“Sur l’équilibre des corps élastiques multiplement connexes,” Ann. Éc. Norm. (3), **24** (1907), 401-517.

ON THE EQUILIBRIUM OF MULTIPLY-CONNECTED ELASTIC BODIES

By VITO VOLTERRA

Translated by: D. H. Delphenich

With 28 figures

(Photographs not reproduced)

TABLE OF CONTENTS

		Page
INTRODUCTION…………………………………………………………….		i
CHAPTER I.	– General theorems on equilibrium……………………………	1
	Note on chapter I……………………………………………	11
CHAPTER II.	– The distortions……………………………………………….	12
CHAPTER III.	– The efforts……………………………………………………	23
CHAPTER IV.	– Distortions and efforts in a symmetric, cyclic body………….	28
CHAPTER V.	– Hollow cylinder of revolution - Distortion of order 6………..	34
CHAPTER VI.	– Hollow cylinder of revolution - Distortion of order 2…………	47
	Note on chapters V and VI……………………………………	58
CHAPTER VII.	– Hollow cylinder of revolution - Distortions of order 1, 3, 4, 5.	61
	Note on chapters V, VI, VII…………………………………..	67
CHAPTER VIII.	– Cyclic systems of pliable, elastic elements……………………	70
CHAPTER IX.	– Planar cyclic systems of pliable, elastic elements……………..	82
BIBLIOGRAPHY……………………………………………………………….		90

INTRODUCTION

I have dedicated this memoir to a systematic study of the equilibrium of multiply-connected elastic bodies.

In the first chapter, I show that there are some cases of equilibrium for multiply-connected bodies that do not present themselves for simply-connected bodies. The point of departure for this research is the group of formulas (I), (I′), (I″) of the first chapter. When one deforms an elastic body, one may calculate the displacements with these formulas if one knows the characteristic elements of the deformation. Formulas (I), (I′), (I″) characterize the polydromy of the displacements and show that under regular deformation a multiply-connected elastic body may preserve the equilibrium of the deformation without the action of external forces. One obtains these states of equilibrium by operations that I have called *distortions.*

In the second chapter, I have studied the elements that characterize the distortions.

The composition of tensions that act upon the elements of an elastic body on which one has performed one or more distortions gives rise to the *efforts* that I have studied in chapter III. One may express the energy of deformation of the elastic body by the characteristics of the distortions and by those of the efforts, or by bilinear forms of two different types of characteristics. I have also given two fundamental propositions in this chapter: *the reciprocity theorem* for the efforts and the *theorem of equivalent cuts.*

Chapter IV is dedicated to the study of multiply-connected elastic bodies that are symmetric with respect to an axis. The symmetry simplifies the expression for energy, and from that simplified expression one may deduce several very singular theorems on the distribution of efforts.

In chapter V, I have begun some particular applications in order to compare the results of calculation with those of experiment, and then continued them in chapters VI and VII.

I have envisioned a hollow cylinder that is a doubly-connected body and I have calculated the forms that it must take when subjected to the six elementary distortions. One may sketch these forms and compare them with the ones that a large, hollow rubber cylinder takes in practice. The sketches that I just spoke of and the photographs of the cylinder are reproduced in these chapters.

Finally, in chapters VIII and IX, I have studied the following problem:

Determine the efforts from knowing the distortions of a system composed of several deformable parts that are rigidly linked together.

One arrives at a theory of the same type as that of Kirchhoff on the distribution of electrical currents in wires.

The first seven chapters are a compilation of some articles that I published, with several repetitions, in the *Comptes rendus de l'Académie dei Lincei*. I have appended the last two chapters, which are unedited.

I have also appended three notes: The first one refers to a proof of formulas (I), (I′), (I″) that was given by Cesàro after the publication of my results. In the second one, I have described the elegant experiments carried out by Rolla in the physics laboratory at the University of Genoa, which is directed by Garbasso. By very ingenious optical experiments, carried out on a hollow cylinder of gelatin, one may distinguish the compressed parts from the dilated ones when one subjects the cylinder to distortions. The third note refers to a method that Almansi just published for determining the deformations of multiply-connected cylinders.

I would like to thank professors Alessandrini and Tranquili for the French translation of this memoir, professors Sella, Pittarelli, and Zambiase for the experiments, the sketches, and the photographs, and the engineer Jona for the rubber models.

___ . ___

CHAPTER I

GENERAL THEOREMS ON EQUILIBRIUM

I.

1. Weingarten has published an interesting note ([1]): *Sur la théorie de l'élasticité*. He remarked that in the case of an elastic body that is subjected to no external action whatever – i.e., one subject to no external forces that act upon its internal points – it may nonetheless be found to not be in the natural state, but in a state of tension that varies in a continuous and regular manner from one point to another.

It is easy to find some practical cases of a body that exist under these conditions: for example, a ring from which one has suppressed a very thin transverse wedge and then welded the two extremities together again.

2. In the note of Weingarten, there is a question that remains unanswered: Outside of the ring and other bodies that occupy multiply-connected spaces, might there exist simply-connected bodies that are found in these conditions?

On first glance, the question is not easily solved; however, intuitively, one will be led to give an affirmative response. Indeed, one will be led to believe that, just as in the case of simply-connected bodies, upon producing a gap and then forcibly introducing a cuneiform element, or similarly, upon welding the two surfaces of the gap, one may obtain equilibrium states without external forces, in which the tensions and deformation vary regularly and without discontinuity from one point to another as in multiply-connected spaces. Weingarten has given conditions that must be verified in this case for them to nevertheless exist.

3. In this chapter, we prove, with the aid of a simple analytical observation, the impossibility of this case when one assumes that the characteristic elements of the deformation ([2]) and their first and second derivatives are continuous.

This establishes a close relationship between the question of elasticity and an analogous question of hydrodynamics.

The theorem of hydrodynamics to which we refer is the following one:

A finite, incompressible fluid that is found between two fixed, rigid walls, and in which there exist no vortices, must remain at rest if the space that it occupies is simply-

---

([1]) *Sur les surfaces de discontinuité dans la théorie de l'élasticité des corps solides* (*Rend. R. Acc. Lincei*, 5th series, vol. X, 1st sem., 1901).

([2]) By the phrase *characteristic elements of a deformation*, we mean the *six elementary deformations*; i.e., the *three dilatations* and the *three shears* (*see* CLEBSCH, *Théorie de l'élasticité des corps solides*, translated by Saint-Venant and Flamant, pp. 46, et seq.). The characteristics that correspond to a deformation are also called the *strain* according to the nomenclature of the English.

connected (acyclic); on the contrary, there may be motion if the space occupied is multiply-connected (cyclic) ([1]).

Now, here are the analogous properties for elasticity:

We say that the deformation of an elastic body is regular if the six characteristics of the deformation are finite, continuous, and monodromic functions that also have finite, continuous, and monodromic derivatives of the first and second order.

We may then state the following theorem:

If an elastic body occupies a finite, simply-connected (acyclic) space, and is subjected to regular deformations, then it will be found in the natural state when it is in equilibrium and not subjected to any external forces.

On the contrary:

An elastic body in equilibrium that occupies a finite, multiply-connected (cyclic) space might not be in the natural state; i.e., it might be found in a state of tension, even when it is not subjected to any external forces and its deformation is regular.

This proposition establishes an essential difference between the properties of elastic bodies that occupy simply-connected (acyclic) spaces and those of the bodies that occupy multiply-connected (cyclic) spaces.

If we refer to the practical cases that we already recalled then what we just said implies that in the simply-connected case the introduction of a cuneiform wedge or the suppression of a very thin cut, followed by welding the surfaces of the vent, will always generate an irregular deformation, or lacuna, in the elastic system, while the opposite property might be verified when the connectivity is multiple.

In general, we may affirm that if there exists a body that is not subject to any external actions, and which is in a state of tension then it must either occupy a multiply-connected space or have some region of irregular deformation.

In this chapter, the second article will be dedicated to the proof of the stated proposition and the one that follows with some analytical examples that relate to the multiply-connected elastic bodies that are found in a state of tension without being subjected to external forces.

II.

1. Let γ_{11}, γ_{22}, γ_{33}, γ_{23}, γ_{31}, γ_{12} represent six functions of the variables x, y, z that are monodromic, finite, and continuous, and also have derivatives of the first and second order that are monodromic, finite, and continuous in a simply-connected, three-dimensional domain S. Draw a regular line s in the interior of the domain S, represent its coordinates by x, y, z, and let x_0, y_0, z_0; x_1, y_1, z_1 denote those of the extremities A_0 and A_1.

([1]) The connectivity of three-dimensional spaces is of two types: *superficially connected* – or *periphaxic* – and *linearly connected* – or *cyclose*. The connectivity that is of interest to us is the cyclose kind (*see* J. Clerk MAXWELL, *Traité d'électricté et de magnétisme*, translated by G. Seligmann-Lui, v. I, pp. 18 et seq.)

Let the positive direction of s point from A_0 to A_1. Let the values of the quantities γ_{rs} at A_0 and A_1 be represented by $\gamma_{rs}^{(0)}$ and $\gamma_{rs}^{(1)}$, respectively. Suppose that $\gamma_{rs} = \gamma_{sr}$. Set:

$$\text{(I)}\qquad \begin{aligned} u = u_0 &+ \frac{1}{2}(\gamma_{21}^{(0)} + r_0)(y_1 - y_0) + \frac{1}{2}(\gamma_{31}^{(0)} - q_0)(z_1 - z_0) \\ &+ \int_s \left\{ \left[\gamma_{11} + (y_1 - y)\frac{\partial \gamma_{11}}{\partial y} + (z_1 - z)\frac{\partial \gamma_{11}}{\partial z} \right] \frac{dx}{ds} \right. \\ &\quad + \left[(y_1 - y)\left(\frac{\partial \gamma_{12}}{\partial y} - \frac{\partial \gamma_{22}}{\partial z} \right) + \left(\frac{z_1 - z}{2} \right)\left(\frac{\partial \gamma_{21}}{\partial z} + \frac{\partial \gamma_{31}}{\partial y} - \frac{\partial \gamma_{23}}{\partial x} \right) \right] \frac{dy}{ds} \\ &\quad \left. + \left[\left(\frac{y_1 - y}{2} \right)\left(\frac{\partial \gamma_{21}}{\partial z} + \frac{\partial \gamma_{31}}{\partial y} - \frac{\partial \gamma_{23}}{\partial x} \right) + (z_1 - z)\left(\frac{\partial \gamma_{13}}{\partial z} - \frac{\partial \gamma_{33}}{\partial x} \right) \right] \frac{dz}{ds} \right\} ds . \end{aligned}$$

$$\text{(I}'\text{)}\qquad \begin{aligned} v = v_0 &+ \frac{1}{2}(\gamma_{32}^{(0)} + p_0)(z_1 - z_0) + \frac{1}{2}(\gamma_{12}^{(0)} - r_0)(x_1 - x_0) \\ &+ \int_s \left\{ \left[\left(\frac{z_1 - z}{2} \right)\left(\frac{\partial \gamma_{32}}{\partial x} + \frac{\partial \gamma_{12}}{\partial z} - \frac{\partial \gamma_{31}}{\partial y} \right) + (x_1 - x)\left(\frac{\partial \gamma_{21}}{\partial x} - \frac{\partial \gamma_{11}}{\partial y} \right) \right] \frac{dx}{ds} \right. \\ &\quad + \left[\gamma_{22} + (z_1 - z)\frac{\partial \gamma_{22}}{\partial z} + (x_1 - x)\frac{\partial \gamma_{22}}{\partial x} \right] \frac{dy}{ds} \\ &\quad \left. + \left[(z_1 - z)\left(\frac{\partial \gamma_{23}}{\partial z} - \frac{\partial \gamma_{23}}{\partial y} \right) + \left(\frac{x_1 - x}{2} \right)\left(\frac{\partial \gamma_{32}}{\partial x} + \frac{\partial \gamma_{12}}{\partial z} - \frac{\partial \gamma_{31}}{\partial y} \right) \right] \frac{dz}{ds} \right\} ds , \end{aligned}$$

$$\text{(I}''\text{)}\qquad \begin{aligned} w = w_0 &+ \frac{1}{2}(\gamma_{13}^{(0)} + q_0)(x_1 - x_0) + \frac{1}{2}(\gamma_{23}^{(0)} - p_0)(y_1 - y_0) \\ &+ \int_s \left\{ \left[(x_1 - x)\left(\frac{\partial \gamma_{31}}{\partial x} - \frac{\partial \gamma_{11}}{\partial z} \right) + \left(\frac{y_1 - y}{2} \right)\left(\frac{\partial \gamma_{13}}{\partial y} + \frac{\partial \gamma_{23}}{\partial x} - \frac{\partial \gamma_{12}}{\partial z} \right) \right] \frac{dx}{ds} , \right. \\ &\quad + \left[\left(\frac{x_1 - x}{2} \right)\left(\frac{\partial \gamma_{13}}{\partial y} + \frac{\partial \gamma_{23}}{\partial x} - \frac{\partial \gamma_{12}}{\partial z} \right) + (y_1 - y)\left(\frac{\partial \gamma_{32}}{\partial y} - \frac{\partial \gamma_{22}}{\partial z} \right) \right] \frac{dy}{ds} \\ &\quad \left. + \left[\gamma_{33} + (x_1 - x)\frac{\partial \gamma_{23}}{\partial x} + (y_1 - y)\frac{\partial \gamma_{33}}{\partial y} \right] \frac{dz}{ds} \right\} ds , \end{aligned}$$

where u_0, v_0, w_0, p_0, q_0, r_0 are constant quantities.

We seek the conditions for u, v, w to not depend upon the line of integration s, but only on the two extremities A_0 and A_1; i.e., upon supposing that A_0 is fixed, we seek the conditions for u, v, w to be functions of x_1, y_1, z_1.

2. To that effect, it suffices to assume that the line s closes by making the points A_0 and A_1 coincide, and determine the conditions for the integrals that are taken along the line s to be zero.

When the line s is closed, Stokes's theorem transforms these integrals into:

$$\int_\sigma \left\{ \left(\frac{y_1 - y}{2} B - \frac{z_1 - z}{2} C \right) \cos nx + \left[(z_1 - z)F + \frac{y_1 - y}{2} A \right] \cos ny \right.$$
$$\left. + \left[(y_1 - y)G + \frac{z_1 - z}{2} A \right] \cos nz \right\} d\sigma$$

$$\int_\sigma \left\{ \left[(z_1 - z)E + \frac{x_1 - x}{2} B \right] \cos nx + \left(\frac{z_1 - z}{2} C - \frac{x_1 - x}{2} A \right) \cos ny \right.$$
$$\left. + \left[(x_1 - x)G + \frac{z_1 - z}{2} B \right] \cos nz \right\} d\sigma$$

$$\int_\sigma \left\{ \left[(y_1 - y)E + \frac{x_1 - x}{2} C \right] \cos nx + \left((x_1 - x)F - \frac{y_1 - y}{2} C \right) \cos ny \right.$$
$$\left. + \left[\frac{x_1 - x}{2} A - \frac{y_1 - y}{2} B \right] \cos nz \right\} d\sigma ,$$

where σ is a surface that has s for its contour and is found in the interior of the domain S; n denotes the normal to σ traced in a convenient direction, and:

$$A = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{31}}{\partial y} + \frac{\partial \gamma_{12}}{\partial z} - \frac{\partial \gamma_{23}}{\partial x} \right) - 2\frac{\partial^2 \gamma_{11}}{\partial z\, \partial y}, \qquad E = \frac{\partial^2 \gamma_{11}}{\partial z\, \partial y} - \frac{\partial^2 \gamma_{22}}{\partial z^2} - \frac{\partial^2 \gamma_{33}}{\partial y^2},$$
$$B = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{12}}{\partial z} + \frac{\partial \gamma_{23}}{\partial x} - \frac{\partial \gamma_{31}}{\partial y} \right) - 2\frac{\partial^2 \gamma_{22}}{\partial x\, \partial y}, \qquad F = \frac{\partial^2 \gamma_{13}}{\partial z\, \partial x} - \frac{\partial^2 \gamma_{33}}{\partial x^2} - \frac{\partial^2 \gamma_{11}}{\partial z^2},$$
$$C = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{23}}{\partial x} + \frac{\partial \gamma_{31}}{\partial y} - \frac{\partial \gamma_{12}}{\partial z} \right) - 2\frac{\partial^2 \gamma_{33}}{\partial y\, \partial x}, \qquad G = \frac{\partial^2 \gamma_{21}}{\partial x\, \partial y} - \frac{\partial^2 \gamma_{11}}{\partial y^2} - \frac{\partial^2 \gamma_{22}}{\partial x^2}.$$

It then follows that the necessary and sufficient conditions for u, v, w to be independent of the line s of integration are:

(II) $$A = B = C = E = F = G = 0.$$

3. Suppose that the preceding conditions are verified; u, v, w will be functions of x_1, y_1, z_1 .

In order to calculate their derivatives with respect to x_1, y_1, z_1, one must remark that these quantities appear explicitly under the integration sign and that they are, at the same time, the coordinates of one extremity of the line of integration. Having made this observation, the ordinary rules of calculation easily lead to:

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial x_1} = \gamma_{11}^{(1)}, & \dfrac{\partial v}{\partial y_1} = \gamma_{22}^{(1)}, & \dfrac{\partial w}{\partial z_1} = \gamma_{33}^{(1)}, \\ \dfrac{\partial v}{\partial z_1} + \dfrac{\partial w}{\partial y_1} = \gamma_{23}^{(1)}, & \dfrac{\partial w}{\partial x_1} + \dfrac{\partial u}{\partial z_1} = \gamma_{31}^{(1)}, & \dfrac{\partial u}{\partial y_1} + \dfrac{\partial v}{\partial x_1} = \gamma_{12}^{(1)}. \end{cases}$$

One infers from this that when the quantities γ_{rs} satisfy the conditions (II), one may find three functions u, v, w that verify equations (1); i.e., that one may consider the quantities γ_{rs} to be the six characteristics of the deformation of an elastic medium. The converse proposition is easily verified.

4. Formulas (I), (I′), (I″) are useful and interesting since each of them gives the means to obtain one of the components of the displacement by a simple quadrature when one is given the characteristics of the deformation.

Kirchhoff ([1]) and Love ([2]) have calculated each of the derivatives of u, v, w by analogous quadratures. One may, with the aid of simple integrations, pass from the formulas of Kirchhoff and Love to (I), (I′), (I″). Six arbitrary constants u_0, v_0, w_0, p_0, q_0, r_0 appear in them; i.e., the values of the components of the displacement at the point A_0 and those of the vector components, which were called the *rotation* by Maxwell. The equalities (II) are only the well-known formulas of Saint-Venant.

5. Equations (II) express the conditions for the values of u, v, w to be given by formulas (I), (I′), (I″) are independent of the line of integration when the space S is simply connected; however, if the space S is multiply connected then these values may depend upon the line of integration while still satisfying the conditions (II). Indeed, recall that one has proved the independence of the line on the values of u, v, w in paragraph 2 where the space was assumed to be simply connected, upon observing that each closed line s of the space may be regarded as the contour of a surface σ belonging to the same space. However, if the space is multiply connected then this fact is no longer verified for each line s, and one then sees that the values of u, v, w may depend on the line of integration. We thus have the following theorem:

An elastic body that occupies a simply-connected space and whose deformation is regular may always be brought to its natural state with the aid of displacements that are finite, continuous, and monodromic at each of its points.

On the contrary, we may say that:

If an elastic body occupies a multiply-connected space and its deformation is regular then the displacements of the points are not necessarily monodromic.

([1]) *Mechanik*, vol. XXVII, § 4.
([2]) *Math. Theory of Elasticity*, vol. I, § 66.

Reduce the cyclic space to a simply-connected one by means of a system of cuts. The displacements that correspond to the given displacement may then be regarded as finite, continuous, and monodromic functions in the sectioned space, but their values may not be continuously attached to the aforementioned cuts. When this comes about, if one desires to bring the body to its natural state then it is necessary to either suppress the connectivity of the matter along the cuts and produce fissures there, or subtract from the matter, or make the two surfaces of the gap slide over each other (*see* the examples of the following article).

6. Now, recall the proof that one makes ([1]) to prove that an elastic body that is not subject to any external forces is found in the natural state. It presupposes implicitly that the points of the elastic body are subject to displacements that are finite, continuous, and monodromic, and that the deformation of the system is regular. This is why, if one knows that the deformation is regular, the body occupies a simply-connected space, and that it is not subject to external forces then one may conclude that the system will not be subject to any internal tension. However, if the body occupies a multiply-connected space then the regular deformation may coexist with a polydromy of displacements, and then the body may be in a state of tension, even if it is not subject to external forces.

This is why one may infer the theorem that we stated in Article 1.

7. One may easily deduce an interesting corollary of this theorem:

When one knows the external forces that act upon an elastic body, the deformation is individualized if the space occupied by the body is simply connected; however, it is not determined if the same space is multiply connected, at least if one only knows, a priori, *that the system may be brought to its natural state by displacements that are finite, continuous, and monodromic.*

The proof of this corollary follows immediately from that of the theorem that one just recalled.

Therefore, the mathematical theory of elasticity must be modified in the case of bodies that occupy multiply-connected spaces, because this theory rests entirely upon the general fact that the external forces determine the deformation of the body. That is the main interest of the proposition that we just stated. The ordinary theory remains the same in the case of bodies that occupy simply-connected spaces, or even when one knows, *a priori*, that the system may be brought to the natural state by means of monodromic displacements.

8. It is easy to infer from formulas (I), (I′), (I″) the nature of the discontinuities that the displacements u, v, w present across the cuts that render the space occupied by the body simply connected. Let u_α, v_α, w_α denote the values on one side of these sections, u_β, v_β, w_β, the values on the other side, and set:

$$u_\beta - u_\alpha = U, \quad v_\beta - v_\alpha = V, \quad w_\beta - w_\alpha = W.$$

([1]) *See*, for example, CLEBSCH, *op. cit.*, pp. 132, et seq.

Upon denoting the six constants across each section by *l, m, n, p, q, r,* we have:

(III) $U = l + ry - qz, \qquad V = m + pz - rx, \qquad W = n + qx - py,$

as Weingarten has proved in another way.

Thus, in the case of a multiply-connected body with all of the cuts that serve to render it simply connected, one may make six constants correspond to it that individualize the polydromy of the displacement that is calculated by means of formulas (I), (I′), (I″).

By analogy with what one does in the theory of functions, these constants may be called the *six constants of each cut.*

The fundamental proposition of the theory of elasticity must then be stated in the following terms:

If an elastic body occupies a multiply-connected space, and if its deformation is regular, then it will be determined by external forces and six constants that relate to each of the cuts that serve to render the space simply connected.

III.

Example I.

1. Set:

$$\gamma_{11} = \frac{\beta x^2 - \alpha y^2}{x^2 + y^2} + \frac{\beta}{2}\log(x^2 + y^2), \qquad \gamma_{23} = 2\gamma\frac{yz}{x^2 + y^2},$$

$$\gamma_{22} = \frac{\beta y^2 - \alpha x^2}{x^2 + y^2} + \frac{\beta}{2}\log(x^2 + y^2), \qquad \gamma_{31} = 2\gamma\frac{xz}{x^2 + y^2},$$

$$\gamma_{33} = \gamma\log(x^2 + y^2), \qquad \gamma_{12} = 2(\alpha + \beta)\frac{xz}{x^2 + y^2},$$

where α, β, γ are constant quantities.

It is easy to verify that equations (II) of de Saint Venant are satisfied. These functions have no other singularities than the one at $x = y = 0$; i.e., along the z-coordinate axis.

Upon thus excluding this singular locus for a cylinder having its axis along the z-axis, in all of the remaining space these quantities may be interpreted as the characteristic of a regular deformation T.

One easily calculates the components u, v, w of the corresponding displacement. It will thus be given (at least up to an arbitrary rigid displacement) by formulas:

$$(2)\qquad \begin{cases} u = \quad \alpha y \arctan\dfrac{y}{x} + \dfrac{\beta}{2} x \log(x^2+y^2), \\ v = -\alpha y \arctan\dfrac{y}{x} + \dfrac{\beta}{2} y \log(x^2+y^2), \\ w = \qquad\qquad\qquad \gamma z \log(x^2+y^2). \end{cases}$$

The functions u and v are polydromic, and the axis of branching is the z-axis.

2. Having said this, imagine a homogeneous, isotropic body C that occupies a space S that is bounded by two cylinders of revolution σ_1 and σ_2 , that have the z-axis for their axes, and whose radii are R_1 and R_2 , and by two planes that are normal to the z-axis. If one supposes that the external forces are zero then the indefinite equations of equilibrium:

$$(3)\qquad \begin{cases} K\Delta^2 u + (L+K)\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0, \\ K\Delta^2 v + (L+K)\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0, \\ K\Delta^2 w + (L+K)\dfrac{\partial}{\partial z}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0 \end{cases}$$

will be satisfied by the functions (2) when the following equation is verified:

$$K\,\alpha + (L + 2K)\,\beta + (L + K)\,\gamma = 0,$$

which, in turn, will be satisfied upon taking:

$$\gamma = 0, \qquad\qquad \beta = -\alpha\frac{K}{L+2K}.$$

The calculation of external forces acting on the surface presents no difficulties. On the surface σ_1, we find a uniform tension normal to σ_1 that is directed towards the interior of the mass, and is given by:

$$T_{\sigma_1} = \alpha(L+K)\left(1 + \frac{2K}{L+2K}\log R_1\right),$$

and similarly on σ_2 there is a uniform normal tension that is directed towards the interior of the mass and is given by:

$$T_{\sigma_2} = \frac{\alpha L}{L+2K}(L + 3K + 2K \log r),$$

where r denotes the distance from the z-axis.

