible that they may serve as the basis for our analysis of stochastic processes.

We therefore suppose that, in principle, these fluctuations are themselves subject to deterministic laws:

1) They are sufficiently complex to be treated statistically. By this, we intend that they are sufficiently chaotic and devoid of correlations for them to be considered as relevant to the calculus of probabilities.

Mathematically, this translates, for example, into the introduction of the theory of fields and random boundary conditions that oscillate in the course of time around the numerical values that are used in the equations that collectively constitute a type of mean abstraction of the true physical conditions.

2) They may be represented with the aid of a pilot fluid that agrees with the pilot fluid that is defined by the calculated φ except for fluctuations (φ is obviously calculated in terms of the mean conditions that act on the system being considered), i.e.:

a) During the fluctuations, the singularity effectively follows a true streamline of the fluid that represents the actual fluctuations.

b) This fluid is conservative, so that one has $\partial\rho/\partial t + \text{div}\,\rho\vec{v} = 0$ during the fluctuations, where ρ and $\vec{v}$ denote its density and velocity at each point, respectively.

c) Everything happens as if the particle singularities actually jump to another trajectory L' in the course of a perturbation of a calculated trajectory L. L and L' belong to the congruence (L) that one calculates without taking into account the possible fluctuations, and corresponds to the φ we calculated previously.

Physically, this is interpreted by saying that in the course of time the fluctuations do not destroy the quantum field that is defined by the calculated φ. This is a very natural hypothesis because if things were otherwise then it would not be possible to define the extended aspect of micro-objects at the microscopic level. *b*) and *c*) then translate into the fact that one may not perturb this field without modifying the trajectory of this particle in a well-defined fashion, if one takes into account the model that is used. This property is essential because will permit us to show, in turn, that this conservation of the field will entail the existence of quantum statistics.

3) They have an origin that is independent of the properties of the system and its position in space, and constitute a continuous aperiodic non-stationary Markoff process. Physically, we intend this to mean that:

a) The probability that these fluctuations will appear depends only on time.

b) They generate "jumps" in the molecules of the pilot-fluid, in such a way that the points of arrival are continuously distributed around the point of departure.

c) No region of space is forbidden in the displacements of type *b*) if one is given a sufficiently large interval of time and starts from an arbitrary point of departure.

If one then follows a particle in its motion through a region in which these fluctuations are produced then we propose to consider that they describe a trajectories of "steps" that we may mathematically consider to be composed of pieces of trajectories that belong to (L) linked together with brief jumps (see fig. 17). Its actual trajectory sweeps out a subset of the congruence (L). Between two spacelike surfaces, such as σ and σ', one has: φ (real) = φ (calculated) + $\delta\varphi$.

Therefore, in principle, one may not calculate the motion of the particle singularity that rigorously gives us the evolution of the ensemble of physical systems that act on it. It is clear that this is not possible – at least unless one wants to attribute something in the nature of sufficiently extensive information to the ensemble, as one does with Maxwell's demon in kinetic theory.

We summarize the results of the preceding discussion as follows:

A. Whenever one confines oneself to the study of the motion of *one* particle singularity, one sees that it behaves as if it conforms, in practice, to the schema that was described in Figure 16. Indeed, by hypothesis, one may neglect the effect of the fluctuations during a finite period, and consider that, in practice, the particle is always confined to one of the streamlines of the causal interpretation that is defined by the wave φ, which is the regular part of u. Therefore, if one abstracts from the phenomena that are attached to the structure of the particle itself then one may consider them to be described by φ, *plus* the streamline on which it is found initially. In this context, an incomplete description of the "state" of the individual particle is comprised of the function φ linked with a definite state of external phenomena that are capable of influencing its motion. This is, moreover, an immediate consequence of the classical hypotheses, which conform to deterministic laws, of the interaction and reciprocal conditioning of the ensemble of physical systems that constitute Nature. The ensemble composed of u and the phenomena that is influences themselves satisfy deterministic laws, as long as the given of φ is coupled to the determination - at least macroscopically – of the systems in a particular state. u then evolves like an isolated system (under the influence of certain potentials) that is in equilibrium with the macroscopic phenomena that surround them. The preceding fluctuations are linked to those properties of these phenomena that do not perturb this equilibrium. In general, they must have a total duration that is negligible with respect to that of the regime that corresponds to φ whenever one may consider this "state" to be microscopically well defined in an arbitrary time interval.

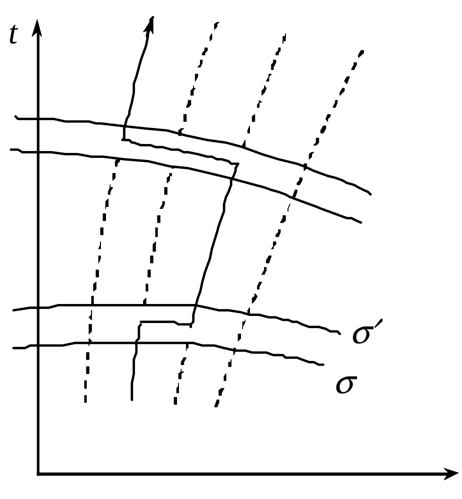

Fig. 17.

B. Nevertheless, if one wants to determine the evolution of a statistical ensemble of particles of the type just considered over a long period of time then it is necessary to take into account all of their actual properties that are deemed to be capable of influencing this evolution, hence, the possible motions that are described in Figure 17.

The preceding considerations permit us to undertake the examination of the statistical laws that are associated with micro-mechanics of the causal interpretation.

The first step along these lines consists of seeking a physical definition of what one may call "a statistical ensemble of particles in a given state." Such a definition is necessary since the quantum ensembles of micro-objects do not reduce to arbitrary collections in the context of the usual probabilistic interpretation.

The considerations that were developed in section two of this chapter immediately suggest the following definition:

Definition: An ensemble (I) of non-interacting particles, which is defined by waves (I = 1, 2, …), whose regular parts we denote by φ_{I}, will be said to exist "in a given state φ" if one may write:

$$\varphi = \varphi_{\mathrm{I}} ,$$

everywhere for any I.

This amounts to saying that an ensemble of particles is in a given state if the waves u_{I} that define them have the same regular part.

This definition has a precise physical sense:

As we have seen, the given of φ corresponds to *one* given macroscopic state of the external physical system that acts on u_{I}, a state that obviously corresponds to what one usually calls the "preparation ([6])" of the statistical system of micro-objects considered. This state does not completely determine the individual objects (since the given of φ_{I} does not define the u_{I} uniquely, nor does it indicate, in particular, on which trajectories the particles are displaced, but restricts their motion in such a fashion that it is possible, as we shall see, to deduce the characteristic statistical properties of the ensemble so defined.

The definition of statistical state then permits us to commence with the statistical mechanics of ensembles of particles, which is essentially the object of this chapter.

We subdivide that study as follows:

A. Let (I) be a statistical ensemble of particles in a given state, which is defined by a function φ.

From the preceding, we know that:

a) Everything happens as if these particles were restricted to following the true streamlines of the pilot fluid that correspond to φ, in which we denote the density and velocity at each point by $\rho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$. One will, moreover, always have the classical relation:

([6]) In the following chapter, after having studied systems of particles in interaction, we will indicate more precisely what the notion of "preparation" signifies at the quantum level.

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \cdot \vec{v} = 0 \,. \tag{4.5}$$

b) This fluid has its density ρ and velocity $\vec{v}$ that are determined at each point and each instant by the function that was defined by equation (4.1), except in certain very brief intervals in which "fluctuations" appear, whose properties we have defined. Before and after these "fluctuations," the fluid is found in the state defined by the wave function and equation (4.1), and its streamlines are defined by the congruence (*L*) that was previously introduced. Therefore, each fluctuation of the fluid briefly transports a streamline *L*, which is defined by (4.1), to a neighboring line. As a result, in the course of time a series of fluctuations transports a fluid along a trajectory in "steps" (whose pieces are formed from fragments of the streamlines and whose jumps are the fluctuations), as described by Figure 17.

We then denote the density of the particle-singularities distributed along the fluid by $P(\vec{x}, t)$.

If there are no fluctuations then it is clear that these particles simply follow the trajectory-streamlines with a density *P* that corresponds to their initial distribution.

Since this density also satisfies the continuity equation:

$$\frac{\partial P}{\partial t} + \operatorname{div} P \cdot \vec{v} = 0 \,, \tag{4.6}$$

one deduces that the ratio $F(\vec{x}, t) = P(\vec{x}, t) / \rho(\vec{x}, t)$, which is equal to $P / |\varphi|^2$ in the Newtonian approximation) remains constant along each trajectory. Indeed, upon comparing (4.6) with (4.5), one infers the relation:

$$\frac{\partial F}{\partial t} + \operatorname{div} F \cdot \vec{v} = 0 \,,$$

which signifies that the derivative of *F* along a trajectory (*L*) is null, as de Broglie remarked in 1927. One therefore has:

$$\frac{dF}{dt} = 0 \,,$$

or furthermore:

$$F(\vec{x}, t) = F(\vec{x}', t') \,,$$

if $(\vec{x}, t)$ and $(\vec{x}', t')$ define points that are situated on the same streamline.

As is well known, if one then introduces the function, $\psi = P^{1/2} \exp(iS/\hbar)$, which does not satisfy the same equation as φ, in general, but an equation with a right-hand side. In the case considered (in which $\varphi = R \exp(iS/\hbar)$, $\rho = R^2$, and $\vec{v} = S/m$), one may write it in the following form:

$$\Delta \psi - \frac{2m}{\hbar^2} V \psi - \frac{2im}{\hbar} \frac{\partial \psi}{\partial t} = \left(\frac{\Delta R}{R} - \frac{\Delta P^{1/2}}{P^{1/2}} \right) \psi \,,$$

which reduces to the Schrödinger equation only if $\Delta R/R = \Delta P^{1/2} / P^{1/2}$ in the initial state.

One concludes from this that in the absence of fluctuations an ensemble of particles that is initially distributed in an arbitrary fashion on (L) may not be represented by a wave ψ that satisfies the usual wave equations.

This is not the case when one takes the actual motions of the particle-singularities into account because the previously described fluctuations transport the fluid, and, as a result, the particles of one line (L) to another, conforming to Figure 17. In this case, the ensemble (I) behaves, in fact, like an ensemble of particles that are displaced on (L) with a density that varies in the course of time.

Indeed, if one abandons the thought of following each isolated particle then, on average, one will never have to be concerned with the particles that follow (L), since, by hypothesis, one may neglect the duration of the fluctuations with respect to that of the regime defined by the calculated φ.

Similarly, if all of the particles are initially concentrated then they are finally distributed on (L) *with a density that varies in the course of time on each trajectory.*

The preceding relations $dF / dt = 0$ and $F(\vec{x}, t) = F(\vec{x}', t')$ are therefore *not* satisfied on (L). $F(\vec{x}, t)$ evolves on each trajectory and no longer depends upon only the actual behavior of the fluctuations.

B. Conforming to the "program" of Blokinzef and Terletski, we then propose to prove the following two statistical laws:

I. On each trajectory L the density $P(\vec{x}, t)$ tends towards a stable limit independently of the initial distribution of the particles. This density plays a role with respect to the previously defined micro-mechanics that is analogous to the one that is played by Maxwell's density (in the kinetic theory of gases) vis-à-vis classical mechanics.

II. This limiting density is nothing but:

$$\begin{aligned} P &= K\varphi^{+}\alpha_4\varphi \\ &= KR^2, \end{aligned}$$

in the Newtonian approximation, where K is a normalization constant.

Mathematically, this signifies that one may represent the limiting statistical density by a wave ψ that has the same phase and an amplitude that is proportional to R, and satisfies the same linear equation.

This conforms completely to the ideas that were developed by de Broglie and the author on the theory of the double solution, because the proof of the preceding laws permits us to establish that an ensemble of particles that is in the most probable state – in the sense of the causal interpretation – behaves exactly like the ensembles of Blokinzef from the statistical point of view.

First, we look at this proof:

It rests on the following lemma, which we shall establish with the aid of methods that are analogous to the procedures that were used by Einstein, Smoluchowski, and Langevin in the theory of Brownian motion:

Lemma: If one lets ρ and $\vec{v}$ denote the density and velocity at each point of a fluid without fluctuations that satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \cdot \vec{v} = 0$$

then any ensemble of particles that is constrained to follow the streamlines (L) ends up by being distributed with a density that is proportional to ρ if one subjects this fluid to particular random fluctuations of the previously described type; moreover, this will be true for any initial distribution of particles that is considered.

First, we clarify the nature of these fluctuations. In order to do this, we let $\Delta(\vec{x}, t)$ denote a very small volume element, each point of which is constrained to follow a streamline of the fluid without fluctuations with the velocity $\vec{v}$. In general, such an element must change form in a very complicated manner in the course of time, but one may choose it to be very small in order for it to remain confined in a small volume for any interval of time considered.

Having said this, the nature of the fluctuations will be defined by four conditions (which are denoted A, B, C, and D) that translate properties 1, 2, 3, 4, of the general conditions that we have enumerated to the mathematical plane.

A. By hypothesis, each element of the pilot-fluid is subject to random perturbations of very short duration Δt at certain instants $\{\tau_k\}$.

Because of condition 3, we therefore first suppose that the distribution of these instants $\{\tau_k\}$ is a Poisson process of density $\nu(t)$ independently of the motion of the fluid and the position of the molecules. This is reasonable, because, by hypothesis, the origin of these fluctuations depends on the external physical conditions that act on it.

In the course of a perturbation at the time t, each molecule will be transported from a position $\vec{x}$ to a new random position.

In an interval of time δt that is arbitrarily small, but long with respect to Δt, one may therefore introduce the probability:

$$\nu(t)\,\delta t\; K(\vec{x}, \vec{x}', t)$$

for an element to pass from the point $\vec{x}$ into the interval $d\vec{x}'$ that is centered at $\vec{x}'$. In general, it depends on the instant t and the positions $\vec{x}$ and $\vec{x}'$.

One obviously has:

$$\int_v K(\vec{x}, \vec{x}', t)\, d\vec{x}' = 1 \tag{4.9}$$

since it must be true that the fluid that leaves from x arrives in some part of space.

B. We know, in turn (see 2.), that each fluctuation conserves the fluid that is found in the state defined by ρ and $\vec{v}$ before and after.

This conservation therefore translates into the equality:

$$\rho(\vec{x},t) = \int_v K(\vec{x},\vec{x}',t) \cdot \rho(\vec{x}',t) d\vec{x}', \tag{4.10}$$

which expresses that the density of the fluid at each point remains unchanged when fluctuations are produced.

This is an essential physical hypothesis on the nature of the fluctuations that are envisioned – that they be collectively constrained to conserve a privileged density ρ among all possible densities that one may associate to (L). Combined with the laws of motion that we adopted, it allows us to show that the density of particles that are transported by the fluid tends to this particular density.

C. Because of condition 3, $K(\vec{x},\vec{x}',t)$ is square-summable and different from zero in a compact domain $D(\vec{x})$ that contains each point $\vec{x}$. Physically, this signifies that the fluctuations allow the "jumps" to be distributed in no particular way around $\vec{x}$ without any closely neighboring point being forbidden.

D. The domains $D(\vec{x})$ are such that it is always possible to pass from a point O_1 to an arbitrary point in space O_N by passing through a finite number of intermediate points O_2 O_3, …, O_{N-1}, such that the corresponding domains $D(O_1)$, …, $D(O_N)$, partially cover the domains of the points that immediately precede them and follow them (see fig. 18). Physically, as we have seen, this hypothesis is equivalent to the fundamental hypothesis that was introduced by Markov in the theory of stochastic examples. It signifies that one may always find an interval of time that is sufficiently large that a fluid element can pass through a very thin current tube to another one that encloses the trajectories (L), and with a probability that is different from zero.

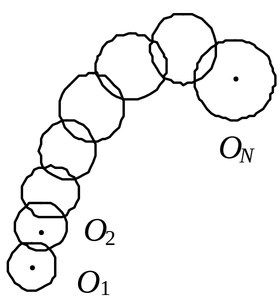

Fig. 18.

Let us move on. In the course of that interval δt, fluctuations are produced that contract and expand the fluid, hence the particles in any volume element of the preceding type $\Delta\omega(\vec{x}, t)$ that we might like to follow along its trajectory.

Thus, if there are no fluctuations then the number of particles that are contained in $\Delta\omega$, namely:

$$\Delta N = P(\vec{x}(t) \cdot \Delta\omega(\vec{x}(t)),$$

will not change and depends only on their density $P(\vec{x}(t))$.

When there are fluctuations, ΔN must vary in the course of time, and one must necessarily study the variation of the distribution of particles. In order to do this, we follow what happens to a given element $\Delta\omega(\vec{x}(t))$.

Let $\delta\Delta N$ denote the variation of ΔN during the time $\delta\tau$. I say that this variation is equal to the number of particles, such that the "jumps" that end in $\Delta\omega(\vec{x}(t))$ diminish the number of particles whose jumps begin in $\Delta\omega(\vec{x}(t))$; what we are calling "jumps" are the motions of the particles away from the trajectories (L) when acted on by the fluctuations considered.

Indeed, the particles whose jumps end in $\Delta\omega(\vec{x}(t))$ are subdivided into two categories, namely:

a) the particles that come from the exterior of $\Delta\omega(\vec{x}(t))$,
b) the particles that come from the interior of $\Delta\omega(\vec{x}(t))$.

Similarly, the particles whose "jumps" start in $\Delta\omega(\vec{x}(t))$ may be decomposed into two categories, namely:

a) the particles that leave $\Delta\omega(\vec{x}(t))$,
b) the particles that remain in $\Delta\omega(\vec{x}(t))$.

If one remarks that the particles of the two categories, *b*) and *c*), are obviously equal in number, since they both represent *the number* of particles that remain in $\Delta\omega(\vec{x}(t))$ in the course of δt, then one sees that the preceding proposition amounts to saying – as is obviously exact - that the variation $\delta\Delta N$ is obtained by subtracting the particles that leave $\Delta\omega(\vec{x}(t))$ from the particles that enter it during the time interval considered.
Obviously, the number of particles in *a*) and *b*) may be then written:

$$v(t)\delta t\int \Delta\omega(\vec{x}(t))\cdot P(\vec{x}'(t))\cdot K(\vec{x}',\vec{x},t)_{x'}\cdot d\vec{x}' . \tag{4.11}$$

The integrand denotes, quite simply, the number of particles that are contained in $d\vec{x}'$ multiplied by their probability of passing into $\Delta\omega(\vec{x}(t))$. The number of particles in categories, *c*) and *d*), is subsequently given by:

$$v(t)\delta t\int \Delta\omega(\vec{x}(t))\cdot P(\vec{x}'(t))\cdot K(\vec{x}',\vec{x},t)_{x'}\cdot d\vec{x}' = v(t)\delta t\cdot\Delta\omega(\vec{x}(t))P(\vec{x}'(t)) , \tag{4.12}$$

on account of (4.9), since one must integrate over all of the possible $\vec{x}'$.

As a consequence, we obtain the expression:

$$\delta\Delta N = v(t)\delta t\Delta\omega(\vec{x}(t))\cdot\left[\int P(\vec{x}'(t))\cdot K(\vec{x}',\vec{x},t)_{x'}\cdot d\vec{x}' - P(\vec{x}'(t))\right],$$

which may also be written (since $\delta P = \delta\Delta N/\Delta\omega$):

$$\frac{\delta P}{\delta t} = v(t)\left[\int P(\vec{x}'(t))\cdot K(\vec{x}',\vec{x},t)_{x'}\cdot d\vec{x}' - P(\vec{x}'(t))\right]. \tag{4.13}$$

This integro-differential equation defines the variation of the number of particles in $\Delta\omega$ due to fluctuations. It permits us to prove the stated lemma.

Indeed, set:

$$F(\vec{x}(t)) = P(\vec{x}(t))/\rho(\vec{x}(t)).$$

One may then write:

$$\delta\Delta N = \delta\{P(\vec{x}(t)\cdot\Delta\omega(\vec{x}(t))\}$$

$$= \Delta\omega(\vec{x}(t)) \cdot \rho(\vec{x}(t))\ \delta F(\vec{x}(t),$$

since, by hypothesis, the product $\rho\,\Delta\omega$ remains constant along a streamline.

Upon equating the two values of $\delta\Delta\omega$, one therefore obtains:

$$\frac{\delta F}{\delta t}(\vec{x}(t)) = v(t)\left\{\int F(\vec{x}'(t)) \cdot \frac{\rho(\vec{x}'(t))}{\rho(\vec{x}(t))} \cdot K(\vec{x}', \vec{x}, t)_{x'}\, d\vec{x}' - F(\vec{x}(t))\right\}$$
$$= v(t)\left\{\int L(\vec{x}', \vec{x}, t) \cdot F(\vec{x}', t)\, dx' - F(\vec{x}(t))\right\}, \tag{4.14}$$

if one sets $L(\vec{x}', \vec{x}, t) = K(\vec{x}', \vec{x}, t) \cdot \rho(\vec{x}', t) / \rho(\vec{x}, t)$ when $\rho(\vec{x}, t)$ is different from zero.

