“Sulle superficie di discontinuità nella teoria della elasticità dei corpi solidi,” Rend. Reale Accad. dei Lincei, classe di sci., fis., mat. e nat., ser. 5, v. 10.1 (1901), 57-60.

On the surface of discontinuity in the theory of elasticity for solid bodies.

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It seems to me that in the theory of equilibrium for elastic solids the only case that has been considered up to now is that of a body whose particles are subjected to some displacements of their natural position that vary from point to point continuously in all of the space that is occupied by the body itself. Under such hypotheses, if it is not acted on by any external forces, either on its contour or its internal space, then the body is not subject to internal tension.

Therefore, there certainly do not exist bodies that are subject to internal tension and which are not subject to external force on either the contour or the interior. To give an imaginary example, take a ring that is not completely closed whose two free plane sections are attached to each other by an infinitely thin stratum that welds them together.

A body that is internally stressed and that is not subject to external forces must necessarily contain one or more surfaces along which its displacement is discontinuous. If the tension that is inside it is continuous in all of the space that is occupied by the body then it will have the character of an isolated body. However, if the tension is not continuous then wherever its displacement is discontinuous the body must regarded as having the character of an aggregate of several distinct bodies. In the latter case, the discontinuity does not yield any new general theorems on the internal properties of the body. On the contrary, no sharp change in the internal tension exists, and the discontinuity in the displacement across the surface is subject to simple and noteworthy laws that I propose to develop in this Note.

Suppose one has a solid in a state of tension that is not subject to external action and is referred to three orthogonal coordinate axes x, y, z. Let u, v, w denote the components, relative to these axes, of the displacement of a point P in its natural position, and consider these components to be functions of the coordinates of that point.

The tension that is developed in the interior of the body is a linear function of the six quantities:

$$\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial w}{\partial z}, \qquad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}, \qquad \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}, \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

which are the coefficients of the arbitrary variations δx, δy, δz in the quadratic form:

$$\delta x\ \delta u + \delta y\ \delta v + \delta z\ \delta w.$$

Since we supposed that the internal tension is a continuous function in all of the space that is occupied by the body, it so happens that these six coefficients enjoy the same property.

Now, let S be the internal surface of discontinuity of the displacement. Distinguish the two sides of it by means of the indices a and i, and denote by u_a, v_a, w_a and u_i, v_i, w_i, respectively, the components of its displacements at the two material points that coincide at the point x, y, z of the surface S. In addition, let α, β, γ be the values of the discontinuity that is experienced by the values of the quantities u, v, w when one traverses the surface S from one side to the other. One can consider α, β, γ to be functions of the coordinates x, y, z, although these coordinates are linked by the equation of the surface. Along it, one must therefore satisfy the following three equations:

$$u_a - u_i = \alpha, \qquad v_a - v_i = \beta, \qquad w_a - w_i = \gamma,$$

and if one refers to the points in the infinitesimal vicinity of the points of the surface by $(x + dx, y + dy, z + dz)$ then one will have:

$$du_a - du_i = d\alpha, \qquad dv_a - dv_i = d\beta, \qquad dw_a - dw_i = d\gamma.$$

We then examine the difference:

$$(dx\, du_a + dy\, dv_a + dz\, dw_a) - (dx\, du_i + dy\, dv_i + dz\, dw_i),$$

and observe that of the two quadratic forms that comprise it, the coefficients of dx, dy, dz must coincide, by virtue of the continuity that we have supposed, in order that the difference between them should be annulled.

We may then write the equation:

$$dx\, d\alpha + dy\, d\beta + dz\, d\gamma = 0$$

for any point of the surface S, from which follows the theorem:

One can consider the three discontinuities α, β, γ at any point of S to be the rectangular coordinates of the points of a new surface that corresponds to S by the orthogonality of linear elements.

In other words, if one subjects the point (x, y, z) of the surface of discontinuity to a geometric displacement that has the components α, β, γ then one will obtain a surface that is infinitely close to S.

This new form of the preceding theorem amounts to the hypothesis that one neglects the powers of u, v, w, and their derivatives higher than the first, as is always done in the theory of elasticity for solids.

The two different particles that coincide at the point (x, y, z) of the surface from the two sides of it have, in the natural state of the body, the coordinates x_a, y_a, z_a and x_i, y_i, z_i, respectively, and verify the equations:

$$x_a = x - u_a\,, \qquad y_a = y - v_a\,, \qquad z_a = z - w_a\,,$$
$$x_i = x - u_i\,, \qquad y_i = y - v_i\,, \qquad z_i = z - w_i\,,$$

and afterwards, the linear elements ds_a and ds_i of the two surfaces that are made to coincide after the deformation of the body from its natural state will have the squares:

$$ds_a^2 = ds^2 - 2(dx\,du_a + dy\,dv_a + dz\,dw_a),$$
$$ds_i^2 = ds^2 - 2(dx\,du_i + dy\,dv_i + dz\,dw_i).$$

In order for the difference $ds_a^2 - ds_i^2$ to be annulled, it must happen that these two surfaces can be mapped onto each other.

There is a well-known theorem on the deformation of surfaces that says that it is not possible for there to exist two distinct mappable surfaces that a have common corresponding line, unless there is no common asymptotic line of the two surfaces. By virtue of this, there exists an essential difference between the surfaces of discontinuity of the bodies that occupy a multiply-connected space and those of the bodies that fill up a simply-connected space. In the former case, the surface can possess a cut of an arbitrary form, and interrupting the material connection along this surface might render the body in its neutral natural state. The slit that this gives rise to will be composed of the two *separate* mappable surfaces. On the contrary, the surface of discontinuity of the simply-connected bodies cannot be composed of slits, insofar as in that case the body splits, must give rise to a slit that is formed of the two mappable surfaces, one over the other, and having a common corresponding line, a condition that cannot be verified, in general, except for the particular case that was recalled previously.

However, even in this exceptional case, they cannot appear distinctly enough to affect the mechanical aspects of an infinitely thin line of soldering that makes the adjacent surface elements coincide at the edge formed by the asymptotic line. Does there exist a gap at this edge, no matter how small it might be?

No matter what one might say in this regard, the considerations that follow are independent of the preceding doubts.

If you cut a multiply-connected body with cuts that do not coincide with the original surface of discontinuity and put it into a new neutral state then the separate surfaces of the bounding slit of the will be mapped onto each other. Now, if, by means of a small elastic deformation of the body, one closes this slit and reattaches its sides to each other with a thin layer of solder then the body will generally be in a new state of tension.

In order for this state to coincide with the original one that was destroyed, the following conditions are necessary: One must have that at the original surface of the discontinuity of the body the discontinuity α, β, γ has one of the values:

$$\alpha = a + qz - ry, \qquad \beta = b + rx - pz, \qquad \gamma = c + py - qx,$$

in which a, b, c, p, q, r are constants.

The three analogous quantities for the surface of discontinuity that come from the new soldering possess the same form. This statement is easily proved.