3. Now, imagine a fictitious body of the same nature as the body C and which occupies the same space, but which is found in the natural state. Without altering the connectivity, subject it to forces T_{σ_1}, T_{σ_2}, and T_ϖ that act upon the bases and the lateral surfaces, respectively. Let u', v', w' indicate the corresponding components of the displacement. They will be finite, continuous, and monodromic functions, and if we take:

$$u'' = u - u' = \alpha\left[\ \ y\arctan\frac{y}{x} - \frac{1}{2}\frac{K}{L+2K}x\log(x^2+y^2)\right] - u',$$

$$v'' = v - v' = \alpha\left[-x\arctan\frac{y}{x} - \frac{1}{2}\frac{K}{L+2K}y\log(x^2+y^2)\right] - v',$$

$$w'' = w - w' = - w'$$

then we obtain a system of displacements of the body C that are not zero and differ by a rigid displacement. To the displacements, u'', v'', w'' there corresponds a deformation that is non-zero and regular, and consequently, an internal tension; however, the external forces are zero. If we indicate by γ'_{is} the characteristics of the deformation Γ' that corresponds to the displacements u', v', w', then those of the deformation Γ'' that corresponds to u'', v'', w'' are:

$$\gamma''_{is} = \gamma_{is} - \gamma'_{is}.$$

4. The functions u'', v'' are polydromic, as well as u, v, and they have the z-axis for their polydromy axis. Let u''_α, v''_α, w''_α denote the values of u'', v'', w'' at a point that is situated in the xz-plane on the positive side of the x-axis. Starting from this point, execute a circuit around the z-axis and take the successive values of u'', v'', w'' that follow by continuity. Letting u''_β, v''_β, w''_β indicate the values when one returns to the point of departure, we will have:

$$u''_\beta - u''_\alpha = 0, \qquad v''_\beta - v''_\alpha = -2\pi\alpha x, \qquad w''_\beta - w''_\alpha = 0.$$

5. It follows that if α is positive then the state of regular deformation Γ'' of the body may be obtained by taking the body that occupies the previously-considered hollow cylinder in the natural state, and then making a cut along the xz-plane on the positive side of the x-axis, and finally placing a very small wedge between the two walls of the cut, whose thickness varies proportional to the distance from the axis.

On the contrary, if α is negative then to in order obtain the corresponding state of tension it is necessary to suppress a very small cut whose thickness varies proportional to the distance from the axis, and then weld the two surfaces of the gap together.

Example II.

5. Set:

$$\gamma_{11} = 0, \qquad \gamma_{22} = 0, \qquad \gamma_{33} = 0,$$
$$\gamma_{23} = \frac{\alpha x}{x^2 + y^2}, \qquad \gamma_{31} = -\frac{\alpha y}{x^2 + y^2} \qquad \gamma_{12} = 0.$$

The equations (II) of de Saint Venant are satisfied and the preceding functions have no other singularity than one along the z-axis.

The corresponding displacements will be (at least, up to a rigid displacement):

$$u = 0, \quad v = 0, \quad w = \alpha \arctan \frac{y}{x}; \tag{4}$$

w is therefore polydromic and has the z-axis for its branching axis.

Imagine a homogeneous, isotropic body that occupies the same space defined by the hollow cylinder S as in the preceding example. The displacements (4) satisfy equations (3), and the external forces that act upon the lateral surfaces σ_1 and σ_2 become zero; whereas the ones that act upon the bases have the following components on one of them:

$$X_\omega = -\frac{\alpha K y}{x^2 + y^2}, \qquad Y_\omega = \frac{\alpha K x}{x^2 + y^2}, \qquad Z_\omega = 0,$$

and:

$$X'_\omega = \frac{\alpha K y}{x^2 + y^2}, \qquad Y'_\omega = -\frac{\alpha K x}{x^2 + y^2}, \qquad Z'_\omega = 0,$$

on the other one, respectively.

Now, take a fictitious body of the same substance that occupies the hollow cylinder S in the natural state and, without changing the connectivity, subject it to the preceding forces of torsion that act upon the two bases.

Let u', v', w' denote the displacements from which it is derived. They are finite, continuous, and monodromic functions, and if one sets:

$$u'' = -u', \qquad v'' = -v', \qquad w'' = w - w'$$

then these displacements correspond to a state of internal tension of the body, while all of the external forces are zero. The deformation will obviously be regular.

7. It is easy to see how one may produce this state of tension. One takes the body that occupies the space defined by the hollow cylinder S in the natural state, cuts it along the xz-plane on the positive side of the z-axis, and then one lightly slides the two surfaces of the cut, one over the other, parallel to the z-axis in such a manner that the cylinder takes on a slightly helicoidal form. Having done this, one welds the two parts to each other along points that are found on its face.

The two bases thus acquire a serration along the xz-plane on the positive x side; however, it is infinitely small, and, without perturbing the conditions of the system, we may imagine removing it and smoothing out the same bases.

NOTE ON CHAPTER I

Cesàro has given a very simple proof ([1]) of formulas (I), (I′), (I″). It is:

Let u, v, w be the components of the displacement of the point (x, y, z) and let:

$$(1) \qquad a = \frac{\partial u}{\partial x}, \qquad b = \frac{\partial v}{\partial y}, \qquad c = \frac{\partial w}{\partial z},$$

$$(2) \qquad f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \qquad g = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad h = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

We suppose that a, b, c, f, g, h and their first and second derivatives are finite, continuous, and monodromic functions.

These conditions might not be verified for u, v, w and by the components of rotation:

$$(3) \qquad p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

In order to calculate u at an arbitrary point M, start with the formula:

$$u = u_0 + \int\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right),$$

u_0 being the value of u at an arbitrary point M_0, and the integral being taken along a line that goes from the point M_0 to the point M_1.

In order to make the characteristics of the deformation appear in the second integral, we may write:

$$\begin{aligned}\int (q\,dz - r\,dy) &= \int [r\,d(y_1 - y) - q\,d(z_1 - z)] \\ &= q_0(z_1 - z_0) - r_0(y_1 - y_0) + \int [(z_1 - z)\,dq - (y_1 - y)\,dr].\end{aligned}$$

Now:

$$(4) \qquad \frac{\partial q}{\partial x} = \frac{\partial a}{\partial z} - \frac{\partial g}{\partial x}, \qquad \frac{\partial r}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial a}{\partial y}, \qquad \frac{\partial q}{\partial y} = \frac{\partial h}{\partial z} - \frac{\partial f}{\partial y}, \ldots$$

As a result:

$$(5) \qquad u = u_0 + q_0(z_1 - z_0) - r_0(y_1 - y_0) + \int (\xi\,dx + \eta\,dy + \zeta\,dz),$$

([1]) *Comptes rendus de la R. Accademia delle Science fisiche e matematiche de Naples*, July and August 1906.

where:

$$\xi = a + (y_1 - y)\left(\frac{\partial a}{\partial z} - \frac{\partial h}{\partial x}\right) + (z_1 - z)\left(\frac{\partial a}{\partial z} - \frac{\partial q}{\partial x}\right),$$

$$\eta = h + (y_1 - y)\left(\frac{\partial h}{\partial z} - \frac{\partial b}{\partial x}\right) + (z_1 - z)\left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right),$$

$$\zeta = g + (y_1 - y)\left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right) + (z_1 - z)\left(\frac{\partial g}{\partial z} - \frac{\partial c}{\partial x}\right).$$

The formulas (5) and the analogous formulas that give v and w coincide with formulas (I), (I′), (I″) of chapter I, but formulas (5) have a very simple and more symmetrical form.

Cesàro, in his memoirs, extended the formulas and theorems that I just gave in chapter I to the case of a non-Euclidian space.

———

CHAPTER II

THE DISTORTIONS

I.

1. In the preceding chapter, I showed that elastic bodies that occupy spaces that are multiply connected may be found in equilibrium states that are very different from the ones that come about when the elastic bodies occupy simply-connected spaces. In these new equilibrium states one has a regular, internal deformation of the body without it nonetheless being subjected to external forces.

Imagine that one makes cuts that render the space that is occupied by the body simply connected. To each of them there correspond six constants that we called the *constants of the cut.* It is easy to establish the mechanical significance of these constants by means of formulas (III) of the preceding chapter.

Indeed, make cuts in the material along the aforementioned sections and let the body return to its natural state. If, in the process of returning to that state, certain parts of the body become superimposed over each other then remove the excess parts. Formulas (III) that we already referred to then show that the pieces that are placed on the two sides of the same section, and which were in contact before the cut, are, by the fact of the cut itself, subjected to a translation and a rotation that is equal for all of the pairs of pieces that are adjacent to the same section.

Upon taking the origin to be the center of reduction, the three components of the translation and the three components of the rotation along the coordinates are the six characteristics of the cut.

Conversely, if the multiply-connected elastic body is taken in the natural state then in order to bring it into a state of tension, one can perform the inverse operation – i.e., the sectioning that will render it simply connected – and then displace the two parts of each cut with respect to each other in such a manner that the relative displacements of the various pairs of pieces (which adhere to each other and which the cut has separated) are the resultants of translations and equal rotations; finally, re-establish the connectivity and the continuity along each cut, by subtracting or adding the necessary matter and welding the parts together. The set of these operations that relate to each cut may be called a *distortion* of the body and the six constants may be called the *characteristic of the distortion.*

In a multiply-connected elastic body whose deformation is regular and which has been subjected to a certain number of distortions, the inspection of the deformation might in no way reveal the places where the cuts and distortions that ensue are produced, and this is by virtue of the regularity itself. One may say, in addition, that the six characteristics of each distortion are not elements that depend upon the location where the cut has been executed.

Indeed, the same process that served to establish formulas (III) for us proved that if one takes two cuts in the body then one may transform the one into the other by a continuous deformation, so the constants that relate to one cut are equal to the constants that relate to the other.

It then follows that the characteristics of a distortion are not elements that are specific to each cut, but they depend exclusively on the geometrical nature of the space that is occupied by the body and the regular deformation to which it has been subjected.

The number of independent distortions to which an elastic body may be subjected is obviously equal to the order of connectivity of the space occupied by the body minus 1.

In conformity with what we have found, when two cuts can be transformed into each other by a continuous deformation they may be called *equivalent*. We also say that a distortion is known when the characteristics are given, along with the one cut or the other equivalent cut that they relate to are given.

2. Having said this, two questions naturally present themselves, namely:

1. For an arbitrarily chosen distortion, will there always correspond an equilibrium state and a regular deformation of the body if one supposes that the external actions are zero?

2. If the distortions are known then what is that state of deformation?

In order to relate these problems to the other ones that we have already solved, we prove the following theorem:

If one takes an arbitrary set of distortions in any multiply-connected, isotropic, elastic body then one may calculate an infinite number of regular deformations of the body that correspond to these distortions and which are equilibrated by external surface forces (which we indicate by T) that have a zero resultant and a zero moment with respect to an arbitrary axis.

Moreover, in order to recognize whether the given distortions in an isotropic body correspond to an equilibrium state, the external forces being zero, it will suffice to see whether the external forces T, with their signs changed and applied to the contour of the body when it is not subject to any distortion, determine a state of regular deformation that equilibrates the forces themselves. If one may effectively calculate this state of deformation then the problem concerning the equilibrium of a body subjected to distortions will be solved.

Indeed, let Γ denote the deformation that relates to the given distortions and the external forces T that are found to act upon the surface, and let Γ' denote the deformation that is determined by these external forces with their signs changed when the body is not subject to any distortion. The deformation Γ'' that results from Γ and Γ' will correspond to the given distortions and to zero external forces.

The aforementioned questions thus come down to seeing whether the deformation Γ' exists and finding it. They thus reduce to problems of elasticity where the distortions do not appear – i.e., to ordinary problems of elasticity.

However, the external forces T that act upon the surface, by virtue of the stated theorem, are such that if the body is rigid then they are equilibrated; it then ensues that they satisfy the fundamental conditions for the existence of the deformation Γ'.

Now, thanks to the foregoing, one can advance considerably using the new methods in the study of the existence theorems for questions of elasticity, which is why one may say that Γ and Γ′ always exist, except for certain conditions that relate to the geometric form of space occupied by the elastic body (conditions that will not specify here).

With these reservations, one may thus respond affirmatively to the first question in the case of isotropic bodies.

The second question that we posed relates to the case where the body is not subject to external actions; however, it may be generalized, and one may suppose that the distortions are given and the body is subject to definite external forces. Then, if the body is isotropic, it suffices to solve the problem by superposing the deformation Γ that is determined by the distortions and the external forces T with the deformation that is determined given external forces and the external forces – T that act upon the surface under the hypothesis that the distortions are absent.

The stated theorem serves in a certain way to eliminate the distortions in any case of isotropy, upon substituting the external surface forces, and it is for this reason that the questions that are attached to the distortions revert to questions of ordinary elasticity.

If the body is anisotropic then one easily sees that the state of deformation Γ is equilibrated by external forces that act upon the surface and external forces that act upon the interior of the body. It is therefore easy, even in this case, to eliminate the distortions, and then the various questions that may present themselves revert to ordinary problems of equilibrium in elastic bodies.

Article II is dedicated to the proof of theorem stated above, and article III examines a particular case.

II.

1. In order to prove the theorem that was stated in the previous article one must first establish certain preliminary formulas ([1]).

Upon the denoting the distance between two points (x, y, z) and (ξ, η, ζ) by r, we set, with Somigliana ([2]):

$$\begin{array}{lll}
u_1 = \dfrac{1}{r} + \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial x^2}, & v_1 = \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial x\,\partial y}, & w_1 = \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial x\,\partial z}, \\[2ex]
u_2 = \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial y\,\partial x}, & v_2 = \dfrac{1}{r} + \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial y^2}, & w_2 = \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial y\,\partial z}, \\[2ex]
u_3 = \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial z\,\partial x}, & v_3 = \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial z\,\partial y}, & w_3 = \dfrac{1}{r} + \dfrac{\alpha}{2}\dfrac{\partial^2 r}{\partial z^2}.
\end{array}$$

The preceding functions have no other singularities than the one at $x = \xi$, $y = \eta$, $z = \zeta$ and are symmetric with respect to the pairs of variables x, ξ; y, η; z, ζ.

([1]) I have presented these formulas for the first time at Pisa in my *Leçons sur la théorie de l'élasticité*, 1892; they have already been cited by professor Lauricella in his dissertation (*Ann. Scuola norm. di Pisa*, 1894).

([2]) *Annali di Matemat.*, 2nd series, t. XVII.

If $\alpha = -\dfrac{L+K}{L+2K}$ then each group of three functions u_s, v_s, w_s verifies the differential equations (3) of the preceding chapter in all of space (except for the singular locus described above), as well as the ones that are obtained from them by substituting ξ, η, ζ for x, y, z. The u_s, v_s, w_s may then be regarded as the components of the displacements of the points of an isotropic, homogeneous, elastic medium that is not subject to external forces that are applied to the interior of the medium, whether one considers these components to be functions of x, y, z or of ξ, η, ζ.

Take an element of the surface $d\Sigma$ that passes through the point ξ, η, ζ, and has a normal n. Let X_s, Y_s, Z_s denote the components of the unitary tension (corresponding to the displacements u_s, v_s, w_s) that is exerted along Σ by the region of the elastic medium that is placed on one side of the normal n or the region placed on the other side of the same normal.

The calculation of X_s, Y_s, Z_s presents no actual difficulty. Now, if u_0, v_0, w_0 are integrals of the differential equations (3) of the preceding chapter, which are regular in the domain S bounded by a surface Σ, and if X_0, Y_0, Z_0 are the components of the corresponding tension that acts on the surface then the Somigliana formulas give:

$$(1)\qquad \frac{1}{4\pi K}\int_\Sigma (X_0 u_1 + Y_0 v_1 + Z_0 w_1)\, d\Sigma + \frac{1}{4\pi K}\int_\Sigma (X_1 u_0 + Y_1 v_0 + Z_1 w_0)\, d\Sigma = u_0(x, y, z),$$

$$(1')\qquad \frac{1}{4\pi K}\int_\Sigma (X_0 u_2 + Y_0 v_2 + Z_0 w_2)\, d\Sigma + \frac{1}{4\pi K}\int_\Sigma (X_2 u_0 + Y_2 v_0 + Z_2 w_0)\, d\Sigma = v_0(x, y, z),$$

$$(1'')\qquad \frac{1}{4\pi K}\int_\Sigma (X_0 u_3 + Y_0 v_3 + Z_0 w_3)\, d\Sigma + \frac{1}{4\pi K}\int_\Sigma (X_3 u_0 + Y_3 v_0 + Z_3 w_0)\, d\Sigma = w_0(x, y, z),$$

upon supposing that the point x, y, z is interior to the domain S and ξ, η, ζ represents the coordinates of the points of the surface Σ. In the calculation of X_s, Y_s, Z_s one must suppose that the normal is directed from the exterior to the interior of the domain S.

On the contrary, if the point x, y, z is external to the domain then the right-hand sides of the preceding equations are zero.

2. Now suppose that in the preceding formulas one has:

$$(2)\qquad u_0 = l + r\,y - q\,z, \qquad v_0 = m + p\,z - r\,x, \qquad w_0 = n + q\,x - p\,y,$$

where l, m, n, p, q, r are constant quantities. Equations (3) of the preceding chapter will be satisfied and X_0, Y_0, Z_0 will be zero as a result.

One will then arrive at the fact that the integrals:

$$U = \frac{1}{4\pi K}\int_\Sigma (X_1 u_0 + Y_1 v_0 + Z_1 w_0)\, d\Sigma,$$

$$V = \frac{1}{4\pi K}\int_\Sigma (X_2 u_0 + Y_2 v_0 + Z_2 w_0)\, d\Sigma,$$

$$W = \frac{1}{4\pi K}\int_\Sigma (X_3 u_0 + Y_3 v_0 + Z_3 w_0)\, d\Sigma$$

will be equal to $l + ry - qz$, $m + pz - rx$, $n + qx - py$, respectively, if the point x, y, z is interior to the space S and will be zero if the point is exterior ([1]). Finally, one sees immediately that upon calculating:

$$\frac{\partial U}{\partial x} = \Gamma_{11}, \qquad \frac{\partial V}{\partial y} = \Gamma_{22}, \qquad \frac{\partial W}{\partial z} = \Gamma_{33},$$
$$\frac{\partial V}{\partial z} - \frac{\partial W}{\partial y} = \Gamma_{23}, \qquad \frac{\partial W}{\partial x} - \frac{\partial U}{\partial z} = \Gamma_{31}, \qquad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = \Gamma_{23},$$

the quantities Γ_{ri} will be zero whether x, y, z are interior or exterior to the space S.

We may thus conclude that the integrals U, V, W are discontinuous upon traversing the surface Σ, while the functions Γ_{rs} have no discontinuities. Upon letting U_i , V_i, W_i denote the values of U, V, W along Σ on the interior side and U_e , V_e, W_e , their values on the exterior side, we have:

$$\begin{aligned} U_i - U_e &= l + ry - qz, \\ V_i - V_e &= m + pz - rx, \\ W_i - W_e &= n + qx - py, \end{aligned}$$

3. Having said this, divide the surface Σ into two parts σ and σ' and set:

$$(3) \qquad \begin{cases} u = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (X_1 u_0 + Y_1 v_0 + Z_1 w_0)\, d\sigma, \\ v = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (X_2 u_0 + Y_2 v_0 + Z_2 w_0)\, d\sigma, \\ w = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (X_3 u_0 + Y_3 v_0 + Z_3 w_0)\, d\sigma, \end{cases}$$

and:

$$(3') \qquad \begin{cases} u' = \dfrac{1}{4\pi K}\displaystyle\int_{\sigma'} (X_1 u_0 + Y_1 v_0 + Z_1 w_0)\, d\sigma, \\ v' = \dfrac{1}{4\pi K}\displaystyle\int_{\sigma'} (X_2 u_0 + Y_2 v_0 + Z_2 w_0)\, d\sigma, \\ w' = \dfrac{1}{4\pi K}\displaystyle\int_{\sigma'} (X_3 u_0 + Y_3 v_0 + Z_3 w_0)\, d\sigma. \end{cases}$$

It is easy to see that u, v, w; u', v', w' enjoy the following properties:

([1]) Upon equating the coefficients of l, m, n, p, q, r in the two sides of the preceding equations, one finds integral relations that are analogous to the Gauss formulas in potential theory. *Cf.*, the memoir cited by Lauricella, chap. III, § 3.

1. The functions u, v, w are finite, continuous, monodromic, and have derivatives of all orders at all points of space, except for the surface σ.

2. The u, v, w satisfy equations (3) of the preceding chapter, except on the surface σ. One may then regard them as the components of the displacements of a homogeneous, isotropic, elastic medium that is not subject to external forces.

3. u', v', w' enjoy the same properties as u, v, w if σ' is substituted for σ;

4. Finally, we will have:

$$U = u + u', \qquad V = v + v', \qquad W = w + w'.$$

Now, u', v', w' are continuous on σ, while U, V, W are discontinuous, so u, v, w will have the same discontinuities on σ as U, V, W. Now, calculate:

$$\begin{array}{ccc}
\dfrac{\partial u}{\partial x} = \gamma_{11}, & \dfrac{\partial v}{\partial y} = \gamma_{22}, & \dfrac{\partial w}{\partial z} = \gamma_{33}, \\[2ex]
\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} = \gamma_{23}, & \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} = \gamma_{31}, & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = \gamma_{12}, \\[2ex]
\dfrac{\partial u'}{\partial x} = \gamma'_{11}, & \dfrac{\partial v'}{\partial y} = \gamma'_{22}, & \dfrac{\partial w'}{\partial z} = \gamma'_{33}, \\[2ex]
\dfrac{\partial v'}{\partial z} + \dfrac{\partial w'}{\partial y} = \gamma'_{23}, & \dfrac{\partial w'}{\partial x} + \dfrac{\partial u'}{\partial z} = \gamma'_{31}, & \dfrac{\partial u'}{\partial y} + \dfrac{\partial v'}{\partial x} = \gamma'_{12}.
\end{array}$$

We will have:

$$\gamma_{rs} + \gamma'_{rs} = \Gamma_{rs} = 0.$$

However, the functions γ'_{rs} preserve their regularity ([1]) upon traversing the surface σ (except for, at most, the contour of σ), so the functions γ_{rs} also enjoy the same property.

Upon substituting the values (2) in the formulas (3) for u_0, v_0, w_0 and ordering the right-hand sides relative to l, m, n, p, q, r one arrives at the following theorem:

Being given a surface σ, set:

$$\begin{cases}
A_{i1}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma X_i\,d\sigma, & A_{i2}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma Y_i\,d\sigma, \qquad A_{i3}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma Z_i\,d\sigma, \\[2ex]
B_{i1}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (\zeta Y_i - \eta Z_i)\,d\sigma, & B_{i2}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (\xi Z_i - \zeta X_i)\,d\sigma, \qquad B_{i3}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (\eta X_i - \xi Y_i)\,d\sigma.
\end{cases}$$

([1]) *See* the cited note in art. I, § 3.

$$\text{(II)}\qquad \begin{cases} u = A_{11}^{(\sigma)}l + A_{12}^{(\sigma)}m + A_{13}^{(\sigma)}n + B_{11}^{(\sigma)}p + B_{12}^{(\sigma)}q + B_{13}^{(\sigma)}r, \\ v = A_{21}^{(\sigma)}l + A_{22}^{(\sigma)}m + A_{23}^{(\sigma)}n + B_{21}^{(\sigma)}p + B_{22}^{(\sigma)}q + B_{23}^{(\sigma)}r, \\ w = A_{31}^{(\sigma)}l + A_{32}^{(\sigma)}m + A_{33}^{(\sigma)}n + B_{31}^{(\sigma)}p + B_{32}^{(\sigma)}q + B_{33}^{(\sigma)}r, \end{cases}$$

l, m, n, p, q, r being arbitrary quantities. One may regard u, v, w as the components of the displacement of an indefinite, homogeneous, isotropic, elastic medium, that has a deformation that is regular everywhere in space except at most the contour L of σ. This medium is devoid of external forces and is in equilibrium; at the same time, the displacements u, v, w are discontinuous on σ. These discontinuities are individualized by the equations:

$$\text{(4)}\qquad \begin{cases} u_i - u_0 = l + ry - qz, \\ v_i - v_0 = m + pz - rx, \\ w_i - w_0 = n + qx - py, \end{cases}$$

where u_0, v_0, w_0 denote the values of u, v, w on one side of the normal to the surface σ and u_i, v_i, w_i denote the values on the other side of the same normal.

One infers from this proposition that upon starting with the characteristics of the preceding deformation and calculating the quantities u, v, w by means of formulas (1), (1′), (1″) of the preceding chapter, they will be polydromic when the surface σ is open. The line – or lines – of branching will be formed from the contour L of σ and the polydromy will be individualized by formulas (4).

4. Now suppose that a body S is n+1-connected. Make n cuts that render it simply connected.

Let σ_1, σ_2, …, σ_n denote the n surfaces that are defined by the aforementioned cuts when prolonged in such a fashion that they extend outside of S. Set:

$$\text{(II′)}\qquad \begin{cases} u = \sum_{i=1}^{n} (A_{11}^{(\sigma)}l_i + A_{12}^{(\sigma)}m_i + A_{13}^{(\sigma)}n_i + B_{11}^{(\sigma)}p_i + B_{12}^{(\sigma)}q_i + B_{13}^{(\sigma)}r_i), \\ v = \sum_{i=1}^{n} (A_{21}^{(\sigma)}l_i + A_{22}^{(\sigma)}m_i + A_{23}^{(\sigma)}n_i + B_{21}^{(\sigma)}p_i + B_{22}^{(\sigma)}q_i + B_{23}^{(\sigma)}r_i), \\ w = \sum_{i=1}^{n} (A_{31}^{(\sigma)}l_i + A_{32}^{(\sigma)}m_i + A_{33}^{(\sigma)}n_i + B_{31}^{(\sigma)}p_i + B_{32}^{(\sigma)}q_i + B_{33}^{(\sigma)}r_i), \end{cases}$$

$$\gamma_{11} = \frac{\partial u}{\partial x}, \qquad \gamma_{22} = \frac{\partial v}{\partial y}, \qquad \gamma_{33} = \frac{\partial w}{\partial z},$$

$$\gamma_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_{31} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad \gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};$$

where l_i, m_i, n_i, p_i, q_i, r_i are arbitrary constants; the deformation $\Gamma \equiv (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12})$ will be regular inside of S and will correspond to arbitrary distortions made along the aforementioned cuts.

If one calculates the external forces acting on the interior of the body then one finds that they are zero, but in general the forces acting on the contour of the body S will not be zero. Now, the body is in equilibrium, which is why the forces must have a zero resultant and a zero moment relative to an arbitrary axis.

The stated theorem of article I is thus proved.

III.

1. Let a finite, simply connected, surface σ be situated in the xz-plane so that it does not encounter the z-axis. While the xz-plane turns around the z-axis, suppose that σ is deformed and displaced in the plane in an arbitrary manner without ever encountering z, but suppose that, after a complete circuit, it reverts to its original configuration. By this motion, the area σ generates a doubly-connected, annular solid linked to the z-axis. Let it be filled with homogeneous, isotropic, elastic matter. Subject a cut formed by a plane passing through z to the most general distortion and study the deformation of the body.

2. One knows that the integrals of equations (3) of the preceding chapter must be bi-harmonic functions; i.e., they must satisfy the double Laplace equation $\Delta^2 \Delta^2 = 0$.