Because of (4.10), we then obtain:

$$\int L(\vec{x}', \vec{x}, t)\, d\vec{x}' = 1\ , \tag{4.15}$$

and if we change the time scale, i.e., if we replace $\delta\tau = v(t)\delta t$ (which assumes that $v(t) \gg \varepsilon > 0$):

$$\frac{\delta F}{\delta\tau} = \int L(\vec{x}', \vec{x}, t)) \cdot F(\vec{x}'(\tau))\, d\vec{x}' - F(\vec{x}(\tau)). \tag{4.16}$$

We therefore recall a classical argument of Markoff. One first sees that F = const. is a solution of (4.16) (because of (4.15)) because it annuls the right-hand side.

We then let $\vec{x}_M(\tau)$ and $\vec{x}_m(\tau)$ be the values of $\vec{x}$ for which $F(\vec{x}, \tau)$ attains its maximum and minimum values, $M(\tau)$ and $m(\tau)$, at the instant τ.

The inequalities (4.15) and (4.16) immediately lead to:

$$\begin{cases} \dfrac{dM(\tau)}{d\tau} + M(\tau) \leq M(\tau) \\ \dfrac{dm(\tau)}{d\tau} + m(\tau) \geq m(\tau) \end{cases} \tag{4.17}$$

if $M(\tau)$ and $m(\tau)$ are absolute extrema in the domain $D(\vec{x}_M)$ and $D(\vec{x}_m)$.

From this, one deduces that the functions $M(\tau)$ and $m(\tau)$ are monotone in the course of time, namely, non-increasing and non-decreasing, respectively.

If we then set $\lambda(\tau) = M(\tau) - m(\tau)$ then we will obviously have $\lim_{\tau\to\infty} \frac{d\lambda}{d\tau} = 0$; by hypothesis, $\lambda(\tau)$ is monotone and non-increasing $\left(d\lambda / dt \leq 0\right)$.

Now, the inequality (4.16) implies:

$$\frac{d\lambda(\tau)}{d\tau} = -\lambda(\tau) + \int_v \{L(\vec{x}', \vec{x}_M, \tau) - L(\vec{x}', \vec{x}_m, \tau)\} F(\vec{x}', \tau)\, d\vec{x}'\ ,$$

or furthermore, if we add a null term:

$$\frac{d\lambda(\tau)}{d\tau} = -\lambda(\tau) + \int_v L(\vec{x}', \vec{x}_M, \tau)(F(\vec{x}', \tau) - m(\tau))d\vec{x}'$$
$$-\int_v L(\vec{x}', \vec{x}_m, \tau)[F(\vec{x}', \tau) - m(\tau)]\,d\vec{x}'.$$

One may therefore finally write:

$$\frac{1}{\lambda(\tau)} \cdot \frac{d\lambda(\tau)}{d\tau} = -1 + \int_v \{L(\vec{x}', \vec{x}_M, \tau) - L(\vec{x}', \vec{x}_m, \tau)\} \cdot \frac{F(\vec{x}', \tau) - m(\tau)}{\lambda(\tau)} \cdot d\vec{x}'. \qquad (4.18)$$

We distinguish two cases:

1. $\lim_{\tau\to\infty} \lambda(\tau) = 0$. In this case, one sees that $F(\vec{x}, \tau) \to$ *const*. everywhere, and is equal to the common value that is taken by M and m.
2. $\lim_{\tau\to\infty} \lambda(\tau) = k \neq 0$. In this case, if we set:

$$\begin{cases} M(\infty) = M_l = const. \\ m(\infty) = m_l = const. \end{cases}$$

then equality (4.18) gives, in the limit:

$$\int L(\vec{x}', \vec{x}_M, \infty)\left\{\frac{F(\vec{x}', \infty) - m}{\lambda}\right\}d\vec{x}' = 1 + \int_v L(\vec{x}', \vec{x}_m, \infty)\left\{\frac{F(\vec{x}', \infty) - m_l}{\lambda_l}\right\}d\vec{x}' \qquad (4.19)$$

This is obviously possible only if one has:

$F(\vec{x}', \infty) = M_l$	everywhere along	$D(\vec{x}_M)$ when $\tau \to \infty$	(4.20*a*)
$F(\vec{x}', \infty) = m_l$	" "	$D(\vec{x}_m)$ " "	(4.20*b*)

because of (4.15) and the fact that $\dfrac{F(\vec{x}', \infty) - m_l}{\lambda_l} \leq 1$ for any τ.

However, relations (4.20) are not compatible, which is contrary to the hypotheses that $\lambda(\tau) \neq 0$ and $M_l = m_l$ everywhere. Indeed, because of hypotheses C and D, one may extend relations (4.20*a*) and (4.20*b*) to all of space by analytic continuation, which is possible only if M_l and m_l are equal.

Therefore, if $\tau(t) \geq \varepsilon > 0$ during the time that is necessary for the limiting distribution to be established (which is natural if one considers that the origin of the fluctuations is external to the fluid considered) then one sees that the density P of particle-singularities tends strongly to $k\rho$ in the course of time (k is a constant that one may take to be = 1 by a suitable renormalization), since $F(\vec{x}, t)$ always tends to 1.

One therefore has: $P(\vec{x}, \tau) = \rho(\vec{x}, \tau)$, which proves the stated lemma. Therefore, an arbitrary distribution of particles will necessarily tend to ρ in the course of time; the density $P = \rho$ constitutes a stable limiting distribution that will no longer be destroyed by the preceding fluctuations.

By virtue of the hypotheses we made, it is clear that this lemma applies to the causal definition of an ensemble of particles in a given state; the pilot-fluid plays the role of the preceding fluid.

Therefore, an ensemble of particles in a given state will necessarily satisfy the two statistical laws that were stated at the beginning of B, laws that we state as follows: The density of an arbitrary stochastic ensemble of micro-objects in a given state φ in the sense of the causal interpretation tends toward a stable limiting distribution that is described by a wave ψ of the same phase and an amplitude that is proportional to the continuous real wave φ that defines the state considered.

This proof is physically interpreted without difficulty by saying that the conservation of the quantum field, without fluctuations, forces the particle-singularities to be distributed with a density that is proportional to $|\varphi|^2$, by reason of the particular relationship between this field and its trajectories.

We conclude this subject by briefly discussing the question of the time that is necessary in order for the preceding equilibrium, in which $P = |\varphi|^2$, to be established. This time obviously depends on the values of $v(t)$ and $K(\vec{x}', \vec{x}, t)$, and may not be specified at the actual point in time.

Nevertheless, as Bohm and Feynman have emphasized, one may, without inconvenience, suppose that our fluctuations are as rare as one likes, because the exact value of this relaxation time is devoid of physical significance in any domain in which the usual interpretation is valid. Indeed, any matter that is used in our experiments is possessed of a practically unlimited time, depending on whether this equilibrium has been established, and we know that once this distribution is established it may not be destroyed by any process to which the usual form of quantum theory applies. This is why, for example, an ensemble of electrons in a metal necessarily satisfies $P = |\varphi|^2$; the same is also true for neutrons in nuclei, etc. In conformity with the program of Blokinzef, the probabilistic theory of the new interpretation therefore recovers the domain of validity of the probabilistic interpretation, with the condition that we suppose that the quantum ensembles that are equivalent to the Bohr ensembles correspond to ensembles of micro-objects that have attained their equilibrium. Since those are the only ones that we are concerned with at this particular moment, this only seems natural.

One may complete this discussion with the following observations that allow us to elaborate upon the significance of the foregoing:

1. If we wish to be rigorous then it is not necessary to introduce the hypothesis that the fluctuations of the type considered are quite infrequent in the preceding proofs (see 2., sec 3).

If one does not do this then the preceding proofs remain valid, but the trajectories (L) lose all physical significance. The density P obviously tends to $|\varphi|^2$ again, but the particle-singularities individually follow the very complicated trajectories that one encounters in Brownian motion. This will certainly be the case in a neighborhood of material bodies and sources where the fluctuations may be sufficiently important as to generate a considerable diffusion of (I).

2. Nothing will change if the fluctuations are sufficiently weak ($D(\vec{x})$ is very small), provided that the ensembles considered have had time to reach their equilibrium states. In this case, the trajectories (L) are actually followed by the particles and possess an important physical significance.

Only experiment is likely to indicate how the two preceding possibilities are actually realized in nature. Personally, we estimate that the second is more realistic (except, perhaps, in the immediate proximity of material aggregates), although there is no proof of this. It is nevertheless suggested by observations in Wilson chambers, in which it clearly seems that the electrons that are associated with plane waves follow the rectilinear trajectories that are predicted by the theory.

3. The statistical model that just discussed permits us to not only prove that $|\varphi|^2$ represents precisely the density of an ensemble (I) of particles in the state of having attained their equilibrium distribution, but also to explain why the measures that were performed on (I) appear, after measurement and by reason of the actual interaction between the measurement apparatus and the micro-objects that are being observed, to have the proper values λ_i that correspond to the operators A that were introduced in the probabilistic interpretation; furthermore, this happens with the probability $| c_i |^2$ that was introduced at the beginning of this chapter. This result, which is due to David Bohm, constitutes a very important step in the theory. In effect, it permits us to integrate all of the results that were postulated in the old theory of measurement into the context of our new interpretation, which eliminates a number of possible objections.

For the moment, we content ourselves with admitting this result, which we study only in the last chapter (because it uses notions that are relevant to the causal theory of particles in interaction, a theory that we have not begun up till now).

4. The set of preceding calculations shows that it is possible to imagine physical processes that are capable of generating the statistical distributions that we observe in nature in the context of the causal interpretation. Contrary to the statements of the advocates of the probabilistic interpretation, these are necessarily neither incomprehensible nor inexplicable, provided that one abandons the positivistic postulates that form the starting point of the Bohr interpretation. Only experiment is likely to show whether the hypotheses that Bohm and myself have made on these processes (fluctuations) are realized in fact. In our opinion, they nevertheless present two advantages: they are simple, reasonable, and agree with the classical hypotheses that one usually makes in statistical theory, and they may subsequently lead the way to experimental research that might clarify the nature of quantum statistics, whose interpretation was forbidden by the old theory, in principle.

§ 4. – We conclude this chapter with several general remarks.

As we saw above, the new interpretation is absolutely opposed to the usual interpretation as far as the significance that it attributes to the notion of probability itself is concerned.

For the Copenhagen School, probability is an irreducible element of the theory that definitively limits our knowledge of nature. For Bohr, Pauli, and Heisenberg, it is, for example, definitely impossible to solve the problem of the motion of the individual micro-objects in the quantum framework; this ruins determinism forever. By its nature, the distribution $P = |\varphi|^2$ will be inexplicable, unanalyzable, and inalterable, no matter what physical conditions apply to the particles.

On the contrary, in the causal interpretation it is not necessary to introduce such a limitation on our knowledge. As we just saw, it is possible to furnish a deterministic "model" of individual micro-objects that permits us to account for the distribution $P = |\varphi|^2$ of an ensemble of such objects in a given state when one takes into account the phenomena that are external to the system of particles considered.

Nevertheless – and it is important to emphasize this – the new interpretation does not constitute a return to the mechanistic determinism of Laplace. We do not think that any model of a particle and a single system of laws will permit us to calculate, once and for all, the evolution of all nature when one is given sufficiently extended initial conditions.

On the contrary – and this is what, in our opinion, constitutes, in part, the interesting aspect of the stochastic theorems that were established in this chapter – we have systematically introduced and utilized the dialectical notion that the character of any theory that relates to the nature of micro-objects must be at the same time absolute (because it conforms to the objective reality of things) and relative (because no model may pretend to exhaust the possibilities).

Neither Bohr nor myself believe that the causal interpretation, even if it passes the test of experiment in a satisfactory manner, constitutes anything but an approximation for the state of matter. It will always exist outside the theory of very complicated phenomena that act as if they were, in fact, governed by chance, no matter how they are determined individually. This is what we introduced in section 2 when we spoke of the "level" of organization of matter.

For us, a set of laws describes a level of modeling, such as, for example, classical mechanics or Mariotte's gas laws. In general, they are valid only at this level, and must be replaced with new laws when one goes to a different level. Experiment has therefore shown that one may thus extend the classical "model" to the individual micro-objects that constitute classical matter, and that it must be introduced into kinetic theory if one would like to take the molecular structure of the gas into account.

It is not precise to say that the old laws constitute an approximation of new laws (which one obtains by "adding decimals," to recall an expression of Kirchhoff). One may only say that they translate very complicated stochastic effects that are subject to completely different subordinate laws. For example, take the case of classical mechanics, which, according to Ehrenfest's theorem, describes a sort of mean motion of micro-objects that are endowed with wavelike properties that are absolutely foreign to the classical model.

Moreover, as David Bohm remarked, this notion of level is not necessarily related to that of dimension (for example, the “quantum level” corresponds to dimensions ranging from 10^{-13} cm. to ones that indicate the beginning of a profoundly new level, etc.). It may also be associated with particular physical conditions, such as large densities, very high energies, or an extraordinary complexity of organization.

In this conception, each level is governed by deterministic laws, and its form is subject to the stochastic laws of chaotic action of the infinite ensemble of levels that constitutes nature itself. Any physical phenomenon therefore seems to interweave causality and chance inextricably, and is, at the same time, a synthesis and an ongoing result of the infinitely complex motions of the matter that it suggests.

There is more: In order to attribute the formation of quantum limit densities to their associated fluctuations amounts to taking a step with respect to quantum theory that is analogous to the one that was taken by Perrin, Einstein, and Smoluchowski when they attributed Brownian motion to a chaotic agitation of molecules that no one has observed up till now instead of assuming that this motion is, by its nature, inexplicable.

They thus led the way not only to the observation of these hypothetical particles, but also to theoretical investigation of numerous phenomena. We similarly hope that the hypotheses that were made in the course of this chapter, hypotheses that amount to attributing the density $P = |\psi|^2$ to the action of subordinate phenomena, lead the way to the study of these phenomena themselves, since the new interpretation raises new problems that were inconceivable in the context of the old theory. Under what conditions might one have $P = |\psi|^2$, for example? What is the time interval that is necessary in order for an ensemble of micro-objects in a given state to attain the limiting density $P = |\psi|^2$? These are the kinds of questions that we hope will lead to a new kind of experiment.

We know that the theory that was proposed in this chapter may not pretend to be definitive; for example, it contains only a very general description of the fluctuations that are capable of generating the quantum probabilities. We nevertheless hope that its development will provoke new research into the properties of matter that are subordinate to the “level” that is actually described by quantum theory.

CHAPTER V

§ 1. – At this stage of our work we have treated only the case of isolated particles in the context of the causal interpretation. One must not stop there, but extend this interpretation to systems of particles in interaction, if one wishes to recover the experimental results that were predicted by the probabilistic interpretation.

Moreover, there is no rigorous physical sense in speaking of isolated micro-objects because such objects do not exist in nature. Any description of an individual micro-object is an abstraction that is valid only to the extent that such isolation can be realized effectively in physical processes. Likewise, one may legitimately assert that in the context considered one must start with a theory of micro-objects in interaction a priori and then deduce only the real behavior of the individual micro-objects.

The essence of the model that corresponds to the causal theory of micro-objects in interaction is found in the article of L. de Broglie in 1927 ([1]). We shall nevertheless discuss the results so obtained here since it has recently been possible to improve them in directions that seem to open up interesting perspectives for the ultimate development of the new interpretation.

We commence by briefly recalling the theory of material points in interaction in the probabilistic interpretation.

This theory, which is due essentially to the work of Schrödinger, rests on the hypothesis that if one starts with a system of N material points, which are denoted by the index, I (I = 1, 2, …, N), and one lets:

$\vec{x}_{\mathrm{I}}(t)$	denote the vector (with components x_{IK}) that locates the position of corpuscle I with mass m_{I} in space.
∇_{I}	be the partial derivative with respect to the coordinates (with components $\partial / \partial x_{IK}$).
F_{I}	the external potentials that act on each particle, and
F_{IJ}	the interaction potential between particles.

One may describe the behavior in the Newtonian approximation with the aid of a continuous wave, $\Phi(\vec{x}_1, \vec{x}_2, , \vec{x}_N, t)$, that propagates in the 3N-dimensional configuration space that one may construct from the 3N coordinates x_{IK} of the N points ([2]).

This wave satisfies the equation:

$$i\hbar \frac{\partial \Phi}{\partial t} = -\hbar^2 \left(\frac{\Delta_1 \Phi}{2m_1} + \cdots + \frac{\Delta_N \Phi}{2m_N} \right) + \left(\sum_{\mathrm{I}} F_{\mathrm{I}} + \sum_{\mathrm{I,J}} F_{\mathrm{IJ}} \right) \Phi \qquad (5.1)$$

and the square of its amplitude gives the expression:

([1]) Cf. *La Physique Quantique restera-t-elle indéterministe?* Gauthier-Villars, Paris, 1953.

([2]) Each position of a representative point in this space obviously corresponds to a given position of N points in the actual space.

$$\Phi^*\Phi\ dv,$$

(with $dv = dv_1 \ldots dv_N$), which designates probability of the presence of the representative point $P(\vec{x}_1, \ldots, \vec{x}_N, t)$ in the volume element at each point of configuration space (which is equivalent to a probability of the presence of N points in N given positions $\vec{x}_\mathrm{I}$ in the current space).

This hypothesis seems natural because equation (5.1) constitutes an immediate generalization of the wave equation for one particle. It reduces to the product of the continuous solutions that correspond to the points if one makes the mutual interactions disappear. This is satisfied since one then sees painlessly that the probabilities of the presence of the N points are completely independent, which conforms to the theorem of composed probabilities.

Moreover, they lead to a great number of theoretical predictions that conform to the experimental results and undoubtedly constitute even greater successes of quantum theory.

One must therefore interpret this success in the context of the causal theory ([3]).

In order to do this, we shall follow steps that are analogous to the ones that we discussed in the first part of the first chapter.

§ 2. – First note, with de Broglie, that it suffices to treat the case of two micro-objects in interaction; the extension, by recurrence, of the reasoning to the case of N micro-objects presents no difficulty, in principle.

Having said this, one may present the causal interpretation in various forms.

One may first – with David Bohm – assume that the behavior of two micro-objects is described with the aid of a function:

$$\Phi(x_1, x_2, t) = R(x_1, x_2, t)\exp(iS(x_1, x_2, t)/\hbar),$$

on configuration space, with the condition that we postulate that:

A_1. Φ satisfies the equation:

$$i\hbar\frac{\partial\Phi}{\partial t} = -\hbar^2\left(\frac{\Delta_1\Phi}{2m_1} + \frac{\Delta_2\Phi}{2m_2}\right) + (F_1 + F_2 + F_{12})\,\Phi\,. \tag{5.2}$$

B_1. The particles are associated with points in space along two particular current lines L_1 and L_2 that belong to congruences (L_1) and (L_2) that are defined by the relations:

([3]) In the preceding form, the probabilistic theory of systems seems, on first glance, even more irreconcilable with the classical ideas than the theory of isolated systems. Since – according to the Copenhagen School – material points may not have trajectories, it seems difficult – and this has been strongly emphasized by L. de Broglie – to give any meaning to the coordinates $\vec{\mathrm{X}}_\mathrm{I}$ by the aid of which one constructs the abstract configuration space. One then sees that the propagation of a wave in such a space is obviously devoid of physical significance, and may no longer be associated with an arbitrary "wavelike aspect" in actual spacetime.

$$\vec{v}_1 = \nabla_1 S / m_1$$
$$\vec{v}_2 = \nabla_2 S / m_2 \tag{5.3}$$

This amounts to saying that the particles are displaced in actual space like the projections of the figurative point on the three dimensional surfaces that are defined by $\vec{x} = \vec{x}_1$.

C_1. As before, $R^2 dv$ represents the probability of the presence of the statistical ensemble of figurative points (which collectively represents all of the possible pairs of material points that are associated with F) in the element dv.

One then painlessly shows that these postulates allow us to recover all of the results that were obtained by the probabilistic interpretation. It is clear that, along the way, one recovers certain elements of the discussion in chapter I.

Postulates A_1 and B_1 introduce a supplementary quantum interaction potential in space:

$$Q = \frac{\hbar^2}{2}\left(\frac{1}{m_1}\frac{\Delta_1 R}{R} + \frac{1}{m_2}\frac{\Delta_2 R}{R}\right),$$

since one easily verifies that everything happens as if the figurative point is displaced under the influence of a force ∇Q. As David Bohm remarked, this amounts to introducing a sort of quantum Van der Waals force in space, which depends on the position of both point-like particles and also on the propagation conditions and boundary conditions (and therefore on the value of the potentials at all the points of configuration space). On each particle, the value of this force is provided by the expressions:

$$\nabla_1 Q(\vec{x} = \vec{x}_1) \qquad \text{and} \qquad \nabla_2 Q(\vec{x} = \vec{x}_2).$$

Moreover, the third postulate C_1 must be deduced from the theory here, on the pain of introducing an unjustifiable statistical element and giving the function R two incompatible meanings (since it serves to define a field Q and a statistical representation).

In fact, we shall immediately show that this postulate is unnecessary and that the distribution $R^2 dv$ may be derived starting with considerations that are analogous to the statistical hypotheses of chapter IV.

Indeed, one sees that if one sets $\Phi = R\exp(iS/\hbar)$ then the complex equation (5.2) subdivides into two equations that correspond to the real and imaginary parts, namely:

$$\frac{\partial S}{\partial t} + \frac{(\nabla_1 S)^2}{2m_1} + \frac{(\nabla_2 S)^2}{2m_2} + F_1 + F_2 + F_{12} - \frac{\hbar^2}{2m_1}\frac{\Delta_1 R}{R} - \frac{\hbar^2}{2m_2}\frac{\Delta_2 R}{R} = 0 \tag{J}$$

$$\frac{\partial R^2}{\partial t} + \frac{1}{m_1}\nabla_1(R^2\nabla_1 S) + \frac{1}{m_2}\nabla_2(R^2\nabla_2 S) = 0 \tag{C}$$

in which the second one (C) is associated with the continuity equation of a real fluid of density R^2 whose molecules follow the trajectories (L) of the representative points in configuration space.

