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Mechanics of solid bodies in the plastically-deformable state

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With 4 figures in the text

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Up to now, the mechanics of continua that are endowed with the general notion of stress that was created by Cauchy has been almost exclusively applied to fluid and solid elastic bodies. Saint-Venant ([1]) has outlined a theory for the domain of plastic or permanent changes of form that, however, does not yield the required number of equations for the determination of the motion. Other occasional attempts in this direction have not arrived at any conclusion, either ([2]).

The following discussion leads to a complete Ansatz for the equations of motion for plastically-deformable bodies in the context of Cauchy’s mechanics and is connected with certain facts of experience that characterize the domain of application.

§ 1. Notations.

Let the stress state at a point in a body be given by the three normal stresses σ_x, σ_y, σ_z and the tangential stresses τ_x, τ_y, τ_z, based upon a rectangular coordinate system. In the matrix:

$$\begin{matrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{matrix} , \tag{1}$$

the quantities in the first row then mean the components of the stress vector $\bar{\sigma}_x$ for an outer surface element whose exterior normal has the direction of the positive x-axis, etc. We shall also briefly refer to the vector structure that is represented by (1), which transforms in the well-known way by means of:

([1]) Comptes rendus, Paris, t. 70, 72, 74. Journ. de math. 1871, pp. 473.

([2]) Haar and v. Kármán, Göttinger Nachr. 1909, derived equations of motion from a new variational principle whose relationship to the rest of mechanics is still not clear.

(2) $$\bar{\sigma}_{x'} = \bar{\sigma}_x \cos(x, x') + \bar{\sigma}_y \cos(y, x') + \bar{\sigma}_z \cos(z, x'),$$

by the *stress dyadic* $\bar{\bar{\sigma}}$.

Analogous constructions lead to the *deformation dyadic* $\bar{\bar{\varepsilon}}$ and the *deformation velocity dyadic* $\bar{\bar{\lambda}}$. If one denotes an infinitely small elastic displacement of a point by ξ, η, ζ then the extensions and angle changes are equal to:

(3)
$$\varepsilon_x = \frac{\partial \xi}{\partial x}, \qquad \varepsilon_y = \frac{\partial \eta}{\partial y}, \qquad \varepsilon_z = \frac{\partial \zeta}{\partial z},$$
$$\gamma_x = \frac{1}{2}\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right), \quad \gamma_y = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right), \quad \gamma_z = \frac{1}{2}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right),$$

and the dyadic $\bar{\bar{\varepsilon}}$ has the matrix:

(4) $$\begin{matrix} \varepsilon_x & \gamma_z & \gamma_y \\ \gamma_z & \varepsilon_y & \gamma_x \\ \gamma_y & \gamma_x & \varepsilon_z \end{matrix} .$$

If one takes u, v, w for the components of the velocity vector instead of ξ, η, ζ then one obtains the extension velocity and displacement velocity:

(5)
$$\lambda_x = \frac{\partial u}{\partial x}, \qquad \lambda_y = \frac{\partial v}{\partial y}, \qquad \lambda_z = \frac{\partial w}{\partial z},$$
$$v_x = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad v_y = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad v_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

and the matrix for the dyadic $\bar{\bar{\lambda}}$:

(6) $$\begin{matrix} \lambda_x & v_z & v_y \\ v_z & \lambda_y & v_x \\ v_y & v_x & \lambda_z \end{matrix} .$$

For any dyadic, there exists at least one coordinate cross for which the matrix reduces to the terms on the main diagonal, and thus, for (1), to the form:

(7) $$\begin{matrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{matrix} .$$

In this, σ_1, σ_2, σ_3 are the "principal stresses," which are determined from the roots of the secular equation, or also by the following three conditions:

$$
(8)\qquad
\begin{aligned}
&\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z ,\\
&\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\,\sigma_1 = \sigma_x\,\sigma_y + \sigma_y\,\sigma_z + \sigma_z\,\sigma_x - (\tau_x^2 + \tau_y^2 + \tau_z^2),\\
&\sigma_1\,\sigma_2\,\sigma_3 = \begin{vmatrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{vmatrix}.
\end{aligned}
$$