Now, if l, m, n, p, q, r are arbitrary constants then the functions:

$$\frac{1}{2\pi}(l - qz + ry)\arctan\frac{y}{x},$$
$$\frac{1}{2\pi}(m - rx + pz)\arctan\frac{y}{x},$$
$$\frac{1}{2\pi}(n - py + qx)\arctan\frac{y}{x}$$

are bi-harmonic and they have the polydromy that corresponds to a distortion that has the characteristics l, m, n, p, q, r.

However, the preceding functions do not satisfy the indefinite equations of elasticity in the case of isotropy.

Therefore, take:

$$u = \frac{1}{2\pi}(l - qz + ry)\arctan\frac{y}{x} + \lambda,$$
$$v = \frac{1}{2\pi}(m - rx + pz)\arctan\frac{y}{x} + \mu,$$
$$w = \frac{1}{2\pi}(n - py + qx)\arctan\frac{y}{x} + \nu,$$

and determine the monodromic functions λ, μ, ν in such a manner that the expressions for u, v, w thus obtained satisfy equations (3).

Set:

$$\lambda = (ax + by + cz + e) \log(x^2 + y^2),$$
$$\mu = (a'x + b'y + c'z + e') \log(x^2 + y^2),$$
$$\nu = (a''x + b''y + c''z + e'') \log(x^2 + y^2).$$

The constants a, b, c, e; a', b', c', e'; a'', b'', c'', e'' are easily calculated, and one finds that:

$$\text{(III)} \quad \begin{cases} u = \dfrac{1}{2\pi}\left[(l - qz + ry)\arctan\dfrac{y}{x} + \left(-m - pz - \dfrac{rK}{L+2K}\,x\right)\log(x^2+y^2)\right], \\ v = \dfrac{1}{2\pi}\left[(m - rx + pz)\arctan\dfrac{y}{x} + \left(\quad l - qz - \dfrac{rK}{L+2K}\,y\right)\log(x^2+y^2)\right], \\ w = \dfrac{1}{2\pi}\left[(n - qz + ry)\arctan\dfrac{y}{x} + \left(px + qy\right)\log(x^2+y^2)\right]. \end{cases}$$

It is easy to recognize that the corresponding deformation is regular and that one may easily obtain the tensions that act upon the surface.

Therefore, for the body in question, one may calculate the deformation Γ and the forces T of article I, no matter what the distortion that the body has been subjected to.

3. The formulas that we have given in article III of the preceding chapter have been deduced as a particular case of the preceding expressions. Indeed, the formulas (2) of the cited chapter are obtained when $\gamma = 0$ by taking:

$$l = m = n = p = q = 0, \qquad r = 2\pi\alpha,$$

and the formulas (4) of the same chapter by setting:

$$l = m = n = p = q = 0, \qquad n = 2\pi\alpha.$$

CHAPTER III

THE EFFORTS

I.

1. In the preceding chapters, I showed that the laws of equilibrium for solid elastic bodies that occupy multiply-connected (cyclic) spaces are very different from those of elastic solids that occupy simply-connected (acyclic) spaces, provided that one assumes regular deformations in the two cases.

Indeed, if the space occupied by the solid is cylic then one may determine a state of tension in the same body even in the absence of external forces when it is subject to distortions. However, the same is not true when the body occupies an acyclic space. This is why, in the case of an elastic solid that occupies a cyclic space, we will have to solve a series of very interesting new problems that do not present themselves in the other case and which consist in calculating the states of tension in the body due to given distortions.

In order to facilitate the solution of these problems, we briefly present some general considerations in this chapter that permit us to easily transform them.

2. First of all, we calculate the energy of an elastic solid that is subject to given distortions.

Represent the characteristics of the tension in a deformed elastic solid (the *stress*, according to the terminology of the English) by t_{11}, t_{22}, t_{33}, t_{23}, t_{31}, t_{12} and the characteristics of the deformation (the *strain*) by γ_{11}, γ_{22}, γ_{33}, γ_{23}, γ_{31}, γ_{12} .

If we let φ denote the unitary elastic potential then φ will be a function that is homogeneous of the second degree in the quantities γ_{rs}, and we will have:

$$\frac{\partial\varphi}{\partial\gamma_{rs}} = t_{rs}\,, \qquad \varphi = \frac{1}{2}\sum t_{rs}\gamma_{rs}\;;$$

the energy of the system will thus be:

$$E = -\frac{1}{2}\int_S \sum t_{rs}\gamma_{rs}\,dS\;,$$

in which S represents the space occupied by the solid.

Suppose that S is multiply connected (cyclic) and the deformation is regular. Imagine that one has made cuts σ_1, σ_2, …, σ_n that render S simply connected. By means of simple integrations, and upon representing by u, v, w the components of the displacements of the points of the elastic solid when starting from the natural state, we will have:

(1) $$E = \frac{1}{2}\int_S \left[u\left(\frac{\partial t_{11}}{\partial x}+\frac{\partial t_{12}}{\partial y}+\frac{\partial t_{13}}{\partial z}\right)+v\left(\frac{\partial t_{21}}{\partial x}+\frac{\partial t_{22}}{\partial y}+\frac{\partial t_{23}}{\partial z}\right)+w\left(\frac{\partial t_{31}}{\partial x}+\frac{\partial t_{32}}{\partial y}+\frac{\partial t_{33}}{\partial z}\right)\right] dS$$

$$+$$

$$+\frac{1}{2}\int_\sigma [u(t_{11}\cos nx + t_{12}\cos ny + t_{13}\cos nz) + v(t_{21}\cos nx + t_{22}\cos ny + t_{23}\cos nz)$$
$$+ w(t_{31}\cos nx + t_{32}\cos ny + t_{33}\cos nz)]\, d\sigma$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{\sigma_i} [\quad (u_\alpha - u_\beta)(t_{11}\cos \nu_i x + t_{12}\cos \nu_i y + t_{13}\cos \nu_i z)$$
$$+ (v_\alpha - v_\beta)\,(t_{21}\cos nx + t_{22}\cos ny + t_{23}\cos nz)$$
$$+ (v_\alpha - v_\beta)\,(t_{21}\cos nx + t_{22}\cos ny + t_{23}\cos nz)]\, d\sigma_i\,,$$

where σ is the contour of S, n is the normal to σ directed towards the interior of S, and v_i is the normal to σ_i ; u_α, v_α, w_α are the values of u, v, w on σ_i on the adjacent side of the region v_i and u_β, v_β, w_β are the values on the other.

Now, let l_i, m_i, n_i, p_i, q_i, r_i denote the six characteristics of the distortion relative to the cut σ_i and let X_i, Y_i, Z_i represent the components of the unitary tension that each element of the section σ_i . Since the external forces are zero, one will have, after integrating by parts:

$$E - \frac{1}{3}\sum_{i=1}^{n}\int_{\sigma_i} \quad [(l_i + r_i y - q_i z)\, X_i + (m_i + p_i z - r_i x)\, Y_i + (u_i + q_i x - p_i y)\, Z_i]\, d\sigma_i$$
$$= \frac{1}{2}\sum_{i=1}^{n}\Big[t_i\int_{\sigma_i} X_i d\sigma_i + m_i\int_{\sigma_i} Y_i d\sigma_i + n_i\int_{\sigma_i} Z_i d\sigma_i$$
$$+ p_i\int_{\sigma_i}(Y_i z - Z_i y)d\sigma_i + q_i\int_{\sigma_i}(Z_i x - X_i z)d\sigma_i + r_i\int_{\sigma_i}(X_i y - Y_i x)d\sigma_i\,.$$

If one sets:

$$L_i = \int_{\sigma_i} X_i d\sigma_i\,, \qquad M_i = \int_{\sigma_i} Y_i d\sigma_i\,, \qquad N_i = \int_{\sigma_i} Z_i d\sigma_i\,,$$
$$P_i = \int_{\sigma_i}(X_i z - Z_i y)d\sigma_i\,, \qquad Q_i = \int_{\sigma_i}(Z_i z - X_i y)d\sigma_i\,,$$
$$R_i = \int_{\sigma_i}(X_i y - Y_i x)d\sigma_i$$

then one will find that:

$$E = \frac{1}{2}\sum_{i=1}^{n} \quad (L_i\, l_i + M_i\, m_i + N_i\, n_i + P_i\, p_i + Q_i\, q_i + R_i\, r_i).$$

Let σ_1, σ_2, …, σ_{6n} denote the $6n$ characteristics of the distortions and let E_1, …, E_{6n} denote the coefficients in the preceding expression that correspond to them. We will then have:

$$E = \frac{1}{3}\sum_{i=1}^{6n} E_i s_i \,.$$

3. For us, the term *elementary distortion* refers to the distortion that corresponds to having all of the quantities $s_i = 0$, except for one that has the value unity.

Suppose that the latter is s_h and let E_{ih} denote the corresponding values of the coefficients E_i . One immediately recognizes that if the values of the characteristics of the distortions are $s_1, \ldots, s_{6n}$ then one has:

$$E_i = \sum_{i=1}^{6n} E_{ih} s_h \,,$$

and, as a consequence:

$$E = \frac{1}{3}\sum_{i=1}^{6n}\sum_{h=1}^{6n} E_{ih} s_i s_h \,.$$

4. It is easy to establish the significance of the quantities E_i and E_{ih} .

To that effect, observe that L_i, M_i, N_i are the components of the resultant force and P_i, Q_i, R_i are the components of the resultant couple of the tensions that act upon the section σ_i when one takes the origin of the axes to be the center of reduction.

We may thus call the coefficients L_i, M_i, N_i, P_i, Q_i, R_i the *efforts that act upon the section* σ_i, or in general we say that $E_1, E_2, \ldots, E_{6n}$ are the *efforts that correspond to the distortion* $s_1, s_2, \ldots, s_{6n}$. The quantity E_{ih} will be called the *effort of order i that is induced by the elementary distortion of order h*. More simply, the coefficients E_{ih} may be called the *coefficients of the efforts.*

II.

1. Green has proved a fundamental proposition in potential theory by an application of Gauss's theorem. By the same process, Betti has discovered an analogous theorem for elasticity ([1]). However, if the potential is polydromic then Green's theorem is inapplicable. Likewise, Betti's theorem is inapplicable if the displacements are polydromic. We shall nonetheless see that even in this case one may recover the fundamental idea, and one is led to a law of reciprocity that is quite interesting.

Envision two distortions $s_1, s_2, \ldots, s_{6n}$ and $s_1', s_2', \ldots, s_{6n}'$ being applied in succession to a multiply-connected, elastic body S that is not subject to any external forces. Let γ_{rs} , γ_{rs}' be the characteristics of the two different deformations that result from it, and let u, v, w; u', v', w' be the components of the displacements, respectively.

One easily finds that:

$$\int_s \sum \frac{\partial \varphi}{\partial \gamma_{rs}} \gamma_{rs}' \, ds = \int_S \sum \frac{\partial \varphi}{\partial \gamma_{rs}'} \gamma_{rs} \, dS \,,$$

([1]) *Teoria della elasticità* (*Nuovo Cimento*, 1872-1873).

where φ' represents the function φ in which one has substituted the quantities γ'_{rs} for the γ_{rs}.

One infers from this that:

$$\begin{aligned}\sum_{i=1}^{n}\int_{\sigma_i}[\quad &(u'_\alpha - u'_\beta)\ (t_{11}\cos \nu_i x + t_{12}\cos \nu_i y + t_{13}\cos \nu_i z)\\
&+ (v'_\alpha - v'_\beta)\,(t_{21}\cos \nu_i x + t_{22}\cos \nu_i y + t_{23}\cos \nu_i z)\\
&+ (w'_\alpha - w'_\beta)\,(t_{31}\cos \nu_i x + t_{32}\cos \nu_i y + t_{33}\cos \nu_i z)]\,d\sigma_i\,.\end{aligned}$$

$$\begin{aligned}= \sum_{i=1}^{n}\int_{\sigma_i}[\quad &(u_\alpha - u_\beta)(t'_{11}\cos \nu_i x + t'_{12}\cos \nu_i y + t'_{13}\cos \nu_i z)\\
&+ (v_\alpha - v_\beta)\,(t'_{21}\cos \nu_i x + t'_{22}\cos \nu_i y + t'_{23}\cos \nu_i z)\\
&+ (w_\alpha - w_\beta)\,(t'_{31}\cos \nu_i x + t'_{32}\cos \nu_i y + t'_{33}\cos \nu_i z)]\,d\sigma_i\,.\end{aligned}$$

where the notations are the same as the ones that were already in formula (1). Therefore:

$$\sum_{i=1}^{6n} E'_i s_i = \sum_{i=1}^{6n} E_i s'_i\,. \tag{2}$$

As a consequence, we have the following theorem:

If two systems of distortions in a multiply-connected elastic body generate two systems of efforts then the sum of the products of the efforts of the first system of distortions with the characteristics of the second system is equal to the product of the efforts of the second system of distortions with the characteristics of the first system.

2. From the equality (2), upon taking into account that $s_1, s_2, \ldots, s_{6n}, s'_1, s'_2, \ldots, s'_{6n}$ are arbitrary quantities, one infers that:

$$E_{ih} = E_{hi} \tag{3}$$

for any values of the indices i and h. Conversely, equation (2) emerges from these equalities as a consequence. The reciprocity theorem that we just gave may thus be stated in the following manner:

The effort of order i that is induced by the elementary distortion of order h is equal to the effort of order h that is induced by the elementary distortion of order i.

By this statement, the theorem takes a form that is similar to the fundamental theorem of electrostatic induction.

More simply, the theorem may be further stated:

The coefficients of the efforts do not change value under a transposition of indices.

3. Being given the numerous applications of the reciprocity theorem, it will be useful to further examine it from another viewpoint.

Take two arbitrary sections σ_1 and σ_2 of an elastic body, which may also coincide.

First of all, perform a distortion that consists of a relative translation T_1 in the direction of h_1 of the elements of the two faces of the cut σ_1. Then determine the projection S_2 in the direction h_2 of the resultant of the tensions that and act upon the section σ_2 .

Finally, instead of the preceding distortion, perform another displacement that consists of a translation T_2 in the direction h_2 of the elements of the two faces of the cut σ_2 and determine the projection σ_1 on h_1 of the resultant of the efforts that act upon the section σ_1 .

The reciprocity theorem gives us:

$$S_2\, T_2 = S_1\, T_1 ,$$

and, as a result:

$$\frac{S_2}{T_1} = \frac{S_1}{T_2} ,$$

i.e., *the projections of the two efforts in the directions of the two translations are proportional to the values of the translations themselves.*

One obtains a theorem that is completely analogous by substituting a rotation T_1 around the straight line h_1 for the translation T_1, provided that one replaces the projection S_1 of the resultant of the tensions that act upon the elements of σ_1 with the moment of these tensions with respect to the straight line h_1 .

Finally, with similar substitutions for T_2 and S_2, one obtains a new theorem that is analogous to the first two.

These three propositions are equivalent to the reciprocity theorem that we have already stated in various forms in the preceding paragraph.

III.

1. By virtue of the equality (3), one has:

$$E_i = \frac{\partial E}{\partial s_i} ,$$

and if one lets e_{ih} denote the coefficients of the reciprocal form of the expression:

$$\sum_i \sum_h E_{ih} s_i s_h$$

then we may express the energy in the system in another fashion by means of the formula:

$$E = \frac{1}{2} \sum_i \sum_h e_{ih} E_i E_h .$$

2. In the preceding chapter we proved that when one is given a deformation of a multiply-connected system the distortions that correspond to its equivalent cuts are equal.

We would now like to complete this proposition and prove that the efforts that correspond to equivalent cuts are equal.

Indeed, envision the section σ_1. By definition, one may reduce it to an equivalent section σ_2 by means of a continuous deformation. While one performs this reduction, the surface σ_1 generates a solid S_1 that constitutes one part of the elastic body S.

The solid S_1 will be bounded by σ_1, σ_2, and a lateral surface ω. We may then imagine S_1 to be in equilibrium under the action of only tensions that act upon σ_1 and σ_2. The equality of the efforts then results from this.

One then concludes that the efforts, like the distortions, are not elements that are specific to each cut, but they depend exclusively upon the geometric nature of the space occupied by the body and the regular deformation that the body is subjected to.

The most fundamental problem that we may propose in the study of multiply-connected, elastic solids will be the following one:

Being given the 6*n distortions, determine the* 6*n efforts if one supposes that the external forces are zero.*

This question amounts to determining the coefficients of the efforts.

CHAPTER IV

DISTORTIONS AND EFFORTS IN A SYMMETRIC, CYCLIC BODY

―――――

I.

1. Starting with the principles that we established in the preceding chapter, we study them in a particular case of distortions. We verify that these principles permit us to go deeper into the mechanism of the distortions and reveal to us some facts that are quite far removed from the ones that one must consider, *a priori*, upon examining the question intuitively. One achieves the goal without recourse to the integration of differential equations, but with the aid of an elementary discussion of the expression for the energy of an elastic system that is subject to given distortions.

In order to briefly give an idea of the results, recall the example that we started with in chapter I.

We supposed that a very thin transverse wedge was suppressed from a ring that varied in thickness proportionally to the distance from the symmetry axis, and then we supposed that we brought the two faces of the cut together and welded them. Left to itself, the body would cease to be in its natural state. It would take on a state of regular deformation and its elements would be subject to elastic forces. One may then demand that they be the actions that are exerted on the welded faces. It might seem obvious that they must be in a state of tension, but things are not that way. There is always a part that is in a state of tension and a compressed part. Moreover, "the sum of the forces of tension is equal to the sum of the forces of compression."

The present chapter is dedicated to this theorem and another analogous one that cast an unexpected light on the distribution of elastic efforts that are generated in the body by the distortions.

2. First of all, we give some definitions. In the preceding chapter, we expressed the elastic energy of a body that is subject to distortions by the formulas:

$$E = \frac{1}{2}\sum_{i=1}^{6n} E_i s_i ,$$

where the efforts are represented by the E_i and the characteristics of the distortions by the s_i. We call E_i the *conjugate effort* to the characteristic s_i of the distortion.

Having chosen the center of reduction, the distortion that is applied to each cut may be decomposed into a translation and a relative rotation of the elements of the faces of the cut. We make use of the same center of reduction and compose the actions that act upon the elements of a face of the same cut as if they were applied to the points of a rigid system. One thus finds a resultant force and a resultant couple. That force and couple constitute the total effort applied to the section (*cf.*, preceding chap., art I., § 4).

By virtue of the preceding definition, the components along the coordinate axes of the resultant force are the conjugate efforts of the projection that correspond to the translation, and the components of the resultant couple are the conjugate efforts of the projections corresponding to the rotation.

If the distortion is elementary then only one of the characteristics and, as a consequence, only one of the preceding projections will be different from zero; the component of the force or the component of the couple that is conjugate to that characteristic may be called the *effort conjugate to the elementary distortion.*

3. A solid of revolution may be generated by the revolution of a connected, planar surface (generating surface) around a straight line in its plane. Let n be the order of the connectivity of the generating surface. If the axis of rotation is external to it then the order of the connectivity of the solid is $n + 1$; however, if the axis is composed of one part of the contour of the generating surface then the order of connectivity of the solid is equal to n.

Reduce the generating surface to a simply connected one by means of $n - 1$ linear cuts. Under rotation, these cuts generate other surfaces that may be regarded as sections of the solid. In the second case, these sections suffice to render the solid simply connected, while in the first case, in order to obtain the simple connectivity one must further make a transversal cut – for example, a cut that coincides with one of the positions that the generating surface takes when it turns around the axis.

This latter cut, or any equivalent cut, will be said to be of the *first kind;* each of the others, or an equivalent cut, will be said to be of the *second kind.*

Let a symmetric solid be doubly connected; two cases may be presented:

1. The surface generated is simply connected and external to the symmetry axis.

2. The surface generated is doubly connected and is partially bounded by the symmetry axis.

In order to reduce the solid to a simply connected one, we make a cut of the first kind in former case and a cut of the second kind in the latter case, and we say, in the former case that the body is *doubly connected of the first kind,* and in the latter case, that it is *doubly-connected of the second kind.*

II.

1. Now, let us study the distortions of a symmetric, elastic body that is doubly-connected of the second kind. In that study, we assume that the symmetry is limited by not only the form, but under the hypothesis of anisotropy, it also persists relative to the constitution of the elastic body.

Suppose that the distortion is performed on a cut σ made along one of the positions that the generating surface takes under rotation.

Place the origin at a point of the symmetry axis and take that axis to be the z-axis.

The energy of the system will be expressed by the formula (*see,* preceding chapter, art. I, § 3):

$$E = \frac{1}{2}\sum_{i=1}^{6}\sum_{h=1}^{6} E_{ih} s_i s_h \,, \tag{1}$$

where:

$s_1 = l$, $s_2 = m$, $s_3 = n$, $s_4 = p$, $s_5 = q$, $s_6 = r$

denote the characteristics of the distortion, according to the notations that were employed in the preceding chapter.

Having said this, observe that, due to symmetry, the energy of the system does not change if, instead of applying the distortion to the original section σ, we apply it to another section that forms an arbitrary angle of θ with the first one.

Now, since the two sections are equivalent the energy of the system will be the same whether we apply the following distortion to the system along the section σ:

$$s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,$$

or we apply the following distortion to the same section:

$$\begin{aligned}
&s_1' = s_1 \cos\theta + s_2 \sin\theta, && s_2' = -s_1 \sin\theta + s_2 \cos\theta, && s_3' = s_3,\\
&s_4' = s_4 \cos\theta + s_5 \sin\theta, && s_5' = -s_4 \sin\theta + s_5 \cos\theta, && s_6' = s_6.
\end{aligned}$$

In other words:

$$E = \frac{1}{2}\sum_{i=1}^{6}\sum_{h=1}^{6} E_{ih} s_i' s_h' \tag{2}$$

will be independent of θ; i.e.:

$$\frac{dE}{d\theta} = 0.$$

But:

$$\begin{aligned}
&\frac{ds_1'}{d\theta} = s_2', && \frac{ds_2'}{d\theta} = -s_1', && \frac{ds_3'}{d\theta} = 0,\\
&\frac{ds_4'}{d\theta} = s_5', && \frac{ds_5'}{d\theta} = -s_4', && \frac{ds_6'}{d\theta} = 0,
\end{aligned}$$

so:

$$\begin{aligned}
0 = \frac{dE}{d\theta} = \quad & (E_{11} - E_{22}) s_1' s_2' + E_{12}(s_2'^2 - s_1'^2) + E_{13} s_2' s_3' - E_{23} s_1' s_3'\\
+ \ & (E_{44} - E_{55}) s_4' s_5' + E_{45}(s_5'^2 - s_4'^2) + E_{46} s_5' s_6' - E_{56} s_4' s_6'\\
+ \ & (E_{14} - E_{25})(s_2' s_4' + s_1'^2 s_5'^2) + (E_{24} + E_{16})(s_2' s_5' - s_4' s_1')\\
+ \ & E_{16} s_2' s_6' - E_{26} s_1' s_6' + E_{34} s_3' s_5' - E_{35} s_3' s_4'.
\end{aligned}$$

Now, the quantities s_1', s_2', s_3', s_4', s_5', s_6' are arbitrary; it then follows that:

$$\begin{gathered}
E_{11} = E_{22}, \qquad E_{44} = E_{55}, \qquad E_{14} = E_{25}, \qquad E_{24} = -E_{15},\\
E_{12} = E_{13} = E_{23} = E_{45} = E_{46} = E_{56} = E_{16} = E_{26} = E_{34} = E_{35} = 0.
\end{gathered}$$

As a consequence, the expression (2) will reduce to:

$$E = \frac{1}{2}[E_{11}(s_1^2 + s_2^2) + E_{33} s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{66} s_6^2 + 2E_{14}(s_1 s_4 + s_2 s_5)$$

$$+ 2E_{24}(s_2\, s_4 - s_1\, s_5) + 2E_{36}\, s_3\, s_6].$$

Take the xz-plane for the section σ and envision the distortion to be of order 6; i.e., a distortion due to a relative rotation of the two faces of a cut σ around the z-axis.

It is obvious that the resulting deformation of the body will be symmetric with respect to the xz-plane and consequently the elasticity ellipsoid and director surface ([1]) at each point of σ will have the xz-plane for its symmetry plane.

In other words, the elastic actions that are exerted on the elements of σ are normal to σ. Upon composing these actions and taking the origin to be the center of reduction, one may obtain only a resultant force that is normal to σ (having the direction y) and a resultant couple whose axis is parallel to σ. It follows that:

$$E_{16} = E_{36} = E_{56} = 0,$$

which is why:

$$E = \frac{1}{2}[E_{11}(s_1^2 + s_2^2) + E_{33}s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{66}s_6^2 + 2E_{14}(s_1 s_4 + s_2 s_5) + 2E_{24}(s_2\, s_4 - s_1\, s_5)]. \tag{3}$$

In the same manner, envision the elementary distortion to be of order 2 – i.e., a distortion that is due to a relative translation of the elements of the two faces of a cut σ that is parallel to the y-axis. The resulting elasticity ellipsoid and director surface will be symmetric with respect to the xz-plane at each point of σ. This is why, with the aid of an argument that is analogous to the one that we just made, one deduces that:

$$E_{12} = E_{32} = E_{52} = 0.$$

However:

$$E_{14} = E_{52}\,,$$

so:

$$E = \frac{1}{2}[E_{11}(s_1^2 + s_2^2) + E_{33}s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{66}s_6^2 + 2E_{24}(s_2 s_4 - s_1 s_5)]. \tag{4}$$

Now, observe that the coefficient of $E_{11} = E_{22}$ cannot be zero, since otherwise the energy due to an elementary distortion of order 1 or 2 would be zero, which is absurd.

It then follows that upon composing all of the actions that act upon σ, by virtue of the elementary distortion of order 2, one must obtain a resultant that is different from zero whose line of action meets the z-axis at a point Ω. Indeed, all of these actions are equivalent to the force E_{22} being applied to the origin and the couple having the moment E_{24} and the x-axis for its axis.

However, if we take the center of reduction to be at the point Ω then we will have $E_{24} = 0$ and, as a result:

$$E = \frac{1}{2}[E_{11}(s_1^2 + s_2^2) + E_{33}s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{66}s_6^2]. \tag{5}$$

([1]) *See* CLEBSCH, *loc. cit.*, chap. I, § 6.

2. Now, let us study the distortions of symmetric bodies that are multiply connected of the second kind. Suppose that the distortions are applied to a cut of the second kind that is symmetric with respect to the symmetry axis of the body.