Since, for us, the material points have completely well-defined coordinates $\vec{x}_1$ and trajectories L_1 and L_2, the system of two moving bodies will be described by a representative point whose six velocity components will be given by relations (5.3). One may therefore compare that system to a particle that is constrained to follow one of the streamlines of the preceding fluid in configuration space.

If one now considers an arbitrary ensemble of such points (which corresponds to an arbitrary initial distribution of the particle positions) that are associated with the same wave Φ in configuration space, I say that these points will tend to distribute themselves in that space with a density $R^2 dv$, if one assumes that the function Φ is subject to stochastic fluctuations that preserve it in the course of time.

This amounts to saying that if one considers a cloud of representative points that are associated with the same state Φ, and constrain it to always follow some trajectories (L), except during time intervals that correspond to external perturbations of the system considered, then the density $P(x_1, x_2, t)$ of this cloud will tend to the density R^2 that is postulated by the probabilistic interpretation if one assumes that these perturbations do not destroy the function Φ; one will thus have $\Phi_{\text{real}} = \Phi_{\text{calculated}}$ before and after.

This theorem is established immediately. Indeed, in order to deduce that $P \to R^2$, it suffices to apply the conclusions of the fundamental lemma of chapter IV to the fluid configuration that corresponds to Φ. Indeed, the conclusions of this lemma do not depend on the number of dimensions to space, but only on the existence of a continuity equation and its conservation law. All of the statistical reasoning we did is thus valid in the case of the six-dimensional fluid that was previously introduced.

Physically, one sees that the jumps performed by the representative points of this system, jumps that make them pass from one trajectory to another that corresponds to a simultaneous displacement of particles 1 and 2 under the influence of an external perturbation that makes them jump from two compatible trajectories L_1 and L_2 to two new trajectories. This is natural because any perturbation acts on 1, as well as 2, due to the classical interaction potential.

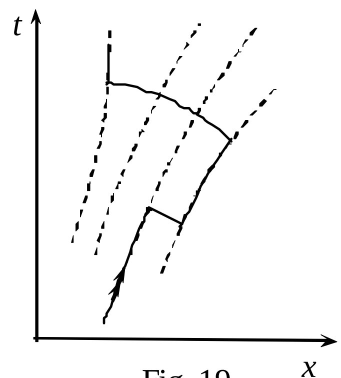

Fig. 19.

§ 3. – The preceding causal theory, when reduced to postulates A_1 and B_1, may not be considered as satisfactory because it does not include a complete description of the behavior of the system in actual spacetime. This leads us to pose the following problem: Is it possible to give a representation of the system of two particles in actual space that permits us to recover the preceding postulates A_1 and B_1?

The solution obviously permits us to complete the desired deterministic model since these postulates suffice, in principle, to recover the results of the probabilistic interpretation when one introduces some natural hypotheses concerning the effect of the external perturbations.

This problem, which was posed for the first time by L. de Broglie, is difficult to treat in full generality. We shall discuss the attempts that were made without pretending to arrive at a complete response that might answer the question.

If one refers to chapter II – in particular, the part that concerns the theory of the double solution – one sees that it is possible to represent each micro-object with the aid of a singular wave u_I that describes both their point-like and extended behavior. These waves may be written in the form:

$$u_{\mathrm{I}} = u_{0\mathrm{I}} + \varphi_{\mathrm{I}} ,$$

in which φ_{I} represents the regular part, which satisfies the usual linear equations and determines the congruence (L_{I}) of streamlines that the center of the singularity might follow.

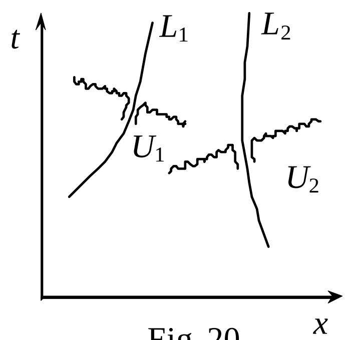

Fig. 20.

The guidance theorem then shows us that if the particles do not interact then everything happens as if one were dealing with N waves u_{I} in actual space, with:

$$u = u_1 + u_2 + \ldots + u_N ,$$

which are superposed without interaction; each singularity is "piloted" by the regular part φ_{I} of its proper wave u_{I} along a trajectory L_1 that is fixed by φ_{I} . When the particles interact by the intermediary of classical potentials – their Coulomb field, for example – this is no longer true. The electromagnetic field at each point then depends on the position of all the particles and the waves u_{I} may not be considered as independent.

This analysis may be extended to the case of micro-objects in interaction. From the model that was proposed in the last section of chapter III, if one recalls the unitary idea that we discussed then one may consider the point-like aspect of micro-objects as having singular regions in a unique wave, and that the particle-singularities remain separated by a distance $> 2r_0$, which satisfies a nonlinear equation in the total electromagnetic field. It will then be possible, in principle, to separate the wavelike phenomena that are associated with the two particles, and one must solve this equation *in toto* if one is to determine the trajectories that are followed by the singular regions ([4]). One may nevertheless remark that, from what we have seen, if one uses the guidance theorem then everything happens as if one may represent each particle by a proper wave u_{I} that moves in the classical total external field (and which comprises the field that is produced by the other particles). Indeed, since one has:

$$u = u_1 + u_2 + \ldots$$

one may write $u \leq u_{\mathrm{I}}$ in the neighborhood of the singular region that is associated with u_{I}; of course, the guidance condition:

([4]) We shall not begin the general problem here. Indeed, it is very difficult, and the calculations one undertakes are also too fragmentary to be published. We only emphasize their theoretical importance. In the context of the preceding ideas, it is indeed somewhat probable that one must study the corpuscular singularities independently in the interior of a very small region of order 10^{-13}, for example, with the aid of linear approximations to equations that are actually more complicated. In a later stage of this work, we are forced to infer results from the proposed nonlinear equations that might confront experiments since only such results permit us to select from all of the possible nonlinear theories the one that constitutes a convenient approximation to the properties of matter in the domain considered.

$$\frac{u^+ \alpha^\mu u}{u^+ u} \simeq \frac{u_I^+ \alpha^\mu u_I}{u_I^+ u} = \frac{\varphi_I^+ \alpha^\mu \varphi_I}{\varphi_I^+ \varphi_I}$$

entails the piloting of the singular part of u_I by the regular part φ_I of the associated wave u_I in the field u.

This suggests that we try, as L. de Broglie did, to represent each micro-object by a singularity in a wavelike phenomenon that takes place in space, and see whether this model might lead, under certain conditions, to the conclusions that were postulated by the probabilistic interpretation in configuration space.

We therefore assume that, in the first approximation, everything happens as if each micro-object acts on the propagation of the wave that is associated with the other object by the intermediary of the action of its classical proper field on the evolution of the wave that it defines. The two particles are thus represented by two waves u_I such that the center of the two singularities are constrained to follow two streamlines that are associated with the two corresponding continuous waves φ_I (I = 1, 2). By neglecting the effects of spin and confining ourselves to the Newtonian approximation (low velocities) one may use distinct Schrödinger equations to define them, in which the external potential that appears in them is the sum of the external potential F_I that acts on the particle and the potential F_IK that represents the action of the other particle on the particle considered.

Physically, this amounts to saying that one may represent the system of two interacting micro-objects with two distinct waves φ_I and two particles that are constrained to follow two streamlines L_1 and L_2 that are defined by these waves. These waves are superposed without direct interaction (such as two systems of ripples on the surface of a pond), but each of them sees its evolution determined by not only the external electromagnetic field, but also by the motion of the particle that it does not influence ([5]).

It remains to show that this schema is coherent, i.e., that it is possible to find a well defined system of such waves when one is given the initial conditions.

One easily sees that this is the case.

Indeed, suppose we are given the values of φ_I, the initial positions P_{01} and P_{02} of the particle singularities, and the values of the external electromagnetic field on a spacelike surface σ_0.

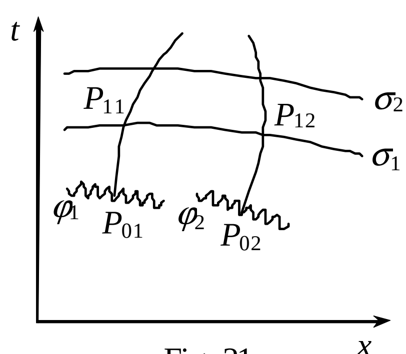

Fig. 21.

The given of the points P_{01} and P_{02} permits us to calculate the value of the value of the fields that act on φ_1 and φ_2 everywhere. On account of the wave equations, these values, combined with the initial conditions, obviously determine the values of φ_1 and φ_2, and therefore the new positions of the particles P_{11} and P_{12} on a closely neighboring spacelike surface σ_I. They represent the new complete initial conditions on σ_I that allow us to calculate the values of the φ_1, as well as the trajectories L_1 (which are composed of the points P_{n1}) of the two singularities everywhere in a

([5]) This is by the intermediary of the classical interaction potential that is attached to that particle.

stepwise fashion. As a consequence, the problem is well defined, and we shall dedicate the next section to discussing the results that were obtained by de Broglie and the author, which show how the preceding viewpoint recovers the classical results of the probabilistic interpretation in configuration space in the Newtonian approximation by recalling postulates A_1 and B_1 of section 2.

§ 4. – To clarify these results, we first briefly recall, as in the example of de Broglie ([6]), what happens in the case of two classical interacting particles.

First assume that the trajectory L_1 of one of the two particles – the second one, for example – is known. Its electromagnetic field is determined at each point and each instant, and there is a corresponding congruence of possible motions for the first one when one is given its initial velocity. One then knows that if one denotes the external potential that acts on particle 1 by $F_1(\vec{x}_1, t)$, and the potential (which is a function of the components $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$ of the distance between the two particles) that represents the action of the second one on the first one, then there exists a Jacobi function S_1 such that one has:

$$m_1 \vec{v}_1 = \nabla S_1 \tag{5.4}$$

in which we have denoted the velocity of particle 1, with the components v_{1i}, by $\vec{v}_1$. The equations of motion may then be written in the Lagrange form:

$$\frac{d}{dt}\left[\frac{\partial \mathfrak{L}_1}{\partial v_{1i}}\right] = \frac{\partial \mathfrak{L}_1}{\partial x_i} \qquad i = 1, 2, 3,$$

in which $\mathfrak{L}_1$ denotes the classical Lagrangian:

$$\mathfrak{L}(1) = \tfrac{1}{2} m_1 v_1^2 - F_1(\vec{x}, t) - F_{12}(\vec{x}, t), \tag{5.5}$$

and the potential F_{12} is a function of the distance between the two particles, therefore of the trajectory that is chosen for the second particle.

By fixing the trajectory L_1 one likewise obtains analogous equations (by permuting the indices, 1 and 2, in the preceding relations) for the ensemble of possible trajectories for particle 2.

One may then make two essential remarks: it is always possible to represent the motions that we just described by constructing a fictitious six-dimensional configuration space that is based on the coordinates x_{Ii} (I = 1, 2 and i = 1, 2, 3) of the two particles. Indeed, to any fixed trajectory L_2 that is given by $x_2(t)$ there correspond trajectories L_1 that are provided by the preceding equations. Any pair of compatible trajectories L_1 and L_2 may then be represented by a unique trajectory of the current point $\vec{x} = (\vec{x}_1, \vec{x}_2, t)$ of the new space. The system of two moving bodies at it will be described at each instant by a representative point $\vec{x}$, whose six velocity components v_{Ii} will be given by equations (5.3). As de Broglie has noted, if one then assumes that the initial velocities are given,

([6]) Cf. op. cit.

but not the initial positions, then one sees that there are diverse trajectories of the representative point that correspond to the diverse hypotheses that we make. As a consequence, the set of all these simultaneous conceivable possibilities is described by a cloud of representative points whose motion satisfies the continuity equations:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\rho\vec{v} = 0 \tag{5.6}$$

in which $\rho(x_1, x_2, t)$ denotes the density of this cloud, and $\vec{v}$, its velocity (with components $v_{\mathrm{I}i}$).

de Broglie has then shown that if one may separate the terms in the Lagrangians L_1 and L_2 that depend on the mutual actions from the ones that do not then one may find a function $S(x_1, x_2, t)$ in configuration space such that one has:

$$m_1 v_{1k} = \frac{\partial S}{\partial x_{1k}}, \tag{5.7}$$

which is equivalent to relations (5.3). The equations of motion are then written in the Lagrange form:

$$\frac{d}{dt}\left[\frac{\partial \mathfrak{L}}{\partial \dot{q}}\right] = \frac{\partial \mathfrak{L}}{\partial q} \qquad \left(\dot{q} = \frac{dq}{dt}\right)$$

in which q denotes any of the six variables x_{1i} and $\mathfrak{L}$ denotes a function in the new space that is obtained by taking the sum of the terms of the second type, added to the half-sum of the terms of the first type, namely:

$$\mathfrak{L} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 - F_1 - F_2 - F_{12}, \tag{5.8}$$

with the condition that one has $F_{12} = F_{21}$, which translates into the principle of action and reaction.

§ 5. – Therefore, let two micro-objects be represented by two waves u and let $\varphi_{\mathrm{I}} = R_{\mathrm{I}} \exp(iS_I / \hbar)$ denote the continuous parts, which satisfy the two Schrödinger equations, by hypothesis. Each of these waves moves in the external field and the electromagnetic field of the particle that it does not influence. These particles are constrained to follow two given streamlines that correspond to the waves that accompany them, conforming to the relations:

$$m_{\mathrm{I}} v_{\mathrm{I}} = \nabla_{\mathrm{I}} S_{\mathrm{I}}.$$

As we have seen, if one assumes that the trajectory of the second corpuscle is known then the Schrödinger equation of the first one, when separated into real and imaginary parts, gives the generalized Jacobi equation:

$$-\frac{\partial S_1}{\partial t} = E_1 = \frac{1}{2m_1}\sum_i \left(\frac{\partial S_1}{\partial x_i}\right)^2 + F_1 + F_{12} + Q_1 \,, \tag{5.9a}$$

for the trajectory of the first one, in which we have let Q_1 denote the quantum potential that is given, in the first approximation, by the equation:

$$Q_1 = -\frac{\hbar^2}{2m_1}\left(\frac{\Delta R_1}{R_1}\right)_{\mathbf{x}_1=\mathbf{x}} .$$

Here, conforming to the ideas of the causal interpretation, S_1 represents the phase of the regular wave that is associated with the first corpuscle in the classical force field that is created by the second, taking into account the corresponding boundary conditions (which might create interference or diffraction phenomena), if there are any; $R_1(\vec{x}_1,\ \vec{x}_2,\ t)$ represents the amplitude of this wave, $\nabla_1 S_1$ signifies $(\nabla S_1)_{x=x_1}$. $F_1(\vec{x}_1,\ t)$ and $F_{12}(\vec{x}_{12},\ t)$ has the meaning that was previously introduced. The energy E_1 is not constant, in general.

On the contrary, if one fixes the trajectory L_1 of the first corpuscle then one obtains the generalized Jacobi equation over all the compatible L_2:

$$-\frac{\partial S_2}{\partial t} = E_2 = \frac{1}{2m_2}\sum_i \left(\frac{\partial S_2}{\partial x_i}\right)^2 + F_2 + F_{12} + Q_2 \,, \tag{5.9b}$$

in which Q_2 is the quantum potential, $Q_2 = -(\hbar^2/2m^2)(\Delta R_2/R_2)_{x=x_2}$, while F_2 and F_{12} have the previously-defined meanings (assuming $F_{12} = F_{21}$, as always). The energy E_2 is no longer constant, in general.

Conforming to the ideas of de Broglie, the generalization of the classical concepts to the case considered is obtained by constructing it from the coordinates $\vec{x}_1$ and $\vec{x}_2$ of the two singularity-particles. Any pair of compatible trajectories L_1 and L_2 will be further represented by a trajectory L of the current point $\vec{x} = (\vec{x}_1, \vec{x}_2)$ in that space since the velocity $\vec{v}$ has the components $\vec{v}_1$ and $\vec{v}_2$ in the two three-dimensional subspaces that are associated with the two particles. It then implies the existence of a generalized Jacobi function $S(\vec{x}_1, \vec{x}_2, t) = S(\vec{x}, t)$ in that configuration space, such that one has:

$$\vec{v}_{\mathrm{I}} = \nabla_{\mathrm{I}} S / m_{\mathrm{I}} = \nabla S_{\mathrm{I}} \,/\, m_{\mathrm{I}} \,, \tag{5.10}$$

at each point. This function plays the role of a classical function and permits us to simultaneously describe the motion of the particles for every possible pair of associated trajectories L_1 and L_2. We are then led to pose the following problems:

1. Is it possible to represent the simultaneous motions of the two particles in configuration space with the aid of function $F = R(\vec{x}, t)\exp(iS/\hbar)$ such that:

a) The trajectories are given by relations (5.8)?

b) The function R allows us to calculate the potential of quantum origin Q that acts on $\mathbf{x}$, while R_2 behaves like a matter density that is constrained to move along the congruence (L)?

2. If yes, what equations must Φ, R, and S satisfy, and what are their relations with φ_1 and φ_2?

§ 6. – The response to the first problem was given by de Broglie with the aid of the following lemma, whose proof is immediate:

Lemma. Let x and y be two variables and let $\omega(x, y)$ be an arbitrary function. The necessary and sufficient condition for one to have the following relations between three functions $G_1(x, \omega)$, $G_2(y, \omega)$, and $G(x, y, \omega)$:

$$\left(\frac{\partial G}{\partial x}\right)_y = \left(\frac{\partial G_1}{\partial x}\right)_y, \qquad \left(\frac{\partial G}{\partial y}\right)_x = \left(\frac{\partial G_2}{\partial y}\right)_x,$$

is that one have:

$$\begin{cases} G_1(x,\omega) = G_{11}(x) + G_{12}(\omega), \\ G_2(y,\omega) = G_{22}(y) + G_{21}(\omega), \\ G(x,y,\omega) = G_{11}(x) + G_{22}(y) + G_{12}(\omega), \end{cases}$$

with $G_{12} = G_{21}$.

If we accept this lemma then one sees that the relations:

$$\begin{cases} m_1\mathbf{v}_1 = \nabla_1 S_1 = \nabla_1 S, \\ m_2\mathbf{v}_2 = \nabla_2 S_2 = \nabla_2 S, \end{cases}$$
$$\begin{cases} \nabla_1 Q_1 = \nabla_1 Q, \\ \nabla_2 Q_2 = \nabla_2 Q, \end{cases}$$

which are necessarily satisfied in order for the preceding problems to have solutions, imply the relations of de Broglie between the phases and the amplitudes:

$$\begin{cases} S_1(\vec{x}_1, \vec{x}_{12}, t) = S_{11}(\vec{x}_1, t) + S_{12}(\vec{x}_{12}, t), \qquad S_{12} = S_{21}, \\ S_2(\vec{x}, \vec{x}_{12}, t) = S_{22}(\vec{x}_2, t) + S_{11}(\vec{x}_{12}, t), \\ S(\vec{x}_1, \vec{x}_2, t) = S_{11}(\vec{x}_1, t) + S_{22}(\vec{x}_2, t) + S_{12}(\vec{x}_{12}, t), \end{cases} \tag{5.11}$$

$$\begin{cases} Q_1(\vec{x}_1,\vec{x}_{12},t)=Q_{11}(\vec{x}_1,t)+Q_{12}(\vec{x}_{12},t), & Q_{12}=Q_{21}, \\ Q_2(\vec{x},\vec{x}_{12},t)=Q_{22}(\vec{x}_2,t)+Q_{11}(\vec{x}_{12},t), & \\ Q(\vec{x}_1,\vec{x}_2,t)=Q_{11}(\vec{x}_1,t)+Q_{22}(\vec{x}_2,t)+Q_{12}(\vec{x}_{12},t). & \end{cases} \tag{5.12}$$

The solution to the second problem is then obtained without difficulty. The equation of continuity in the configuration space is written:

$$\frac{\partial R^2}{\partial t}+\sum_{\mathrm{I}}\sum_k \frac{\hbar}{m_{\mathrm{I}}}\frac{\partial}{\partial x_{\mathrm{I}k}}\left(R^2\frac{\partial S}{\partial x_{\mathrm{I}k}}\right)=0,$$

in which I = 1, 2, denotes the particles, and k = 1, 2, 3, their coordinates. Since:

$$R^2 = \Phi^*\Phi$$

and:

$$\Phi^*\frac{\partial\Phi}{\partial x_{\mathrm{I}k}}-\Phi\frac{\partial\Phi^*}{\partial x_{\mathrm{I}k}}=-\frac{2i}{\hbar}R^2\frac{\partial S}{\partial x_{\mathrm{I}k}},$$

by substitution, one obtains the equality:

$$\Phi\left(\frac{\partial\Phi^*}{\partial t}-\frac{\hbar}{2i}\sum_{\mathrm{I},k}\frac{1}{m_{\mathrm{I}}}\frac{\partial^2\Phi^*}{\partial x_{\mathrm{I}k}^2}\right)+conj.=0,$$

which is valid at every point.