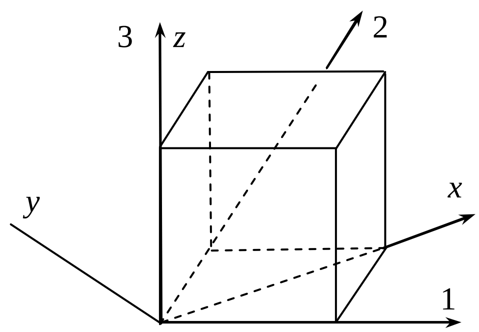

Figure 1.

If one constructs a coordinate cross such that the z-axis coincides with the third principal axis, while the x-axis and y-axis bisect the angle between the first two principal axes (Fig. 1) then, as a result of (2), this yields the following matrix:

$$
(9)\qquad
\begin{matrix}
\dfrac{\sigma_1+\sigma_2}{2} & \dfrac{\sigma_2-\sigma_1}{2} & 0 \\
\dfrac{\sigma_2-\sigma_1}{2} & \dfrac{\sigma_1+\sigma_2}{2} & 0 \\
0 & 0 & \sigma_3
\end{matrix} .
$$

Likewise, one can see that the values of τ that appear here are *extrema* of the tangential stress; i.e., one always finds the absolutely largest and smallest tangential stress among the three quantities:

$$
(10)\qquad \tau_1 = \frac{\sigma_3-\sigma_2}{2}, \qquad \tau_2 = \frac{\sigma_1-\sigma_3}{2}, \qquad \tau_3 = \frac{\sigma_2-\sigma_1}{2}.
$$

The simplest of all stress dyadics is that of the ideal fluid $-\overline{\overline{p}}$. In any coordinate system, it has the matrix:

(11) $$\begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} .$$

If one subtracts a stress state of the form (11) from the stresses that are represented by (1) then the *tangential stresses remain unchanged*, and there is a matrix:

(12) $$\begin{matrix} \sigma'_x & \tau_z & \tau_y \\ \tau_z & \sigma'_y & \tau_x \\ \tau_y & \tau_x & \sigma'_z \end{matrix} ,$$

with

(13) $$\sigma'_x = \sigma_x + p, \quad \sigma'_y = \sigma_y + p, \quad \sigma'_z = \sigma_z + p.$$

The dyadic (12) has the same principal directions as (1) and principal values σ'_1, σ'_2, σ'_3 are the principal values of (1), reduced by – *p*. With that, it follows from (10) that the principal tangential stresses are identical for (12) and (1).

All of these relations are naturally valid for the deformation dyadic $\bar{\bar{\varepsilon}}$ or for $\bar{\bar{\lambda}}$, as well.

We now give a formula that will be employed in what follows that arises from combining (10) with (8). It is:

(14) $$\begin{aligned} \tau_1^2 + \tau_2^2 + \tau_3^2 &= \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{1}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= \tfrac{1}{2}(\sigma_1 + \sigma_2 + \sigma_3)^2 - \tfrac{3}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= \tfrac{1}{2}(\sigma_x + \sigma_y + \sigma_z)^2 - \tfrac{3}{2}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \tfrac{3}{2}(\tau_x^2 + \tau_y^2 + \tau_z^2) \\ &= \left(\frac{\sigma_z - \sigma_y}{2}\right)^2 + \left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tfrac{3}{2}(\tau_x^2 + \tau_y^2 + \tau_z^2). \end{aligned}$$

§ 2. Experimental basis.