The energy of the system will always have the form (1) and, if we take the z-axis to be the symmetry axis then the expression for this energy must not change if we rotate the x and y axes through an angle of θ in their plane. Therefore, even in this case the expression (2) must be independent of θ as a result, and E must take the form (3). However, by virtue of symmetry E must not vary if one changes s_6 into $-s_6$ when one supposes that $s_1 = s_2 = s_4 = s_5 = 0$; thus, $E_{36} = 0$. Even if one exchanges the x and y axes the energy will not vary; i.e., the quantity E must be conserved if one substitutes, at the same time, s_1 for s_2 and s_4 for $-s_5$. It then results that $E_{14} = 0$, and as a consequence the energy E must have the form (4).

An argument that is analogous to the one that we made in the preceding paragraph proves that upon conveniently choosing the origin to be a point Ω one may make E_{24} equal to zero. As a consequence, even in the case where the double connectivity is of the second kind, the expression for the energy may be reduced to formula (5).

The point Ω will be called the *central point of the symmetry axis.*

3. When one takes into account the principle of equivalent cuts, formula (5) contains the following theorem:

In a doubly-connected, symmetric, elastic body, each elementary distortion generates only the conjugate effort when one takes the center of reduction to be the central point of the symmetry axis.

The following corollary ensues from this theorem:

The total effort that is generated by a distortion that consists of a relative translation of the elements of the faces of the cut is a force whose line of action passes through the central point of the symmetry axis.

The total effort that is generated by a distortion that consists of a relative rotation of the elements of the faces of the cut around an axis that passes through the central point of the symmetry axis is a couple.

It is then simple to prove that:

If an elastic body has a symmetry plane that is normal to the symmetry axis then the central point is the point of intersection of the symmetry axis with the symmetry plane.

4. Now, examine the case where the double connectivity is of the first kind and the distortion has order 6. The effort is then reduced to a couple that has the symmetry axis for its axis. Thus, if we consider the elastic actions that act upon a face of the cut then their resultant is zero. From the theorem stated in § 1 of article I, it is easy to complete this theorem by showing that the moment of the forces of tension with respect to the symmetry axis surpasses that of the forces of compression and is precisely equal to the quantity E_{66} .

In an analogous manner, suppose that the cut is made in the *xz*-plane and consider the distortion to be of order 2. The induced effort will be a force normal to the cut whose line of action will meet the symmetry axis. Thus, in this case there must also exist elements of the faces of the cut that are compressed, while the others are in a state of tension.

Returning to the example of paragraph 1, we may state the following proposition:

If we suppress a wedge of uniform size from the ring (instead of a wedge that is proportional in size to the distance from the symmetry axis), and if we then weld the faces of the gap together then some parts of these faces will be in a state of tension and others will be compressed. The tensions exceed the pressures (and are precisely equal E_{11}*), but the moment of the former will be equal to the moment of the latter with respect to the symmetry axis.*

One very easily deduces from the preceding results that if one suppresses a wedge from the ring whose thickness is given by:

$$s_2 - s_6 x,$$

letting x denote the distance from the symmetry axis, and then welds the faces of the gap together then one generates a normal effort to the section whose line of action is directed along the symmetry axis of:

$$h = \frac{s_6}{s_2}\frac{E_{66}}{E_{11}}.$$

One thus sees that, upon conveniently choosing the ratio s_6 / s_2 , one may arrange that this line of action is an arbitrary distance from the symmetry axis.

In the first chapter, we examined the distortion that consists of sliding the two faces of the cut relative to each other in the direction of the symmetry axis in such a manner that it gives the ring a slightly helicoidal form and then welding these two faces together.

This distortion corresponds to a distortion of order 3. As a consequence, the corresponding effort has the symmetry axis for its line of action; this is why the elements of one face of the cut will be carried along, one in the direction of the sliding and the other in the opposite direction; moreover, the moment of the former action will be equal to that of the other with respect to an axis that is normal to the section, and which meets the symmetry axis.

We shall not stop to discuss some other particular cases that are not, just the same, without interest, but which give rise to some considerations and conclusions that are analogous to the ones that we just developed and formulated.

CHAPTER V

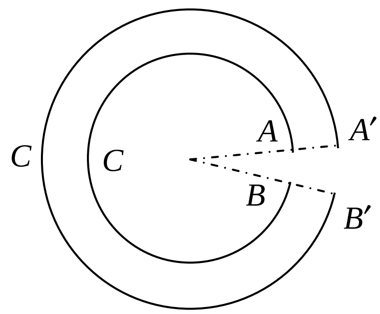
HOLLOW CYLINDER OF REVOLUTION – DISTORTION OF ORDER 6

I.

1. One of the results that I arrived at in the preceding chapter was the following one:

Let a ring be symmetric with respect to an axis (Fig. 1). Subtract a thin slice $AA'BB'$ from it whose size varies with the distance to the axis (we call this operation *making a radial fissure*). Then, reconnect the faces AA' and BB' of the fissure, solder them and let the ring be free.

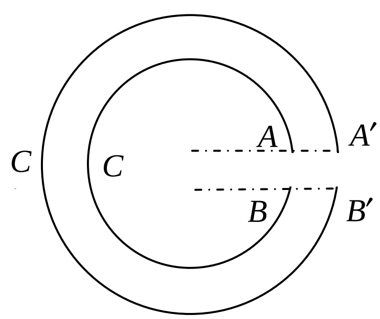
Figure 1



The soldered faces are not under simple tension, but part of them is under tension and part of them is compressed, and the sum of the forces of compression is equal to the sum of the forces of tension (*see* Chap. IV, art. I, § 1, art II, § 4).

Upon subtracting a small slice from $AA'BB'$ (Fig. 2) whose faces are parallel and equidistant from the axis of the ring (*uniform fissure*), one will again find, after soldering the two faces and releasing the body itself, that the faces are partly in a state of tension and partly compressed. Nevertheless, the conditions of the body in equilibrium are essentially different in the two cases (Chap. IV, art. II, § 4).

Figure 2.



In the first case, the state of deformation of the body is symmetric relative to the axis, in such a way that one will obtain the same state by performing the radial fissure at another arbitrary axial section of the ring [for example, at the diametrically opposite place CC' (Fig. 1)] and then soldering the faces. In the second case, in order to obtain the same state of deformation by performing a distortion in the region that is diametrically opposite to $AA'BB'$, one must make a cut at CC' and interpose a cut of uniform thickness (*see* the principle of equivalent cuts; Chap. II, art. I, § 4).

Moreover, the distribution of efforts is entirely different in the two cases.

In the first case, if we examine the actions that AA' exerts on BB', after soldering, and if we compose all of the efforts of compression between them and then all of those of tension then find that the line of action of the resultant of the former efforts is situated towards to the interior region of the ring – i.e., the side AB – and the line of action of the other efforts, towards the exterior region – i.e., the side $A'B'$. By virtue of symmetry, one will find an analogous resultant in each axial section of the ring.

Indeed, we have found (*see* Chap. IV, art. II, § 4) that the resultant of the efforts of tension is equal to the intensity of that of the efforts of compression, but that the moment of the former resultant relative to the axis of symmetry exceeds the moment of the other resultant.

On the contrary, upon making a similar composition in the second, one finds that the line of action of the resultant of the efforts of compression that acts on the face BB', after soldering, is situated towards the exterior region of the ring – i.e., the side $A'B'$ – whereas the line of action of the efforts of tension is situated towards the interior region – i.e., the side AB. However, one finds the opposite to be true for the opposite side CC': There, the line of action of the resultant of the efforts of compressions is situated towards the interior side – i.e., towards C – and the line of action of the resultant of the tensions is towards the opposite side – i.e., C'.

Indeed, we have proved (Chap. IV, art. II, § 4) that in the case of a uniform cut, the efforts of tension at each transverse section of the ring exceed those of compression, and the resultants of the both of them meet orthogonally to the axis of symmetry of the ring.

The results that we just stated are easily deduced from either the principle of equivalent cuts or the law of composition of the efforts (*see* the preceding chapters). One will arrive at these results intuitively with difficulty, *a priori*; they seem unexpected to us. One may account for them by remarking that daily experience has accustomed us to imagining the deformations of the body when it is subject to known external efforts. However, in the present case, no external effort is imposed upon the elastic body. The efforts that it experiences are internal, and, so to speak, hidden from the observer, in such a way that they, just like the deformation, appear to be unknowns of the problem.

2. In order to have experimental conformation of some of the results that were obtained, I have experimented on solids made of rubber, with which it is easy to obtain very reasonable deformations.

In order to make a comparison between the results of calculation and experiment, in this chapter, I will commence by going into the first example that was developed in Chapter I in detail – i.e., the case that corresponds to the distortion of order 6 (*see* Chap. IV) – which is due to a radial fissure in a hollow cylinder of revolution, a case that presents less difficulties from the analytical viewpoint. The distortions of other orders will be examined in the following chapters.

II.

1. Formulas (a) of Chapter I, in which one supposes that $\gamma = 0$, express the displacements that correspond to a distortion of order 6 (radial fissure) when the cylinder is subject to uniform actions along the cylindrical surfaces that comprise the lateral contour of the body and tensions that act upon the base, respectively. One easily eliminates the former by composing the displacements (2) that were deduced in said chapter with the displacements:

$$u = \lambda\,\frac{x}{r^2} + \mu\,x, \qquad v = \lambda\,\frac{y}{r^2} + \mu\,y, \qquad w = 0,$$

and by choosing the constants λ and μ suitably.

Upon doing this, one arrives at the displacements:

$$\text{(I)}\qquad \begin{cases} u = -\alpha\left[y\arctan\dfrac{y}{x} - \dfrac{1}{2}\dfrac{K}{L+2K}\,x\log r^2 + \dfrac{L+K}{L+2K}R_1^2R_2^2\,\dfrac{\log R_1^2 - \log R_2^2}{R_1^2 - R_2^2}\,\dfrac{x}{r^2}\right. \\ \qquad\left. + \dfrac{x}{2}\left(1 + \dfrac{K}{L+2K}\,\dfrac{R_1^2\log R_1^2 - R_2^2\log R_2^2}{R_1^2 - R_2^2}\right)\right], \\ v = -\alpha\left[x\arctan\dfrac{y}{x} - \dfrac{1}{2}\dfrac{K}{L+2K}\,y\log r^2 + \dfrac{L+K}{L+2K}R_1^2R_2^2\,\dfrac{\log R_1^2 - \log R_2^2}{R_1^2 - R_2^2}\,\dfrac{y}{r^2}\right. \\ \qquad\left. + \dfrac{y}{2}\left(1 + \dfrac{K}{L+2K}\,\dfrac{R_1^2\log R_1^2 - R_2^2\log R_2^2}{R_1^2 - R_2^2}\right)\right], \end{cases}$$

which correspond to the hypothesis of a distortion (of order 6) that is due to a radial fissure whose angular opening is $2\pi\alpha$, while the only action that acts upon the body is caused by just two actions that act upon the two bases. These actions keep the aforementioned bases planar and at the original distance ([1]).

One can easily calculate the six characteristics of tension (i.e., the strains) that correspond to the displacements (I), and one has:

(1) $t_{11} =$

$$\frac{\alpha(L+K)K}{L+2K}\left[\log r^2 + \frac{2y^2}{r^2} - \frac{R_1^2R_2^2(\log R_1^2 - \log R_2^2)}{R_1^2 - R_2^2}\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) - \frac{R_1^2\log R_1^2 - R_2^2\log R_2^2}{R_1^2 - R_2^2}\right],$$

(2) $t_{22} =$

([1]) In these formulas, as in all of the following ones, the logarithms are Napierian ones.

$$\frac{\alpha(L+K)K}{L+2K}\left[\log r^2 + \frac{2x^2}{r^2} - \frac{R_1^2 R_2^2(\log R_1^2 - \log R_2^2)}{R_1^2 - R_2^2}\left(\frac{1}{r^2} - \frac{2y^2}{r^4}\right) - \frac{R_1^2 \log R_1^2 - R_2^2 \log R_2^2}{R_1^2 - R_2^2}\right],$$

(3) $$t_{33} = \frac{\alpha LK}{L+2K}\left(1 + \log r^2 - \frac{R_1^2 \log R_1^2 - R_2^2 \log R_2^2}{R_1^2 - R_2^2}\right),$$

(4) $$t_{23} = 0,$$

(5) $$t_{31} = 0,$$

(6) $$t_{12} = -\frac{2\alpha(L+K)K}{L+2K}\frac{xy}{r^2}\left[1 - \frac{R_1^2 R_1^2(\log R_1^2 - \log R_2^2)}{R_1^2 - R_2^2}\frac{1}{r^2}\right].$$

The equalities $t_{23} = 0$ and $t_{31} = 0$ prove that the forces act normally to the bases.

That action on the bases, when referred to the unit of surface, must be considered to be positive when it is directed from the exterior to the interior of the cylinder and negative in the opposite case. Here is its expression:

(II) $$P_\omega = t_{33} = \frac{\alpha LK}{L+2K}\left(1 + \log r^2 - \frac{R_1^2 \log R_1^2 - R_2^2 \log R_2^2}{R_1^2 - R_2^2}\right).$$

We thus have the following theorem:

A hollow cylinder of revolution that is subject to a distortion (of order 6) that is due to a radial fissure of opening $2\pi\alpha$ keeps its bases planar and at the original distance with the aid of normal forces that act upon the same bases. These forces are given by the preceding formula (II), *in which R_1 and R_2 represent the radii of the lateral cylindrical surfaces and r, the distance from the axis to the various points of the bases.*

2. Having said that, we calculate the actions that are exerted on the elements of the section σ of the cylinder. That section is made by half of a plane that is detached from the axis of the cylinder and which forms the angle β with the *xz*-plane.

Equations (1), (2), and (6) immediately provide the components along the axes of the unitary action relative to each element of the section. These components are:

$$-F\sin\beta, \qquad F\cos\beta, \qquad 0,$$

in which:

$$F = -\frac{2\alpha(L+K)K}{L+2K}\left[1 + \log r - \frac{R_1^2 R_2^2(\log R_1 - \log R_2)}{R_1^2 - R_2^2}\frac{1}{r^2} - \frac{R_1^2 \log R_1 - R_2^2 \log R_2}{R_1^2 - R_2^2}\right].$$

This proves that each element of σ is acted upon by a normal force F of unit magnitude.

An elementary calculation gives us:

$$\int_{R_2}^{R_1} F\,dr = 0.$$

It then follows that one finds a zero resultant force upon composing all of the actions that are exerted on the elements of σ.

This result verifies the general theorem that was proved in the preceding chapter, § 6 in the particular case that we treated.

Indeed, it proves that the sum of the compressions that act upon σ is equal in absolute value to the sum of the tensions. This condition must obviously continue to persist, even when we no longer act upon the bases of the hollow cylinder with forces P_ω.

One can easily calculate the moment of the actions that act upon the elements of σ with respect to the axis z. Upon denoting the height of the cylinder by h, this moment will be:

$$h\int_{R_2}^{R_1} rF\,dr = -\frac{2\alpha(L+K)K}{L+2K}\left[\frac{R_1^2-R_2^2}{4}-\frac{R_1^2R_2^2(\log R_1-\log R_2)}{R_1^2-R_2^2}\right]h.$$

3. Set:

$$f(r) = 1 + \log r - \frac{R_1^2R_2^2(\log R_1-\log R_2)}{R_1^2-R_2^2}\frac{1}{r^2}-\frac{R_1^2\log R_1-R_2^2\log R_2}{R_1^2-R_2^2}. \tag{7}$$

The function $f(r)$ is increasing, and since:

$$\int_{R_2}^{R_1} f(r)\,dr = 0,$$

in then results that equation:

$$f(r) = 0$$

has just one root ρ_1 between R_2 and R_1 , and that $f(r)$ is negative for the values of r that are between R_1 and ρ_1, and positive for values between ρ_1 and R_1 .

This proves that the circular fibers of the cylinder that have the axis of the cylinder for their axis, and whose radius is between R_1 and ρ_1 are compressed, while the ones whose radius is between ρ_1 and R_1 are in a state of tension. The neutral fibers have radius ρ_1 .

From the equation $f(r) = 0$, one deduces that:

$$\log\frac{\rho_1}{\sqrt{R_1R_2}} = \frac{\dfrac{R_1}{R_2}\log\dfrac{R_1}{R_2}}{\dfrac{R_1^2}{R_2^2}-1}\,\frac{R_1R_2}{\rho_1^2}+\frac{\dfrac{R_1}{R_2}\log\sqrt{\dfrac{R_1}{R_2}}-\log\sqrt{\dfrac{R_2}{R_1}}}{\dfrac{R_1^2}{R_2^2}-1}-1,$$

and upon setting:

$$\frac{R_1}{R_2} = \varepsilon, \qquad \log\frac{\rho_1}{\sqrt{R_1 R_2}} = \varphi,$$

we will have:

(8) $$\varphi = \frac{\varepsilon\log\varepsilon}{\varepsilon^2 - 1}e^{-2\varphi} + \frac{1}{2}\frac{\varepsilon^2+1}{\varepsilon^2-1}\log\varepsilon - 1.$$

If we let:

$$\gamma = 1 - \frac{1}{\varepsilon} = \frac{R_1 - R_2}{R_1}$$

then we will have $0 < \gamma < 1$. In the expression (8), we develop the coefficient of $e^{-2\varphi}$ and the successive terms in a series in increasing powers of γ. We get:

$$\varphi(\gamma) = \frac{1}{2}\left(1 - \frac{1}{6}\gamma^2 + \cdots\right)e^{-2\varphi} - \frac{1}{2}\left(1 - \frac{1}{3}\gamma^2 + \cdots\right),$$

so one has:

$$\varphi(0) = 0, \qquad \varphi'(0) = 0, \qquad \varphi''(0) = \frac{1}{12},$$

which makes:

$$\varphi(\gamma) = \frac{1}{24}\gamma^2 + \ldots$$

and

$$\frac{\rho_1}{\sqrt{R_1 R_2}} = e^{\varphi(\gamma)} = 1 + \frac{1}{24}\gamma^2 + \ldots,$$

i.e., upon neglecting the powers of $\frac{R_1 - R_2}{R_1}$ that are higher than the second, one gets:

$$\rho_1 = \sqrt{R_1 R_2}\left[1 + \frac{1}{24}\left(\frac{R_1 - R_2}{R_1}\right)^2\right].$$

If the thickness of the hollow cylinder is small compared to the radius and if the ratio of the thickness to the exterior radius is envisioned to be a first-order quantity then the radius of the neutral fibers will thus be:

$$\rho_1 = \sqrt{R_1 R_2},$$

if one neglects the second-order quantities.

4. From formula (7) and the equation:

$$0 = 1 + \log \rho_1 - \frac{R_1^2 R_2^2 (\log R_1 - \log R_2)}{R_1^2 - R_2^2} \frac{1}{\rho^2} - \frac{R_1^2 \log R_1 - R_2^2 \log R_2}{R_1^2 - R_2^2},$$

one deduces that:

$$f(r) = \log \frac{r}{\rho_1} + \frac{R_1^2 R_2^2 \log \dfrac{R_1}{R_2}}{R_1^2 - R_2^2} \left(\frac{r^2 - \rho_1^2}{r^2 \rho_1^2} \right).$$

Consider $(R_1 - R_2) / R_1$ to be a very small quantity of the first order. By some simple calculations, upon setting $r = \rho_1 + \xi$ and neglecting the second-order quantities, we obtain:

$$f(r) = \frac{2\xi}{\sqrt{R_1 R_2}} \left[1 - \frac{1}{4} \frac{R_1 - R_2}{\left(\dfrac{R_1 + R_2}{2} \right)} \right].$$

This is why:

$$F = -\frac{2\alpha(L+K)K}{L+2K} \frac{2\xi}{\sqrt{R_1 R_2}} \left[1 - \frac{1}{4} \frac{R_1 - R_2}{\left(\dfrac{R_1 + R_2}{2} \right)} \right].$$

Let E be the modulus of elasticity and let η be the Poisson coefficient. We will have:

$$\frac{(L+K)K}{L+2K} = -\frac{E}{4(1-\eta^2)},$$

and if we let θ denote the angular opening of the radial cut then α will be equal to $\theta / 2\pi$; therefore:

(III) $$F = \frac{E}{4(1-\eta^2)} \frac{\theta}{2\pi} \frac{\xi}{\rho} \left(1 - \frac{1}{4} \frac{s}{\rho} \right),$$

where ρ is the (arithmetic or geometric) mean of the radius and s is the difference between the radii – i.e., the thickness of the hollow cylinder.

5. We now pass on to the examination of the law of distribution of the forces P_ω on the bases of the cylinder.

Set:

(9) $$\psi(r) = 1 + \log r^2 - \frac{R_1^2 \log R_1^2 - R_2^2 \log R_2^2}{R_1^2 - R_2^2};$$

by an elementary calculation, one proves that:

(10) $$\int_{R_2}^{R_1} r\,\psi(r)\,dr = 0,$$

and it follows that the equation $\psi(r) = 0$ must have a root between R_2 and R_1, and since $\psi(r)$ is an increasing function, that root will be unique. If we call it ρ_2 then we will have that $\psi(r)$ will be negative if the variable r is between R_2 and ρ_2 and positive if r is between ρ_2 and R_1 .

It is easy to show that $\rho_2 > (R_1 + R_2) / 2$.

Indeed, we have:

(11) $$1 + \log \rho_2^2 - \frac{R_1^2 \log R_1^2 - R_2^2 \log R_2^2}{R_1^2 - R_2^2} = 0,$$

so:

$$\log \frac{2\rho_2}{R_1 + R_2} = \frac{R_2^2 \log \frac{R_1}{R_2}}{R_1^2 - R_2^2} + \log \frac{R_1}{R_1 - R_2} - \frac{1}{2} + \log 2,$$

and upon setting:

$$\frac{R_1}{R_2} = \varepsilon, \qquad \log \frac{2\rho_2}{R_1 + R_2} = \chi(\varepsilon),$$

it then follows that:

$$\chi'(\varepsilon) = \frac{\varepsilon - \frac{1}{\varepsilon} - 2\log\varepsilon}{\frac{1}{\varepsilon}(\varepsilon^2 - 1)^2} = \frac{\varpi(\varepsilon)}{\frac{1}{\varepsilon}(\varepsilon^2 - 1)^2}.$$

If one writes, as we did previously (§ 3), $\gamma = 1 - 1/\varepsilon$, and if one develops the logarithm, and then $\chi'(\varepsilon)$ in a series in powers of γ then one will have:

$$\chi'(\varepsilon)_{\varepsilon=1} = 0, \qquad \chi(\varepsilon)_{\varepsilon=1} = 0.$$

However:

$$\varpi'(\varepsilon) = \left(1 - \frac{1}{\varepsilon}\right)^2,$$

which is why $\varpi(\varepsilon)$, and consequently $\chi(\varepsilon)$, are positive for $\varepsilon > 1$. Therefore, $\chi(\varepsilon)$ is an increasing function for $\varepsilon > 1$. However, $\lim_{\varepsilon=1} \chi(\varepsilon) = 0$, so:

$$\log \frac{2\rho_2}{R_1 + R_2} > 0,$$

or rather:

$$\rho_2 > \frac{R_1 + R_2}{2}.$$

Since $\chi(\varepsilon)_{\varepsilon=1} = 0$, one deduces from formula (12) that:

$$\chi(\varepsilon) = \frac{1}{24}\gamma^2 + \ldots,$$

and, in turn:

$$\frac{2\rho_2}{R_1 + R_2} = e^{\frac{1}{24}\gamma^2+\cdots} = 1 + \frac{1}{24}\gamma^2 + \ldots,$$

so

$$\rho_2 = \frac{R_1 + R_2}{2}\left[1 + \left(\frac{R_1 - R_2}{2}\right)^2\right],$$

if we neglect powers of $(R_1 + R_2) / 2$ that are higher than the second in the expression for ρ_2 .

We may thus conclude that the circle that separates the region in a state of tension from the compressed region at each base point has a radius that is the arithmetic or geometric mean of the extreme radii, up to second-order quantities.

6. From equations (9) and (11), it then results that:

$$\psi(r) = 2 \log \frac{r}{\rho_2},$$

so, for formula (II), one has:

$$P_\omega = \frac{2\alpha LK}{L+2K}\log\frac{r}{\rho_2},$$

and upon introducing the modulus of elasticity, the Poisson coefficient, and the angular opening θ of the radial fissure (*see* § 4), one sets:

$$\text{(II}'\text{)} \qquad P_\omega = -\frac{E\eta}{1-\eta^2}\frac{\theta}{2\pi}\log\frac{r}{\rho_2}.$$

One infers from this that in order to keep the two bases of the cylinder planar and at the original distance one must compress them in the region between these circles of radius R_2 and ρ_1 and stretch them in the region between the circles of radius ρ_2 and R_1 .

From the formula (10), one deduces that the algebraic sum of all the forces that act upon a radial band that is found on one of the bases is equal to zero; i.e., the resultant of the tensions has the same intensity as the resultant of the pressures. The totality of all the forces that act upon the radial band is then equivalent to a couple.

Upon setting $r = \rho_2 + \xi$, we will have $| \xi / \rho_2 | < 1$ when $\rho_2 > (R_1 + R_2) / 2$ (*see* § 5). It will thus be possible to develop the function $\log r / \rho_2$ in a series in powers of ξ / ρ_2, and formula (II′) will be written:

$$P_\omega = -\frac{E\eta}{1-\eta^2}\frac{\theta}{2\pi}\left(\frac{\xi}{\rho_2} - \frac{1}{2}\frac{\xi^2}{\rho_2^2} + \frac{1}{3}\frac{\xi^3}{\rho_3^2} + \cdots\right).$$

If the thickness of the cylinder is small then upon neglecting terms of second order, we will have:

$$P_\omega = -\frac{E\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{\xi}{\rho_2}. \tag{II″}$$

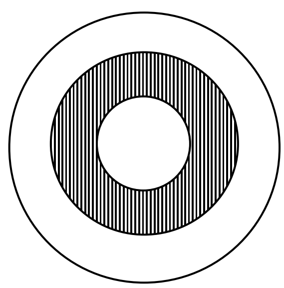
III.

1. Now suppose that the two bases of the cylinder are no longer subject to the actions P_ω, but are left free, and look for the form that the cylinder will take by virtue of only the distortion when no external force acts upon it.

To that effect, it will suffice to apply the general principles that we stated in Chapter II, article I, paragraph 2 – i.e., that one must superpose the deformation (I) with that one that is due to the forces $-P_\omega$ that act upon the bases of the cylinder. However, the deformation (I) preserves the form of the body as a cylinder of revolution, so it will suffice to examine the deformation that a cylinder undergoes when subjected to actions $-P_\omega$ on the two bases.