If one then duplicates an argument that we already used (which is developed in Appendix III), by multiplying the preceding equality by a real function, such that the product:

$$\int f\left\{\Phi\left(\frac{\partial\Phi^*}{\partial t}-\frac{\hbar}{2i}\sum_{\mathrm{I},k}\frac{1}{m_{\mathrm{I}}}\frac{\partial^2\Phi^*}{\partial x_{\mathrm{I}k}^2}\right)+conj.\right\}dvdt,$$

is integrable, and integrating, one shows that the expression:

$$\left\{\frac{\partial}{\partial t}-\frac{\hbar}{2i}\sum_{\mathrm{I},k}\frac{1}{m_{\mathrm{I}}}\frac{\partial^2}{\partial x_{\mathrm{I}k}^2}\right\}\Phi,$$

is an anti-Hermitian operator that commutes with the $x_{\mathrm{I}k}$.

One finally has:

$$\frac{\partial\Phi}{\partial t}-\frac{\hbar}{2i}\sum_{\mathrm{I},k}\frac{1}{m_{\mathrm{I}}}\frac{\partial^2\Phi}{\partial x_{\mathrm{I}k}^2}=\frac{i}{\hbar}K\cdot\Phi,$$

in which K is a function of only $\vec{x}_1$, $\vec{x}_2$, and t that multiplies Φ due to the linear character of the wave equation.

Since, by hypothesis, the motions that are described must be identified with actual motions, K is well defined because it suffices to decompose the preceding equation into real and imaginary parts in order to recover, on the one hand, equation (C), and the generalized Jacobi equation (J), on the other. This permits us to write:

$$K = F_1 + F_2 + F_{12}\,,$$

since one must recover the classical equation when $\hbar \to 0$.

The preceding argument has the consequence that if one starts with the proposed model in which the micro-objects are represented by waves in real space then one may describe their behavior with the aid of a wave in configuration space that is constructed from the coordinates of the singularity-particles only if one satisfies two conditions:

1. Relations (5.11) and (5.12) of de Broglie must be satisfied on the trajectories.

2. Φ must obey the Schrödinger equation in configuration space.

This is satisfied since the passage from R_1 and Q_2 to Q is done as in classical mechanics. The passage from E_1 and E_2 to E takes the mutual energy term only once.

Indeed, if one compares (J_1), (J_2), and (J) then one obtains the relations:

$$E = E_1 + E_2 - F_{12} + Q - Q_1 - Q_2\,,$$

by taking relations (5.11) into account. Relations (5.12) then give:

$$\begin{aligned} E &= E_1 + E_2 - F_{12} + Q_{12}\,, \\ &= \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 + F_1 + F_2 + F_{12} + Q_1 + Q_2 + Q_{12}, \end{aligned}$$

which symmetrically treats the ordinary and quantum interaction potentials F_{12} and Q_{12}.

We conclude with a physical remark. In real space, the wave u_1, for example, may be decomposed into two parts, such as:

$$u_1 = u_{01} + \varphi_{11}\,.$$

u_{01} is non-trivial only inside a tube Γ_1 of radius r_{01} that surrounds the trajectory L_1. This tube is physically important. As we have seen, the guidance relations, for example, must be valid inside it and on the boundaries of Γ_1, and not just on the trajectory L_1, since L_1 constitutes a limiting representation of the singular region when one confines oneself to considering φ_1.

Having said this, we return to relations (5.11), which we proposed to call the de Broglie conditions. The geometric significance is clear ([7]). They simply express that if one fixes $x_2(t)$ on L_2 in configuration space, then the isophase surfaces S and S_1 are tangent to the points $x_1(t)$ along L_1, and they have a contact of order 1 (equality of the first derivatives with respect to x_1 on L_1) in the subspace that corresponds to particle 1. In

([7]) By differentiating S and S_1 in (5.11) with respect to $\vec{x}_1$, one sees that $\nabla_1 S = \nabla_1 S_1$; $\vec{x}_2$ is fixed.

summation, this assumes that particle 1 reduces to a moving point on the trajectory L_1. Such conditions are the only ones that are possible; one may seek to generalize them. For example, one may assume that the de Broglie conditions are valid up to higher-order infinitesimals, not only on L, but also in its immediate neighborhood – for example, the interior of the tube Γ_1. Geometrically, this amounts to assuming that the preceding contact between φ_1, S_1, and S is of order higher than one at the point. For example, one might assume only that one substitutes the equality of first and second derivatives, namely:

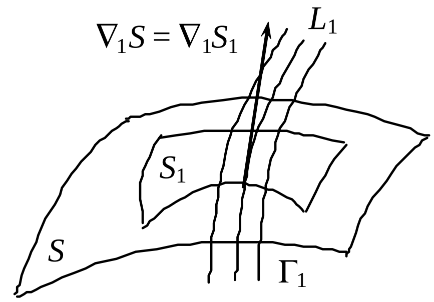

Fig. 23.

$$\begin{cases} \nabla_1 S_1 = \nabla_1 S, \\ \dfrac{\partial^2 S}{\partial x_{1i}^2} = \dfrac{\partial^2 S_1}{\partial x_{1i}^2}, \end{cases} \tag{5.13}$$

which corresponds to a contact of order two, for relations (5.11). With these conditions, the de Broglie conditions remain valid to the approximation considered in the entire region that surrounds L_1, in which one may content oneself with the first two terms of a Taylor series development of S, namely:

$$S = S_1(x_{10}, t) + (h\nabla_x S_1 + k\nabla_y S_1 + l\nabla_z S_1)_{\vec{x}=\vec{x}_{01}} + \frac{1}{2!}(h^2\nabla_x^2 S + \cdots),$$

in which we have denoted a point of L_1 by $\vec{x}_{01}$, and the infinitesimal components of the distance to neighboring points by h, k, and l. For the given dimensions of Γ_1, if one wishes to impose stronger conditions then one is led to choose an order of contact that corresponds to the powers of r_{01} that one has neglected. In the case envisioned, for which $r_{01} = 10^{-13}$ cm., one sees that one can legitimately neglect the quantities of order r_{01}^2, and thus content oneself with a contact of order two on L_1 between S_1 and S. The same argument obviously applies to particle 2.

In other words: If, instead of de Broglie's mathematical conditions, we consider the guidance condition from the physical viewpoint then we see that we must account for the fact that a linear trajectory, such as L_1, obviously constitutes the mathematical abstraction of an extremely thin tube that is described by an extended singularity that is bounded by a surface (S). When one considers, as is necessary, not only the linear trajectory, but the tube itself, one sees that the de Broglie conditions (contact along L_1) do not suffice, and that the contact must be defined – up to a sufficient approximation – over the entire interior of the tube, which imposes a contact of order higher than one.

§ 7. – In our opinion, the preceding conditions are quite interesting in that they specify the mathematical nature of the relationship that unites the waves φ_1 and φ_2 that propagate in actual spacetime with the amplitude and phase of the wave Φ in the fictitious configuration space. Even so, they are not completely satisfactory for two reasons:

1. We have not proved that any system of two waves φ_1 and φ_2 (the regular parts of the waves u_1 and u_2) that is associated with interacting particles gives correlated pairs of trajectories L_1 and L_2 that are representable with the aid of streamlines of a wave Φ. This is a very difficult problem that we have not treated up till now. In the opinion of de Broglie, one may attempt to prove that conditions (5.11) and (5.12) are always satisfied by using a system of coordinates that is centered at the center of gravity of the two corpuscles. We have not succeeded in this attempt, as of yet. The principal difficulty lies in the fact that these conditions must be satisfied by actual initial conditions, even if it may be shown that they are effectively satisfied in all of the physically meaningful cases that have been treated by equation (5.2). In fact, I am not certain that this is true. It may be that in certain real cases it is not legitimate to pass to the intermediary of configuration space in order to describe the behavior of the system envisioned. It will then be necessary to solve a system of simultaneous equations in real spacetime instead of starting with the Schrödinger equation in configuration space. The use of configuration will then be less general than our model.

2. One must show that any solution Φ of this Schrödinger equation in configuration space gives pairs of actual trajectories L_1 and L_2 that are associated with waves φ_1 and φ_2 satisfy two Schrödinger equations in real space, conforming to our model. This is necessary because one knows that a number of actual problems in wave mechanics may be treated with the aid of such solutions. If our model is exact then it must apply to the set of these cases, at minimum. That proof is therefore necessary to establish the equivalence of the two interpretations.

One may present it as follows:

We start with a particular solution Φ ([8]) of the Schrödinger equation that correctly accounts for the behavior of an actual physical system. One then needs to know whether the given of Φ will determine two waves φ_1 and φ_2 that accompany two particles, conforming to the proposed model, and consequently satisfy the de Broglie conditions.

In order to do this, we first demand to know what indications give us knowledge of Φ relative to φ_1 and φ_2 in the context of the preceding ideas.

According to our model, to any streamline L that is defined by Φ there correspond two correlated trajectories L_1 and L_2 that are actually followed by the two particles. We therefore know:

a) On L_1 and L_2, the manner in which the two particles displace in time - namely, $\vec{x}_1(t)$ and $\vec{x}_2(t)$ – in which $\vec{x}_1$ and $\vec{x}_2$ denote their coordinates in actual space.

([8]) Which is defined by given initial conditions.

b) At $\vec{x}_1(t)$ and $\vec{x}_2(t)$, the values of the phases S_1 and S_2, and their first four derivatives, $\nabla_\mu S_1$ and $\nabla_\mu S_2$; S_1 and S_2 are determined only up to an additive constant.

This results from the hypotheses that we made. For example, for $\partial S_1/\partial x$, $\partial S_1/\partial y$, and $\partial S_1/\partial z$ this results from the relations:

$$\nabla S_1(\vec{x}_1(t)) = \nabla_1 S(\vec{x}_1, \vec{x}_2, t),$$

which expresses the law of motion we assumed, the derivative $\partial S_1/\partial t$ is given by the Jacobi equation (J_1) for corpuscle 1, which gives this quantity as a function of quantities that are assumed to be known.

c) At $x_1(t)$ and $x_2(t)$, the values of the quantities:

$$Q_1 = -\frac{\hbar^2}{2m_1}\frac{\Delta R_1}{R_1} \qquad \text{and} \qquad Q_2 = -\frac{\hbar^2}{2m_2}\frac{\Delta R_2}{R_2},$$

as well as their spatial derivatives ∇Q_1 and ∇Q_2. This also results from the proposed model since one has, by hypothesis:

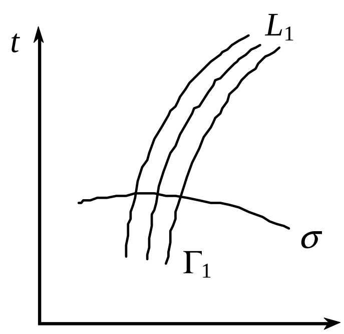

Fig. 24.

$$\begin{cases} \nabla Q_1 = \nabla_1 Q, \\ \nabla Q_2 = \nabla_2 Q. \end{cases}$$

At this stage, one then sees that the problem is not determined by the de Broglie conditions (5.11) and (5.12), in general.

Indeed, in our model, from the Cauchy-Kovalewska theorem, the determination of φ_1 necessitates, in principle, the knowledge of the Cauchy conditions that correspond to the Schrödinger equation:

$$i\hbar\frac{\partial\varphi_1}{\partial t} = -\hbar^2\frac{\Delta\varphi_1}{2m_1} + (F_1 + F_{12})\varphi_1,$$

on a tube Γ_1 that surrounds L_1 (since F_{12} is determined by the knowledge of the motions on L_2).

In the particular case, these Cauchy conditions obviously reduce to the knowledge of S_1, ∇S_1, R_1, and ∇R_1, provided that the tube is not tangent to the bicharacteristics of the wave equation, i.e., to the light cone (which is not the case in the Newtonian approximation that we considered).

Now, if we know all of the values of S_1 and ∇S_1 on Γ_1 because of *b*, etc., then the same is not the case for R_1 and ∇R_1, which are simply constrained to satisfy the equation:

$$-\frac{\hbar^2}{2m_1}\frac{\Delta R_1}{R_1} = Q_1$$

in Γ_1; since Q is determined by the first two terms of the Taylor development on Γ_1 (since one knows ∇Q_1 for each value of t).

Mathematically, there thus exists, in principle, an infinitude of possible values of R_1 and ∂R_1 that constitute acceptable Cauchy conditions on Γ_1. One may obviously say that the physical conditions select from among all of these solutions, the ones that give values of φ_1 on a surface σ that correspond to the initial physical givens, but I do not think that such a determination is satisfactory. Instead of the de Broglie conditions, one must adopt condition (5.13), which corresponds to a contact of order 2 and determines the waves φ_1 and φ_2 completely.

Indeed, under that hypothesis, we know not only S_1 on Γ_1, but also ∇S_1 and $\nabla^2 S_1$, since one has:

$$\frac{\partial^2 S}{\partial x_{Ij}^2} = \frac{\partial^2 S_I}{\partial x_{Ij}^2}, \qquad I = 1, 2, \qquad j = 1, 2, 3, \tag{5.14}$$

on L.

We shall see that this determines R_1 and ∇R_1 on Γ_1. In order to do this, we first prove the following lemmas:

Lemma I.

Let $G(u) = \Delta u + c \cdot u$ be an equation of elliptic type in three-dimensional space (in which c designates a continuous function of the variables) that must be satisfied inside a sufficiently small domain E that is bounded by a surface (S). One may always uniquely determine an integral of the preceding equation that is regular in E and takes given values on (S).

The proof of this property is found in numerous treatises on analysis ([9]), so we shall not develop it here.

Lemma II.

If one is given the values of ∇S and $\nabla^2 S$ on a trajectory L of the causal interpretation, which is defined as a streamline of a solution $\varphi = R\exp(iS/\hbar)$ to the Schrödinger equation, then it is possible to calculate R and ∇R up to a multiplicative constant on this trajectory.

The proof rests on the use of the continuity equation (C). Indeed, return to the hydrodynamical equation, and consider, as we did in the stochastic proof of chapter IV, a

([9]) Cf. GOURSAT, *Traité d'analyse*, t. III, pp. 513; HILBERT and COURANT, *Meth. Der Math. Physik*, pp. 280.

volume element $\Delta\omega$ each point of which follows the fictitious fluid that is associated with φ. In the course of a time interval δt we have, due to equation (C):

$$\delta(R^2\Delta\omega) = 0,$$

namely:

$$\Delta\omega\,\delta R^2 + R^2\,\delta\,\Delta\omega = 0.$$

We then denote the arc of the curve L by s. We then have:

$$\begin{cases} \delta R^2 = 2R\dfrac{\partial R}{\partial s}\dfrac{ds}{dt}dt \\ \qquad = \dfrac{2R}{m}\dfrac{\partial R}{\partial s}\dfrac{\partial S}{\partial s}dt = \dfrac{2R}{m}\dfrac{\partial R}{\partial s}\dfrac{dS}{ds}dt \end{cases}$$

since the fluid velocity along L is given by $dS / ds / m$. Since one has, on the one hand:

$$\delta\Delta\omega = \text{div}\ \frac{\nabla S}{m}\Delta\omega\,dt,$$

by hypothesis, the preceding formula becomes:

$$\Delta\omega\cdot\frac{R}{m}\left(2\frac{\partial R}{\partial s}\frac{dS}{ds} + R\,\text{div}\,\nabla S\right) = 0\,,$$

which determines the evolution of R along L by the expression:

$$R = R_0\ exp\left\{-\frac{1}{2}\int_{s_0}^{s}\left[\frac{div\,\nabla S}{\frac{dS}{ds}}\right]ds\right\},$$

which is calculable, in principle, since one knows the values of div ∇S and dS / ds on the trajectory L.

Let us continue.

Due to Lemma II, one first sees that the given of conditions (5.13) determines $R_1(t)$ on L_1 up to a multiplicative constant, provided that one chooses a trajectory on which R_1 is non-null ([10]).

We then attribute a very small radius r_{01} to Γ_1 (which we ultimately make go to zero), and cut it by constant time sections, and assume that R_1 takes the previously determined values $R_1(t)$ on these sections.

([10]) Which is the general case, because in the contrary case Q_1 becomes infinite, which may happen only for particular trajectories or in regions that the particles do not enter.

One then easily sees that R_1 is likewise determined on the interior of Γ_1, since it must satisfy the relations:

$$-\frac{\hbar^2}{2m_1}\Delta R_1 = Q_1 R_1 ,$$

in which Q_1 is fixed by the Taylor development in Γ_1 (since we now know Q_1 and ∇Q_1 on L_1), and must take the preceding values $R_1(t)$, on Γ_1; this is what we replace in the conditions of Lemma I.

We thus know R_1 everywhere in Γ_1; hence, the value of its derivatives, as well. If one then assumes – as is natural – that these derivatives ∇R_1 do not experience discontinuities when they cross the surface of the tube then one sees that one knows the set of Cauchy conditions on Γ_1 (relating to the Schrödinger equation) that are necessary in order to determine a solution $\varphi_1(r_{01})$ outside of Γ_1 (indeed, one knows R_1, ∇R_1, S_1, and ∇S_1) in spacetime.

Finally, it suffices to make r_{01} go to zero in order for this solution to go to the desired solution φ_1 in the limit ([11]).

Since this reasoning is valid for φ_2, one concludes from this that the hypothesis that the de Broglie conditions remains valid in a neighborhood of L_1 and L_2 (which corresponds to the physical notion of guidance) and a second-order contact of the surfaces S_1, S_2, and S^0 suffices to uniquely determine φ_1 and φ_2.

We may therefore state the following theorem:

Theorem.

To any solution Φ in configuration space there bijectively corresponds two continuous waves φ_1 and φ_2 in real space, which conforms to the previously proposed model and which satisfies the conditions that were proposed by de Broglie in a neighborhood of L_1 and L_2.

This suffices to prove the physical equivalence of the two interpretations for all of the cases that are actually known.

§ 8. – We shall now reproduce an argument of de Broglie that relates to quantum statistics in the case of particles of the same nature ($m_1 = m_2 = m$).

As one knows in the usual wave mechanics, it is necessary to assume that the wave Φ of the system in configuration space is either symmetric or anti-symmetric when the regions possible existence overlap each other if one takes the experimental facts into account. If one then refers to the preceding ideas, in which these particles are represented by two waves u, then one sees that it is natural to assume that if these waves overlap then they are superposed, and finally form a single wave:

$$u = f \exp(iS/\hbar) ,$$

([11]) An operation that is always meaningful here, since one only considers $\varphi(r_{01})$ that are continuous and bounded.

for which the amplitude f has two distinct singularities. With the preceding notations, we then have:

$$S_1(\vec{x}_{\mathrm{I}},\ \vec{x}_{\mathrm{II}},\ t) = S_2(\vec{x}_{\mathrm{I}}, \vec{x}_{\mathrm{II}},\ t).$$

Namely, from formulas (5.11):

$$\begin{cases} S_{11}(\vec{x}_1, t) = S_{22}(\vec{x}_1, t) \\ S(\vec{x}_1, \vec{x}_2, \vec{x}_{12}, t) = S_{11}(\vec{x}_1, t) + S_{11}(\vec{x}_2, t) + S_{12}(\vec{x}_{12}, t) \end{cases} \tag{5.18}$$

in which S designates the phase of the system in configuration space.

One may make the same argument for amplitudes. One obtains:

$$R_{11}(\vec{x}_1,\ \vec{x}_{12},\ t) = R_{22}(\vec{x}_1, \vec{x}_{12},\ t),$$

which gives, due to (5.12), the following equalities for the quantum potentials:

$$Q_{11}(\vec{x}_1,\ t) = Q_{22}(\vec{x}_1,\ t),$$
$$Q(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_{12},\ t) = Q_{11}(\vec{x}_{\mathrm{I}},\ t) + Q_{11}(\vec{x}_2,\ t) + Q_{12}(\vec{x}_{12},\ t),$$

which, like formulas (5.19), translate into the mathematical fact that the two singularities may be exchanged without anything being modified in the wavelike state. From this, one concludes that the quantum potential in configuration space:

$$Q(\vec{x}_1, \vec{x}_2, \vec{x}_{12},\ t) = -\frac{\hbar^2}{2m}\frac{(\Delta_1 + \Delta_2)R(\vec{x}_1, \vec{x}_2, \vec{x}_{12}, t)}{R(\vec{x}_1, \vec{x}_2, \vec{x}_{12}, t)},$$

must be symmetric in $\vec{x}_1$ and $\vec{x}_2$.

If R is then the amplitude of an arbitrary (asymmetric) solution of the wave equation in configuration space then one must form a linear combination of the form:

$$R' = CR + RC,$$

in which C and D are complex constants, such that the quantity:

$$\frac{C\Delta R + D\Delta R}{CR + DR}\,\varphi\, Q \qquad (\Delta = \Delta_1 + \Delta_2),$$

is insensitive to permutations $\vec{x}_1$ and $\vec{x}_2$, which translates into exchanging the positions of the singularities. By writing this condition explicitly, one easily finds $C^2 = D^2$, namely, $C = |\,D\,|$, or furthermore:

$$2 \arg C = 2 \arg D + 2n\pi,$$

$$\begin{cases} C = |C| e^{i\alpha}, \\ D = \pm |C| e^{i\alpha} = \pm C, \end{cases}$$

which expresses that one may allow only symmetric or anti-symmetric forms for the wave Φ in configuration space; this conforms to the classical results of the probabilistic interpretation as it applies to the particles of the same nature. On the other hand, here, as de Broglie has pointed out, we find that this result has been deduced from the model of the theory of the double solution, instead of being postulated *a priori*.

