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ON THE ELECTRODYNAMICS OF THE VACUUM ON THE BASIS OF THE QUANTUM THEORY OF THE ELECTRON

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One of the most important results in the new development of the theory of the electron is the possibility of converting electromagnetic field energy into matter. A light quantum, for example, can, by the existence of other electromagnetic fields in empty space, be absorbed and converted into matter, in the form of a pair of electrons with opposite charges.

The conservation of energy demands that the absorbing field must be static so that the absorbed light quantum will necessarily impart energy to the electron pair that is produced. Its frequency must therefore satisfy the relation $h\nu = 2\,mc^2 + \varepsilon_1 + \varepsilon_2$, in which mc^2 is the rest energy of the electron, and ε_1 and ε_2 are the remaining energies of the two electrons. We must consider this case, for example, in the production of electron pairs by a γ-quantum in the Coulomb field of an atomic nucleus.

Absorption can also be found in fields that arise from other light quanta, in which the latter can carry the energy of the electron pair, such that in this case the energy $2\,mc^2 + \varepsilon_1 + \varepsilon_2$ of the two electrons must be equal to the sum of all the light quanta that are absorbed in this process.

The phenomenon of the absorption of light in vacuo represents an essential deviation from MAXWELLian electrodynamics. Namely, the vacuum shall be independent of the fields in it for a light wave that freely penetrates it, since fields can be superimposed independently in it as a result of the linearity of the MAXWELL equations themselves.

It is already understandable without the introduction of the special theory of relativity that in fields that do not possess the necessary energy for the creation of an electron pair deviations from MAXWELLian electrodynamics must arise: if high-frequency light can be absorbed into electromagnetic fields then so must one expect the scattering or deflection of a light ray whose frequency is not enough for pair creation, analogous to the scattering of light by an atom whose smallest absorption frequency is larger than that of the light. The light in its passage through the electromagnetic fields will thus behave as if the vacuum took on a dielectric constant that differs from unity as a result of the action of the fields.

In order to represent these phenomena one must attribute certain properties with the theory of empty space that would produce the desired deviation from MAXWELLian electrodynamics. In fact, the relativistic wave mechanics of electrons also leads to such

consequences, if one uses the states of negative kinetic energy that follow from the DIRAC wave equation for the description of the vacuum.

The basic assumption of DIRAC's theory of the positron is that the physical behavior of the vacuum can be described, in a certain sense, by the behavior of an infinite set of electrons – the vacuum electrons – that are found in states of negative energy and collectively define its state. It is self-explanatory that the determination cannot be complete, since the vacuum electrons must possess infinite charge and current density, which must have no physical meaning. However, it shows that, for example, pair creation (and the opposite process) can be regarded as a jump of a vacuum electron into a state of positive energy under the influence of electromagnetic fields, so it appears to be a real electron, whereas the vacuum around a negative electron is poorer, which must follow from the appearance of a positive electron. The calculations for pair creation and annihilation that come out of this picture show good agreement with experiments.

The calculation of most other effects that follow from the theory of positrons always runs into the problem of the degree to which the behavior of vacuum electrons is to actually be regarded as that of the vacuum. This problem is complicated by the fact that the charge, current, and energy densities of the vacuum electrons are infinite, such that one must generally break off a finite piece of this infinite sum and associate it with reality. The solution of this problem was carried out by DIRAC and HEISENBERG, who gave a method for determining the physically meaningful part of the effects of the vacuum electrons that was free of contradictions. In what follows, it will be shown that this determination is completely of any arbitrariness, since it consequently assumes that only the following properties of the vacuum electrons are physically meaningless:

$$(1)\qquad \begin{cases} 1) & \text{The energy of the vacuum electrons in field-free space.} \\ 2) & \text{The charge and current densities of the vacuum electrons in field-free space.} \\ 3) & \text{A spatially and temporally constant field-independent electric and magnetic polarizability of the vacuum.} \end{cases}$$

These quantities [1] relate only to a field-free vacuum, and it may be regarded as self-explanatory that they can have no physical meaning. All three quantities prove to be divergent sums of contributions from all vacuum electrons. It must be further added that a constant polarizability will be in no way established, but only total charge and field strength values, multiplied by a constant factor.

In the next section, we will compute the physical properties of the vacuum that are slowly varying in time and space on the basis of these assumptions. We understand this to mean such fields F that vary only slightly over distance of length[2] $\frac{h}{mc}$ and time intervals of length $\frac{h}{mc^2}$, and thus satisfy the conditions:

[1] In the sequel, the assumption that 1), 2), or 3) is to be regarded as meaningless will be denoted by I_1, I_2, or I_3, resp.