Figure 3 represents one of the bases of the cylinder.

Figure 3.

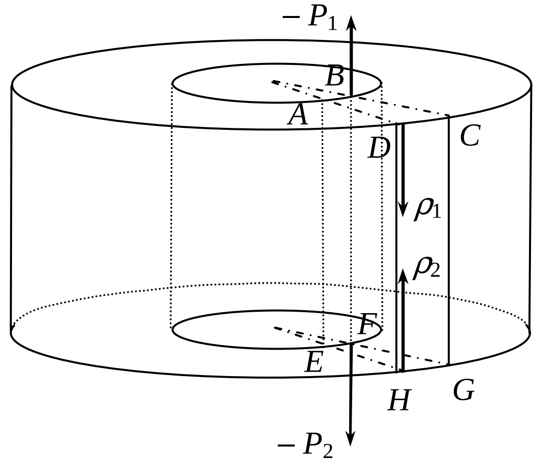


The large circle and the small circle are the two boundaries of the base; the patterned circle is the line of separation between the region that must be in a state of tension under the forces $-P_\omega$ (it has been lined) from the region that must be compressed by the forces $-P_\omega$ (it has been left white).

Now consider (Fig. 4) an infinitely small longitudinal slit *ABCDEFGH* in the hollow cylinder and imagine that is has been detached from the rest of the body. From what we found in paragraph 6 of the preceding article, the sum of the compressions that act upon the upper base *ABCD* will be equal to the sum of the tensions; the same will be true for the lower base *EFGH*, so the two bases will be acted upon by the couples $P_1, -P_1$; $P_2, -P_2$, respectively.

It then follows that the slit bends in such a manner that the generators of the face *DCGH* curve and take on a concave form. The generators of the face *ABFG* likewise curve, but become convex. At the same time, the region of the upper base *ABCD* that is adjacent to *AB* will be raised and the region of the same base that is adjacent to *CD* will be lowered. The inverse is true for the lower base.

Figure 4.



It is easy to calculate the lift, the lowering, and the bending of flexion relative to the slit, when considered by means of the usual formulas of flexure. We refer to the plane that is normal to the axis that leads through the medium with the same axis; we will have:

Lifting of the points on the upper base:

(13)
$$w' = -\frac{1}{E}(-P_\omega)\frac{h}{2} = -\frac{\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{\xi h}{2\rho_2}.$$

Lowering of the points of the lower base:

(13′)
$$w'' = -\frac{1}{E}(-P_\omega)\frac{h}{2} = -\frac{\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{\xi h}{2\rho_2}.$$

Bending of flexion:

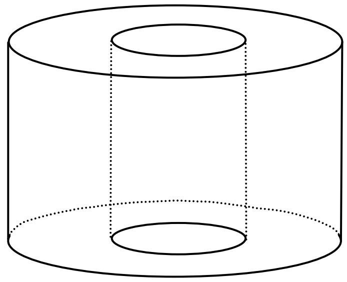
(14)
$$g = -\frac{P_\omega}{E\xi}\frac{h^2}{8} = \frac{\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{h^2}{8\rho_2},$$

where h represents the height of the cylinder.

One will get the same result for any other infinitely thin longitudinal band on the cylinder if one assumes that it is separated from the rest of the body. The mutual couplings between the various bands will change the aforementioned raising and lowering if one repeats them and, above all, reduces the bending of flexion; however, the process of deformation will obviously remain unaltered and the corrections that must be made to the values that are found will be even smaller such that the cylinder will be lower and the thickness, relative to the radii of the bases, will be smaller ([1]).

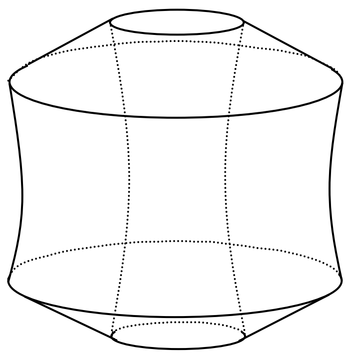
([1]) One may obtain this result with great ease; it suffices to formulate the problem in an equation by means of the equations of elasticity, when transformed into cylindrical coordinates.

Figure 5.



The original cylinder, which is represented by Figure 5, will then, by virtue of the distortion, take on the form that is represented by Figure 6, where we have exaggerated the deformations in order to make them more visible.

Figure 6.



According to formulas (13), (13′), (14), and upon taking $\rho_2 = (R_1 + R_2) / 2$, the total height of the lateral surface that bounds the solid internally, after distortion, will be equal to:

$$h + \frac{\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{R_1 - R_2}{R_1 + R_2}h,$$

and the total height of the lateral surface that bounds the solid externally will become:

$$h - \frac{\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{R_1 - R_2}{R_1 + R_2}h.$$

Therefore, the difference between the heights of the two surfaces that bound the interior and exterior of the cylinder will be:

$$H = \frac{2\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{R_1 - R_2}{R_1 + R_2}h, \tag{15}$$

and the bending of flexion will be:

$$g = \frac{2\eta}{1-\eta^2}\frac{\theta}{2\pi}\frac{h^2}{4(R_1+R_2)}. \tag{16}$$

2. I have done experiments with a hollow rubber cylinder with the following dimensions:

$$R_1 = 28 \text{ mm}, \quad R_2 = 12 \text{ mm}, \quad h = 28 \text{ mm},$$

and I made a radial cut of $68^{\circ}\,30'$.

After the distortion, all of the peculiarities that were predicted by the calculations were manifested: The difference between the heights of the surfaces that bounded the solid internally and externally were measured and found to be equal to 2.1 mm and the length of the bending of flexure was 0.35 mm. After having done the calculations by means of formulas (15) and (16) and taking $\eta = 1/2$, I obtained:

H = difference between the heights = 2.6 mm,
g = bending of flexion = 0.53 mm.

The agreement between the calculations and the direct measurements is therefore very satisfactory.

Jona, an engineer at the Pirelli establishment in Milan, has kindly prepared a hollow rubber cylinder with the following dimensions:

$$R_1 = 5 \text{ cm}, \qquad R_2 = 2.95 \text{ cm}, \quad h = 13 \text{ cm}.$$

He made a radial cut with an angular magnitude of 78°.

Since the soldering tended to open it up, in order to fix the form of the deformed solid, I made a plaster cast, which is reproduced photographically in Figure 7 (*).

This solid obviously showed all of the peculiarities that the calculations predicted – i.e., the internal elongation, the external shrinking, and the lateral flexion – as is indicated in Figure 8. In order to make the phenomena more obvious, the photograph was taken with a set square resting against the left side of the cylinder. In figures 7 and 8, one sees quite well the places where the cuts and then the solders were made.

Figure 9 represents a photograph of the plaster cast of the core of the cylinder. Having placed a ruler on the left side, the internal curvature becomes clearly visible.

Due to the great height of the cylinder when compared with the radius of the base, formulas (15) and (16) are not applicable in this case.

(*) Translator’s note: The photographs in original article (Figures 7, 8, 9) were not reproduced in this translation.

CHAPTER VI

HOLLOW CYLINDER OF REVOLUTION – DISTORTION OF ORDER 2

I.

1. In the preceding chapter, I began by pointing out (art. 1, § 1) the essentially different conditions that present themselves when one imposes a distortion that is due to a radial cut (distortion of order 6) or a uniform cut (distortion of order 1) on a hollow cylinder. I then expanded upon the former case and showed that the body, after distortion, does not preserve its cylindrical form: The internal boundary of the two bases is swollen and lifted, while the external boundary is contracted and the middle part of the cylinder has shrunk (*see* Figures 6, 7, 8, 9 of the preceding chapter). The deformations that are produced in the case of a uniform cut are more reasonable and also more singular, since the body ceases to be symmetric after the distortion. I propose to develop this case in the present chapter, in spite of the fact that the calculations are very complicated. Since the results that were predicted by calculation in the preceding case are quite well confirmed by experiment, the case constitutes an instructive example in the domain of elasticity. Indeed, mere intuition, unguided by calculation or experiment, cannot predict, even in a gross, qualitative manner, what the deformation will be that is produced in the body by the distortion.

One will thus arrive at the following very curious result: If one suppresses a slit in a symmetric ring that has the form of a hollow cylinder then it is impossible to preserve the cylindrical form of the ring after soldering the faces of the cut. Indeed, if one makes a radial cut then the body takes the form that was indicated in Figure 6 of the preceding chapter. If the cut is uniform then the body ceases to be symmetric and takes the form that is indicated in Figure 6 of the present chapter.

By means of a cut that one may consider to be the result of a cut or radial corner and a cut or corner with parallel faces, one may always arrive at a state of deformation where the symmetry with respect to the axis is lost.

In practice, the metalworkers that must narrow down a tube in which there is a slit first make a radial cut. Then, after bringing the faces of the cut together, they file the internal part in such a fashion as to make the one fit onto the other one exactly and with as little effort as is possible ([1]). Finally, they solder them. But then, since the cut is no longer radial, the tube does not keep the form of a solid of revolution.

([1]) In order for us to get an idea of the magnitude of these actions, suppose that our hollow cylinder is a symmetric steel ring with a rectangular section whose mean diameter is 5 cm and whose thickness is 1 cm. Apply formula (III) of the preceding chapter upon taking E = 19459 (kg. per square millimeter; Wertheim), η = 0.3, ρ = 2.5, s = 1. Likewise, take $\theta / 2\pi$ = 1 / 360 (upon supposing that the angular amplitude of the radial fissure is 1°), ξ = 0.5, in order to calculate the pressure in the regions that are adjacent to the external surface. One will then obtain F = 10.7; i.e., the calculated pressure or tension will be 10.7 kg. per square millimeter, and for each degree of angular amplitude of the radial fissure that was made in the ring. One obtains these efforts when one supposes that the bases are acted upon by actions that keep them planar and at the original distance. Upon calculating these actions by means of formulas (II″) of the preceding chapter, one finds that they take the value 3.6 kg. per square millimeter at the boundaries of the bases.

2. We recall Figure 2 of the preceding chapter and seek the formulas that relate to the uniform fissure.

Suppose that the z-axis is the symmetry axis and that the cut has been performed along the *xz*-plane on the positive *x*-axis. Upon setting:

$$l = n = p = q = r = 0$$

in the formulas of paragraph 2 of article III of Chapter II we will have:

$$(1) \qquad u = -\frac{1}{2}\frac{m}{2\pi}\log(x^2 + y^2), \qquad v = \frac{m}{2\pi}\arctan\frac{y}{x}.$$

These formulas correspond to the uniform cut of size m. Meanwhile, the body will be subject to surface tensions that equilibrate between themselves (*see* art. I of Chap II).

Represent the internal radius and external radius of the hollow cylinder that constitutes the ring by R_1 and R_2, respectively. By a simple calculation, we find the six characteristics of deformations and the tensions, which will be zero on the two bases, while the unit tensions that act upon the lateral surfaces will be parallel to the x-axis and equal to $Km / \pi R_1$ on the external surface and $-Km / \pi R_2$ on the internal surface, respectively.

One must now eliminate these lateral tensions. One can go about achieving this goal in the following manner: Abstract from the z coordinate and in place of the body in question, substitute an elastic strip (*lame*) that is bounded by two circles of radius R_2 and R_1. Begin by eliminating the tensions that act upon the internal circumference C_2. For this, suppose that the strip is not bounded by the external circumference C_1, but that is extends indefinitely in all directions externally to C_2. The question is then presented in a manner that is perfectly analogous to a problem in an elastic medium that is external to a sphere that I solved in a course that I gave in Pisa in 1893 and that Professor Tedone ([1]) has recently taught again.

In other words, we will eliminate the tensions in C_2 if we compose the displacements (1) with the displacements:

$$(2) \qquad \begin{cases} u' = \dfrac{m}{2\pi}\dfrac{L+3K}{L+2K}\left[\log r + \dfrac{L+K}{2(L+3K)}(r^2 - R_2^2)\dfrac{\partial^2 \log r}{\partial x^2}\right], \\[2ex] v' = \dfrac{m}{2\pi}\dfrac{L+K}{2(L+2K)}(r^2 - R_2^2)\dfrac{\partial^2 \log r}{\partial x\,\partial y}, \end{cases}$$

in which $r = \sqrt{x^2 + y^2}$ and L and K denote the elasticity constants, as in the preceding chapters.

However, upon composing the preceding tensions that act upon C_1 by virtue of the displacements (1) with the tensions that are generated on C_2 by the displacements (2), one finds tensions on C_1 that have the components:

([1]) *Comptes rendus du Cercle mathématique de Palermo*, t. XVII, 1903, pp. 259.

$$-\frac{mK(L+K)}{\pi(L+2K)}\frac{R_1^2-R_2^2}{R_1^2}\cos 2\theta,$$

$$-\frac{mK(L+K)}{\pi(L+2K)}\frac{R_1^2-R_2^2}{R_1^2}\sin 2\theta,$$

where $\theta = \arctan y / x$ – i.e., θ represents the angle that the radius vector forms with the x-axis.

Now, these latter tensions may be eliminated, either by means of the displacements:

$$\text{(3)}\qquad \begin{cases} u'' = \dfrac{2A}{4K(L+K)}[(3L+5K)y^2+(L-K)x^2], \\ v'' = -\dfrac{2A}{4K(L+K)}(L+3K)xy, \end{cases}$$

or by means the displacements:

$$\text{(3}'\text{)}\qquad \begin{cases} u'' = B\dfrac{\partial^2 \log r}{\partial x^2}, \\ v'' = B\dfrac{\partial^2 \log r}{\partial x\,\partial y}, \end{cases}$$

upon suitably choosing the constants A and B, or by a linear combination of these displacements.

We may now take advantage of the arbitrary character of that linear combination to make the resultant displacement of the displacements that are represented by formulas (1), (2), (3), (3′) generate zero tensions, not only on C_1, but also on the circle C_2. In this manner, one easily arrives at the formulas:

$$\text{(I)}\qquad \begin{cases} U = \dfrac{m}{2\pi}\left\{\dfrac{K}{L+2K}\log r + \dfrac{L+K}{2(L+2K)}\left(r^2 - \dfrac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\dfrac{\partial^2 \log r}{\partial x^2}\right. \\ \qquad\qquad \left. + \dfrac{1}{2(L+2K)(R_1^2+R_2^2)}[(3L+5K)y^2+(L+K)x^2]\right\}, \\ V = \dfrac{m}{2\pi}\left\{\arctan\dfrac{y}{x} + \dfrac{L+K}{2(L+2K)}\left(r^2 - \dfrac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\dfrac{\partial^2 \log r}{\partial x\,\partial y} - \dfrac{L+3K}{(L+2K)(R_1^2+R_2^2)}xy\right\}. \end{cases}$$

3. If the one adjoins:

$$\text{(I}'\text{)}\qquad W = 0$$

to the preceding formulas then one will obtain the components of the displacements that are due to a distortion that is generated by a uniform fissure of size *m*, under the hypothesis that the two bases are acted upon by forces that are capable of keeping them planar and at their original distance.

It is easy to calculate the characteristics of tension that correspond to these displacements. They are given by the following formulas:

$$
(4) \qquad t_{11} = \frac{mK}{\pi}\left\{\frac{L}{L+2K}x\left(\frac{1}{r^2}-\frac{2}{R_1^2+R_2^2}\right)+\frac{K}{L+2K}\frac{\partial \log r}{\partial x}\right.
$$

$$
\left. + \frac{L+K}{2(L+2K)}\left[2x\frac{\partial^2 \log r}{\partial x^2}+\left(r^2-\frac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\frac{\partial^2 \log r}{\partial x^2}\right]+\frac{L-K}{(L+2K)(R_1^2+R_2^2)}x\right\},
$$

$$
(5) \qquad t_{22} = \frac{mK}{\pi}\left\{\frac{L}{L+2K}x\left(\frac{1}{r^2}-\frac{2}{R_1^2+R_2^2}\right)+\frac{x}{x^2+y^2}\right.
$$

$$
\left. + \frac{L+K}{2(L+2K)}\left[2y\frac{\partial^2 \log r}{\partial x\,\partial y}+\left(r^2-\frac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\frac{\partial^2 \log r}{\partial x\,\partial y}\right]-\frac{L+3K}{(L+2K)(R_1^2+R_2^2)}x\right\},
$$

$$
(6) \qquad t_{33} = \frac{mKL}{\pi(L+2K)}x\left(\frac{1}{r^2}-\frac{2}{R_1^2+R_2^2}\right),
$$

$$
(7) \qquad t_{12} = \frac{mK}{2\pi}\left\{\frac{K}{L+2K}\frac{\partial \log r}{\partial y}-\frac{y}{x^2+y^2}+\frac{L+K}{2(L+2K)}\right.
$$

$$
\left.\times\left[2y\frac{\partial^2 \log r}{\partial x^2}+2x\frac{\partial^2 \log r}{\partial x\,\partial y}+2\left(r^2-\frac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\frac{\partial^3 \log r}{\partial x^2\,\partial y}\right]+\frac{2(L+K)}{(L+2K)(R_1^2+R_2^2)}y\right\},
$$

$$
(8) \qquad t_{23} = t_{31} = 0.
$$

From these formulas, one infers that:

$$
t_{11}\,x + t_{12}\,y = \frac{mK}{2\pi}\frac{L+K}{L+2K}\frac{(r^2-R_1^2)(r^2-R_2^2)}{R_1^2+R_2^2}\frac{\partial^2 \log r}{\partial x^2},
$$

$$
t_{21}\,x + t_{22}\,y = \frac{mK}{2\pi}\frac{L+K}{L+2K}\frac{(r^2-R_1^2)(r^2-R_2^2)}{R_1^2+R_2^2}\frac{\partial^2 \log r}{\partial x\,\partial y},
$$

which are quantities that are annulled for $r = R_1$, $r = R_2$. One thus verifies that the external actions are annulled on the lateral surfaces of the hollow cylinder.

One then has:

$$\Theta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{mK}{\pi(L+2K)}\,x\left(\frac{1}{r^2} - \frac{2}{R_1^2 + R_2^2}\right)$$

for the value of the cubic dilatation.

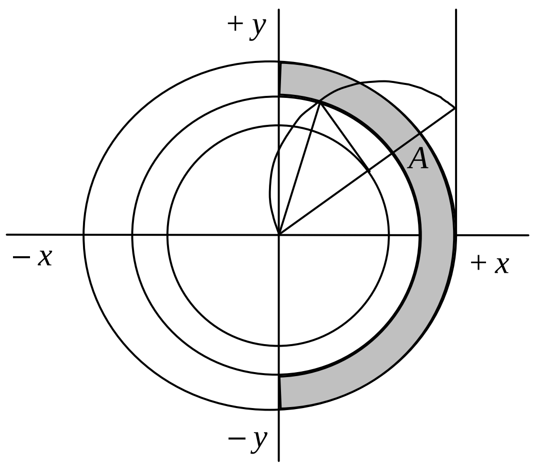
We may thus establish the division between the dilated part and the compressed part of the elastic body.

To that effect, draw the circle of radius:

$$r = \sqrt{\frac{R_1^2 + R_2^2}{2}}\,,$$

which is the intermediate circumference in Figure 10 that is found between the two extreme circumferences of radii R_1 and R_2 . Then, trace the y-axis.

Figure 10.



These two lines divide the circular rim (*couronne*) into four regions that we have distinguished in Figure 10 by white and dark shading. The white regions represent the projections onto the xy-plane of the dilated parts of the elastic body and the dark regions represent the projections on the same plane of the compressed parts.

In the figure, we have indicated the construction that must be made in order to obtain the intermediate circumference. It is sufficiently obvious that it requires no explanation.

II.

1. We now go on to the determination of the form that is taken by the elastic body after the distortion, while always assuming that the two bases are kept planar and at their original distance.

For this, it will suffice to see how the bases deform. By means of formulas (1), we may calculate the values of U and V on the circumferences σ_1 and σ_2 that define the original contour of the two bases and have radii R_1 and R_2, respectively.

Upon representing these values by the same letters U and V, to which we add the indices σ_1 and σ_2, we will have:

$$U_{\sigma_1} = \frac{m}{2\pi}\left(\frac{K}{L+2K}\log R_1 + \frac{L+K}{L+2K}\frac{R_1^2}{R_1^2+R_2^2} - \frac{R_1^2}{R_1^2+R_2^2}\cos 2\theta\right),$$

$$V_{\sigma_1} = \frac{m}{2\pi}\left(\theta - \frac{R_1^2}{R_1^2+R_2^2}\sin 2\theta\right),$$

$$U_{\sigma_2} = \frac{m}{2\pi}\left(\frac{K}{L+2K}\log R_2 + \frac{L+K}{L+2K}\frac{R_2^2}{R_1^2+R_2^2} - \frac{R_2^2}{R_1^2+R_2^2}\cos 2\theta\right)$$

$$V_{\sigma_2} = \frac{m}{2\pi}\left(\theta - \frac{R_2^2}{R_1^2+R_2^2}\sin 2\theta\right).$$

The displacements U_{σ_1} and V_{σ_1} can be decomposed into three elementary displacements (a), (b), (c) that have the components:

$$(a) \qquad \begin{cases} U'_{\sigma_1} = \dfrac{mK}{2\pi(L+2K)}\left(\log R_1 - \dfrac{R_1^2}{R_1^2+R_2^2}\right) \\ V'_{\sigma_1} = 0, \end{cases}$$

$$(b) \qquad \begin{cases} U''_{\sigma_1} = \dfrac{m}{\pi}\dfrac{R_1^2}{R_1^2+R_2^2}\sin^2\theta, \\ V''_{\sigma_1} = -\dfrac{m}{\pi}\dfrac{R_1^2}{R_1^2+R_2^2}\sin\theta\cos\theta, \end{cases}$$

$$(c) \qquad \begin{cases} U'''_{\sigma_1} = 0, \\ V'''_{\sigma_1} = \dfrac{m}{2\pi}\theta, \end{cases}$$

respectively.

The first displacement (a) consists of a translation parallel to the x-axis and consequently it does not change the form of the circumference σ_1.

One then has, in turn:

$$U''_{\sigma_1}\cos\theta + V''_{\sigma_1}\sin\theta = 0,$$

$$U''_{\sigma_1}\sin\theta - V''_{\sigma_1}\cos\theta = \frac{m}{\pi}\frac{R_1^2}{R_1^2+R_2^2}\sin\theta.$$

This proves that the under the second displacement (b) each point of the circumference σ_1 is displaced tangentially to the same circumference by the quantity:

$$\frac{m}{\pi}\frac{R_1^2}{R_1^2+R_2^2}\sin\theta.$$

Under the second displacement, the points of the circumference σ_1 always remain on it, up to second-order quantities that one can neglect.

By virtue of the third displacement (c), each point of the circumference σ_1 moves parallel to the y-axis by a quantity that is proportional to the arc of the circle σ_1 that is between the origin of the arcs and the point itself.

One then sees that if one neglects the second-order quantities then the form of the circle σ_1 after deformation will be obtained by taking only the displacement (c) into account.

One may decompose the displacements U_{σ_1}, V_{σ_1} in an analogous manner. One thus obtains the following three elementary displacements:

$$(a') \qquad \begin{cases} U'_{\sigma_1} = \dfrac{mK}{2\pi(L+2K)}\left(\log R_2 - \dfrac{R_2^2}{R_1^2+R_2^2}\right), \\ V'_{\sigma_1} = 0, \end{cases}$$

$$(b') \qquad \begin{cases} U''_{\sigma_1} = \dfrac{m}{\pi}\dfrac{R_2^2}{R_1^2+R_2^2}\sin^2\theta, \\ V''_{\sigma_1} = -\dfrac{m}{\pi}\dfrac{R_2^2}{R_1^2+R_2^2}\sin\theta\cos\theta, \end{cases}$$

$$(c') \qquad \begin{cases} U'''_{\sigma_1} = 0, \\ V'''_{\sigma_1} = \dfrac{m}{2\pi}\theta. \end{cases}$$

In order to obtain the form that is taken by σ_2 after deformation, one can neglect the displacements (a') and (b') and take into account only the third displacement (c'), which is perfectly analogous to the preceding displacement (c).

The two displacements (a) and (a') consist of two translations. Their difference will be:

$$\delta = \frac{mK}{2\pi(L+2K)}\left(\log\frac{R_1}{R_2} - \frac{R_1^2-R_2^2}{R_1^2+R_2^2}\right).$$

Upon setting $(R_1 - R_2) / R_1 = \gamma$ and developing the preceding expression in powers of γ, one will obtain:

$$\delta = \frac{mK}{2\pi(L+2K)}\left(\frac{1}{3}\gamma^2 + \cdots\right);$$

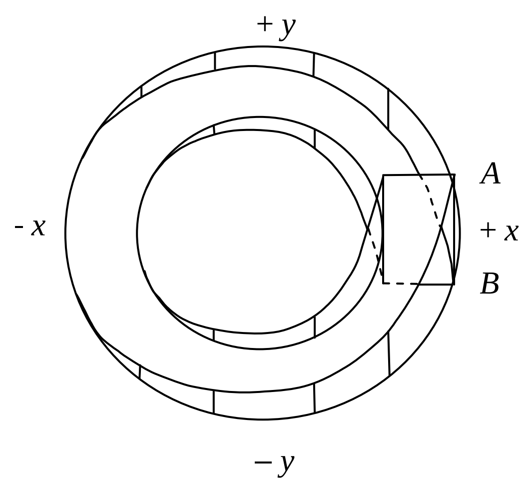
i.e., upon introducing the modulus of elasticity E and the Poisson coefficient η (*see* the preceding chapter, § 6), one will have:

$$\delta = \frac{m}{2\pi}\frac{(1-2\eta)}{2(1-\eta)}\left(\frac{1}{3}\gamma^2 + \cdots\right);$$

therefore, if the thickness of the ring is small relative to its external radius then the difference δ of the two translations will be negligible.

In Figure 11, we have constructed the contour of the deformed bases by taking the origin of the arcs of the two circles σ_1 and σ_2 to be their intersection with the negative side of the x-axis. The two circumferences that are represented by thin lines are the original contours of the two bases. The two thicker lines represent the contours of the deformed bases. The rectilinear lines are the displacements that the points of the contour have experienced by virtue of the displacements (c) and (c'). The line AB represents the size of the cut. The difference δ was neglected.

Figure 11.