In the sequel, we shall not account for spin effects. Nonetheless, the preceding calculation that it might also be possible to derive the Pauli principle from the theory if one proceeds to show that for fermions the wave u may be composed of only one singularity, whereas in the case of bosons it may be composed of several.

§ 9. – The preceding considerations allow us to finally respond to the partial criticism of the causal interpretation that was raised by Pauli ([12]). We formulate it as follows: the introduction of exact positions $\vec{\xi}$ into the theory of the point-like aspects of micro-objects is necessarily devoid of all experimental significance because if one analyzes the effects of these parameters then two cases present themselves:

1. They never physically manifested themselves and consequently amount to considering a "metaphysical" character.

2. They are physically manifested by modifying the wave functions in a manner that depends on their precise values and, at the same time, destroys the validity of the Bose-Einstein or Fermi-Dirac statistics, even in the domains where they have verified by experiment.

We respond to this criticism by showing that point 2 of Pauli is based on an arbitrary concept, and unjustified in the manner by which the hidden parameters $\vec{\xi}$ may be experimentally manifested.

We commence with a preliminary remark:

A number of people have imagined that the validity of the Bose-Einstein and Fermi-Dirac statistics signifies that particles of the same nature are necessarily indiscernible from each other. According to them, if one may enumerate them, as will be the case if they have continuous trajectories, then one will be obliged to use the statistics of Boltzmann regardless of whether the wave function is symmetric or anti-symmetric.

This inexact concept is applied to a very widely used interpretation of the "Gibb's paradox" in classical statistical mechanics. In order to obtain a correct value for the entropy of a system of n particles one must divide the classical volume of phase space by $n!$, which is interpreted by saying that the exchange of two particles does not lead to a new state of the system. From this, one obviously concludes that in order to obtain the

([12]) In the work: LOUIS de BROGLIE, *Physicien et Penseur.*

Bose-Einstein or Fermi-Dirac statistics one must assume that the particles are indiscernible, in the sense that it is not possible to attribute distinct identities to them.

The weak point in this argument is that it rests on the hypothesis that all of the elements of phase space that have the same energy are equally probable when one applies the laws of classical mechanics to the motions. As a consequence, it only applies to classical particles.

Since we know that the classical laws do not account for the behavior of micro-objects in the context of the causal interpretation, one sees that one must calculate this probability in phase space and introduce forces of quantum origin, which then leads to the statistical conclusions that were predicted by the probabilistic interpretation in all of the cases.

In order to see this, one must first note that in the causal theory particles that have well-defined trajectories are, in principle, identifiable. From this, one concludes that the exchange of two trajectories actually leads to a new state of the system considered. However, having said this, it results from the preceding sections that the external fluctuations generate the distribution $P = |\varphi|^2$, for an ensemble of systems. This distribution - whether symmetric or anti-symmetric – will therefore not be destroyed by the exchange of the parameters $\vec{\xi}_{(\mathrm{I})}$ and $\vec{\xi}_{(\mathrm{J})}$. Now, as Bohm has shown, the distribution $P = |\psi|^2$ suffices to establish that the new interpretation gives back all of the statistical results that are associated with the ideas of the Copenhagen School ([13]).

In domains where the usual form of the theory ceases to be valid, i.e., at distances less than 10^{-13} cm., it will obviously be possible, in principle, to exhibit the processes that are capable of destroying the distribution $P = |\varphi|^2$, and the validity of the Bose-Einstein and Fermi-Dirac statistics. This will be the case, for example, if a measure of the *localization* of the individual micro-objects becomes possible at this level. However, even in this case, the external perturbations that were described in chapter II quickly re-establish the distribution.

Objection 2 of Pauli consists of affirming that it is not possible to obtain manifestations of $\vec{\xi}$ at this or that level without destroying the symmetry properties of the wave functions, even in the levels where we know that they account for experimental phenomena. To prove this, Pauli implicitly assumed that the only possible physical manifestations for $\vec{\xi}$ appear through their action on the wave function.

If one then recalls, for example, the hypothesis that Bohm made, that it is possible to introduce terms into the Schrödinger equation that depend on the position $\vec{\xi}$ of the particles as in the case of two particles (I = 1, 2):

([13]) If one has $P = |\psi|^2$ then one sees, for example, that the predictions of statistical mechanics depend uniquely on the values of the energy levels and the occupation probabilities for each level. Since the values of these levels (which depend only on the wave function that is common to the two interpretations) and their statistical treatment is identical on the common domain of validity of these two theories, one is then led to the same occupation probability $\exp(-E_n/KT)$ for each level (in which E_n designates the value of the energy and K, Boltzmann's constant). One concludes from this that the two interpretations lead to the same statistical results.

$$i\hbar\frac{\partial\varphi}{\partial t} = -\hbar^2(\Delta_1 + \Delta_2)\varphi + V\varphi + F(\vec{x}_1,\ \vec{x}_2,\ \vec{\xi}_1, \vec{\xi}_2),$$

one sees that if F does not have special properties then, in time, this will lead to considerable changes in ψ that destroy its symmetry, even if Φ represents a perturbation that has only the effect of altering things at the level of 10^{-13} cm.

This argument is not valid for two reasons:

In the first place, there is no reason for F to not have the same symmetry properties as ψ, which excludes the appearance of cumulative effects that capable of destroying this symmetry. Pauli did not explain this point clearly. He seemed to believe that this hypothesis will further prevent the particles from manifesting themselves individually, even if the $\vec{\xi}$ remain "metaphysical," and that it will be vain to attribute a distinct identity to the micro-objects considered. In our opinion, this is not exact because they then manifest themselves collectively with a behavior that is different from the behavior that is predicted by the usual interpretation that makes it necessary to introduce the $\vec{\xi}$ into the theory.

In the second place, it is not necessary that the $\vec{\xi}$ manifest themselves uniquely through their influence on the field ψ, as Pauli assumed. In a recent article ([14]), for example, Bohm has envisioned the introduction of potentials $V(\vec{\xi}_{\mathrm{I}})$ into the theory that act on the particles directly without touching the field ψ. Such an interaction allows us to observe the individual $\vec{\xi}_{(\mathrm{I})}$ with a unlimited precision without altering ψ. The $\vec{\xi}_{(\mathrm{II})}$ may therefore manifest themselves without destroying the validity of the Bose-Einstein or Fermi-Dirac statistics at the level where quantum theory is valid, provided that one lets an interval of time pass that is sufficient for the distribution $P = |\varphi|^2$ to be re-established.

([14]) *Progress of Theoretical Physics,* Vol. 9, No. 3, pp. 273, 1953.

CHAPTER VI

§ 1. – The theory of interacting micro-objects that was discussed in the preceding chapter plays an essential role in the new interpretation. It has allowed David Bohm to define the basis for a causal theory of measurement that agrees with the postulates ([1]) of Bohr and Heisenberg on the results of measurements in probabilistic theory. One may therefore address the principal objections that were raised by the adversaries of determinism to the causal interpretation, and, with the same stroke, understand one of the essential properties of matter in the context of the organization considered: the exchange of energy by quanta between systems of interacting micro-objects.

We shall give a brief summary of it and refer the reader to the works of David Bohm for more details.

We first analyze what happens for an isolated micro-object when the wave φ is a superposition of two plane waves, namely:

$$\varphi = C_1\varphi_1(\vec{x}_1)\exp(-iE_1 t/\hbar) + C_2\varphi_2(\vec{x}_2)\exp(-iE_2 t/\hbar),$$

in which C_1 and C_2 are real. Setting $\varphi_1 = R_1$, $\varphi_2 = R_2$, $\varphi = R\exp(iS/\hbar)$, one immediately has:

$$R^2 = C_1 R_1^2 + C_2 R_2^2 + 2C_1 C_2 R_1 R_2 \cos[(E_1 - E_2)t/2\hbar],$$

with:

$$\tan\left\{\frac{S + (E_1 - E_2)t/2}{\hbar}\right\} = \frac{C_2 R_2 - C_1 R_1}{C_2 R_2 + C_1 R_1}\tan\left\{\frac{(E_1 - E_2)t}{2\hbar}\right\}.$$

These relations show that everything happens as if the singularity-particle is subject to a quantum potential that fluctuates with an angular frequency $\omega = (E_1 - E_2)/\hbar$. The energy $E = -\partial S/\partial t$, and the moment of the particle then oscillate with the same frequency along a very complicated trajectory in an apparent state of Brownian motion ([2]). In the absence of any interaction, these oscillations persist indefinitely, which is reasonable since a transition from one state to another demands an exchange of energy with the external systems.

Therefore consider an exchange of this type – for example, the Franck-Hertz experiment – i.e., the inelastic collision of a planetary electron that belongs to a hydrogen atom in a stationary state E_0 with an incident electron.

Before the collision, these two particles are represented by the wave $\varphi_1 = \varphi_0 \exp(-iE_0 t/\hbar)$ and the wave-packet:

$$f_0(\mathbf{y}, t) = \int e^{\vec{k}\vec{y}} f(\vec{k} - \vec{k}_0)\exp(-i\hbar k^2 t/m)\,d\vec{k},$$

([1]) That were discussed at the beginning of chapter IV.

([2]) Cf. Takabayasi, *loc. cit.*

respectively, in which **y** designates the coordinates of the incident particle. The center of this packet coincides with the extremum of the phase – a function of k – and is therefore found at the point $\vec{y} = \hbar\vec{k}_0 t/m$.

As one knows, in the absence of interaction, which is the present case, the corresponding motions are represented by a unique function in configuration space that depends on $\vec{x}$ and $\vec{y}$ as a product of the preceding two functions. One therefore has:

$$\Phi_i = \varphi_0(\vec{x})\,\exp(-iE_0 t/\hbar)\,f_0(\vec{y}, t),$$

and the two particles move in space independently.

During the collision, the two micro-objects interact, and the wave function in configuration space – a solution of the corresponding Schrödinger equation – may be written:

$$\Phi = \varphi_i + \sum_n \varphi_n(\vec{x})\exp(-iE_n t/\hbar)\,f_n(\vec{y}, t),$$

in which the f_n are the coefficients of the development of Φ into the φ_n, which correspond to the energy levels E_n of the planetary particles. The motion of the two particles is then very complicated, and governed by proper and interaction (classical and quantum) potentials.

From the interaction, one verifies without difficulty that the function Φ tends to the asymptotic form:

$$\Phi = \varphi_i(\vec{x}, \vec{y}) + \sum_n \varphi_n(\mathbf{x})\exp(-iE_n t/\hbar)\cdot$$
$$\int f(\vec{k}-\vec{k}_0)\frac{\exp[i\vec{k}_n\vec{r} - (\hbar k_n^2/2m)t]}{r}\,g_n(\theta, \Phi, \vec{k})\,d\vec{k},$$

with $\hbar^2 k_n^2/2m = (\hbar^2 k_0^2/2m) + E_0 - E_n$ (which corresponds to the conservation of energy). The preceding terms in the $\sum$ sign in the right-hand side represent diffuse wave packets (in which the particle takes the velocity $\hbar\vec{k}_n/m$) correlate with the functions $\varphi_n(\vec{x})$ that represent the corresponding states of the atomic electron. The center of the n^{th} packet is given by $\vec{r}_n = (\hbar\vec{k}_n/m)t$, and one easily sees that these packets end up being separated by a classically describable distances because their velocity depends on the quantum number n.

In the causal theory, the function Φ allows us to calculate both the density $\Phi\Phi^*$ and the trajectories $m_j\vec{v}_j = \nabla_j S$ of an ensemble of particles that are pairwise associated with the state considered. As a consequence, if one is given the initial positions of two electrons and the initial values of the state functions then one may predict the motion of the micro-objects, in principle. From our description, it is clear that two particles follow very complicated trajectories during and after the interaction, even though the emerging wave packets are superposed in space. This is no longer the case during a certain time in which the function Φ tends to the preceding asymptotic form (6.1). The complex behavior of the system then subsides progressively. When the packets are macroscopically separated, one sees that the incident electron is necessarily in one of them (the one that is characterized by a given value of n) since it does not penetrate into the regions where the function is negligible. The planetary electron itself is finally found in the energy state that corresponds to the asymptotic development. Since the incident electron remains thenceforth in the emerging packet considered, one may practically treat the ultimate evolution of the system by limiting the state function to the corresponding term of the development ([3]), namely:

$$\Phi_n = \varphi_n(\vec{x})\exp(-iE_n t/\hbar)\cdot$$
$$\int f(\vec{k}-\vec{k}_0)\frac{\exp\{i[\vec{k}_n\vec{r}-(\hbar k_n^2/2m)t]\}}{r}g_n(\theta,\Phi,\vec{k})d\vec{k}\ ,$$

in which n defines the packet that actually contains the particle. This function represents an atomic electron in the n^{th} quantum state and an emerging electron of energy $\hbar^2 k_n^2/2m$ that evolves in an independent fashion since it reduces to a product of functions of $\vec{x}$ and $\vec{y}$.

The energy of that atomic electron is E_n because its wave function may be written in the form $\varphi(\vec{x})\exp(-iE_n t/\hbar)$; this proves precisely that energy is always transferred by quanta of magnitude $E_n - E_0$ in inelastic collisions of the preceding type.

We have therefore obtained a causal description in terms of the new mechanics of the exchanges of energy by quanta without resorting to the postulates of Bohr and Heisenberg.

Such exchanges take place in a continuous (although rapidly varying) fashion during the collision, and are quantized only by interacting with systems. The probability of each process is obviously equal to the one that one obtains in the probabilistic interpretation.

§ 2. – It finally remains for us to discuss the theory of measurement in the new interpretation. In order to facilitate its comprehension, we first recall the essential principles of that theory in the interpretation of Bohr, which we state in the same order as they are usually presented in ([4]).

1. According to the Copenhagen School, the notion of measurement must be placed at the basis for any theory of micro-phenomena. Indeed, as we saw in the preceding chapter, Bohr and Heisenberg estimated that such a theory does not have the objective of giving a representation (however approximate) of everything that happens in nature objectively, but only that of giving a mathematical symbolism that allows us to predict the results of measurements that are performed by an observer.

([3]) Indeed, one may show that the other packets may not interfere with the packet considered, by reason of their interaction with the macroscopic systems that surround the system

([4]) Cf. the beginning of chapter IV.

2. The results are not determined in advance, in general, by reason of the fact that the interactions between the measuring apparatus and the micro-objects are unpredictable, in principle.

The probabilistic interpretation therefore describes such processes by associating a Hermitian operator A (which admits a spectrum of functions and proper values φ_i and λ_i) to every measuring apparatus that is supposed to measure a certain magnitude, and a wave function ψ to every micro-object.

During a measurement, the preceding interaction, which is unpredictable, by definition, then generates one of the values that are defined by the operator A, and this happens with a probability $| C_i |^2$ that corresponds to the square of the Fourier coefficient C_i of the development of the characteristic function ψ of the micro-object that is being measured ($\psi = \sum C_i \varphi_i$).

According to Niels Bohr and Heisenberg, the wave ψ is therefore a sort of representation for the ensemble of potentialities of the measurement of the particle, with their respective probabilities. It is no longer a "predictive element," to recall the expression of Destouches, which is capable of being briefly modified when the observer acquires new information.

This "reduction of the wave packet by the measurement" that was described by Heisenberg suffices to show the non-physical and purely subjective character of the wave ψ. The Copenhagen School therefore opposed the determinism of micro-phenomena in an irremediable fashion because if it is true that such probability waves evolve in a rigorous fashion between measurements then one easily sees that any observation of the information that it carries interrupts the course of this determinism of the probabilities.

The causal interpretation takes the counterpoint of the preceding positions. Indeed, according to it, micro-objects and apparatuses, which are themselves composed of micro-objects, exist independently of us in nature. They may be described, at least at the levels considered, by waves u that give an approximate representation. As we have seen, these waves, by the intermediary of their regular parts, also allow us to calculate the statistical density of particle ensembles in a given state that have attained their equilibrium state. The "hidden parameters" characterize both the description of the micro-object *and* the measuring apparatus.

Before the interaction, the micro-object and the apparatus evolve independently, and their state functions are governed by distinct Hamiltonians.

During the measurement, there is an interaction between the apparatus and the micro-object; it is represented by the introduction of an interaction Hamiltonian H_1. The apparatus and the micro-object mutually influence each other, and the original states of the micro-object and the apparatus are obviously perturbed. That interaction is characterized by the measurement process because in the causal interpretation the observed variations in the apparatus may be related to actual states of the micro-object and the apparatus before the measurement.

After the measurement, if the interaction ceases then the apparatus and the micro-object must evolve independently in states that differ, in general, from the states in which they were previously found. One may predict them, in principle, when one is given the initial states of the micro-objects and the distribution of the observation.

We remark that these ideas constitute a natural extension of the classical ones on the nature of the measurement process. In full rigor, such processes always perturb observed systems; however, at the classical level one admits that such perturbations are negligible. The same is not true at the quantum level, where the interactions are such that they do not leave such systems in the state in which they found before the measurement, except for exceptions. This is, moreover, a fundamental property of the observations at the level considered: They must permit the causal theory of measurement to explain the experimental success that was attained by the probabilistic interpretation and to clarify the new physical significance that one must attribute to the Heisenberg Uncertainty principle.

§ 3. – The general analysis of measurement processes allows us to single out a number of properties, which we summarize as follows:

a) The measurement of an arbitrary variable must be performed by means of an interaction between the observed system and a convenient part of the measurement apparatus. In order to give it a precise macroscopic significance, the measurement apparatus must be constructed in such a fashion that a given state of the observed micro-system ultimately corresponds, at the classical level, to a certain interval of states of the apparatus that is used. During a measurement the interaction introduces a correlation between the state of the system that is being observed and the state of the apparatus, with a precision that depends on the preceding interval.

b) Then consider the measurement of a certain observable magnitude Q that is associated with a given micro-object with the aid of a certain apparatus. Let $\vec{x}$ be the position coordinates of that object and y the coordinate (or coordinates) of the apparatus that is associated with that observable. One may show, as David Bohm did, by analysis of the physical properties of the apparatus that is used, that one may confine oneself to the use of apparatuses of the "impulsive" type; i.e., such that the duration of the interaction between the object and the apparatus is sufficiently brief that one may neglect the evolution that one subjects the micro-object and apparatus to during this time interval. We may thus suppress the parts of the Hamiltonian that are associated the apparatus and the isolated micro-object during this interaction, and keep only H_I. That Hamiltonian will obviously depend on the observable Q that one must measure for the micro-object and also the operators that act on y; this is necessary for the system observed and the apparatus to be coupled.

Having said this, we shall show, by way of example ([5]), that the apparatuses act objectively like "spectral analyzers," i.e., they ultimately decompose the state function φ that is associated with the micro-object into distinct wave packets that correspond to the proper functions and values of Q. As in the case of the Frank-Hertz experiment, the micro-object will enter into one of the two with a probability gives us the results that were postulated by the probabilistic interpretation.

([5]) Cf. D. Bohm, loc. cit. An analysis of the apparatuses that were actually used shows that this property is valid for all types of apparatuses. The proof that follows therefore has a completely general character when one uses real apparatuses at the level considered.

By way of illustration, we treat the Hamiltonian $H_I = -aQp_y$, in which a is a constant and p_y is the momentum conjugate to y.

In the causal interpretation, as we have seen, the evolution of two systems during the interaction is represented by a function of $\vec{x}$, y, and t that allows us to describe the motion of the singularity-particle and the system in configuration space.

This function ψ satisfies the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = a/\hbar Qp_y \cdot \psi = (-ia/\hbar^2)Q\frac{\partial\psi}{\partial y}\ . \tag{6.2}$$

It may be developed into a series of proper functions $\psi_q(\vec{x})$, of the operator Q, where q designates a proper value of Q, namely:

$$\psi(\vec{x},\ \vec{y},\ t) = \sum_q \psi_q(\mathbf{x})\, f_q(y,t)\,. \tag{6.3}$$

Since $Q\psi_q(\vec{x}) = q\psi_q(\vec{x})$, one obtains the following equality for each value of q:

$$i\frac{\partial f}{\partial t}(y,t) = (-ia/\hbar^2)Q\frac{\partial f}{\partial y}(y,t)\,,$$

which admits the value:

$$f_q(y,\ t) = f_q^0(y - aqt/\hbar^2)$$

as a solution, in which the index 0 refers to the initial values.

It then obviously follows that ψ, which may be written:

$$\psi(\vec{x},\ y,\ t) = \sum_q \psi_q(x)\cdot\ f_q^0(y - aqt/\hbar^2)\,,$$

must separate into distinct wave packets in the space of y. Indeed, one has, initially:

$$\psi_0(\vec{x},\ y) = \varphi_0(\vec{x})g_0(y) = g_0(y)\sum_q c_q\psi_q(\vec{x})\,, \tag{6.4}$$

in which the φ refers to the state function of the micro-object, since in the absence of interaction the function ψ reduces to a product of functions of $\vec{x}$ and y alone. The c_q are generally unknown coefficients in the development of the state function of the micro-object into the ψ_q; g_0 is the initial state function of the coordinate y of the apparatus, a function that takes the form of a packet of size Δy.

By comparing (6.3) and (6.4), one sees that $f_q^0(y) = c_q g_0(y)$; by substituting this result into (6.2), this gives:

$$\psi(\vec{x}, y, t) = \sum_q c_q \psi_q(\vec{x}) g_0(y - aqt/\hbar^2) .$$

This relation, which is analogous to (6.1), allows us to repeat the argument that we made in the context of the Frank-Hertz experiment.