We now cite some facts of experience that will explain the calculations in the following equations of motion. We shall not claim to produce an axiomatic structure; i.e., we abstain from employing a precise minimum of assumptions.

a) *All solid bodies behave like elastic ones for sufficiently small stresses: a one-to-one correspondence exists between stresses and deformations.*

With this postulate, we limit the solid bodies to inviscid ones. An example of a "solid" would be ductile wax, which admittedly already yields to minor external pressures, or iron, which first reaches its elasticity limit at very high pressure. By contrast, pitch, or the like, is not plastically-deformable at normal temperatures, but fluid.

We will discuss the meaning and form of the elasticity limit later on.

As is known, the mathematical theory of elasticity assumes that the connection between the stress dyadic $\bar{\bar{\sigma}}$ and the deformation dyadic $\bar{\bar{\varepsilon}}$ is *linear:*

(15) $$\bar{\bar{\sigma}} = L(\bar{\bar{\varepsilon}}) .$$

The most general linear relationship under which no direction in space is preferred consists of the one for which the two dyadics have the same principal directions and their principal values are coupled by:

(16) $$\sigma_1 = \alpha\,\varepsilon_1 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \qquad \sigma_2 = \alpha\,\varepsilon_2 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$
$$\sigma_3 = \alpha\,\varepsilon_3 + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

In this, α and β are elastic constants. In a well-known way, (16) can be converted in such a way that relations arise between the components relative to arbitrary axes.

b) *If the elasticity limit is reached then the solid body behaves essentially like a viscous, almost incompressible fluid.*

The behavior of the fluid that we intend here is characterized by the fact that it is not the deformation *state* that generates stresses, as for elastic bodies, but the deformation *process*. However, one cannot simply assume that the stress dyadic $\bar{\bar{\sigma}}$ is a function of the deformation velocity dyadic $\bar{\bar{\lambda}}$ now, but one must observe that a volume element experiences no finite deformation velocity under an external *pressure that is the same for all time*. The *change in volume* that therefore comes about always comes from the order of magnitude of the elastic distortions, as observation shows accordingly.

It then follows, in turn, that in the mechanics of viscous fluids one must subtract a part $-\bar{\bar{p}}$ from the stress dyadic $\bar{\bar{\sigma}}$ that corresponds to a pressure that is the same for all time. The remainder $\bar{\bar{\sigma}}'$ [cf., (12) in § 1] can then be described as a linear function:

(17) $$\bar{\bar{\sigma}}' = L(\bar{\bar{\lambda}}) .$$

If one observes the same symmetry as above then, analogous to (16), one has:

(18) $$\sigma_1' = k\,\lambda_1 + k'\,(\lambda_1 + \lambda_2 + \lambda_3), \ \ldots$$

However, the expression in parentheses measures precisely the divergence – or change in volume – that was mentioned above, such that it is negligibly small compared to λ_1. One thus gets:

(19) $$\sigma_1' = k\,\lambda_1 , \qquad \sigma_2' = k\,\lambda_2 , \qquad \sigma_3' = k\,\lambda_3 .$$

These equations say nothing more than the fact that $\bar{\bar{\sigma}}'$ emerges from $\bar{\bar{\lambda}}$ when one multiplies each component of $\bar{\bar{\lambda}}$ by k, thus:

(20) $\sigma'_x = \sigma_x + p = k\,\lambda_x\,,\quad \sigma'_y = \sigma_y + p = k\,\lambda_y\,,\quad \sigma'_z = \sigma_z + p = k\,\lambda_z\,,$
$$\tau_x = k\,\nu_x\,,\qquad \tau_y = k\,\nu_y\,,\qquad \tau_z = k\,\nu_z\,.$$

These are precisely the same equations that the *Navier-Stokes* theory of viscous fluids led to. An essential difference will arise when we examine the meaning of the quantity k more closely. That would lead to the following, decisive empirical postulate:

c) *If one changes the velocity with which a motion proceeds, while preserving all relationships between absolute values, then for plastically-deformable bodies the work that is needed to arrive at a certain change in form does not change.*

We justify this postulate on the basis of all the observed materials that have been examined up to now in the realm of permanent changes of form, namely, in engineering. For the most part, engineering employs formulas for the work done that omit the influence of velocity from the outset. Whenever this influence is especially noticeable, it proves to be slight ([1]). One will have to regard the constancy that is stated in the postulate c) as similar to the constancy of friction coefficients under varying normal pressures under the sliding friction of solid bodies. At the very least, the assumption in c) specifies an *ideal case* that admits a well-defined theory, and which necessarily implies a approximation in the actual behavior of bodies.