2. Formula (6) gives the characteristic t_{33} . Upon introducing the modulus of elasticity and the Poisson coefficient it will be written:

$$t_{33} = -\frac{m}{2\pi}\frac{E\eta}{1-\eta^2}x\left(\frac{1}{r^2} - \frac{2}{R_1^2 + R_2^2}\right).$$

Upon taking this formula into account, along with the preceding results, we can state the following theorem:

A hollow cylinder of revolution that has experienced a distortion (of order 2*) that is due to a uniform fissure keeps its bases planar and at their original distance by subjecting them to normal forces that are given by:*

$$P = -\frac{m}{2\pi}\frac{E\eta}{1-\eta^2}x\left(\frac{1}{r^2}-\frac{2}{R_1^2+R_2^2}\right),$$

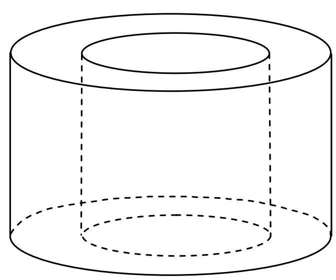
where one imagines the actions that are directed towards the interior of the body to be positive and ones that are directed in the opposite sense to be negative. At the same time, the bases deform according to the previously established laws (see Fig. 11).

Figure 10 can thus be interpreted in another manner: Upon supposing that the circular rim represents one of the bases in the original form, the shaded region will represent the part of the base that will be compressed on the outer side after the distortion and the white region will indicate the part that must be stretched to the outside in order to keep the bases planar and at the original distance.

III.

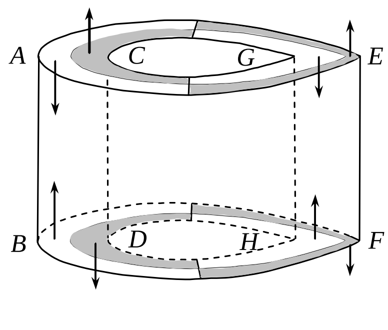
4. Figure 12 represents the cylinder before the distortion and Figure 13 represents the same cylinder after the distortion when the bases are kept planar and at the original distance.

Figure 12.



The bases themselves are divided into four regions that are white and shaded, respectively. The shaded regions are the ones that are compressed from the outside and the white regions are the ones that are stretched. The sense of these external actions is obtained by inverting the direction of the arrows that are traced in the figure.

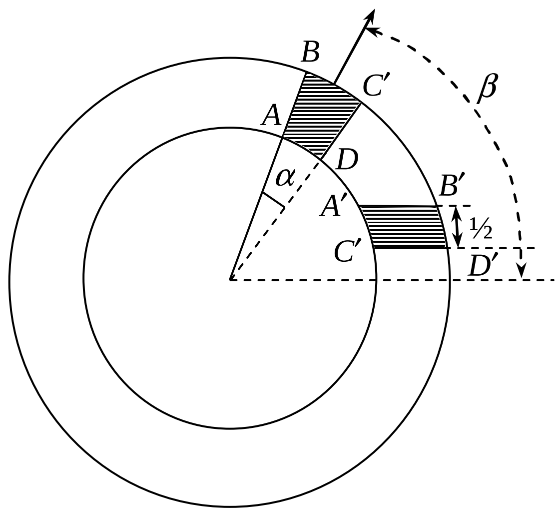
Figure 13.



It is easy to compose these actions that act upon the bases.

First, consider a radial band *ABCD* on one of the bases with an angular opening α and whose middle line forms an angle β with the *x*-axis (see Fig. 14).

Figure 14.



We calculate the resultant of the actions *P* that act upon the band *ABCD*. By a simple calculation, one obtains:

$$-\frac{m}{\pi}\frac{E\eta}{1-\eta^2}\frac{(R_1-R_2)^2 a}{3(R_1^2+R_2^2)(R_1+R_2)}\cos\beta\left(\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right),$$

where *a* represents the surface area of the band.

If the band is infinitely thin then one can substitute unity for $\sin\frac{\alpha}{2}/\frac{\alpha}{2}$, and one will obtain:

$$-\frac{m}{\pi}\frac{E\eta}{1-\eta^2}\frac{(R_1-R_2)^2}{3(R_1^2+R_2^2)(R_1+R_2)}a\cos\beta\,.$$

Upon setting:

$$-\frac{m}{\pi}\frac{E\eta}{1-\eta^2}\frac{(R_1-R_2)^2}{3(R_1^2+R_2^2)(R_1+R_2)} = M,$$

one will have the following expression for the resultant action:

$$M\,a\cos\beta;$$

i.e., the resultant action will be proportional to the surface area of the infinitely thin band and to the cosine of the angle that it forms with the *x*-axis.

Now consider a band $A'B'C'D'$ of thickness h in the circular rim that is between two lines that are parallel to the x-axis. The resultant P of the forces that act upon $A'B'C'D'$ will be:

$$-\frac{m}{\pi}\frac{E\eta}{1-\eta^2}\left(\log\frac{R_1}{R_2}-\frac{R_1^2-R_2^2}{R_1^2+R_2^2}\right)h;$$

i.e., that resultant will be proportional to the thickness of the band.

If we develop the preceding expression in powers of γ (*cf*, art. II, §1) then we obtain:

$$-\frac{m}{\pi}\frac{E\eta}{1-\eta^2}\left(\frac{1}{3}\gamma^2+\cdots\right)h.$$

If we suppose that the ring is thin and we neglect the powers of γ that are higher than the first then this expression, as well as that of M, will become negligible quantities, and the expression for P can then be written:

$$P = \frac{2m}{\pi}\frac{E\eta}{1-\eta^2}\frac{\xi}{R_1^2+R_2^2}\cos\theta,$$

in which the radius vector is:

$$r = \sqrt{\frac{R_1^2+R_2^2}{2}} + \xi,$$

and we let θ denote the angle that the radius vector forms with the x-axis.

Under these hypotheses, each radial band of the bases can be regarded as being approximately subject to a couple.

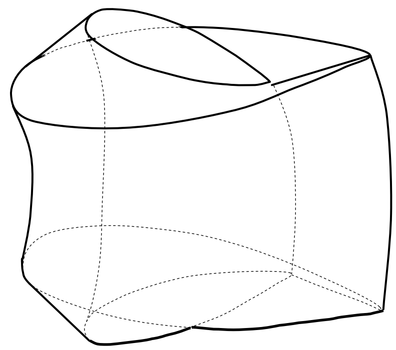
2. We now look for the form that the cylinder will take when one no longer subjects the bases to the actions P, but rather one leaves them free; i.e., we seek the form that the cylinder takes by virtue of just the distortion when it is not acted upon by any external force.

For this, it will suffice to apply the principles that we established in Chapter II, article, paragraph 2 (also *see* the preceding chapter, art. III) and then study the deformation of a body that has the form that is represented in Figure 13 in the natural state and is subject to the actions – P on the two bases. One must then suppose that the body is stretched in the shaded regions of the bases and, on the contrary, it is compressed in the white regions. In other words, one must suppose that the bases are subject to forces that are represented by the arrows in Figure 13.

Here, we may proceed in the same fashion as in the preceding chapter (*cf*, art. III) and suppose that the body is divided into radial wedges. The couples that act upon the bases will flex the wedges that are situated to the left in such a manner as to raise the internal boundary at C and lower it at D (see Fig. 13), while they will lower the external boundary at A and raise it at B. At the same time, the generators AB will curve and take on a concave form, whereas the generators CD will become convex. The contrary case will exist on the right; however, if one takes into account the resistance that is presented by the edge *EF* then the curvature of the generators *EF* and *GH* will be less reasonable.

The body will thus take on the form that is represented in Figure 15, where the deformations have been exaggerated in order to make them more visible.

Figure 15.



Thanks to the hospitality of Mr. Jona, an engineer at the Pirelli factory in Milan, I have been able to confront the results of calculation with the experiments.

He procured a large hollow rubber cylinder for me that was roughly 7.7 cm in height and whose internal and external radii were 2.95 cm and 5 cm, respectively; he cut a wedge in the cylinder whose faces were parallel and of thickness 2.3 cm, and then soldered the faces of the cut. The cylinder was forcibly coupled by means of a string, and when one untied it, that tended to open it up along the soldering on the interior side, while the two extreme boundaries of the soldering were forcibly compressed, one against the other. In this way, the exactitude of the predictions of the calculation of the distribution of the tensions along the cut was verified. Since the cylinder, when left to itself, tended to open up, I made a plaster cast in order to keep its form. Figures 16 and 17 reproduce the photographs in two different positions (*). Upon comparing them with Figure 15, one sees the perfect analogy with the form that was indicated by the calculations.

____

NOTES ON CHAPTERS V AND VI.

1. Mr. Rolla, doctor of physical sciences, has sought to verify the results that we found in the preceding chapters. He proposed to find a method of verification that one can show in a course of lectures. To that effect, he has employed an optical method.

Dr. Rolla has carried out his research in the physics laboratory of the University of Genoa, under the direction of professor Garbasso, and he has published it in the *Comptes rendus de l'Académie des Lincei* (t. XVI, 1st semester 1907). In this note, we shall present Rolla's research.

(*) Translator's note: The photographs in Figures 17 to 21 were not reproduced in this translation.

The chosen substance was gelatin, and the deformation was determined by observing the birefringence that it produced, which one can compensate for by a well-known method, with a strip of the same deformable substance in a known manner.

2. In the two preceding chapters, I have envisioned the distortions of order 6 and 2 on a hollow cylinder of revolution and I have compared the results of calculation with those of experiment. The optical method, in turn, permits one to establish the comparison, but under conditions that are more similar to the hypotheses of the calculations.

First of all, gelatin is cast into a cylindrical mold in tin-plate with a height of 6 cm and an exterior radius of 5 cm and an interior radius of 2 cm. The mold has a radial fissure of about 56° and is endowed with four small mobile brass cylinders that are arranged in such a manner as to produce four holes in the solidified gelatin cylinder that penetrate up to the middle of its thickness. Three of these holes are open on the exterior face of the cylinder and the fourth one, on the interior face (Fig. 18).

The position and direction of the holes are calculated in such a fashion that the faces of the gap ([1]), once soldered, will have two of the exterior holes in a straight line and the third one will correspond to the interior one.

At the bases of the cylinder, after soldering the fissure, one clearly sees (Fig. 19) the deformations that my calculations have predicted.

Now, if one carefully attaches the two bases along their entire surfaces to two wooden planes, in such a manner that they remain planar and at the original distance then the phenomena of accidental double refraction must appear.

Indeed, in red light between two crossed Nicol prisms one easily observes that the light does not fade as it traverses the two exterior holes, and likewise does not have a minimum intensity. On the contrary, if one observes that the light has traversed the exterior hole and its corresponding interior hole then the gelatin will prove to be isotropic. Upon destroying one of the holes – i.e., upon observing how the polarized light traverses just one hole by means of an analyzer – the birefringence returns.

All of this conforms to the theory perfectly. Indeed, the two regions that are compressed and dilated, respectively, are symmetric relative to the axis of the hollow cylinder, and they are separated by a coaxial cylinder that has a radius of the arithmetic mean of the extreme radii of the original cylinder. The nature of the deformation can be recognized with the aid of the method of paragraph 2, but it always proves to conform to the predicted deformation.

3. In the case of a uniform cut, the experiment is entirely similar to the one that we just described, although the results are different. The mold (Fig. 20) has a fissure of 6 cm and four small cylinders that are disposed as before. After attaching the faces of the gap, the cylinder takes the form that is represented in Figure 21.

([1]) In order to make the soldered joint, one coats the faces of the cut with a little of the dissolved gelatin and then pastes them together by an arbitrary means; for example, one applies any object until the gelatin is solidified. The adhesion happens rapidly, given the high viscosity of the gelatin when prepared according to the method that was described in paragraph 2.

The distribution of tensions and compressions is deduced immediately upon attaching the bases to two wooden planes, as in the preceding case, and observing the polarized light that traverses the various regions of the cylinder with a Nicol prism.

Upon observing the light that traverses the exterior holes, the gelatin proves to be isotropic. Upon observing the way it traverses an exterior hole and an interior one, it exhibits birefringence of the highest degree. Finally, upon observing the traverse of just one hole, the birefringence always remains very obvious. In the latter case, it is easy to establish the sign of the deformation.

4. The experiments described can be made visible to a large audience by means of a projector, and for this reason, are quite suitable for a lecture demonstration.

CHAPTER VII

HOLLOW CYLINDER OF REVOLUTION – DISTORTION OF ORDER 1, 3, 4, 5.

———

I.

1. In the preceding two chapters, I have considered the distortions of a hollow cylinder of revolution that are due to a radial fissure and a cut with parallel faces – i.e., the distortions of orders 6 and 2. Now, in order to consider all of the possible distortions, we must examine the ones of order 1, 3, 4, 5.

However, the distortions of order 1 can be reduced to ones of order 2 by a simple change of coordinate axes. Similarly, the distortions of orders 4 and 5 can be transformed into each other by an analogous change of axes. It thus remains for us to study the distortions of order 3 and 4. We first observe that the calculation of the deformation of the cylinder has been carried out by eliminating all of the actions along the lateral surfaces and keeping only the actions on the bases. Now, in the formulas that I gave in Chapter I for the distortions of order 3, the lateral actions were already eliminated. All that remains for us to do is then to examine the case of distortions of order 4 in detail.

In this chapter, we will show that this case can be reduced to that of the distortion of order 2 ([1]). We may then say that that the problem of the deformation of a hollow cylinder of revolution that is subjected to the most general distortion and whose bases are not acted upon is solved.

In order to obtain the form that the cylinder takes by virtue of just the distortion without being acted upon by anything external, one must eliminate the actions on the bases. One can carry out this elimination in an approximate manner, as we have already seen in the cases that we treated in the preceding chapters.

2. In the formulas that we found in article III of chapter II, we successively set:

$$t = n = p = q = r = 0$$

and

$$l = m = n = q = r = 0;$$

we obtain the following values for the right-hand sides:

([1]) The method that is followed is analogous to the one that was employed by professor Almansi in his memoir: *Sur la déformation des cylindres sollicités lateralement* (*Comptes rendus de la R. A. des Lincei*, sessions on 5 and 19 May 1901).

$$\text{(1)}\qquad \begin{cases} -\dfrac{1}{4\pi} m \log(x^2+y^2), \\ \dfrac{1}{2\pi} m \arctan\dfrac{y}{x}, \\ 0, \end{cases}$$

$$\text{(2)}\qquad \begin{cases} -\dfrac{1}{4\pi} pz \log(x^2+y^2), \\ \dfrac{1}{2\pi} pz \arctan\dfrac{y}{x}, \\ \dfrac{1}{2\pi} pz \arctan\dfrac{y}{x} + \dfrac{1}{4\pi} pz \log(x^2+y^2), \end{cases}$$

respectively.

Formulas (1) give the displacements that correspond to a distortion of order 2 and formulas (2) give the displacements that correspond to a distortion of order 4. It is now easy to recognize that the first two expressions (2) can be inferred from the corresponding expressions (1) by multiplying the latter by p / mx. On the other hand, in the preceding chapter (art. 1, § 2), where we envisioned the uniform fissure, we showed that in the case of a hollow cylinder of revolution whose lateral surfaces have radii R_1 and R_2 , one can eliminate the tensions along the lateral surfaces by adding the quantities:

$$\text{(3)}\qquad \begin{cases} u' + Au'' + Bu''', \\ v' + Av'' + Bv''', \\ 0 \end{cases}$$

to the expressions (1), respectively, and choosing the constants A and B suitably. We then seek to eliminate the lateral tensions in the case of the distortion of order 4 by taking the components of the displacement that are given by:

$$\text{(4)}\qquad \begin{cases} u = \dfrac{p}{m} z\left[-\dfrac{1}{4\pi} m \log(x^2+y^2) + u' + Au'' + Bu'''\right] = zU, \\ v = \dfrac{p}{m} z\left(\dfrac{1}{2\pi} m \arctan\dfrac{y}{x} + v' + Av'' + Bv'''\right) = zV, \\ w = -\dfrac{1}{2\pi} py \arctan\dfrac{y}{x} + \dfrac{1}{4\pi} px \log(x^2+y^2) + \Phi(x, y) = W + \Phi(x, y), \end{cases}$$

where $\Phi(x, y)$ is a regular, unknown function that must be determined.

Upon substituting p for the letter m in formulas (I) of the last chapter, one will obtain:

$$(5)\qquad \begin{cases} U = \dfrac{p}{2\pi}\left\{\dfrac{K}{L+2K}\log r + \dfrac{L+K}{2(L+2K)}\left(r^2 - \dfrac{R_1^2 R_2^2}{R_1^2+R_2^2}\right)\dfrac{\partial^2 \log r}{\partial x^2}\right. \\ \qquad\qquad \left. + \dfrac{1}{2(L+2K)(R_1^2+R_2^2)}[(3L+5K)y^2 + (L+K)x^2]\right\}, \\ V = \dfrac{p}{2\pi}\left\{\arctan\dfrac{y}{x} + \dfrac{L+K}{2(L+2K)}\left(r^2 - \dfrac{R_1^2 R_2^2}{R_1^2+R_2^2}\right)\dfrac{\partial^2 \log r}{\partial x\,\partial y} - \dfrac{L+3K}{(L+2K)(R_1^2+R_2^2)}xy\right\}. \end{cases}$$

Then, upon setting $r^2 = x^2 + y^2$ we will have:

$$(5')\qquad W = -\frac{1}{2\pi}py \arctan\frac{y}{z} + \frac{1}{2\pi}px \log r.$$

3. If we replace these expressions (4) in the indefinite equations for elastic equilibrium:

$$K\,\Delta^2 u + (L+K)\frac{\partial\theta}{\partial x} = 0,$$

$$K\,\Delta^2 v + (L+K)\frac{\partial\theta}{\partial y} = 0,$$

$$K\,\Delta^2 w + (L+K)\frac{\partial\theta}{\partial z} = 0,$$

where

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

then we will easily see that the first two equations are satisfied. The third equation becomes:

$$(6)\qquad K\,\Delta^2\,\Phi + (L+K)\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0.$$

Call the characteristics of tension that correspond to the displacements (4) t_{11}, t_{22}, t_{33}, t_{23}, t_{31}, t_{12}. One easily verifies that along the lateral surfaces of the hollow cylinder one has:

$$t_{11}\cos nx + t_{12}\cos ny + t_{13}\cos nz = 0,$$
$$t_{21}\cos nx + t_{22}\cos ny + t_{23}\cos nz = 0,$$
$$t_{31}\cos nx + t_{32}\cos ny + t_{33}\cos nz$$

$$= \left(U + \frac{\partial W}{\partial x} + \frac{\partial \Phi}{\partial x}\right)\cos nx + \left(V + \frac{\partial W}{\partial y} + \frac{\partial \Phi}{\partial y}\right)\cos ny,$$

where n denotes the normal to the contour. Moreover, in order that the displacements (4) correspond to zero lateral tensions, it will be necessary and sufficient that:

$$\left(U + \frac{\partial W}{\partial x} + \frac{\partial \Phi}{\partial x}\right)\cos nx + \left(V + \frac{\partial W}{\partial y} + \frac{\partial \Phi}{\partial y}\right)\cos ny = 0. \tag{2}$$

II.

1. By virtue of formulas (6) and (7) of the preceding article, the problem that we are proposing amounts to the determination of the function $\Phi(x,y)$ in the space ω between two circumferences σ_1 and σ_2 of radii R_1 and R_2 that have their centers at the origin. In this field, the function Φ satisfies the differential equation:

$$\Delta^2 \Phi = -\frac{L+K}{K}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right), \tag{6$'$}$$

and on the contour, the condition:

$$\frac{\partial \Phi}{\partial n} = -\frac{\partial W}{\partial n} - (U \cos nx + V \cos ny). \tag{7$'$}$$

Now, by simple calculations, one transforms the equality (6′) into:

$$\Delta^2 \Phi = -p\frac{L+K}{\pi(L+2K)}\,x\left(\frac{1}{r^2} - \frac{2}{R_1^2 + R_2^2}\right), \tag{6$''$}$$

and the conditions (7′) into:

$$\frac{\partial \Phi}{\partial n} = \begin{cases} -\dfrac{p}{2\pi}\left(\dfrac{L+3K}{L+2K}\log R_1 + 1 - \dfrac{K}{L+2K}\dfrac{R_1^2}{R_1^2+R_2^2}\right)\cos\theta & \text{on } \sigma_1, \\[2ex] -\dfrac{p}{2\pi}\left(\dfrac{L+3K}{L+2K}\log R_2 + 1 - \dfrac{K}{L+2K}\dfrac{R_2^2}{R_1^2+R_2^2}\right)\cos\theta & \text{on } \sigma_2, \end{cases} \tag{7$''$}$$

where

$$\cos\theta = \cos nx .$$

One easily verifies that:

$$\int_\omega \Delta^2 \Phi \, d\omega + \int_{\sigma_1} \frac{\partial \Phi}{\partial n} d\sigma_1 + \int_{\sigma_2} \frac{\partial \Phi}{\partial n} d\sigma_2 = 0,$$

when one supposes that the normal n is directed towards the interior of the field ω. Indeed, the three integrals of the preceding formula are zero when taken separately. It then follows that conditions (6′) and (7′) are mutually compatible.

2. If one makes:

$$\Phi = -\frac{p}{2\pi}\frac{L+K}{L+2K}\left(\log r - \frac{1}{2}\frac{r^2}{R_1^2+R_2^2}\right)x + \Psi \tag{8}$$

then equation (6″) will be transformed into:

$$\Delta^2\Psi = 0,$$

and conditions (7″) become:

$$\frac{\partial\Phi}{\partial n} = \begin{cases} -\dfrac{p}{2\pi}\dfrac{K}{L+2K}\left(1+2\log R_1 + \dfrac{3L+K}{2K}\dfrac{R_1^2}{R_1^2+R_2^2}\right)\cos\theta & \text{on } \sigma_1, \\ -\dfrac{p}{2\pi}\dfrac{K}{L+2K}\left(1+\log R_2 + \dfrac{3L+K}{2K}\dfrac{R_2^2}{R_1^2+R_2^2}\right)\cos\theta & \text{on } \sigma_2. \end{cases}$$

Therefore:

$$\Psi = Mx + N\,\frac{x}{r^2},$$

in which M and N are constants. They may be calculated easily, and one finds:

$$\Psi = -\frac{p}{2\pi}\frac{Kx}{L+2K}\left[\left(\frac{3}{2}\frac{L+K}{K} + \frac{R_1^2\log R_1^2 - R_2^2\log R_2^2}{R_1^2-R_2^2}\right) + \frac{R_1^2R_2^2}{r}\left(\frac{\log R_1^2 - \log R_2^2}{R_1^2-R_2^2} + \frac{3L+K}{2K}\frac{1}{R_1^2+R_2^2}\right)\right], \tag{9}$$

so, upon combining formulas (5′), (8), and (9) one easily infers the value of w.

3. Upon taking formulas (4) and (5) into account, we will then have:

$$(A)\quad \begin{cases} u = \dfrac{pz}{2\pi}\left\{\dfrac{K}{L+2K}\log r + \dfrac{L+K}{2(L+2K)}\left(r^2 - \dfrac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\dfrac{\partial^2 \log r}{\partial x^2}\right. \\ \qquad\left. + \dfrac{1}{2(L+2K)(R_1^2+R_2^2)}[(3L+5K)y^2+(L+K)x^2]\right\} \\ v = \dfrac{pz}{2\pi}\left[\arctan\dfrac{y}{x} + \dfrac{L+K}{2(L+2K)}\left(r^2 - \dfrac{R_1^2R_2^2}{R_1^2+R_2^2}\right)\dfrac{\partial^2 \log r}{\partial x\,\partial y} - \dfrac{L+3K}{(L+2K)(R_1^2+R_2^2)}xy\right], \\ w = -\dfrac{py}{2\pi}\arctan\dfrac{y}{x} - \dfrac{px}{2\pi}\dfrac{K}{L+2K}\left[\dfrac{3}{2}\dfrac{L+K}{K} + \dfrac{R_1^2\log R_1^2 - R_2^2 \log R_2^2}{R_1^2-R_2^2} - \log r\right. \\ \qquad \left. + \dfrac{R_1^2R_2^2}{r^2}\left(\dfrac{\log R_1^2 - \log R_2^2}{R_1^2-R_2^2} + \dfrac{3L+K}{2K}\dfrac{1}{R_1^2+R_2^2} - \dfrac{1}{2}\dfrac{L+K}{K}\dfrac{r^2}{R_1^2+R_2^2}\right)\right], \end{cases}$$

so one infers that:

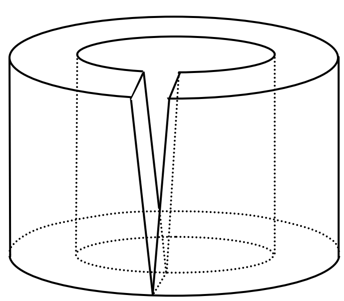
$$(B)\quad \begin{cases} t_{33} = \dfrac{pLK}{\pi(L+K)}xz\left(\dfrac{1}{r^2} - \dfrac{2}{R_1^2+R_2^2}\right), \\ t_{13} = 2K\left(U + \dfrac{\partial w}{\partial x}\right), \\ t_{23} = 2K\left(V + \dfrac{\partial w}{\partial y}\right). \end{cases}$$

One thus determines the tensions that act upon the two bases.

III.

1. In order to obtain a distortion of order 4 in practice, it suffices to make a uniform fissure in the hollow cylinder, as is indicated in Figure 22, in such a manner that the two faces of the fissure meet along a radius of one of the two bases (e.g., the *x*-axis). Having done this, one brings the two faces of the fissure together and solders them. If the faces of the fissure are equally inclined to the base then the form that deformed solid takes after soldering is symmetric with respect to a plane perpendicular to the base.

Figure 22.



Upon applying the results that we just found and employing reasoning that is analogous to what we followed in the preceding two chapters, one can get a rough idea of the form that the cylinder takes when it is subject to only a distortion of order 4 – i.e., when one supposes that the tensions on the bases have been eliminated. However, we suppress that discussion and we confine ourselves to presenting (Fig. 23 and 24) (*) an image of a rubber cylinder that has been acted upon by a distortion of order 4.

The two photographs (Fig. 23 and 24) of the plaster cast of the deformed solid, as seen from two different sides, clearly show the form of the two bases. The edge corresponds to the soldering. The rubber cylinder was measured before the distortion to have an exterior diameter of 10.6 cm, and interior diameter of 6 cm, and a height of 5.9 cm. The angular opining of the wedge was around 38^{o}.

2. In order to complete the images of the six elementary distortions of a hollow cylinder of revolution, we reproduce the photographs of the plaster casts here of three large rubber tubes that were subjected to distortions of order 1, 3, 5, respectively.