Before the interaction, the micro-object and the apparatus evolve independently, and may be described by distinct wave functions.

During the interaction, the wave function ψ is very complicated. The micro-object and the apparatus are subject to violent oscillations that are analogous to the previously described processes.

After the interaction, one sees that their behavior stabilizes after a certain length of time because the packets $g_0(y - aqt/\hbar^2)$ that correspond to the different values of q cease to be superposed in the space of y. Indeed, the q^{th} packet is centered in that space at the point $q = \hbar^2 y / at$, and the adjacent packets are obviously separated by an interval:

$$\delta y = at\delta q/\hbar^2 ,$$

which (if q and t are sufficiently large) may be made much larger than Δy.

It then results from this analysis that the packets are classically separated in the space of y.

When one observes a given micro-object under these conditions, one sees that the variable y of the apparatus enters naturally into one particular wave packet, which is determined by the initial conditions, in principle, once the observation is completed. The final result of the measurement is therefore determined by the initial form of the state function $\psi_0(\vec{x}, y)$, and the initial positions $\vec{x}$ and y of the particle and the variable that characterizes the apparatus.

As before, one may therefore eliminate the other packets (which no longer act on the quantum potential or the moments p_x and p_y of the micro-object and the apparatus) of the state function that thenceforth may be written:

$$\psi(\vec{x}, y, t) = \psi_q(\vec{x}) g_0(y - aqt/\hbar^2) ,$$

in which q corresponds to the packet that actually contains the variable y. This expression shows that the apparatus and the micro-object evolve in a manner that is independent of the measurement. If one then obtains the approximate value of the coordinate of the apparatus with a precision $\Delta y \approx \delta y$ then one sees that the wave function of the particle will be ψ_q after the interaction, and the observable Q will assume the numerically definite value q. (As in the probabilistic interpretation, if the product $at\delta q/\hbar^2$ is less than Δy then no precise measurement will be possible.) We thus recover the first part of the postulates of N. Bohr.

If one then observes an ensemble of micro-objects in the same state, in the sense of the causal interpretation, then one obtains an ensemble of results of the preceding type that is statistically represented by $|\psi|^2$. We then seek to evaluate the probability of obtaining each particular value q. It is obviously obtained by integrating y, which is normalized over all the $\vec{x}$ and y in a neighborhood of the q^{th} packet. Since these packets are separated in space, one concludes that it suffices to integrate the expression $C_q \psi_q(\vec{x})$ $g_0(y - aqt/\hbar^2)$. Since ψ_g and g_0 are normalized, the probability that the particle and the variable observed enter into the q^{th} packet is consequently given by the equality $P_q = |C_q|^2$; i.e., precisely the value that is postulated by the probabilistic interpretation.

We have therefore proved that the new interpretation recovers all of the postulates of the old theory by attributing to the measurement an objective sense of interaction that is absolutely foreign to the positivistic concepts that govern the Copenhagen interpretation.

It is clear that the preceding theory may not pretend to answer the question of measurement at the present time. We confine ourselves to the following remarks, which precisely clarify the orientation of the research undertaken.

1. Whenever one is reduced to observing micro-objects with the aid of macroscopic apparatuses that are made of a very complicated ensemble of objects of the same type, powerful interactions are inevitable. The values of the physical quantities that are attached to the micro-objects that are given by such measurements will therefore be distinct real values, the values before measurement.

2. In principle, the preceding theory must be repeated for each class of apparatus that is associated with a given magnitude since it comes down to a physical description of the actual process of interaction. All of the Hermitian operators that do not strongly correspond to the observations have a physical significance as in the probabilistic interpretation. This general correspondence, which was postulated by N. Bohr, is obviously not provable, in principle. In particular, in the usual analysis of the effects of known apparatuses (cf., D. Bohm, *Quantum Theory*, pp. 594), one will find the extension of the preceding theory to measurements of spin and moments. As David Bohm has remarked, the canonical invariance and the theory of transformations do not play an essential role in the new interpretation; however, this is not necessary on the physical plane, provided that one may causally account for the properties that are verified by experiments, namely:

- the existence of discontinuous energy levels in matter;
- the quantization of electromagnetic energy;
- the appearance of integer quanta of energy in the photoelectric effect, even when the electromagnetic wave is macroscopically extended in space.
- The appearance of interference phenomena, even when the photons are introduced into the system separately and independently (cf. the experiments of VAVILOV);
- The analogous phenomena for particles, namely the transfer of energy by quanta, even when the force of interaction is weak (see the Franck-Hertz experiment) and the appearance of diffraction phenomena when the particles are introduced separately and independently into the apparatus.

In particular, the causal theory that gives a spatio-temporal description of the preceding phenomena considers the general theory of operators to be a mathematical procedure that is devoid of any physical significance.

The only known observables at the moment are position, momenta, angular momenta, several functions of position (such as dipole and quadrupole moments), energy, and spin.

As all of these quantities are now included in the causal interpretation one deduces from this that the two interpretations recover the experimental results. Bohr's general theory of transformations, which assumes the existence of physical operations that allow us to observe an arbitrary Hermitian operator, obviously does not rest upon any experimental justification, and may therefore not be used as an argument against the model that we have proposed in this work.

§ 3. The existence of powerful interactions between the apparatus and the objects observed is not contradictory in itself if the theory allows us to calculate, in principle, the values of the magnitudes that exist before by starting with the values that are observed after the measurement. We take only one example. A photographic screen or plate that registers the impact of an electron or a photon obviously perturbs the micro-object that is observed (which must likewise be annihilated), but gives a relatively precise indication of the position of that object at the instant of its measurement. In particular, the observation of spectral lines that are due to the photons emitted by a hydrogen atom annihilates these photons, but gives exact indications about the energy state of the atom in question. For us, the apparatus here acts only on a part of the system observed – the photon – and is not responsible for the phenomenon of emission in the slightest. The measurement even permits us to deduce the initial and final state of the atom observed.

It is not the same in the probabilistic interpretation, where, in full rigor, it is the provisional interaction between the apparatus and the atom that makes it pass from a proper value E_n of the energy to another E_m with a probability that is described by the wave function that is associated with the planetary electron.

When, as is the case in a number of experiments, the atom is found on the Sun and the observer is on the Earth, this interaction has a macroscopic character. If there is no logical difficulty in this, as Bohr has shown (because, according to him, it is not possible to describe this interaction, in principle) then one cannot help but feel a certain malaise, at least if one believes in the actual existence of physical processes.

We shall return to this particular point later on in the context of angular correlations.

§ 4. – It is clear that the preceding theory transforms the significance of the uncertainty principle.

Indeed, as we have seen, the actual interaction between the apparatus and the object measured makes the primitive state ψ of the micro-object practically pass into one of the proper functions of the observable Q being measured. The value q that is obtained by this measurement is, as a consequence, automatically reproduced if one repeats the experiment considered.

Then suppose that one performs two successive measurements on a micro-object that correspond to two observables Q and P that do not commute: the second one is defined by a variable z. The state function, after the two interactions, will be transformed into the expression:

$$\psi(\vec{x}, z, t) = \sum_p a_{pq} \Phi_p(\vec{x}) g_0(z_0 - apt/\hbar^2),$$

in which Φ_p is a proper function of P_0 that corresponds to the value p, and a_{qp} is the coefficient defined by:

$$\psi_q(\vec{x}) = \sum_p a_{pq}\Phi_p(\vec{x}).$$

Since the packets that correspond to different values of p separate from each other in the course of time in the space of t, one further deduces that this function may be finally replaced by:

$$\psi = a_{pq}\Phi_p(\vec{x})\, g_0(z - apt/\hbar^2),$$

in which p represents the packet in which the coordinate z enters. The probability of obtaining the value p by starting with an ensemble of such measurements that are associated with an ensemble of micro-objects in the same state may be therefore written:

$$|a_{pq}|^2,$$

exactly as in the probabilistic interpretation.

From this, one concludes that if P and Q do not commute then it is impossible to make a simultaneous measurement that gives two values p and q with precision. Conforming to the proposed model, the perturbation of the micro-object that is introduced by the apparatus makes one precise measurement incompatible with the other since the measurement of P transforms the state function ψ into ψ_p: This is a function that may be given a value q with certainty only if it is also a proper function of Q. (As one knows, this is impossible if P and Q do not commute.) It then follows that in the causal theory the uncertainty principle does not express a fundamental, forever inexplicable, limitation of the precision with which one may simultaneously measure the position and momentum of one micro-object. In effect, in that theory, the principle does not apply to measurements that are performed on the individual micro-objects (whose motions are, in principle, perfectly defined and describable), but only on ensembles of measurements. One may formulate this as follows: In the context in question, the interaction between the measurement apparatus and the quantum ensembles of micro-objects being observed obligates the uncertainties Δp and Δq that are associated with an ensemble of measurements of the complementary observables to satisfy the relations:

$$\Delta p \cdot \Delta q \geq \hbar.$$

The Heisenberg uncertainties thus simply express a statistical property of measurements that one actually performs. As we do not know how to determine the true properties of the micro-objects (positions and momenta of the particles, etc.) by such measurements one is constrained, from the statistical viewpoint, to limit oneself to them and to consider those properties that characterize both the object and the apparatus to be provisionally hidden parameters. The observables that we are concerned with are meaningful with respect to the objects observed only at the classical level where one may neglect the effects of the apparatus. At the quantum level, they do not characterize these objects alone since the results obtained result from an interaction that is impossible to neglect, in general.

§ 5. – As D. Bohm has emphasized, this leads one to see whether it is possible to measure, not the usual "observables," whose physical significance is ambiguous, but quantities that are actually hidden, such as the position or velocity of the particle-singularities. We shall not treat that question in this work. It obviously raises very delicate theoretical and experimental problems that would be premature to discuss here ([6]). We nevertheless note that the unobservable character of the elementary trajectories at the present time does not signify, as we have already remarked, that the causal and probabilistic interpretations are physically equivalent. For example, one owes Schrödinger for the description of an experiment that makes this distinction clear. With the same stroke, he makes certain difficulties of the probabilistic interpretation stand out as they relates to the classical difficulties with the instantaneous action at a distance.

Consider two charged particles, 1 and 2, of different masses that arrive with known probability distributions as wave trains of limited dimensions.

The directions **1** and **2** intersect in a region *V* outside of which we may neglect their interactions.

When leaving *V*, the incidence directions **1** and **2** correspond to probable directions of refraction that are pairwise coupled ($\mathbf{1}'$ and $\mathbf{2}'$, $\mathbf{1}''$ and $\mathbf{2}''$, …), and calculable in advance.

Assume that we place a detector *D* at A' that registers the arrival of 1 in that region. We will then have that 2 is found in B'.

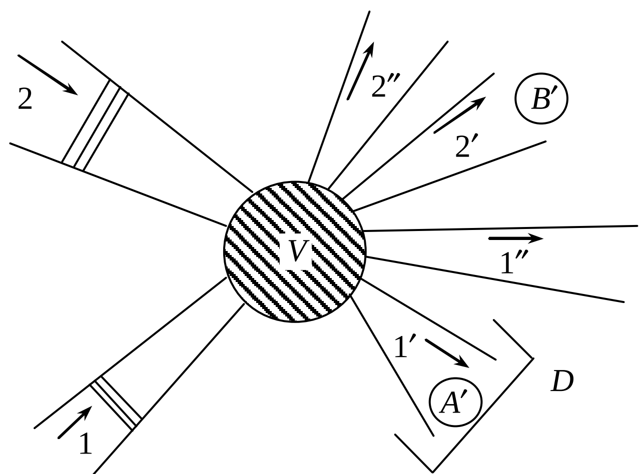

Fig. 25.

In the probabilistic interpretation, one says that the action of *D* (which is indescribable, by definition) on the system of two particles that are observed leaving *V* will oblige $1'$ to go into A' and $2'$ to go into B'. As it is possible to separate V, A', and B' by macroscopic distance, this signifies, as Schrödinger remarked, that *D* acts on the particles 1 and 2 instantaneously, even if the second one is separated from it by macroscopic distances, and this happens in the same fashion for any detector employed (plate, counter, etc.). This is the "magic" in the expression of Schrödinger because if one assumes the actual existence particles outside the observer, it implies the existence of

([6]) For example, in an article of H. RENNINGER, *Zeit. F. Physik*, t. 136, Heft 3, pp. 18, one will find the schema for an experiment that is capable of exhibiting both the corpuscular and the extended character of photons.

physical actions of a new type that are unknown in Nature up till now and run contrary to relativistic thinking, moreover.

In the causal interpretation, one says that the two particles have followed actual trajectories that are linked in probability by the manner in which they entered *V*. If the trajectory of particle 1 leads into A' then the associated trajectory of 2 will lead into B', etc. The introduction of D into A', which permits us to confirm the presence of 1, does not act on 2, which is found in B' since the interaction between 1 and 2 occurs when they have traversed *V*.

We have recovered exactly those essential traits that opposed the two interpretations here.

The Copenhagen School denied any possibility of knowing objects outside of the effects of their action on the measurement apparatus. It likewise assumed, a priori, the forever-incomprehensible character of such an action, whose exact mechanism definitively escapes the analysis of human endeavor. One is therefore logically confined, as de Broglie has noted, to "a sort of 'subjectivism' appearing in idealism, in the philosophical sense…that tends to deny the existence of a physical reality that is independent of the observer."

On the contrary, the causal interpretation, which starts with the objective existence of micro-objects independently of any observation, affirms the possibility of giving deterministic "models" that are valid at the different levels considered. It therefore necessarily indicates a spatio-temporal analysis of measurement processes that are realized in Nature, as well as an exact description of the interactions that occur between the measurement apparatuses and the micro-objects that are being observed. The analysis that was previously sketched out makes no pretension of completeness; in our opinion, it marks progress towards returning a real character to measurement processes that is describable, in principle, in the framework of the proposed theory. At the level of micro-phenomena, any description of such processes must take into account both the macroscopic (therefore complex) and microscopic character of the apparatuses used (since, in the final analysis, they reduce to ensembles of micro-objects that belong to the same level as the objects observed). In general, one is therefore concerned with powerful interactions that are impossible to neglect and, according to us, explain the statistical character that was given up till now for the quantum theory of measurement.

In summation, the new interpretation justifies the essential role that is attributed to stochastic phenomena in the old theory by the importance that it attached to interactions and to the reciprocal conditioning of the micro-phenomena. This importance, which gives the causal theory a somewhat peculiar aspect, constrains us to consider Nature to be an extraordinarily complex continuous ensemble of micro-processes in a state of interaction and perpetual transformation.

Moreover, it is in the theory of measurement, when one succeeds in accounting for the complex motion of things, that the new micro-mechanics, which, in a sense, extends ([7]) the mechanistic materialism of classical theory, passes to the quantum level to bring about the dialectical evolution of matter in motion.

([7]) Since it preserves two essential traits: the objective reality of the external world and the determinism of phenomena.

APPENDIX I

§ 1. Any ordered set of n real independent variables x_i (in which i takes the values 1, 2, …n) may be considered as defining a point in an n-dimensional space V_n.

If $\varphi^i(x^i, \ldots, x^n)$, with $i = 1, 2, \ldots, n$ represents real functions whose Jacobian is not zero then the equations:

$$x'^i = \varphi^i(x^1, \cdots, x^n) \tag{I.1}$$

define a change of coordinates in V_n.

Let x^i be the coordinates of a point M. The coordinates of points that are infinitely close to M may be obtained by giving the coordinates x^i arbitrary infinitesimal increases dx^i. If M' is such a point and $x^i + dx^i$, its coordinates, then we say that the points M and M' define an infinitesimal vector that is attached to the point M and the components dx^i.

We then perform the coordinate change (I.1). Taking into account the usual convention on summation over repeated indices, we get:

$$dx^i = \frac{\partial x^i}{\partial x'^\alpha} dx'^\alpha, \tag{I.2}$$

which indicates that the components of MM' are transformed by a linear substitution.

We then say that the set of vectors that are collinear with the infinitesimal vectors that are attached to the point M define an affine vector space attached to the point, which we may call the affine tangent space to V_n at M.

In this space, the notion of contravariant, covariant, or mixed tensor is immediately introduced with the aid of the usual formulas.

For example, the expressions λ^i and λ'^i represent the contravariant components of a vector if:

$$\lambda^i = \lambda''^\alpha \frac{\partial x^i}{\partial' x^\alpha}. \tag{I.3}$$

A covariant vector field will be given by its components η_i, which satisfy the transformation formulas:

$$\eta'_i = \frac{\partial x^k}{\partial x'_i} \cdot \eta_k, \tag{I.4}$$

or by:

$$\eta_i = \frac{\partial x'^k}{\partial x^i} \cdot \eta'_k,$$

and, more generally, the notion of a mixed tensor field will be related to transformation formulas such as:

$$R^{\dots mn}_{\dots kh} = R'^{\dots rs}_{\dots lp} \frac{\partial x^l}{\partial x'^k} \frac{\partial x^p}{\partial x'^h} \cdots \frac{\partial x'^r}{\partial x^m} \frac{\partial x'^s}{\partial x^h} \cdots \tag{I.5}$$

§ 2. We have thus defined the general notion of tensor at each point. It remains for us to provide the means to compare the values of their components at different points. To do this, one introduces the notion of coefficients of a connection that will permit us to generalize the covariant derivative of ordinary Riemannian spaces.

The differentiation of (I.3):

$$\frac{\partial \lambda^i}{\partial x^j} = \frac{\partial \lambda'^\alpha}{\partial x'^k} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^j} + \lambda'^\alpha \frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} \frac{\partial x'^\beta}{\partial x^j}, \tag{I.6}$$

shows, because of the last term in the right-hand side, that the partial derivatives of λ^i and λ'^i are not the components of a tensor.

By contrast, one remarks that if one introduces the three-index expressions Γ^i_{kl} and Γ'^i_{kl}, which are functions of x and x', and satisfy the equations:

$$\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + \Gamma^i_{jk} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^k}{\partial x'^\beta} = \Gamma'^\gamma_{\alpha\beta} \frac{\partial x^i}{\partial x'^\gamma}, \tag{I.7}$$

then the quantities $\lambda^i_{,j}$ and $\lambda'^\alpha_{,\beta}$, which are defined by:

$$\begin{cases} \lambda^i_{,j} = \dfrac{\partial \lambda^i}{\partial x_j} + \lambda^h \Gamma^i_{hj} \\ \lambda'^\alpha_{,\beta} = \dfrac{\partial \lambda'^\alpha}{\partial x'^\beta} + \lambda'^\gamma \Gamma^\alpha_{\gamma\beta} \end{cases} \tag{I.8}$$

satisfy the relations:

$$\lambda^i_{,j} = \lambda'^\alpha_{,\beta} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^j}, \tag{I.9}$$

and behave like the components of tensor.

More generally, one may show that if the quantities, $a^{r_1 \cdots r_m}_{s_1 \cdots s_p}$, are the components of a tensor then the quantities:

$$a^{r_1 \cdots r_m}_{s_1 \cdots s_p, i} = \frac{\partial a^{r_1 \cdots r_m}_{s_1 \cdots s_p}}{\partial x^i} + \sum_{\alpha}^{1-m} a^{r_1 \cdots r_{\alpha-1} J k_{\alpha+1} \cdots r_m}_{s_1 \cdots s_p} \Gamma^{r_\alpha}_{Ji} - \sum_{\beta}^{1 \cdots p} a^{r_1 \cdots r_{\alpha-1}}_{s_1 \cdots s_{\beta-1} k s_{\beta+1} \cdots s} \Gamma^k_{s_\beta i} \tag{I.10}$$

are the components of a tensor of order $m+p+1$.

Therefore, the introduction of connection coefficients $\Gamma^i_{k\rho}$ that satisfy (I.7) (which are therefore not tensors) permits us to generalize the notion of covariant derivative, and notably, to gradually compare the components of tensors that are attached to different tangent affine spaces. These spaces, as well as the givens of the Γ^i_{kl}, define the most general affine space.

By starting with the Γ^i_{kl}, one constructs the usual generalized curvature tensor:

$$R^l_{srk} = \frac{\partial \Gamma^l_{kr}}{\partial x^s} - \frac{\partial \Gamma^l_{ks}}{\partial x^r} + \Gamma^q_{kr}\Gamma^l_{qs} - \Gamma^q_{ks}\Gamma^l_{qr}, \tag{I.11}$$

which plays the usual role when one parallel displaces a vector or tensor along a closed contour.

To define such an infinitesimal contour passing through a point $O(x^i)$, one will arbitrarily define two infinitesimal vector fields by two systems of differentials d_x and δ_x, which define two points $A(x^i + dx^i)$ and $B(x^i + \delta x^i)$. At A, the vector field δ defines a vector AM; at B the field d defines a vector BM'. The coordinates of M are:

$$x^i + dx^i + \delta x^i + d\delta x^i,$$

and those of M' are:

$$x^i + \delta x^i + dx^i + \delta d x^i.$$

By virtue of $d\delta x^i = \delta dx^i$, M and M' agree, and we have a closed contour. Parallel displace a vector λ^i along this closed contour. One then finds that the variations $\Delta\lambda^i$ of the λ^i have the expression:

$$\Delta\lambda^i = -\lambda^p R^i_{srp}\, dx^s \delta x^r, \tag{I.12}$$

or, more generally, for a two-index tensor $f_i^{\ k}$:

$$\Delta f_i^{\ k} = (-f_i^{\ p} R^i_{srp} + f_p^{\ k} R^p_{sri})\, dx^s \delta x^r. \tag{I.13}$$

If we let the symbols $(\alpha\beta)$ and $[\alpha\beta]$, denote the symmetric and antisymmetric parts of a quantity with two indices $\alpha\beta$, as usual ([1]):

$$\begin{aligned} f^{(\alpha\beta)} &= \frac{1}{2}(f_{\alpha\beta} + f_{\beta\alpha}) \\ f^{[\alpha\beta]} &= \frac{1}{2}(f_{\alpha\beta} - f_{\beta\alpha}), \end{aligned} \tag{I.14}$$

then, with Cartan, one may call the expressions $\Gamma^i_{[kl]}$ the "components of the torsion of space."