The second work done per second per unit volume is generally given by:

(21) $\sigma'_x\lambda_x + \sigma'_y\lambda_y + \sigma'_z\lambda_z + 2\,\tau_x\,\nu_x + 2\,\tau_y\,\nu_y + 2\,\tau_z\,\nu_z$
$$= k\,(\lambda_x^2 + \lambda_y^2 + \lambda_z^2 + 2\nu_x^2 + 2\nu_y^2 + 2\nu_z^2)\,.$$

If one multiplies all velocities by a factor c then this expression changes by a proportionality factor kc^2. Likewise, however, the duration of the deformation process is shortened by the ratio 1 : c, so the total work will then be proportional to kc. Therefore, the proportionality factor k that was introduced into (20) will be inversely proportional to the velocity. One can also say: The stress dyadic $\bar{\bar{\sigma}}'$ remains the same when all components of $\bar{\bar{\lambda}}$ will be changed in the same ratio.

From the latter formulation, it follows that the stresses in a plastically-deformable body must vary in a domain *of slight multiplicity* when compared to, perhaps, an elastic one. It is clear that this domain cannot be anything but the *elastic limit*; i.e., our postulate c) can also be stated:

c′) *The stresses always remain at the elastic limit under plastic deformations.*

This rule implies the demand that the elastic limit must be independent of an additive contribution of the form (11). (cf., *infra*)

One can verify c′ immediately by observation. In the one-dimensional case of the tensile loading of a rod, from c′), the stress-extension diagram must have the form in Fig. 2: First, one has an inclined line for the elastic state that converges to the velocity-

([1]) This is detailed, along with references, in my encyclopedia article IV 10, no. 5, pp. 187.

independent limiting stress in the plastic domain. Observation now shows that for iron, steel, and similar materials, in fact, a horizontal piece connects to the inclined line, but it soon turns into a weakly-increasing line. This goes back to a process that is linked to the crystalline nature of the body and strongly thermally-influenced, and which one calls "solidification." Thus, our theory does not account for this solidification. However, one must consider that the actual domain of application for plasticity theory lies in the domain of the load pressure (viz., positive *p*). It is still not sufficiently clear whether such a solidification also exists for pressures in iron, etc. In any event, it does not seem unlikely that "solidification" plays a very minor role for slightly ductile bodies – like wax, et al.

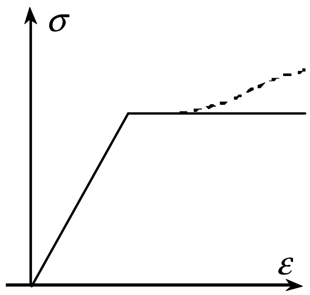

Figure 2.

We turn to a last postulate that concerns the nature of the elastic limit:

d) *In a coordinate system that has the principal tangential stresses for its coordinates, the elastic limit appears as a closed curve in the plane that includes the origin:*

(22) $$\tau_1 + \tau_2 + \tau_3 = 0.$$

As is known, one can thank O. Mohr for the first thorough examination of the elastic and fracture limits ([1]). For Mohr, only the largest and smallest of the three principal stresses entered in – say, σ_1 and σ_2 . In a coordinate system:

(23) $$x = \frac{\sigma_1 + \sigma_2}{2}, \quad y = \frac{\sigma_1 - \sigma_2}{2} = -\tau_3 ,$$

when one considers not only the work of Mohr, but also the recent research of von Kármán ([2]), the fracture limit appears as something like Fig. 3. The big difference