Figure 25 refers to a distortion of order 1 of a hollow cylinder. It was obtained by making an axial cut (*x*-plane, *z*, on the positive side of the *x*-axis) and then sliding the two faces of the cut over each other in a direction that is normal to the axis of the cylinder (viz., the *z*-axis).

Figure 26 refers to the distortion of order 3 (cf., chap. I, art. III, § 7). The cylinder was cut as in the preceding case, and one then slides the two faces of the cut over each other in the sense of the axis of the cylinder (viz., the *z*-axis).

Finally, Figure 27 represents a hollow cylinder that was subject to a distortion of order 5. After making the cut, one rotates the two faces with respect to each other around the perpendicular (viz., the *y*-axis) to the two faces, guided by the middle of the cylinder axis. The origin is therefore situated at the middle of the cylinder axis. We observe that in order to make the construction of the model for the distortion of order 4 easier, we have chosen the origin to be the center of one of the bases.

NOTE ON CHAPTER V, VI, VII.

1. Almansi has dedicated two notes ([1]) to the study of regular deformations of the cylinder when the displacements are polydromic.

In the first note, with the *z*-axis parallel to the generators of the cylinder, Almansi imagined the case where the characteristics of the tensions were independent of *z*, while in the second note, he imagined the general case.

(*) Translator's note: The photographs depicted in Figures 24-27 were not reproduced in this translation.

([1]) *Sopra una classe particolare di deformazioni a spostamenti polidromi dei solidi cilindrici (Rend. d. R. Accademia dei Lincei*, January 1907).

Sulle defomazioni a spostamenti polidromi dei solidi cilindrici (Rend. d. R. Istituto Lombardo, 1907).

2. Let an elastic cylinder be in the natural state. Suppose that it is deformed by forces that act upon the bases, and suppose that upon composing the forces that act upon each base, one finds that the resultant force and resultant couple are zero. Almansi then called the deformation of the cylinder a *deformation of type* D_6. He remarked that in the problem of De Saint-Venant one finds the deformation of a cylinder that is acted upon by given forces that act upon the bases by neglecting a deformation of type D_6. Now, in the problem of De Saint-Venant one envisions only the case where the displacements are monodromic. Almansi proposed the following problem: *Given a homogeneous, isotopic, multiply-connected, elastic cylinder that is not acted upon by external forces, determine the most general deformation of the cylinder, while neglecting a deformation of type* D_6.

3. He began by proving the following theorem: *In the case envisioned, one can always represent the characteristics of the tensions by linear functions of z.*

Take the x and y axes to be the principal axes of inertia of a normal section of the cylinder. He then proved that *one can calculate the characteristics of the tensions in the case envisioned by the formulas:*

$$t_{11} = z\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 V}{\partial y^2}, \qquad t_{31} = -\eta\frac{\partial U}{\partial x} + \frac{\partial W}{\partial y},$$

$$t_{22} = z\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 V}{\partial x^2}, \qquad t_{32} = -\eta\frac{\partial U}{\partial y} - \frac{\partial W}{\partial x},$$

$$t_{12} = z\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 V}{\partial x^2}, \qquad t_{33} = \eta(z\,\Delta^2 U + \Delta^2 V),$$

where η *denotes the Poisson coefficient* (*see* Chap. V) *and* U, V, W *do not depend upon* z *and are regular, biharmonic functions of the variables* x *and* y (*see* Chap. II, art. III, § 2).

The following relations must exist between U *and* W:

$$\frac{\partial \Delta^2 W}{\partial x} = (1-\eta)\frac{\partial \Delta^2 U}{\partial y}, \qquad \frac{\partial \Delta^2 W}{\partial y} = -(1-\eta)\frac{\partial \Delta^2 U}{\partial x}.$$

Let σ *be the base of the cylinder, and suppose that the contour of that base is formed by several closed lines* $\sigma_1, \sigma_2, \ldots, \sigma_n$. *On each line* σ_1, *one must have:*

$$U = a_i\,x + b_i\,y + c_i, \qquad V = g_i\,x + h_i\,y + l_i,$$

$$\frac{\partial U}{\partial v} = \frac{\partial(a_i x + b_i y + c_i)}{\partial \nu}, \qquad \frac{\partial V}{\partial v} = \frac{\partial(g_i x + h_i y + l_i)}{\partial \nu},$$

$$W = \eta(a_i\,x - b_i\,y + k_i),$$

where the quantities a_i, b_i, c_i, g_i, h_i, l_i are constants and v denotes the normal to the line s_i that is directed interior to the area σ.

4. If the function U is zero then the characteristics of the tensions are independent of z. In this case, one has:

$$\Delta^2 W = \text{const.}$$

and the characteristics of the tensions are given by the formulas:

$$t_{11} = \frac{\partial^2 V}{\partial y^2}, \qquad t_{21} = \frac{\partial W}{\partial y},$$

$$t_{22} = \frac{\partial^2 V}{\partial x^2}, \qquad t_{32} = -\frac{\partial W}{\partial y},$$

$$t_{12} = -\frac{\partial^2 V}{\partial x \partial y}, \qquad t_{33} = \eta\, \Delta^2 V.$$

CHAPTER VIII

CYCLIC SYSTEMS OF PLIABLE, ELASTIC ELEMENTS

I.

1. At the end of Chapter III, I stated the *fundamental problem* that is posed in the theory of distortions of multiply-connected elastic solid bodies in the following terms: *Being given the distortions of the elastic system, determine the efforts.* In this chapter, I would like to present the principles of the solution of this problem in one case that presents an especial interest ([1]).

2. In order to fix ideas, consider a rectilinear rod whose transverse dimensions are very small relative to its length.

Imagine particles A and B that form the extremities of the thin rod. We intend that A and B represent the two extreme *end faces* of the thin rod, whose widths have the same order of magnitude as the transverse dimensions.

When the body is deformed, the relative displacements of A and B are, in general, very large relative to the pure deformation of the particles themselves, as well as any other element of the body whose dimensions are of the same order as the transverse dimensions of the thin rod.

We may thus consider A and B approximately to be two rigid elements whose relative displacement will be the resultant of a translation and a rotation. We also suppose that the relative displacements of A and B are such that one may neglect the powers higher than the first of the components of the aforementioned rotations and translations.

3. Now assume the exterior forces that act upon the interior of the rod are negligible. Suppose that the exterior forces are applied only to the particles A and B, and that the system is in equilibrium. Under that hypothesis, imagine an arbitrary transverse section σ that divides the thin rod into two parts S_a and S_b, the first of which contains particle A and the second of which contains B, and compose the actions that the part S_b exerted on the part S_a along σ, after taking an arbitrary point O to be the center of reduction. It is obvious that if one keeps this point fixed and changes the section σ in any way then the resultant force and couple are independent of the section. They will also be respectively equal to the resultant force and couple that one would obtain upon composing the forces applied at B, and will be equal and opposite to the force and the couple that one would find upon composing the forces applied at A, while O is always the center of reduction.

4. Suppose, to consider the simplest case, that the thin rod is isotropic and that the natural state has the form of a cylinder of revolution of length l and radius R.

([1]) *See* CLEBSCH, *loc. cit.*, Chap. VIII.

Take the origin O to be in the center of the base adjacent to the particle A and take the z-axis to be the axis of the cylinder. Upon choosing the origin O to be the center of reduction, let:

$$X_1^{(ab)},\ X_2^{(ab)},\ X_3^{(ab)}$$

represent the components of the resultant force of the exterior actions applied at B, and let:

$$X_4^{(ab)},\ X_5^{(ab)},\ X_6^{(ab)}$$

represent the components of the resultant couple.

Let:

$$x_1^{(a)},\ x_2^{(a)},\ x_3^{(a)}$$

denote the components of the translation that is experienced by A relative to the natural state, and let:

$$x_4^{(a)},\ x_5^{(a)},\ x_6^{(a)}$$

denote the components of the rotation that is experienced by the same particles. Let:

$$x_1^{(b)},\ x_2^{(b)},\ x_3^{(b)},\ x_4^{(b)},\ x_5^{(b)},\ x_6^{(b)}$$

be the analogous quantities for the particle B. The components of the translation and rotation of B relative to A will be:

$$x_1^{(b)} - x_1^{(a)},\ x_2^{(b)} - x_2^{(a)},\ x_3^{(b)} - x_3^{(a)},\ x_4^{(b)} - x_4^{(a)},\ x_5^{(b)} - x_5^{(a)},\ x_6^{(b)} - x_6^{(a)}$$

respectively. Between the forces $X_i^{(ab)}$ and the quantities $x_i^{(b)} - x_i^{(a)}$, there exist the following relations:

$$(A)\qquad \begin{cases} x_1^{(b)} - x_1^{(a)} = \dfrac{1}{E}\dfrac{l^2}{\mu}\left(-X_5^{(ab)} + \dfrac{2}{3}X_1^{(ab)} l\right), \\ x_2^{(b)} - x_2^{(a)} = \dfrac{1}{E}\dfrac{l^2}{\mu}\left(+X_4^{(ab)} + \dfrac{2}{3}X_2^{(ab)} l\right), \\ x_3^{(b)} - x_3^{(a)} = 0; \end{cases}$$

$$(A')\qquad \begin{cases} x_4^{(b)} - x_4^{(a)} = \dfrac{1}{E}\dfrac{l}{\mu}\left(2X_4^{(ab)} + X_2^{(ab)} l\right), \\ x_5^{(b)} - x_5^{(a)} = \dfrac{1}{E}\dfrac{l}{\mu}\left(2X_5^{(ab)} - X_1^{(ab)} l\right), \\ x_6^{(b)} - x_6^{(a)} = \dfrac{2(1+\eta)}{E}\dfrac{l}{\mu}X_6^{(ab)}, \end{cases}$$

where E denotes the modulus of elasticity, η is the elastic constant that was already introduced in the preceding chapter, and $\mu = \pi R^4/2$ is the moment of inertia of the circular section of the thin rod relative to its center. In the preceding formulas, we have supposed that the forces $X_i^{(ab)}$ are of the same order of magnitude and we have neglected the terms of order higher than the ones that figure in them ([1]).

5. These formulas prove that if one arbitrarily chooses the three components of the relative rotation of B with respect to A and the two components of the relative translation in the normal sense to the thin rod then one can always find some exterior forces that are capable of generating them. The relative translation in the sense of the axis of the thin rod is, on the contrary, of the order of negligible quantities.

However, it will be simple to slightly modify the conditions of the system in such a way as to make it likewise possible to have a relative translation in the sense of the axis. Indeed, suppose that the exterior forces are applied to two small sliding loops (*coulants*) that are capable of sliding along the cylinder in the longitudinal sense and held against the cylinder by two springs such that efforts that have the same order of magnitude as the ones that produce the flexions and torsions of the thin rod induce relative displacements in the two loops in the sense of the axis and of the same order of magnitude as the former ones. If one supposes that the two loops are situated at the extremities of the thin rod and if one calls them A and B then formulas (A) and (A') are not altered. Only the third formula must be replaced with this one:

$$x_3^{(b)} - x_3^{(a)} = mX_3^{(ab)}, \tag{1}$$

where m is a positive quantity of the same order of magnitude as the coefficients of the quantities $X_i^{(ab)}$ in the preceding formulas.

6. The elastic energy of the deformed system is:

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^{6}(x_i^{(b)} - x_i^{(a)})X_i^{(ab)} \\ &= \frac{1}{E}\frac{l}{\mu}\left[\frac{1}{3}(X_1^{(ab)}l)^2 + \frac{1}{3}(X_2^{(ab)}l)^2 + \frac{m}{3}(X_3^{(ab)}l)^2 + (X_4^{(ab)})^2 + (X_5^{(ab)})^2 + (1+\eta)(X_6^{(ab)})^2\right. \\ &\qquad \left. - X_1^{(ab)}X_5^{(ab)}l + X_2^{(ab)}X_4^{(ab)}l\right], \end{aligned}$$

which is a positive-definite form if m is non-zero. However, if m is zero (as in the case where the loops are missing) then H is a positive form that may be annulled regardless of whether $X_3^{(ab)}$ is zero. However, in order for H to be zero it is necessary that $X_1^{(ab)}$, $X_2^{(ab)}$, $X_4^{(ab)}$, $X_5^{(ab)}$, $X_6^{(ab)}$ be zero. *It thus suffices that just one of the quantities* $x_i^{(b)} - x_i^{(a)}$ *be different from zero in order for H to also be non-zero.*

([1]) One may obtain the preceding formulas in several ways – for example, by employing the method of De Saint-Venant (*See* KIRCHHOFF, *Vorl. über Math. Physik; Mechanik,* 27, 28 Vorl.).

II.

4. One may imagine an infinite number of other cases in which bodies of widely varied forms have properties analogous to the ones that we just examined. The thin rod – with or without loops – may be regarded as a typical case. Desiring to place ourselves at a general viewpoint, we envision bodies to which we *attribute* certain properties *in an absolute manner*. These properties will be the same ones that were verified approximately in the case examined in the preceding article:

1. There exist two particles A and B of the body that we shall call its *extremities*, whose deformations are negligible with respect to the relative translations and rotations that they are subjected to.
2. If one assumes that the exterior actions are applied only to the extremities A and B and the body is in equilibrium then the components of the translations and rotations of B with respect to A may be represented linearly by means of the components of the resultant force and couple of the exterior action applied at B.
3. The elastic energy of the deformed system (which is always positive) may be annulled only in the case where all of the components of the relative translation and rotation of B with respect A are zero.

We call bodies that have the stated properties *pliable elastic elements*, and we distinguish them into two categories:

1. *Freely pliable, elastic elements*, i.e., ones such that if one arbitrarily chooses the three components of the translation and the three components of the rotation of one extremity with respect to the other one then one may always find exterior actions that are capable of generating them (of the same type as the thin rod with loops).
2. *Pliable elastic elements that are subject to constraints*, such that the components of the rotation and translation of one extremity with respect to the other one are related by one or more linear relations (of the same type as the simple thin rod).

2. We indicate the pliable, elastic element by AB and the components of the resultant force and couple of the exterior action applied at B by:

(2) $$X_1^{(ab)},\ X_2^{(ab)},\ X_3^{(ab)};\quad X_4^{(ab)},\ X_5^{(ab)},\ X_6^{(ab)}.$$

By virtue of equilibrium, the components of resultant force and couple of the exterior action applied at A will be:

$$-\ X_1^{(ab)}, -X_2^{(ab)}, -X_3^{(ab)};\qquad -X_4^{(ab)}, -X_5^{(ab)}, -X_6^{(ab)},$$

which we represent by:

$$X_1^{(ba)},\ X_2^{(ba)},\ X_3^{(ba)};\quad X_4^{(ba)},\ X_5^{(ba)},\ X_6^{(ba)}$$

respectively.

If one imagines an arbitrary transverse section σ that divides the body into two parts S_a and S_b , the first of which possesses the *extremity B* and the second of which, the *extremity A*, then the quantities (2) will be the components of the resultant force and couple that one will obtain upon composing the actions that the part S_b exerts on S_a along σ ([1]). One must assume that the same center of reduction has been chosen for all of these compositions of forces, and that it is the origin of the coordinate axes.

The quantities (2) are called the *characteristics of the efforts* – or simply, the *efforts* – that the element *AB* is subject to. Denote the components of the translation by $x_1^{(a)}$, $x_2^{(a)}$, $x_3^{(a)}$, and the components of the rotation of the *extremity A* by $x_4^{(a)}$, $x_5^{(a)}$, $x_6^{(a)}$.

We call these quantities the *characteristics of the displacement* of the point *A*.

We denote the analogous quantities for the *extremity B* by $x_1^{(b)}$, $x_2^{(b)}$, $x_3^{(b)}$; $x_4^{(b)}$, $x_5^{(b)}$, $x_6^{(b)}$.

The linear relations that link the relative displacements of the two extremities to the efforts may be written in general:

$$x_i^{(b)} - x_i^{(a)} = \sum_{s=1}^{6} A_{is}^{(ab)} X_s^{(ab)} \qquad (i = 1, 2, \ldots, 6), \tag{3}$$

and one will obviously have:

$$A_{is}^{(ab)} = A_{is}^{(ba)}.$$

3. The quantities $A_{is}^{(ab)}$ depend only upon the nature of the body and its position with respect to the axes. They are called the *direct constants of the element.* It is easy to see the values that they take if one changes position of the body with respect to the axes. To that effect, suppose that the preceding *constants* are known when the body is referred to a certain system of axes and then suppose that the axes are changed. Some well-known equations of statics give us the relations that exist between the forces and the analogous quantities $X_i^{(ab)}$ that one finds if one changes the directions of the axes and the origin, which is the center of reduction of the forces. Likewise, some elementary formulas of kinematics give us the relations that exist between quantities $x_i^{(a)}$ and $x_i^{(b)}$ and the analogous quantities referred to new axes and a new center of reduction.

Some simple operations of substitution in the formulas (3) thus suffice to give the linear relations that exist between the coefficients $A_{is}^{(ab)}$ and the corresponding coefficients relative to new systems of axes.

4. If the elastic element is freely pliable then the equalities (3) will be invertible, and we will have:

$$X_s^{(ab)} = \sum_{i=1}^{6} a_{is}^{(ab)} (x_i^{(b)} - x_i^{(a)}). \tag{3′}$$

([1]) If the body is multiply connected (*see*, for example, Art. V, § 3) then the section σ that divides the body into two parts might be formed from distinct parts.

The determination of the coefficients a_{is} and their variations, if the axes change, presents no difficulty.

We call the coefficients $a_{is}^{(ab)}$ the *inverse constants* of the element *AB*.

When the pliable, elastic element is subject to constraints it is impossible to invert the equations (3).

5. The elastic energy of deformed system will be given by:

$$H = \frac{1}{2}\sum_{i=1}^{6} X_i^{(ab)}(x_i^{(b)} - x_i^{(a)}) = \frac{1}{2}\sum_{i=1}^{6}\sum_{s=1}^{6} A_{is}^{(ab)} X_i^{(ab)} X_s^{(ab)} .$$

The preceding form will thus be a *positive form*, and it will be *definite if the elastic system is freely pliable; on the contrary, it will not be definite if the system is pliable, but subject to constraints.*

In the former case, we again have:

$$H = \frac{1}{2}\sum_{i=1}^{6}\sum_{s=1}^{6} a_{is}^{(ab)}(x_i^{(b)} - x_i^{(a)})(x_s^{(b)} - x_s^{(a)}) .$$

III.

1. The preliminary considerations that were presented in the preceding articles will now serve as the basis for us in the study of the distortions of a cyclic system composed of several pliable elements. Indeed, imagine that they are united by an arbitrary number of pliable, elastic elements *that rigidly unites the extremities*, in such a way that they form a *cyclic* set, all of whose parts are in the natural state. We shall study the effect of the distortions in this system.

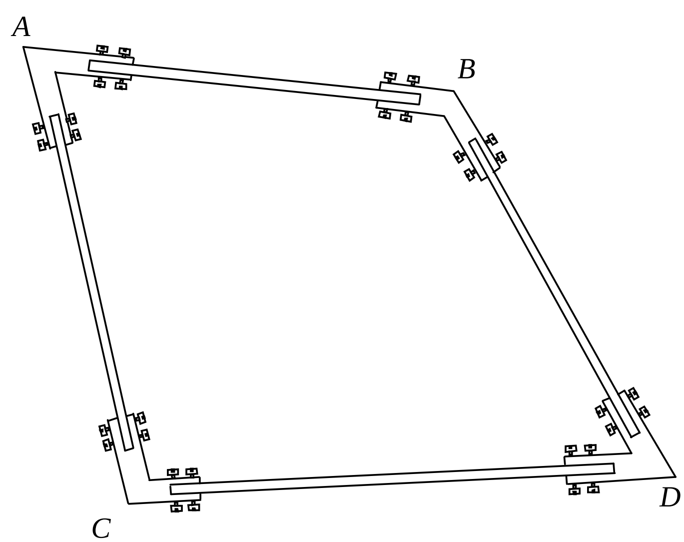


Figure 28.

2. In order to fix ideas, suppose that we have four thin, rectilinear rods. Connect their extremities pair-wise by fixing four of them rigidly with hand clamps in such a way that the rods form the edges of a quadrilateral *ABCD* and the four hand clamps form its summits, as is indicated in Figure 28. One then makes a cut into one of the edges and performs a distortion along the cut. We shall see how the system is deformed and what efforts are induced in it.

3. Assume, in general, that the pliable, elastic elements amount to *n*, in total, and that the extremities are rigidly united at *m* nodes. Suppose, in addition, that the system is devoid of any external action.

First consider an arbitrary element that links the nodes *A* and *B*, whose extremities are then *A* and *B*; denote that element by *AB*.

By making use of the same notations that we employed in the preceding article, we will find that if the element is not subject to any distortion then the following relations subsist:

$$x_i^{(b)} - x_i^{(a)} = \sum_{s=1}^{6} A_{is}^{(ab)} X_i^{(ab)} \qquad (i = 1, 2, \ldots, 6). \tag{3}$$

However, if the element is subjected to a distortion with the characteristics $\alpha_1^{(ab)}$, $\alpha_2^{(ab)}$, $\alpha_3^{(ab)}$, $\alpha_4^{(ab)}$, $\alpha_5^{(ab)}$, $\alpha_6^{(ab)}$ then the preceding equations must be replaced with:

$$x_i^{(b)} - x_i^{(a)} - \alpha_i^{(ab)} = \sum_{s=1}^{6} A_{is}^{(ab)} X_i^{(ab)} \qquad (i = 1, 2, \ldots, 6). \tag{I}$$

Thus, if, in general, one executes a distortion at each element then *one will have six equations that are analogous to the preceding ones for each element.*

Now consider a node, which we denote by the letter *A*, that bounds and rigidly unites the extremities *A* of the elements *AB*, *AC*, *AD*, …

For equilibrium, we must have the six equations:

$$X_i^{(ab)} + X_i^{(ac)} + X_i^{(ad)} + \ldots = 0 \qquad (i = 1, 2, \ldots, 6). \tag{II}$$

One will thus have six equations that are analogous to the preceding one for each node.

4. *Suppose that the constants of each element and the characteristics of each distortion are known* – i.e., all of the coefficients $A_{is}^{(ab)}$ and all of the characteristics $\alpha_i^{(ab)}$ – and suppose that the components of the translations and rotations of each extremity and the efforts that each element are subjected to are unknown. We will have 6*n* + 6*m* unknowns that verify the 6*n* + 6*m* linear equations (I) and (II).

Nonetheless, observe that six of the equations in this system follow from the other ones. Indeed, upon adding both sides of the equalities (II) that correspond to the same index *i* we will find that the left-hand side is identically zero, because each term $X_i^{(ab)}$

that appears in one equation is eliminated by the term $X_i^{(ba)}$ that appears in another. This result is easily explained because it is obvious that the three components of the translation and the three components of the rotation of a node are arbitrary.

5. We now prove the following fundamental theorem:

In any cyclic system of pliable, elastic elements, if the constants of each element and the distortions performed at each of them are known then the relative translations and rotations of all the nodes will be determined, as well as the efforts that act upon all of the elements of the system that are freely pliable.

In order to simplify this, suppose that the three components of the translation are zero and the components of the rotation that corresponds to a node are chosen arbitrarily. Suppose, moreover, that to the same system of values of characteristics $\alpha_i^{(ab)}$ and coefficients $A_{is}^{(ab)}$ there correspond two systems of values for the quantities $x_i^{(a)}$ and $X_i^{(ab)}$, which we denote by $\bar{x}_i^{(a)}$ and $\bar{X}_i^{(ab)}$, $\bar{\bar{x}}_i^{(a)}$ and $\bar{\bar{X}}_i^{(ab)}$, respectively. Write:

$$\bar{\bar{x}}_i^{(a)} - \bar{x}_i^{(a)} = \xi_i^{(a)},$$
$$\bar{\bar{X}}_i^{(ab)} - \bar{X}_i^{(ab)} = \Xi_i^{(ab)}.$$

These quantities verify the equations:

$$\xi_i^{(b)} - \xi_i^{(a)} = \sum_{s=1}^{6} A_{is}^{(ab)} \Xi_s^{(ab)}, \tag{4}$$

$$\Xi_i^{(ab)} + \Xi_i^{(ac)} + \Xi_i^{(ad)} + \ldots = 0. \tag{5}$$

Multiply the two sides of equation (4) by $\Xi_i^{(ab)}$ and the two sides of equation (5) by $\xi_i^{(a)}$ and add the corresponding sides of all of the equations that we just obtained. The left-hand side will be identically zero as a result, so:

$$\sum_{ab} \sum_{i=1}^{6} \sum_{s=1}^{6} A_{is}^{(ab)} \Xi_i^{(ab)} \Xi_s^{(ab)} = 0.$$

One intends $\sum_{ab}$ to mean a sum of n terms relative to all of the elastic elements that constitute the system.

However, each form:

$$\sum_{i=1}^{6} \sum_{s=1}^{6} A_{is}^{(ab)} \Xi_i^{(ab)} \Xi_s^{(ab)}$$

is positive; thus, by virtue of the preceding equations, we will have:

(6) $$\sum_{i=1}^{6}\sum_{s=1}^{6} A_{is}^{(ab)}\Xi_i^{(ab)}\Xi_s^{(ab)} = 0,$$

and consequently:

$$\xi_i^{(b)} - \xi_i^{(a)} = 0,$$

so:

$$\xi_i^{(a)} = 0 \qquad \text{and} \qquad \bar{\bar{x}}_i^{(a)} = \bar{x}_i^{(a)}.$$

Thus, the components of the translations and rotations of the nodes cannot differ between the two solutions.

From formula (6), one infers that if the element *AB* is freely pliable then the quantities $\Xi_i^{(ab)}$ might be zero, and, as a result $\bar{\bar{X}}_i^{(ab)} = \bar{X}_i^{(ab)}$. As a consequence, the efforts relative to the element (*AB*), if it is freely pliable, cannot differ between the two solutions.

The stated theorem is thus proved.

6. The following corollary follows immediately from the preceding theorem:

In a cyclic system of freely pliable elastic elements in which one knows the constants, the efforts are determined by the distortions, and one may obtain them by solving a system of equations of the first degree.

Upon always supposing that the elements are freely pliable, we have found that the formulas (4) and (5) have no other solutions than $\xi_i^{(a)} = 0$, $\Xi_i^{(ab)} = 0$ if we assume that the quantities ξ are zero for a given node.

This proves that the equations (I) and (II) are always mutually compatible, in such a fashion that one can choose the $\alpha_i^{(ab)}$ arbitrarily, so:

In a cyclic system of freely pliable elastic elements the distortions may be chosen in a completely arbitrary fashion.