([1]) Ed. Note: I did not have the original symbols used by Vigier, so I substituted alternative symbols.

Unlike the Γ^i_{ke}, the $\Gamma^i_{[ke]}$ are the components of a tensor.

The geometric significance is clear: When one infinitesimally parallel displaces around a closed circuit, a vector experiences not only the usual Riemannian rotation, but also a displacement that corresponds to torsion.

More precisely, if one displaces OA along OB then it becomes BA'. If one displaces OB along OA then it becomes AB'. If O is taken as the origin then the coordinates of A' are:

$$\delta x^i + dx^i + dx^p \Gamma^i_{pn} \delta x^n ,$$

and those of B' are

$$dx^i + \delta x^i + \delta x^r \Gamma^i_{rp} dx^p .$$

$B'A'$ is therefore a second order infinitesimal and has the components:

$$\delta f^i = \Gamma^i_{[pr]} dx^p \delta x^r .$$

§ 3. The Riemannian notion of a geodesic line is immediately generalized to affine spaces. It is characterized by the fact that its tangent remains parallel to itself for any displacement along the line itself.

Any geodesic line will therefore satisfy the second order differential equation:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{pr} \frac{dx^p}{dt}\frac{dx^r}{dt} + \lambda \frac{dx^i}{dt} = 0 , \tag{I.16}$$

in which λ is an arbitrary scalar function that one may eliminate by a change of parameter t.

Therefore, these lines depend only upon the $\Gamma^i_{[pk]}$, and do not change if one modifies the torsion of space without changing the symmetric components of Γ^i_{pk}.

§ 4. In all of the foregoing, we have not introduced the notion of a metric (or that of the length of a vector). As usual, this will be related to the definition of the scalar product (x, y) of two vectors $x(\xi_i)$ and $y(\eta_i)$.

To do this, we introduce a symmetric second order covariant tensor field g_{ik} that we call the fundamental tensor.

The scalar product of x and y will be given by the formula:

$$(x, y) = g_{ik}\, \xi^i\, \eta^k. \tag{I.17}$$

By definition, (x, x) corresponds to the scalar "square of the length" of the vector.

Other than these covariant components g_{ik} one may also introduce the "normalized" minors of the determinant g of the elements g_{ik}, i.e., the contravariant components g^{ik} such that one has:

$$g_{i\rho} g^{\rho k} = \delta^k_i , \tag{I.18}$$

in which the δ^i_k have the usual significance.

This permits us to define the operations of lowering or raising an arbitrary index. For example, one sets:

$$\begin{cases} \xi_i = g_{ik}\xi^k \\ \eta^i = g^{ik}\xi_k \end{cases} \tag{I.19}$$

and one does not consider two tensors to be distinct if they can be deduced from each other by raising or lowering an index.

After a change of variables, the square root of the determinant of g_{ik} then satisfies the equality:

$$\sqrt{-g'} = \sqrt{-g}\,\frac{D(x)}{D(x')}, \tag{I.20}$$

and therefore transforms as a scalar density, which gets multiplied by the functional determinant of the transformation.

The introduction of the g_{ik} then permits us to define a simple solution to (I.7), which permitted Riemann to define the covariant derivative in his metric spaces. One obtains them by setting:

$$\Gamma^i_{kl} = \begin{Bmatrix} i \\ kl \end{Bmatrix} = \frac{1}{2} g^{is}\left(\frac{\partial g_{ks}}{\partial x^l} + \frac{\partial g_{sl}}{\partial x^k} - \frac{\partial g_{lk}}{\partial x^s}\right). \tag{I.21}$$

This solution has the essential property of conserving length under parallel displacement. Moreover, the most general Γ^i_{kr} that satisfy this condition must satisfy the condition:

$$\frac{\partial g_{ik}}{\partial x^r} - g_{il}\Gamma^l_{kr} - g_{lk}\Gamma^l_{ir} = 0, \tag{I.22}$$

which may also be written:

$$g_{ik,r} = 0.$$

It admits the following expression – which leaves the torsion indeterminate – as general solution (which is obtained by permuting i, j, k, and combining the relations so obtained):

$$\Gamma^s_{ik} = \begin{Bmatrix} s \\ ik \end{Bmatrix} + \frac{1}{2} g^{rs}(\Gamma_{[ir],k} + \Gamma_{[ik],r} + \Gamma_{[kr],i}), \tag{I.23}$$

in which one has set:

$$\Gamma_{[ir],k} = g_{lk}\Gamma^l_{[ir]}.$$

Conversely, the existence conditions (which are deduced from (I.22)) for a metric whose length is conserved by parallel displacement in an affine space may be written:

$$g_{ip}R^p_{srk} + g_{pk}R^p_{sri} = 0. \tag{I.24}$$

Weyl has proposed to give a meaning to not only the fundamental tensor, but also the notion of a scale.

This amounts to considering the fundamental tensor to be a field of quadrics such that when the quadric of the field g_{ik} placed at M is parallel transported along an arbitrary path to N is transported into a homothetical quadric of the quadric of the field that is placed at N.

This definition leads us to substitute the relation:

$$g_{ik,r} = - g_{ik}\, \varphi_r \,, \tag{I.25}$$

for (I.22), in which φ_r is a well-defined covariant vector at each point of space.

Then consider a change of scale:

$$g'_{ir} = \lambda g_{ik}\,, \tag{I.26}$$

in which λ is an arbitrary scalar; φ_r must be replaced by φ'_r, such that:

$$g'_{ik,r} = -g'_{ik}\varphi'_r\,, \tag{I.27}$$

from which one deduces:

$$\varphi'_r = \varphi_r - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x^r}\,, \tag{I.28}$$

which leads us to replace (I.23) by the expression:

$$\Gamma^s_{ir} = \begin{Bmatrix} s \\ ir \end{Bmatrix} + \frac{1}{2} g^{ks}(\Gamma_{[ik]r} + \Gamma_{[rk]i} + \Gamma_{[ir]k}) + \frac{1}{2}(\delta^s_i \varphi_r + \delta^s_r \varphi_i - g_{ir}\varphi^s)\,.$$

We conclude this subject with the definition of the notion of the weight of a tensor: one says that a tensor has weight n if its components are multiplied by λ^n under the scale change (I.26).

If we then introduce the symbol $a_{\ldots/\mathrm{r}} = a_{\ldots n;r} + \varphi_n$ then (I.25) is written simply:

$$g_{in/r} = 0.$$

§ 5. We conclude our brief summary of the essential notion of geometry with an affine connection with several considerations on the deformations of spaces V_n.

A deformation is obviously defined by a modification of the Γ^i_{kl} – namely $\delta\Gamma^i_{kl}$ – that entails corresponding modifications of all of the possible derived tensors.

One may then define a certain number of simple deformations that we will use in the sequel, namely:

– the isometric deformation:

which corresponds to a change of affine connection without a change of metric.

Let Λ_{irk} be an arbitrary covariant tensor that is anti-symmetric in the first two indices; taking (I.13) into account, the corresponding isometric deformation, may be written:

$$\delta\Gamma_{ik}^{s} = g^{rs}(\Lambda_{ir,k} + \Lambda_{kr,i} + \Lambda_{ik,r}), \tag{I.29}$$

– the projective deformation:

which conserves the geodesic lines of the given affine connection.

As we have already seen, these lines do not depend on torsion and one may set $\delta\Gamma_{k\rho}^{i} = \Lambda_{[k\rho]}^{i}$ arbitrarily. It therefore remains for us to look for the modifications that relate to the symmetric part of the $\Gamma_{k\rho}^{i}$, and obey the stated condition.

They may satisfy the property:

$$\delta\Gamma_{k\rho}^{i}\xi^{k}\xi^{\rho} = \lambda\xi^{i}, \tag{I.30}$$

in which $x(\xi^{i})$ is an arbitrary vector, and λ is an arbitrary scalar. From (I.30), one deduces:

$$\begin{gathered}(\delta\Gamma_{pr}^{i}\xi^{k} - \delta\Gamma_{pr}^{k}\xi^{i})\xi^{p}\xi^{r} = 0 \\ (\delta_{q}^{k}\delta\Gamma_{pr}^{i} - \delta_{q}^{i}\delta\Gamma_{pr}^{k})\xi^{p}\xi^{r}\xi^{q} = 0,\end{gathered} \tag{I.31}$$

which is identically annulled if one has:

$$\delta\Gamma_{pr}^{i} = \delta_{p}^{i}\varphi_{r} + \delta_{r}^{i}\varphi_{p},$$

in which φ_r is an arbitrary covariant vector.

The general projective deformation is therefore given by:

$$\delta\Gamma_{ik}^{s} = \delta_{i}^{s}\varphi_{k} + \delta_{k}^{s}\varphi_{i} + \Lambda_{[ik]}^{s}. \tag{I.32}$$

– the conformal deformation:

which conserves the metric tensor up to a scalar.

We have already seen this case when we discussed change of scale. If we abstract to an arbitrary isometric deformation then the conformal transformation is thus written:

$$\delta\Gamma_{ik}^{s} = \delta_{i}^{s}\Phi_{k} + \delta_{k}^{s}\Phi_{i} - g_{ik}\Phi^{s},$$

in which Φ_r is an arbitrary tensor.

– the conformal projective deformation:

which is both conformal and projective. It is obviously obtained by comparing the expressions:

$$\delta\Gamma^s_{ik} = \delta^s_i \varphi_k + \delta^s_k \varphi_i - g_{ik}\varphi^s + g^{rs}(\Lambda_{[ir]k} + \Lambda_{[kr]i} + \Lambda_{[ik]r})$$

and

$$\delta\Gamma^s_{ik} = \delta^s_i \Phi_k + \delta^s_k \Phi_i + \Lambda^s_{[ik]},$$

which gives, by addition and circularly permuting the *ikr*:

$$\varphi_i = 2\Phi_i$$

and

$$\Lambda_{[ir]k} = g_{ik}\Phi_r - g_{ki}\Phi_i + \Pi_{irk},$$

in which Π_{irk} is arbitrary, except that: $\Pi_{irk} = -\Pi_{rik} = -\Pi_{ikr}$.

The desired deformation is therefore finally written:

$$\delta\Gamma^s_{ik} = \delta^s_i \varphi_k + g^{rs}\Pi_{ikr}.$$

From this, we deduce that the simplest affine space that is projectively conformal to an ordinary Riemann space will be defined by:

$$\Gamma^i_{kr} = \begin{Bmatrix} i \\ kr \end{Bmatrix} + \varepsilon\delta^i_k A_r,$$

in which ε is an arbitrary constant and A_r is an arbitrary vector.

APPENDIX II

The introduction of spinors into affine theory may be effected in the following manner:

One begins by defining a system of orthogonal axes at each point P of spacetime, which are in the affine tangent spacetime to the space V_4 considered.

This system will be determined by the components $h^i_{(\alpha)}$ of the unitary vectors $C_{(\alpha)}$ that are collinear with these axes (called "Beingrössen" by Einstein). The index () indicates the ordering number of the vector and the one without the () indicates the component. Both may take the values 1 … n and we use the usual summation convention on repeated indices in both cases.

These "Beingrössen" obviously satisfy the relations:

$$\begin{cases} h_i^{(\alpha)} h^i_{(\beta)} = \delta^\alpha_\beta \\ h_i^{(\alpha)} h^j_{(\alpha)} = \delta^i_j \end{cases} \tag{2.4}$$

One then introduces a system of N complex functions that are attached to the preceding system of axes at each point. One represents them by the notation $\psi^{(\alpha)}$, $\alpha = 1, \ldots, n$ and denotes their Hermitian conjugates by $\psi^{+(\alpha)}$.

These functions define a "semi-vector" – or spinor – that is attached to the system considered. If one designates it by ψ then one may show that it is always possible to also introduce n N-dimensional Hermitian matrices α^μ in this system, such that the magnitudes:

$$\psi^+ D \alpha^\mu \, \psi = u^\mu$$

(in which D designates a scalar matrix that we shall define later on) behave like the components of a vector.

As various authors have shown ([1]), these ψ and the preceding α^μ permit us to construct all of geometry from sub-tensors that are related to the theory of groups. We content ourselves by recalling several classical relations that are associated with the possible changes of axis.

Suppose we change coordinates in such a way that the chosen system of axes is rotated (Lorentz transformation in the case of the usual spacetime), which we symbolically write ([2]):

$$x'_\mu = t(x_\mu) = t^\nu_\mu x_\nu \, .$$

It corresponds to a unitary linear transformation $T^+ = T^{-1}$, of the spinor, which may be written:

$$\psi' = T\psi \, ,$$

([1]) Einstein & Mayer, Infeld & Van den Waerden, Schrödinger, etc…

([2]) In this paragraph, we systematically use the notations and results that were presented by BHABHA (*Th. Of Particles*). Rev. Mod. Phys. 1949.

such that one has:

$$\alpha'_\mu = t_\mu^\nu \alpha_\nu = T^{-1}\alpha_\mu T$$

in order for $\psi^+ D\alpha^\mu \psi$ to behave like a tensor.

We then remark that any product of the α matrices is transforms according to the relation:

$$\alpha'_\mu \alpha'_\nu \alpha'_\rho = t_\mu^\theta t_\nu^\Phi t_\rho^\psi \alpha_\theta \alpha_\Phi \alpha_\psi = U^{-1}\alpha_\mu \alpha_\nu \alpha_\rho \cdots U\ .$$

Moreover, transformations of this type obey the symbolic composition law:

$$\begin{matrix} u \leftrightarrow U \\ t \leftrightarrow T \end{matrix} \qquad \rightarrow \qquad ut \leftrightarrow UT.$$

For the sake of simplicity, we shall now confine ourselves to the usual case of four-dimensional spacetime (the line of reasoning is immediately generalized to the case of V_n).

The rotations considered t reduce to the Lorentz transformation. One knows that the orthochronous group (which reverses the time axis) may be based on the infinitesimal transformations,

$$\begin{cases} x'_\mu = x_\mu + u_{\mu\nu} x^\nu \\ u = 1 - \dfrac{1}{2} u^{\rho\sigma} I_{\rho\sigma} \end{cases} \tag{2.11}$$

in which the $u_{\mu\nu}$ are the antisymmetric infinitesimal magnitudes ($u_{\mu\nu} + u_{\nu\mu} = 0$), and the $I_{\rho\sigma}$ are likewise antisymmetric, since they are the matrices that correspond to the elementary Lorentz rotations.

The condition $u\alpha_\mu = \alpha_\mu u$ gives:

$$(\alpha_\mu \times I_{\rho\sigma}) = g_{\mu\rho}\alpha_\sigma - g_{\mu\sigma}\alpha_\rho\ . \tag{2.12}$$

As usual, (A, B) represents the commutator of the matrices A and B, namely, $AB - BA$.

In the same fashion, the condition $t^{-1}ut \rightarrow T^{-1}UT$ gives:

$$(I_{\mu\nu} I_{\rho\sigma}) = -\, g_{\mu\rho} I_{\mu\rho} - g_{\nu\sigma} I_{\mu\rho} + \ g_{\mu\sigma} I_{\nu\rho} + g_{\nu\rho} I_{\mu\sigma}\ . \tag{2.13}$$

These commutation relations ultimately permit us to specify the representations chosen.

Finally, suppose there exists a matrix D such that:

$$\begin{cases} (\alpha^\mu)^+ D = D\alpha^\mu \\ U^+ D U = D \\ U^+ D = D U^{-1} \end{cases} \tag{2.14}$$

in which U is arbitrary, and $(\alpha^{\mu})^{+}$ is the Hermitian conjugate matrix of α^{μ}. One therefore obtains:

$$I_{\rho\sigma}^{+} D + D I_{\rho\sigma} = 0\,, \tag{2.15}$$

and also

$$\eta_0^{+} D = D\eta_0\,,$$

if η_0 is the matrix that reverses the spatial axes.

Under these conditions, any given expression of the form:

$$S_{\mu\nu\rho}\ldots = \psi^{+} D\alpha_{\mu}\,\alpha_{\nu}\,\alpha_{\rho}\ \ldots\ \psi \tag{2.16}$$

behaves like a tensor with indices $\mu\nu\rho$.

In particular, if we are given 4 fundamental matrices α_i that satisfy (2.5), (2.12), (2.13), etc., then one may find a representation of the α_i such that:

1. $\psi^{+} D\alpha_i\psi$ transforms like the components of a vector,
2. $\psi^{+} D\alpha_4\psi$ and $\psi^{+} D\alpha_1\alpha_2\alpha_3\alpha_4\psi$ transform like scalars,
3. One has the Bhabha relation:

$$I_{\mu\nu} = (\alpha_{\mu}\,, \alpha_{\nu}). \tag{2.17}$$

After having summarized the major aspects of the theory of spinors, which are valid in every Galilean space that is tangent to V_4 , we shall define a procedure for passing to spinors that are attached to a tangent space that is infinitely close to the first.

To do this, we shall first reproduce the results obtained by Fock and Weyl by using the notations used by Fock in his fundamental memoir (Zeitschrift für Physik, v. 57, pp. 500).

We verify that the definitions used are equivalent to a supplementary postulate that defines a mode of variation of the entities (the spinors) that were introduced in space.

Fock introduced operators C_l that serve to define a covariant variation of ψ under a displacement of the components δs_l in the tangent space.

By definition, one will therefore have:

$$\begin{cases} \delta\psi = C_l\, ds^l\, \psi \\ \delta\psi^{+} = \psi^{+} C_l^{+}\, ds^l \end{cases} \tag{2.18}$$

namely, if we set: $\xi_i = \psi^{+} D\alpha_i\psi$ then:

$$\begin{cases} \delta\xi_i = \delta\psi^{+} D\alpha_i\psi + \psi^{+} D\alpha_i\delta\psi \\ \qquad = (\psi^{+} C_l^{+} D\alpha_i\psi + \psi^{+} D\alpha_i C_l\psi)\, ds^l \end{cases} \tag{2.19}$$

Fock then determines the C_l by postulating:

1. This variation is identical to the usual covariant variation, which gives:

$$\delta\xi_i = \gamma_{il}^k \xi_k ds^l , \tag{2.19 cont.}$$

from which one infers:

$$\alpha_i C_l + C_l^+ \alpha_i = \gamma_{il}^k \alpha_k \tag{2.20}$$

in which the γ_{il}^k designate the usual Ricci rotation coefficients:

$$\gamma_{ikl} + \gamma_{kil} = 0.$$

2. The C_l are compatible with the existence of the matrix. As a consequence, the $\psi^+ D\alpha^j \psi g_{i\nu}$ satisfy the usual formulas of covariant derivation; this immediately furnishes the condition that was written for the first time by Weyl:

$$\begin{cases} \dfrac{\partial \alpha_i}{\partial x_l} - \Gamma_{il}^n \alpha_n - C_l^+ \alpha_i + \alpha_i C_l = \dfrac{1}{2} g_{in;l} \alpha^n \\ g_{ij;k} = \dfrac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik}^n g_{nj} - \Gamma_{ik}^n g_{jn} \end{cases} \tag{2.21}$$

Equations (2.20) and (2.21) then admit the following expression as a general solution:

$$C_l = \frac{1}{2} \gamma_{rsl} I^{rs} + i\Phi_l ,$$

which was calculated by Fock and Weyl (in which Φ_l is an arbitrary vector).

Therefore, the variation of ψ is written:

$$\psi_{;l} = \frac{\partial \psi}{\partial s_l} - C_l \psi$$

in the tangent space, namely:

$$\psi_{;\sigma} = \frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi$$

in arbitrary coordinates (with $\Gamma_\sigma = h_{k\sigma} C^k$).

The definition of semi-vector and the corresponding expressions for their covariant variation permit us to introduce a covariant procedure ([3]) for decomposing expressions such as:

$$\psi^+ D\alpha_i \psi \qquad \text{and} \qquad (\psi^+ D\alpha_i \psi)_{/j} .$$

Indeed, consider a spinor with N components that is defined as before. To each Lorentz transformation matrix (generalized rotation), there corresponds a matrix U. The set of

([3]) Cf. BHABHA, "*Theory of Elementary Particles,*" Rev. Mod. Phys., 1949.

these matrices constitutes a representation R of the Lorentz group. A classical theorem then tells us that it is possible to subdivide this representation into a sum of irreducible representations R_γ, by a convenient change of axes, namely:

$$R = \sum_\gamma C_\gamma R_\gamma . \tag{2.32}$$

In the case of the Lorentz group, it results from the work of Bhabha that when:

$$\mathrm{I}^{jk} = \alpha^j\alpha^k - \alpha^k\alpha^j,$$

and the relations (2.12) and (2.13) are satisfied, these representations are isomorphic to the irreducible representations of the restricted Lorentz group in five dimensions, which may be written:

$$R_5\{\lambda_1, \lambda_2\} \qquad \text{in which} \qquad \lambda_1 \leq \lambda_2 \leq 0.$$

These λ are both integers (including 0) or half-integers (excluding 0). One then has:

$$R = \sum_{\lambda_1\lambda_2} R_5\{\lambda_1;\lambda_2\}, \tag{2.33}$$

as one knows that for $\lambda = \frac{1}{2}$ there is only one possible representation of degree 4: $R_5\left\{\frac{1}{2}, \frac{1}{2}\right\}$ (the Dirac matrices). For $\lambda = 1$, there are two: $R_5\{1, 1\}$, of degree 10, and $R_5\{1, 0\}$, of degree 5, which correspond to the matrices of vector mesons and scalars, and so on, with increasing degrees.