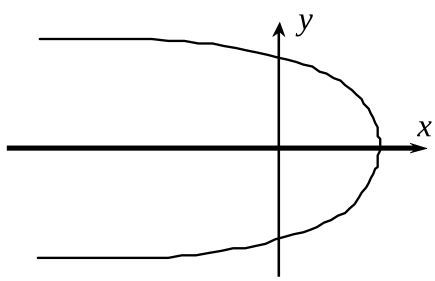

Figure 3

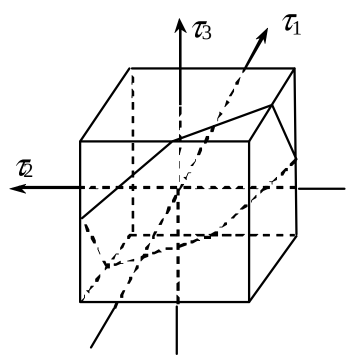

Figure 4

([1]) O. Mohr, Abhandl. a. d. Bebiete der techn. Mechanik, Berlin, 1906, pp. 187.
([2]) Zeitschr. d. Vereines deutscher Ingenieure 1911, pp. 1749.

between the behavior for positive x (tension) and negative x (pressure) is based in the fact that there is tearing when there is tension on all sides, but no crushing under pressure that is the same on all sides. It is not very likely that the analogous situation also exists for the limit of elastic behavior. Since we have, moreover, dealt with states of large mean pressure, first and foremost, it will be permitted for us to regard the horizontal asymptotes in Fig. 3 as the essential limit. This notion, which is also repeatedly advocated, leads to the elastic limit:

$$|\tau_1| \leq K, \qquad |\tau_2| \leq K, \qquad |\tau_3| \leq K. \tag{24}$$

The cube (24) will be cut by the plane (22) in a regular hexagon (Fig. 4), such that our condition d) is fulfilled.

However, we would like to modify Mohr's Ansatz in yet another direction. Only the *vertices* of the hexagon (22), (24) are established by the attempt up to this point, which are states for which one of the τ is zero, while the absolute values of the other two are equal. The rectilinear connection arises from the assumption that the middle principal stress (the smallest principal tangential stress, resp.) is not involved at all. This assumption does not seem very plausible, since one may not try to replace the hexagon with a simpler structure, such as a *circumscribed circle.* In place of the cube (24), one would then have the cone:

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = 2K^2. \tag{25}$$

In any event, (25) would admit a much simpler analytical treatment, except that the difference compared to (24) would be larger than the scope of the present attempt.

§ 3. Equations of motion.

We let ρ denote the specific mass of the body, and let κ_x, κ_y, κ_z denote the components of the specific volume force (gravity, etc.). In any case, the equations of motion then read:

$$\text{(I)} \qquad \begin{aligned}
\rho \frac{du}{dt} &= \kappa_x - \frac{\partial p}{\partial x} + \frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z}, \\
\rho \frac{dv}{dt} &= \kappa_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_z}{\partial x} + \frac{\partial \sigma_y'}{\partial y} + \frac{\partial \tau_x}{\partial z}, \\
\rho \frac{dw}{dt} &= \kappa_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_x}{\partial y} + \frac{\partial \sigma_z'}{\partial z}.
\end{aligned}$$

The six stress components σ_x', …, τ_z are expressed, from (20) and (5), in terms of the three velocity quantities u, v, w as follows:

(II)
$$\sigma'_x = k\frac{\partial u}{\partial x},\ \sigma'_y = k\frac{\partial v}{\partial y},\ \sigma'_z = k\frac{\partial w}{\partial z},$$
$$\tau_x = \tfrac{1}{2}k\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\ \tau_y = \tfrac{1}{2}k\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\ \tau_z = \tfrac{1}{2}k\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

As in hydrodynamics, the elimination of p gives the continuity equation:

(III)
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0.$$

Incompressibility is assumed in this, corresponding to postulate b) and the remarks connected with it, although the more general case can also be treated in the context of our theory with no further assumptions.