7. If the elastic elements are not all freely pliable then formulas (4) and (5) might not admit solutions where the unknowns $\Xi_i^{(ab)}$ are not all zero. The same situation may be encountered in the case where the formulas (4) and (5) are satisfied only for zero values of the quantities $\Xi_i^{(ab)}$. In the first case, the distortions cannot be chosen arbitrarily, whereas in the second case the distortions are arbitrary. In addition, in the first case, the efforts are not determined, while in the second case, they are.

One immediately sees that if the elastic elements are simple, rectilinear rods then we will have the latter or the former case, respectively, according to whether the system is or is not statically determinate.

IV.

1. Equations (I) and (II) present close analogies with the Kirchhoff equations for the propagation of currents in a system of conducting wires that form a network; however, in our case there are six Kirchhoff equations. The components of the efforts appear in equations (I) and (II) like elements analogous to the current intensities, the components of the translations and rotations of the nodes, appear like elements analogous to the electric potentials at the nodes of the network, and the *characteristics of the distortions* replace the *electromotive forces.* The relations (I) replace the equation that expresses Ohm's law. The constants of the elastic elements have the same role as the inverses of electric resistances.

2. Once this analogy has been established, it is easy to examine some cases that present themselves in a manner that is analogous to the Wheatstone bridge in the study of electricity, and to profit from them by determining the constants of the electric elements.

3. The *principle of equivalent cuts* (Chap. II, Art. I, § 1) permits us to substitute a distortion that is carried out on a given section with another one that is performed in a section that is obtained from the first one by continuous deformation.

One understands that in practice one will have a simple means of obtaining distortions in a cyclic system of pliable elements when one performs them at the nodes, which one may do in the same manner by which one attached the extremities of the elements between them.

V.

1. Before passing on to the next chapter, where we propose to treat a particular case, we would like to prove the following general theorem and several other propositions:

The direct constants $A_{is}^{(ab)}$ (*see* Art. II) *of each element verify the equations:*

$$A_{is}^{(ab)} = A_{si}^{(ab)} .$$

Indeed, suppose that the *efforts* $X_i^{(ab)}$ correspond to the displacements $x_i^{(a)}$ and $x_i^{(b)}$ (i = 1, 2, …, 6) of the extremities A and B of the element AB and that the *efforts* $\Xi_i^{(ab)}$ correspond to the displacements $\xi_i^{(a)}$ and $\xi_i^{(b)}$ (i = 1, 2, …, 6). We call the two deformations that the element AB is subjected to the *first* and *second* deformations, respectively.

Consider the quantity:

$$\sum_{i=1}^{6} (x_i^{(b)} - x_i^{(a)}) \Xi_i^{(ab)} .$$

It is the work done by the tensions that generate the second deformation of the element AB by virtue of the first deformation of the element. The quantity:

$$\sum_{i=1}^{6}(\xi_i^{(b)} - \xi_i^{(a)})X_i^{(ab)}$$

is the work done by the tensions that generate the first deformation of the element *AB* by virtue of the second deformation.

However, by virtue of a general principle of elasticity [Betti's theorem (*see* Chap. III, Art. II, § 1)] that we extend to pliable, elastic bodies, these two works are equal and, as a consequence:

$$\sum_{i=1}^{6}(x_i^{(b)} - x_i^{(a)})\Xi_i^{(ab)} = \sum_{i=1}^{6}(\xi_i^{(b)} - \xi_i^{(a)})X_i^{(ab)}$$

or:

$$\sum_{i=1}^{6}\sum_{s=1}^{6}A_{is}^{(ab)}X_s^{(ab)}\Xi_i^{(ab)} = \sum_{i=1}^{6}\sum_{s=1}^{6}A_{is}^{(ab)}\Xi_s^{(ab)}X_i^{(ab)} ;$$

this is why:

$$A_{is}^{(ab)} = A_{si}^{(ab)} .$$

Upon passing from the direct constants to the inverse ones $a_{is}^{(ab)}$ (Art. II, § 4), one obviously finds that the analogous relation:

$$a_{is}^{(ab)} = a_{si}^{(ab)}$$

is verified.

2. When one has *n* pliable, elastic elements $A_1 A_2$, $A_2 A_3$, …, $A_n A_{n+1}$, one may unite the one with the other in such a way that two consecutive elements $A_{i-1} A_i$, $A_i A_{i+1}$, have the common extremities A_i rigidly attached between them. One will thus obtain a unique elastic element $A_1 A_{n+1}$. One will call this constraint *a series composition of the given elements* and the element thus obtained, the *element composed by series.*

Call $A_{is}^{(a_h a_{h+1})}$ the direct constants of each element composing $A_h A_{h+1}$ and $A_{is}^{(a_1 a_{n+1})}$, the direct constants of the composed element, which are always referred to the same system of axes. By virtue of equations (3), we will have:

$$A_{is}^{(a_1 a_{n+1})} = \sum_{h=1}^{6} A_{is}^{(a_h a_{h+1})} ;$$

i.e.:

The direct constants of an element that is composed by series are obtained by adding the corresponding constants of the composed elements.

This theorem corresponds to the proposition that one encounters in the theory of electric conduction – viz., the resistance of several conductors connected in series is the sum of the electrical resistances of each conductor (*see* § 1 of the preceding Art.).

3. The constraint on several pliable elastic elements that they must define a composed element may also be accomplished in another way. Indeed, take n pliable elements $(AB)_1$, $(AB)_2$, …, $(AB)_n$ that have the same extremities A and B in their natural states, and suppose that they rigidly link the n extremities that are at A, as well as the n extremities that are at B. This constraint will be called a *composition of the given elements by derivation* – or *in parallel.* The composed element AB will be called the *element composed by derivation*.

Suppose that each composed element is freely pliable, and call $a_{is}^{(AB)_h}$, the inverse constant of each of them and $a_{is}^{(AB)}$, the inverse constant of the composed element; due to equations (3′), we will have:

$$a_{is}^{(AB)} = \sum_{h=1}^{n} a_{is}^{(AB)_h} ;$$

i.e.:

The inverse constants of an element that is composed by derivation are obtained by adding the corresponding constants of the composed elements.

This theorem also corresponds to a theorem on electric conduction. Indeed, the electric conductivity of a conductor formed from the union of several conductors in parallel is the sum of the conductivities of each component conductor.

___ . ___

CHAPTER IX

PLANAR CYCLIC SYSTEMS OF PLIABLE ELASTIC ELEMENTS

I.

1. When a pliable, elastic element *AB* is planar and subject to forces that are situated in its plane, if one takes it to be the first plane coordinate then one will find that the three characteristics of the efforts:

$$X_3^{(ab)}, \; X_4^{(ab)}, \; X_5^{(ab)}$$

are zero. Likewise, the characteristics of the displacements of the extremities:

$$x_3^{(a)}, \; x_4^{(a)}, \; x_5^{(a)}, \; x_3^{(b)}, \; x_4^{(b)}, \; x_5^{(b)}$$

will be zero.

In order to simplify, let x, y represent the coordinate axes, let $X^{(ab)}$, $Y^{(ab)}$, $M^{(ab)}$ represent the efforts, $X_1^{(ab)}$, $X_2^{(ab)}$, $X_6^{(ab)}$, resp., and let $x^{(a)}$, $y^{(a)}$, $r^{(a)}$ represent the characteristics, while $x_1^{(a)}$, $x_2^{(a)}$, $x_6^{(a)}$ are the displacements of the extremity *A*. At the same time, let $x^{(b)}$, $y^{(b)}$, $r^{(b)}$ indicate the quantities that correspond to $x_1^{(b)}$, $x_2^{(b)}$, $x_6^{(b)}$.

2. Having said this, we prove the following theorem:

If a pliable, planar, elastic element AB is subjected to forces in its plane then one will always find a pair of orthogonal axes x, y, in this same plane such that:

$$\text{(1)} \qquad \begin{cases} x^{(b)} - x^{(a)} = \lambda X^{(ab)}, \\ y^{(b)} - y^{(a)} = \mu X^{(ab)}, \\ r^{(b)} - r^{(a)} = \nu X^{(ab)}. \end{cases}$$

Indeed, we will have, in general, that:

$$\text{(2)} \qquad \begin{cases} x^{(b)} - x^{(a)} = a_{11} X^{(ab)} + a_{12} Y^{(ab)} + a_{13} M^{(ab)}, \\ y^{(b)} - y^{(a)} = a_{21} X^{(ab)} + a_{22} Y^{(ab)} + a_{23} M^{(ab)}, \\ r^{(b)} - r^{(a)} = a_{31} X^{(ab)} + a_{32} Y^{(ab)} + a_{33} M^{(ab)}, \end{cases}$$

in which $a_{rs} = a_{sr}$.

Upon transporting the origin to the point with coordinates ξ, η, without altering the direction of the axes, and upon distinguishing the quantities relative to the new system of axes by a suffix, we will have:

$$x_1^{(b)} - x_1^{(a)} = x^{(b)} - x^{(a)} + \eta(r^{(b)} - r^{(a)}),$$

$$\begin{aligned}
y_1^{(b)} - y_1^{(a)} &= y^{(b)} - y^{(a)} - \xi(r^{(b)} - r^{(a)}), \\
r_1^{(b)} - r_1^{(a)} &= r^{(b)} - r^{(a)},
\end{aligned}$$

$$\begin{aligned}
X_1^{(ab)} &= X^{(ab)}, \\
Y_1^{(ab)} &= Y^{(ab)}, \\
M_1^{(ab)} &= M^{(ab)} - X^{(ab)}\eta + Y^{(ab)}\xi;
\end{aligned}$$

as a result, formulas (2) will become:

$$(2') \quad \begin{cases}
x_1^{(b)} - x_1^{(a)} = (a_{11} + 2\eta a_{31} + \eta^2 a_{33})X_1^{(ab)} + (a_{12} + \eta a_{32} - \xi a_{31} - \xi\eta a_{33})Y_1^{(ab)} \\
\qquad\qquad + (a_{13} + \eta a_{33})M_1^{(ab)}, \\
y_1^{(b)} - y_1^{(a)} = (a_{21} - \xi a_{31} + \eta a_{23} - \xi\eta a_{33})X_1^{(ab)} + (a_{22} - 2\xi a_{23} + \xi^2 a_{33})Y_1^{(ab)} \\
\qquad\qquad + (a_{23} - \xi a_{33})M_1^{(ab)}, \\
r_1^{(b)} - r_1^{(a)} = (a_{31} + \eta a_{33})X_1^{(ab)} + (a_{32} - \xi a_{33})Y_1^{(ab)} + a_{33}M_1^{(ab)}.
\end{cases}$$

It will thus suffice to take:

$$\eta = -\frac{a_{13}}{a_{33}}, \qquad \xi = \frac{a_{23}}{a_{33}},$$

which is always possible if $a_{33} \neq 0$, in order to make the preceding equations become:

$$(2'') \quad \begin{cases}
x_1^{(b)} - x_1^{(a)} = \dfrac{a_{11}a_{33} - a_{13}^2}{a_{33}} X_1^{(ab)} + \dfrac{a_{12}a_{33} - a_{23}a_{13}}{a_{33}} Y_1^{(ab)}, \\
y_1^{(b)} - y_1^{(a)} = \dfrac{a_{12}a_{33} - a_{13}a_{23}}{a_{33}} X_1^{(ab)} + \dfrac{a_{22}a_{33} - a_{23}^2}{a_{33}} Y_1^{(ab)}, \\
r_1^{(b)} - r_1^{(a)} = a_{33}M_1^{(ab)}.
\end{cases}$$

If a_{33} is zero then a_{13} and a_{23} will also be zero (Chap. VIII, Art. II, § 1, 3rd property), and then the formulas (2) will originally have the form (2″).

Upon now changing the orientation of the axes – viz., upon choosing them to be the principal axes of the conic:

$$(a_{11}a_{33} - a_{13}^2)\, x^2 + 2(a_{12}\, a_{33} - a_{23}\, a_{13})\, xy + (a_{22}\, a_{33} - a_{23}^2)\, y^2 = a_{33},$$

we may reduce the formulas (2″) to the form (1).

3. We call the origin of the axes x, y the *center of the elastic element* for which formulas (1) are verified, and these axes themselves will be the *principal axes of the element*. The coefficients λ and μ are called the *coefficients of traction* and ν is the *coefficient of flexion*.

It is easy to prove the following theorem:

If the elastic element admits two symmetry axes then they are the principal axes of the element.

It is also easy to calculate the *constants of an element* relative to arbitrary axes when one knows the *coefficients of traction and flexion*.

Let ξ and η denote the coordinates of the center of the element relative to the axes x, y and let x', y' denote the principal axes. Let the table of cosines of the two systems of axes be:

	x'	y'
x	α	β
y	γ	δ

and let λ and μ be the coefficients of traction with respect to the x' and y' axes. Then formulas (2), relative to the axes x, y, take the form:

$$(3)\quad \begin{cases} x^{(b)} - x^{(a)} = (\lambda\cos^2\theta + \mu\sin^2\theta + \nu\eta^2)X^{(ab)} + [(\lambda-\mu)\sin\theta\cos\theta - \nu\xi\eta]Y^{(ab)} \\ \qquad\qquad - \nu\eta M^{(ab)}, \\ y^{(b)} - y^{(a)} = [(\lambda-\mu)\sin\theta\cos\theta - \nu\xi\eta)X^{(ab)} + [\lambda\sin^2\theta + \mu\cos^2\theta + \nu\xi^2]Y^{(ab)} \\ \qquad\qquad + \nu\xi M^{(ab)}, \\ y^{(b)} - y^{(a)} = -\nu\eta X^{(ab)} + \nu\xi Y^{(ab)} + \nu M^{(ab)}. \end{cases}$$

4. Let $A_1 A_2$, $A_2 A_3$, …, $A_n A_{n+1}$ be n planar elements, and suppose they are rigidly linked pairwise by common extremities A_2, A_3, …, A_n – i.e., suppose that they are composed in series (*see* Chap. VIII, Art. V).

Mark the quantities λ, μ, ν, ξ, η, θ by the index i when they are referred to the i^{th} element. The formulas relative to the planar element generated by means of the constraint between the given elements will then be:

$$\begin{aligned} x^{(A_{n+1})} - x^{(A_1)} = &\sum_{i=1}^{n}(\lambda_i\cos^2\theta_i + \mu_i\sin^2\theta_i + \nu_i\eta_i^2)X^{(A_1A_{n+1})} \\ &+ \sum_{i=1}^{n}[(\lambda_i - \mu_i)\sin\theta_i\cos\theta_i - \nu_i\xi_i\eta_i)Y^{(A_1A_{n+1})} \\ &- \sum_{i=1}^{n}\nu_i\eta_i M^{(A_1A_{n+1})}, \end{aligned}$$

$$
\begin{aligned}
y^{(A_{n+1})} - y^{(A_1)} = \; & \sum_{i=1}^{n} [(\lambda_i - \mu_i)\sin\theta_i \cos\theta_i - \nu_i \xi_i \eta_i) X^{(A_1 A_{n+1})} \\
& + \sum_{i=1}^{n} (\lambda_i \sin^2\theta_i + \mu_i \cos^2\theta_i + \nu_i \xi_i^2) Y^{(A_1 A_{n+1})} \\
& + \sum_{i=1}^{n} \nu_i \xi_i M^{(A_1 A_{n+1})} ,
\end{aligned}
$$

$$
r^{(A_{n+1})} - r^{(A_1)} = - \sum_{i=1}^{n} \nu_i \eta_i X^{(A_1 A_{n+1})} + \sum_{i=1}^{n} \nu_i \xi_i Y^{(A_1 A_{n+1})} + \sum_{i=1}^{n} \nu_i M^{(A_1 A_{n+1})} .
$$

However, if the axes x, y are the principal axes of the composed element $A_1 \, A_{n+1}$ then one will have:

$$
\sum_{i=1}^{n} \nu_i \xi_i = 0, \qquad \sum_{i=1}^{n} \nu_i \eta_i = 0,
$$

$$
\sum_{i=1}^{n} (\lambda_i - \mu_i)\sin\theta_i \cos\theta_i - \nu_i \, \xi_i \, \eta_i = 0.
$$

In addition, the coefficients of traction and the coefficients of flexion will be:

$$
\begin{aligned}
A &= \sum_{i=1}^{n} (\lambda_i \cos^2\theta_i + \mu_i \sin^2\theta_i + \nu_i \eta_i^2) , \\
B &= \sum_{i=1}^{n} (\lambda_i \sin^2\theta_i + \mu_i \cos^2\theta_i + \nu_i \xi_i^2) , \\
N &= \sum_{i=1}^{n} \nu_i ,
\end{aligned}
$$

respectively, from which, the following theorems fall out:

The center of a composed element is the center of gravity of the centers of the component elements if one supposes that at each of them there is a concentrated mass equal to the coefficient of flexion.

From the center of an element that is composed in series, draw unitary segments that are normal to the axes of each component element, and at the extremity of each of them concentrate a mass equal to the corresponding coefficient of traction, and, at the same time, consider masses equal to the coefficients of flexion of each component element that are concentrated at the centers, respectively. The axes of inertia of this system of masses are the principal axes of the composed element and its principal moments of inertia of it are the coefficients of traction.

The coefficient of flexion of an element that is composed in series is the sum of the coefficients of flexion of each component element.

II.

1. Now, consider an arbitrary planar elastic body that is doubly connected, and subject it to distortions that keep it planar. If the axes x, y are situated in the same plane, and if we indicate the six characteristics of the distortions by l, m, n, p, q, r then we will have (*see* Chap. III):

$$n = p = q = 0$$

and, upon representing the efforts by L, M, N, P, Q, R one will have:

$$N = P = Q = 0,$$

while the relations between the characteristics and the efforts become (*see* Chap. III, Art. II, § 3):

$$\begin{aligned} L &= E_{11}\, l + E_{12}\, m + E_{16}\, r, \\ M &= E_{21}\, l + E_{22}\, m + E_{26}\, r, \\ N &= E_{61}\, l + E_{62}\, m + E_{66}\, r. \end{aligned}$$

Upon transporting the origin to the point with coordinates ξ, η without altering the direction of the axes, we will have:

$$\begin{aligned} L_1 &= E_{11}\, l_1 + E_{12}\, m_1 + (E_{16} - E_{11}\eta + E_{12}\,\xi)\, r_1\,, \\ M_1 &= E_{11}\, l_1 + E_{12}\, m_1 + (E_{16} - E_{11}\eta + E_{12}\,\xi)\, r_1\,, \\ R_1 &= (E_{61} - E_{11}\eta + E_{21}\,\xi)\, l_1 + (E_{12} - E_{12}\,\eta + E_{22}\,\xi)\, m_1 \\ &\qquad + (E_{66} - 2E_{16}\eta + 2E_{26}\,\xi + 2E_{26}\,\xi^2 - 4E_{12}\xi\eta + 2E_{22}\,\xi^2)\, r_1\,, \end{aligned}$$

where $L_1 = L$, $M_1 = M$, R_1; l_1, m_1, $r_1 = r$ are the efforts and the characteristics relative to the new system of axes.

However, we may choose the coordinates ξ and η in such a manner that the coefficients of r_1 in the expressions for L_1 and M_1 are annulled, and as a result also the coefficients of l_1 and m_1 in the expression of R_1 .

Upon then conveniently orienting the axes, we may reduce the relations between the characteristics and the efforts to the following form:

$$\begin{aligned} L &= E_{11}\, l, \\ M &= E_{22}\, m, \\ R &= E_{66}\, r; \end{aligned}$$

i.e.:

Being given a doubly-connected, planar system that is subjected to distortions that preserve the plane, there exists a system of axes in that plane such that each elementary distortion produces only the conjugate effort relative to that system.

2. Now, suppose that the doubly-connected, planar system is formed from pliable elements A_1A_2 , $A_2\,A_3$, …, $A_n\,A_1$ that are rigidly linked pair-wise between themselves by the common extremities – i.e., are obtained from a series composition – and that in the natural state the first and last extremity A_1 coincide and are rigidly linked.

In order to obtain the axes that we spoke of in the preceding paragraph, it will suffice to apply the rules given in § 4 of the preceding article to find the center and principal axes of the element that is composed in series from the elements A_1A_2, A_2A_3, …, A_nA_1. The three coefficients of the efforts are obtained by calculating the inverses of the coefficients of traction and the coefficient of flexion of the composed element.

3. In the preceding chapter (Art. IV), we compared the theory of distortion of a system composed of pliable elastic elements with the theory of Kirchhoff on the propagation of currents in wires. The results that we just obtained shed a new light on the relationships that exist between the two theories.

Indeed, the theorem that we just proved in § 4 of the preceding article – i.e., *the coefficients of flexion of the composed circuit is the sum of coefficients of flexion of each element* – corresponds to this proposition: *The electrical resistance of a circuit is the sum of the resistances of all of the parts which, when connected in series, define the circuit itself* (*see* Chap. VIII, Art. V, § 2). However, the rule for obtaining the coefficients of traction is much more complicated, and does not have an analogue in the theory of electrical conduction. In addition, the consideration of the center and the principal axes, which is fundamental in the present theory, is missing completely from the electrical theory.

III.

1. In article I, we treated the case of a planar system of pliable, elastic elements connected in series, and determined the axes and coefficients of traction and flexion of the composed element if one knows the analogous axes and coefficients of the component elements.

We now propose to solve the same question by studying a composition of elements in parallel (by derivation) (*see* the preceding Chap., Art. V, § 3).

2. Upon supposing that we have a freely pliable planar element, we will have, upon referring to its principal axes [*see* formula (1), Art. 1]:

$$X^{(ab)} = \frac{1}{\lambda}(x^{(b)} - x^{(b)}),$$

$$Y^{(ab)} = \frac{1}{\mu}(y^{(b)} - y^{(b)}),$$

$$M^{(ab)} = \frac{1}{\nu}(r^{(b)} - r^{(b)}).$$

With the aid of very simple calculations, one finds that if the principal axes x', y' form angles with the x and y axes whose table of cosines is:

	x'	y'
x	α	β
y	γ	δ

and if the center of the element has the coordinates ξ, η then the formulas that express the efforts by means of the displacements will be:

$$X^{(ab)} = \left(\frac{1}{\lambda}\alpha^2 + \frac{1}{\mu}\beta^2\right)(x^{(b)} - x^{(a)}) + \left(\frac{1}{\lambda}\alpha\gamma + \frac{1}{\mu}\beta\delta\right)(y^{(b)} - y^{(a)})$$
$$+ \left(\frac{1}{\lambda}\eta\alpha - \frac{1}{\mu}\xi\beta\right)(r^{(b)} - r^{(a)}),$$

$$Y^{(ab)} = \left(\frac{1}{\lambda}\alpha\gamma + \frac{1}{\mu}\beta\delta\right)(x^{(b)} - x^{(a)}) + \left(\frac{1}{\lambda}\gamma^2 + \frac{1}{\mu}\delta^2\right)(y^{(b)} - y^{(a)})$$
$$+ \left(\frac{1}{\lambda}\eta\gamma - \frac{1}{\mu}\xi\delta\right)(r^{(b)} - r^{(a)}),$$

$$M^{(ab)} = \left(\frac{1}{\lambda}\eta\alpha - \frac{1}{\mu}\xi\beta\right)(x^{(b)} - x^{(a)}) + \left(\frac{1}{\lambda}\eta\gamma - \frac{1}{\mu}\xi\delta\right)(y^{(b)} - y^{(a)})$$
$$+ \left(\frac{1}{\nu} + \frac{1}{\lambda}\eta^2 + \frac{1}{\mu}\xi^2\right)(r^{(b)} - r^{(a)}).$$

3. Compose the n elements $(AB)_1$, $(AB)_2$, …, $(AB)_n$. Upon distinguishing the quantities that relate to the element $(AB)_h$ with an index h, we will have the following formulas with respect to the composed element:

$$X^{(ab)} = \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\alpha_h^2 + \frac{1}{\mu_h}\beta_h^2\right)(x^{(b)} - x^{(a)}) + \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\alpha_h\gamma_h + \frac{1}{\mu_h}\beta_h\delta_h\right)(y^{(b)} - y^{(a)})$$
$$+ \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\eta_h\alpha_h - \frac{1}{\mu_h}\xi_h\beta_h\right)(r^{(b)} - r^{(a)}),$$

$$Y^{(ab)} = \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\alpha_h\gamma_h + \frac{1}{\mu_h}\beta_h\delta_h\right)(x^{(b)} - x^{(a)}) + \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\gamma_h^2 + \frac{1}{\mu}\delta_h^2\right)(y^{(b)} - y^{(a)})$$
$$+ \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\eta_h\gamma_h - \frac{1}{\mu_h}\xi_h\delta_h\right)(r^{(b)} - r^{(a)}),$$

$$M^{(ab)} = \sum_{h=1}^{n}\left(\frac{1}{\lambda_h}\eta_h\alpha_h - \frac{1}{\mu_h}\xi_h\beta_h\right)(x^{(b)} - x^{(a)}) + \left(\frac{1}{\lambda_h}\eta_h\gamma_h - \frac{1}{\mu_h}\xi_h\delta_h\right)(y^{(b)} - y^{(a)})$$

$$+\sum_{h=1}^{n}\left(\frac{1}{\nu_h}+\frac{1}{\lambda_h}\eta_h^2+\frac{1}{\mu_h}\xi_h^2\right)(r^{(b)}-r^{(a)})\,.$$

The following theorem emerges from these formulas:

From an arbitrary point, draw arbitrary unitary segments that are parallel to the axes, and, at the extremity of each of them, concentrate a mass equal to the inverse of the corresponding coefficient. The axes of inertia of that set of masses are parallel to the principal axes of the composed element and the principal moments of inertia are the inverses $1/\lambda$ *and* $1/\mu$ *of the coefficients of traction.*

Upon considering these axes of inertia to be coordinate axes, the coefficients of $r^{(b)} - r^{(a)}$ *in the expressions for* $X^{(ab)}$ *and* $Y^{(ab)}$ *will be respectively equal to the coordinates* η *and* ξ *of the center of the composed element, multiplied by* $1/\lambda$ *and* $-1/\mu$.

Bibliography

In addition to the works cited in this article, one must add:

MAGGI, *Sull' interpretazione del nuovo theorema di Volterra sulla teoria dell' elasticà* (*Rend. Acc. dei Lincei*, vol. XIV, 2nd sem.).

TIMPE, *Probleme der Spannungsverteilung in ebenen Systemen einfach gelöst mit Hilfe der Airyschen Function.* [*Inaugural Dissertion, Göttingen* (Leipzig, 1905). – *Zeitschrift für Math. und Phys.*, Bd. LII. – See *Rend. Acc. dei Lincei*, vol. XV, 1st sem., pp. 521.]