The decompositions just stated are obtained immediately. Let ψ be a semi-vector with N components, and let α_i be the corresponding matrices; one performs successive changes of axis until formula (2.33) is satisfied.

The matrices α and u may then be exhibited in diagonal form:

$$\alpha^j = \alpha_i\{\lambda_1, \lambda_2\}, \tag{2.34}$$

in which only the cross-hatched parts, which correspond to the preceding irreducible representations, contain terms different from zero.

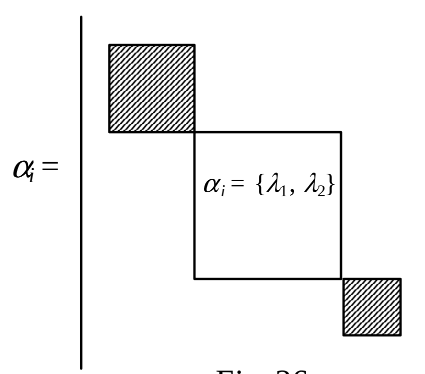

Fig. 26.

As a consequence, the expressions $\psi^+ D\alpha_i\psi$ and $(\psi^+ D\alpha_i\psi)_{/j}$ are separated into a sum of distinct parts that correspond to the irreducible representations $R_5\{\lambda_1, \lambda_2\}$:

$$\psi^+ D\alpha_i\psi = \sum_{\lambda_1\lambda_2} \psi^+ D\alpha_i(\lambda_1 \mid \lambda_2)\psi, \tag{2.35}$$

in which we have taken the decomposition effected into account (in which $\alpha_i\psi = \sum_{\lambda_1,\lambda_2}\alpha_i\{\lambda_1\lambda_2\}\psi$ and the products $\alpha_i\{\lambda_1,\lambda_2\}\alpha_j\{\lambda_1'\lambda_2'\}$ are null because of (2.34)).

APPENDIX III

§ 1. Let $\theta_{\mu\nu}$ be a symmetric energy-momentum tensor, as in Belinfante-Rosenfeld, that is deduced from an invariant function $L(\psi, \psi^*)$ of the functions $\psi^{(\alpha)}$, $\psi^{*(\alpha)}, \frac{\partial}{\partial x^k}\psi^{(\alpha)}, \frac{\partial}{\partial x^k}\psi^{*(\alpha)}$, and A_ρ .

One knows that L satisfies the following conditions:

a) If λ is a real constant, then one has $L(\psi, \psi^*) = L(e^{i\lambda}\psi, e^{-i\lambda}\psi^*)$ (gauge invariance of the first type).

b) $$\frac{\partial L}{\partial \psi^{*(\alpha)}} = \left(\frac{\partial L}{\partial \psi^{(\alpha)}}\right)^*, \qquad \frac{\partial L}{\partial \frac{\partial \psi^{*(\alpha)}}{\partial x^k}} = \left(\frac{\partial L}{\partial \frac{\partial \psi^{(\alpha)}}{\partial x^k}}\right)^* .$$

$\frac{\partial L}{\partial \frac{\partial \psi^{(\alpha)}}{\partial x^k}}$ and $\frac{\partial L}{\partial \psi^{(\alpha)}}$ depend linearly on ψ, ψ^*, and their derivatives.

c) If one replaces the operators $\frac{\partial}{\partial x^k}$ with the operators:

$$\partial_k = \frac{\partial}{\partial x^k} - i\varepsilon A_k$$

then one has:

$$L(\psi, \psi^*, A_k) = L\left(e^{i\lambda}\psi, e^{-i\lambda}\psi^*, A_k - \frac{i}{\varepsilon}\frac{\partial \lambda}{\partial x^k}\right)$$

(gauge invariance of the second type).

We then introduce the expressions:

$$f_{ik} = \frac{\partial}{\partial x^i}A_k - \frac{\partial}{\partial x^k}A_i ,$$

$$R\psi^{(\alpha)} = \partial_k^*\left(\frac{\partial L}{\partial_k \psi^{(\alpha)}}\right) - \frac{\partial L}{\partial \psi^{(\alpha)}}$$

$$T_{ik} = \sum_\alpha \left(\frac{\partial L}{\partial\left(\frac{\partial \psi^{(\alpha)}}{\partial x^k}\right)}\right)\partial_i \psi^{(\alpha)} - L\varepsilon_{ik}$$

$$s_k = i\sum_\alpha \left(\psi^{(\alpha)} \frac{\partial L}{\partial\left(\dfrac{\partial \psi^{(\alpha)}}{\partial x^k}\right)} - \psi^{*(\alpha)} \frac{\partial L}{\partial\left(\dfrac{\partial \psi^{*(\alpha)}}{\partial x^k}\right)} \right),$$

and also

$$\begin{cases} F_{\lambda\mu\nu} = \frac{1}{2}(f_{\nu\lambda\mu} + f_{\mu\lambda\nu} + f_{\nu\lambda\mu}) \\ f_{\lambda\mu\nu} = -f_{\mu\lambda\nu} = \mathrm{Re}\left\{ \sum_\alpha \left(\dfrac{\partial L}{\partial\left(\dfrac{\partial \psi^{(\alpha)}}{\partial x_\nu}\right)} \cdot I_{\mu\lambda} \psi^{(\alpha)} \right) \right\}. \end{cases}$$

The tensor $\theta_{\mu\nu}$ is classically written as:

$$\theta_{\mu\nu} = T_{\mu\nu} + \frac{\partial}{\partial x^\lambda} F_{\lambda\mu\nu}.$$

We therefore have the equality:

$$\frac{\partial}{\partial x^\mu}\theta^{\mu\nu} = \frac{\partial}{\partial x^\mu} T^{\mu\nu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} F_{\lambda\mu\nu} - \frac{\partial}{\partial x^\mu}\theta^{[\mu\nu]} \tag{3.24}$$

which we shall transform by taking the preceding definitions into account.

Indeed, if one remarks that the operators satisfy the relations:

$$\begin{aligned} &(\partial_i, \partial_k) = -i\varepsilon f_{ik} \\ &(\partial_i^*, \partial_k^*) = i\varepsilon f_{ik} \\ &\frac{\partial}{\partial x_k}(f * g) = \partial_k^* f * g + f * \partial_k g \end{aligned}$$

when f^* and g are two functions, which are multiplied by e^{-il} and e^{il}, respectively, under a gauge transformation of the first type. One obtains the following equalities without difficulty:

$$\begin{aligned} &\frac{\partial}{\partial x^\mu} T_{\mu\nu} = -f_{\nu\mu} S^\mu + \sum_\alpha (R\psi^{(\alpha)} \partial_k \psi^{(\alpha)} + conj.) \\ &\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} F_{\lambda\mu\nu} = 0 \qquad \text{(since } F_{\lambda\mu\nu} = -F_{\mu\lambda\nu}\text{)} \\ &\frac{\partial}{\partial x^\mu}\theta_{[\mu\nu]} = -\sum_\alpha \left[\frac{\partial}{\partial x^\mu}(R\psi^{(\alpha)} I_{\mu\nu} \psi^{(\alpha)}) + conj. \right] \end{aligned} \tag{3.25}$$

and finally:

$$\frac{\partial}{\partial x^{\mu}}\theta_{(\mu\nu)} = -f_{\nu\mu}s^{\mu} + \sum_{\alpha}\left\{\left[R\psi^{(\alpha)}\partial_{\nu}\psi^{(\alpha)} - \frac{\partial}{\partial x^{\mu}}(R\psi^{(\alpha)}I_{\mu\nu}\psi^{(\alpha)})\right] + conj.\right\} . \quad (3.26)$$

§ 2. Similarly, one finds that the divergence $\frac{\partial s^k}{\partial x^k}$ has the value:

$$\frac{\partial s^k}{\partial x^k} = -\sum_{\alpha}\varepsilon\{iR\psi^{(\alpha)}\psi + conj.\} \quad (3.27)$$

on account of the gauge invariance of the first type.

These two properties permit us to effect the stated proof.

Indeed, introduce the $g_{\mu\nu}$ that are defined by (3.21). Because of the preceding results, it necessarily satisfies (3.22), which may be written:

$$\frac{\partial}{\partial x^{\mu}}\theta_{(\mu\nu)} = -f_{\nu\mu}s^{\mu} - A_{\nu}\frac{\partial s^k}{\partial x^{\mu}} \quad (3.28)$$

and furthermore, because of (3.26) and (3.27), we have:

$$K(\psi_1^*\psi) = \sum_{\alpha}\left[\left\{R\psi^{(\alpha)}\partial_{\nu}\psi^{(\alpha)} - \frac{\partial}{\partial x_{\mu}}(R\psi^{(\alpha)}I_{\mu\nu}\psi^{(\alpha)} - i\varepsilon A_{\nu}\cdot R\psi^{(\alpha)}\cdot)\psi^{(\alpha)}\right\} + conj.\right]$$
$$= 0 . \quad (3.29)$$

This relation is valid at each point. We set, as usual:

$$<\varphi\,|\,\psi> = \sum_{\alpha}\int\varphi^{*(\alpha)}\psi^{(\alpha)}dx_1\cdots dx_n$$

when φ and ψ are square-summable functions of $x_1, \ldots, x_n$. We shall multiply K by if, where f represents an arbitrary real square-summable function, which is such that product Kf is also square-summable. By integrating, we then obtain:

$$<\left(i\partial_{\nu}f\cdot R - I_{\mu\nu}^{+}\cdot f\frac{i\partial}{\partial x^{\mu}}R - \frac{i\partial}{\partial x^{\mu}}I_{\mu\nu}^{+}\cdot fR - fA_{\nu}R\right)\psi\,|\,\psi> - conj. = 0 . \quad (3.30)$$

From this, it results that the operator:

$$\left(i\partial_{\nu}^{*}f - I_{\mu\nu}^{+}\cdot fi\frac{\partial}{\partial x^{\mu}} - i\frac{\partial}{\partial x^{\mu}}I_{\mu\nu}^{+}f - \varepsilon fA_{\nu}\right)R$$

is Hermitian. Since R is also Hermitian, one deduces from this that the operators:

$$J = \left(i\partial^*_\nu f - I^+_{\mu\nu} \cdot fi\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial x^\mu} I^+_{\mu\nu} f - \varepsilon f A_\nu \right)$$

and R necessarily commute for any f.

Since R is independent of f, this is impossible, in general, except when $R\psi = C\psi$, where C is a real constant since R is Hermitian.

Indeed, consider a complete normed sequence of orthogonal functions ψ_j. One may always find f such that one has:

$$(J, R)\psi_j = 0$$

for any J; since the relations $(J, R)\psi = 0$ eliminate only a denumerable set of functions among the infinitude of possible f. In general, one will infer:

$$(J, R)\psi_j \neq 0$$

for any development, $\psi = \sum_j c_j \psi_j$.

Therefore, if $(J, R) = 0$ then it is necessary that

either: $R = C_1$,
or: $J = C_2$,

in which C_1 and C_2 are two constants.

We examine the first case.

If $R\psi = C\psi$ then it suffices to substitute this into (2.39) in order to have:

$$C\left[\left\{\psi^+\partial_\nu\psi - \frac{\partial}{\partial x_\mu}(\psi^+ I_{\mu\nu}\psi) - i\varepsilon A_\nu \psi^+\psi\right\} + conj.\right] = 0 \ . \tag{3.31}$$

Two cases then present themselves:

1. $C = 0$, in which case, ψ satisfies the relations: $R\psi = 0$.
2. $C \neq 0$, in which case, one has:

$$K'(\psi^+\psi) = \left\{\psi^+\partial_\nu\psi - \frac{\partial}{\partial x_\mu}(\psi^+ I_{\mu\nu}\psi)\right\} + conj. = 0 \,, \tag{3.32}$$

since the terms in $i\varepsilon A_\nu \psi^+\psi$ disappear, as they are pure imaginary. If we then apply the same reasoning to K' as we did to K, then one deduces the equality:

$$< \left(i\partial_\nu f - I_{\mu\nu} \cdot fi\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial x^\mu} I_{\mu\nu} \cdot f \right)\psi \mid \psi > - conj. = 0\,, \qquad (3.33)$$

which may also be written:

$$< \left(i\partial_\nu - iI_{\mu\nu}\frac{\partial}{\partial x^\mu} \right) f\psi \mid \psi > - conj. = 0\,,$$

(because the term, $< I_{\mu\nu}\, fi\frac{\partial}{\partial x^\mu}\psi \mid \psi >$, may be written $< \psi \mid i\frac{\partial}{\partial x^\mu} fI_{\mu\nu}\psi >$, i.e.:

$$< I_{\mu\nu}\, fi\frac{\partial}{\partial x^\mu}\psi \mid \psi >).$$

As in the foregoing, one deduces from this that the operator $\frac{\partial}{\partial x^\nu} + \partial_\mu I^\mu{}_\nu$ commutes with f for any f, and it therefore reduces to constant terms, which we write in the most general form:

$$\left(\frac{\partial}{\partial x^\nu} + \partial_\mu I^\mu{}_\nu - C'\alpha_\nu - C''_\nu \right)\psi = 0\,,$$

in which C' is a constant and C''_ν are the components of a constant vector. If we then make C enter into the constant term of R (which is legitimate, since it is also indeterminate) then we therefore obtain, in any case, equations for ψ that are expressed as:

$$R\psi = 0; \qquad (3.36)$$

i.e., precisely the usual linear wave equations that are written in Bhabha form:

$$(\alpha^\nu\partial_\nu - \mu)\psi = 0\,, \qquad (3.37)$$

which are subject to satisfy the auxiliary relations (3.35), in a certain case.

For them to be compatible with (3.37), they must be written:

$$\left(\frac{\partial}{\partial x^\nu} + \partial_\mu I^\mu{}_\nu + \mu\alpha_\nu \right)\psi = 0\,, \qquad (3.38)$$

i.e., auxiliary conditions that are precisely similar to the ones that were postulated by Dirac in his theory of particles with spin.

The second case, for which $J = C$, gives the same result. The conclusion is therefore general.

As one knows, this likewise applies to the Klein-Gordon, Dirac, and Petiau-Kemmer equations.

APPENDIX IV

The use of nonlinear Lagrangians permits us to introduce stationary singularities with spatial symmetry into electromagnetic theory in which the proper potential A_μ^0 does not go to infinity like $1/r$.

Introduced for the first time by Born and Infeld, this concept may be introduced without difficulty into the unitary theory. We shall recall several classical results:

a) In the first case, B. Hoffmann has shown that the most general metric solution with spherical symmetry for the equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \cdot R = -8\pi\gamma T_{\mu\nu}$$

may be written in the form:

$$ds^2 = A dt^2 - A^{-1} dr^2 - r^2 \, (d\theta^2 + \sin^2\theta \; d\varphi^2)$$

in cylindrical coordinates, where we have written:

$$A(r) = 1 - \frac{2\gamma}{r} [(m_0 + m(r)] \, ,$$

in which m_0 is a constant, with:

$$m(r) = 4\pi \int_0^r T_\mu^\mu r^2 dr = \int 4\pi r^2 T_4^4 dr \, ,$$

because of a theorem of v. Laue. $m(r)$ represents the totality of all mass of electromagnetic origin. At a distance, one has: $m = (m_0 + m(r))_{r=\infty}$ (proper mass of the particle).

b) It remains for us to calculate the form of the electromagnetic field, which is symmetric around the singular world-line L and centered at the gravitational singularity.

One knows that the most general spatially spherically-symmetric solution of Det | $g_{\mu\nu}$ + $f_{\mu\nu}$| may be written:

$$\text{Det} \, | g_{\mu\nu} + f_{\mu\nu} | = \begin{vmatrix} -\alpha & 0 & 0 & iw \\ 0 & -\beta & irv \sin\theta & 0 \\ 0 & -irv \sin\theta & -\beta \sin^2\theta & 0 \\ -iw & 0 & 0 & \sigma \end{vmatrix} \qquad (4.1)$$

in which α, β, s, v and w are functions of r alone, and the $g_{\mu\nu}$ figure only on the principal diagonal.

If we use the preceding values for the $g_{\mu\nu}$ then one sees that:

$$\sqrt{-g} = 1,$$
$$\alpha = \frac{1}{\sigma} = A,$$
$$\beta = r^2,$$

namely:

$$g_{11} = -A^{-1}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \quad g_{22} = A, \quad g_{\mu\nu} = 0, \text{ if } \mu \neq \nu;$$

hence:

$$g^{11} = -A, \quad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2 \sin^2\theta}, \quad g^{44} = -A^{-1},$$

expressions that can be substituted in Det $| g_{\mu\nu} + f_{\mu\nu} |$ while leaving F and G unchanged. From this, one concludes that a singularity of $g_{\mu\nu}$ of type (4.1) does not modify the value of the field $f_{\mu\nu}$ that is obtained from the Galilean $E_{\mu\nu}$. This remark enormously simplifies the calculations and has permitted Infeld ([1]) to completely solve the problem.

If we first assume, with Einstein and Schrödinger, that the space components F represent the electric field then the magnetic field corresponds to the time components([2]). If we then set:

$$P^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial f^{\mu\nu}} \tag{4.2}$$

then we shall have, with the usual symbols:

$$\begin{gathered} (f_{23},\ f_{31},\ f_{12}) \rightarrow E \\ (f_{14},\ f_{24},\ f_{34}) \rightarrow B \\ \\ \begin{cases} (P_{23}, P_{31}, P_{12}) \rightarrow D \\ (P_{14}, P_{24}, P_{34}) \rightarrow H \end{cases} \end{gathered} \tag{4.3}$$

In the preceding particular case of the polar system of axes that are related to the particle, one sees that $B = H = 0$, while E and D depend only on r.

Moreover, one has ([3]): $G = 0$; $L = -\frac{b^2}{2} \log\left(1 + \frac{1}{b^2} F\right)^{\frac{1}{2}}$, which, because of the field equations $\frac{\partial}{\partial x_\nu} \frac{\partial L}{\partial f_{\mu\nu}} = 0$, may be then written:

$$\begin{cases} \operatorname{rot} E = 0 \\ \operatorname{div} D = 0. \end{cases} \tag{4.4}$$

([1]) INFELD and HOFFMANN, *Phys. Review*, 51, pp. 765.
([2]) Ed. Note: This seems to be the opposite of what one would expect.
([3]) B is a constant that depends on the units chosen.

This has a spherically-symmetric solution of the Infeld-Hoffmann form:

$$D_r = \frac{e}{4\mu}\frac{1}{r^2} \qquad \text{in which} \qquad e = \text{integration constant,}$$

$$E_r = \frac{e}{4\mu r^2}\cdot\frac{x^2}{(1+x^4)} \qquad \text{with } x = \frac{r}{r_0}; \quad r_0 = \sqrt{\frac{e}{4\pi b}}. \tag{4.5}$$

The electric potential may be written:

$$\begin{aligned}\varepsilon(r) &= \frac{e}{4\pi r_0}\int_0^\infty \frac{r^2 dr}{1+r^4} \\ &= \frac{e}{4\pi r_0}\left\{\log\frac{1+\sqrt{2}r+r^2}{1-\sqrt{2}r+r^2} - 2\tan^{-1}\frac{\sqrt{2}r}{1-r^2}\right\}\end{aligned} \tag{4.6}$$

and its value for $r = 0$ is:

$$\varepsilon(0) = \frac{e}{4\pi r_0}\sqrt{2}\cdot\frac{\pi}{4}\ , \tag{4.7}$$

by virtue of a general relation that was proved by Born for the quadratic theory; e is a constant that depends on the chosen units. One therefore has:

$$\begin{cases} W = 4\pi\int_0^\infty T_4^4 r^2 dr = \dfrac{2}{3}\varepsilon(0) \\ \quad = \dfrac{e^4}{4\pi r_0}\dfrac{\sqrt{2}\cdot\pi}{6} \\ \quad = \dfrac{e^4}{4\pi r_0}\cdot 0.741. \end{cases} \tag{4.8}$$

In order to determine the Born constant, we make the classical hypothesis that the mass of the electron is essentially of electromagnetic origin.

More exactly, one sets:

$$m_0 \text{ (electron)} \times c^2 = \mu_0 c^2 \text{ (infinitely small mass)} + \frac{e^2}{4\pi r_0}\cdot 0.741,$$

which gives, if e is the charge of the electron and one neglects μ_0:

$$\begin{cases} r_0 = \dfrac{0.741}{1.2361} 3.47\cdot 10^{-13}\ cm. \\ b = \dfrac{0.741}{1.2361} 3.96\cdot 10^{15}\ ues \end{cases} \tag{4.9}$$

which is an acceptable value for the classical radius of the electron ($r \sim 2\times10^{-13}$ cm.).

The upper limit on fields is so great that it plays no practical experimental role (except for making the electromagnetic divergence disappear in the classical theory).