The Ansatz that is comprised of (I) to (III) agrees completely with that of the viscous fluid, except that in that theory the quantity k is the given viscosity, while for us it is a *reaction* quantity that can be first calculated from a knowledge of the motion itself. To that end, it will suffice to state that the stress remains at the elastic limit during plastic deformation.

If one chooses the boundary to be a circle in the form (25) and substitutes the value (14) into it then one gets:

(26)
$$(\sigma'_x+\sigma'_y+\sigma'_z)^2 - 3(\sigma'_x\sigma'_y+\sigma'_y\sigma'_z+\sigma'_x\sigma'_z)+3(\tau_x^2+\tau_y^2+\tau_z^2) = 4K^2.$$

Then, from the final form of the expression (14), it follows that one can also replace the σ with the σ'. However, if one adds the first three of eq. (II) and observes (III) then one finds:

(27)
$$\sigma'_x+\sigma'_y+\sigma'_z = 0,$$

such that (26) reduces to:

(IV)
$$\frac{4K^2}{3} = \tau_x^2+\tau_y^2+\tau_z^2-(\sigma'_x\sigma'_y+\sigma'_y\sigma'_z+\sigma'_x\sigma'_z).$$

If one substitutes the value from (II) here then one has the desired equation for k. Equations (I) to (IV) are the *complete system of equations of motion for plastically-deformable bodies.*

Here, one must introduce the boundary condition: The given of the *velocity components u*, *v*, *w* for each outer surface point. However, this can be replaced with the given of the *outer surface stress* on the entire outer surface or a part of it.

In the case of a planar motion, our Ansatz reduces to that of *Saint-Venant*. In part, this is based upon the fact that in the planar case the difference between the elastic limits obtained from either (24) (hexagon) or (26) (circle) vanishes. One therefore now has only two principal tangential stresses τ_1, τ_2 with:

(28) $$\tau_1 + \tau_2 = 0,$$

such that $\tau_1^2 + \tau_2^2 \leq 2K^2$ says the same thing as $|\tau_1| \leq K$, $|\tau_2| \leq K$.

One can write eq. (I) to (IV) very simply with the use of vector symbolism. If $\bar{v}$ denotes the velocity vector and $\bar{\kappa}$ denotes the specific force vector then one has:

(I′) $$\rho \frac{d\bar{v}}{dt} = \bar{\kappa} - \text{grad}\, p + \nabla \bar{\bar{\sigma}}',$$

(II′) $$\bar{\bar{\sigma}}' = k\bar{\bar{\lambda}},$$

(III′) $$\text{div}\, \bar{v} = 0,$$

(IV′) $$-(\bar{\bar{\sigma}}')_2 = \frac{4K^2}{3}.$$

In this, the symbol ∇ in (I′) means the differentiation that is performed on the dyadic that is determined by (I). The index 2 in (IV′) shall imply that the second of the orthogonal invariants that were written down in § 1, eq. (8) is to be taken.

One can easily eliminate $\bar{\bar{\sigma}}'$ from (I′) to (IV′), and one gets:

(a) $$\rho \frac{d\bar{v}}{dt} = \bar{\kappa} - \text{grad}\, p + \nabla(k\bar{\bar{\lambda}}),$$

(b) $$\text{div}\, \bar{v} = 0,$$

(c) $$k^2 = -\frac{4K^2}{3(\bar{\bar{\lambda}})_2}.$$

If one scalar multiplies (I′) by $\bar{v}$ and integrates over the volume then one finds, after a corresponding conversion, that the dissipation function will be represented by (21), with which the agreement between the present Ansatz and our Ansatz c) in § 2 is proved.

Strassburg i. E., 4 October 1913.