[2] h is PLANCK's constant divided by 2π.

$$(1) \qquad \frac{h}{mc}\,|\,\text{grad}\,F\,| << |\,F|\,, \qquad\qquad \frac{h}{mc^2}\left|\frac{\partial F}{\partial t}\right| << |\,F\,|\,.$$

In general, the presence of such fields will create no pairs, since the light quanta that appear have too little energy. We would like to exclude the extreme cases in which the radiation density is so high as to allow the collective effect of very many quanta or in which electrostatic fields with potential differences of greater than $2mc^2$ are present (in these cases, pairs will be created, on the grounds of the KLEIN paradox). Under these circumstances the electromagnetic properties of the vacuum may be represented by a field-independent electric and magnetic polarizability of empty space, which leads to, for example, the splitting of light by electric fields or the scattering of light by light. For weak fields, the dielectricity and permeability tensor of the vacuum has the following approximate form ($\vec{E}, \vec{B}, \vec{D}, \vec{H}$ are the four electromagnetic field quantities [1]):

$$D_i = \sum_k \varepsilon_{ik} E_k\,, \qquad\qquad H_i = \sum_k \mu_{ik} B_k\,,$$

$$(2) \qquad \varepsilon_{ik} = \delta_{ik} + \frac{e^4 h}{45\pi m^4 c^7}\left[2(E^2 - B^2)\delta_{ik} + 7B_i B_k\right]$$

$$\mu_{ik} = \delta_{ik} + \frac{e^4 h}{45\pi m^4 c^7}\left[2(E^2 - B^2)\delta_{ik} - 7E_i E_k\right] \qquad \delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

The computation of these quantities has already been carried out by EULER and KOCKEL [2], as well as HEISENBERG and EULER [3]. In the next section, a somewhat simpler method will be employed. In addition, the properties of the vacuum shall be calculated on the basis of the scalar relativistic wave equation for the electron of KLEIN and GORDON. In PAULI and WEISSKOPF, this wave equation yielded the existence of positive and negative particles, as well as their creation and annihilation by electromagnetic fields, without any further particular assumptions. Thus, these particles possess no spin and obey Bose statistics, so this theory is not applicable to real-world electrons. It is therefore worthy of note that this theory also leads to properties of the vacuum for which no physical meaning can be attached. One thus obtains, for example, a likewise infinite spatially and temporally constant field-independent polarizability of the vacuum. By neglecting the corresponding terms, one arrives at results that are similar to those of DIRAC's positron theory. The physical properties of the vacuum originate in the "zero-point energy" of matter, which also depends on absent particles through the external field strengths and therefore contributes an additional term to the purely MAXWELLian field energy.

In section 3, we will treat the consequences of the DIRAC theory of positrons for the case of general external fields, and we will show that on the basis of the aforementioned three assumptions concerning the effects of vacuum electrons one always comes to finite and unique results. The HEISENBERG subtraction prescription proves itself to be

[1] In the sequel, arrows will be placed over vector quantities when confusion is possible.
[2] H. EULER and B. KOCKEL, Naturwiss., **23**, 246, 1935; H. EULER, Ann. d. Phys., v. **26**, 393.
[3] W. HEISENBERG and H. EULER, ZS. f. Phys. **38**, 714, 1936.

identical with these three assumptions and thus appears somewhat less arbitrary than was assumed in the prior literature.

None of the following calculations explicitly consider the interactions of the vacuum electrons, but exclusively consider a single vacuum electron under the influence of a given field. However, by this choice of path the opposite effect is not completely neglected since one can by no means separate the external field from the field that is created by the vacuum electrons themselves, such that the field that enters into the calculations implicitly partially includes the action of the other vacuum electrons. This process is analogous to the HARTREE calculation of the electron orbits of an atom in the field that is produced by the electrons themselves. For the explicit calculation of the interaction one must employ quantum electrodynamics, i.e., perform the quantization of the wave fields. As is well known, even without the assumption of infinitely many vacuum electrons this already leads to divergences, and shall not be pursued any further in what follows.

II.

In this section, the electrodynamics of the vacuum shall be treated for fields that satisfy conditions (1). The field equations are established by being given the energy density U as a function of the field strengths. We determine them from the energy density $\tilde{U}$ of the vacuum electrons, which shall be definitive of the behavior of the vacuum.

It is advantageous to recall the Lagrange function L of the electromagnetic field since it is already completely established by the requirement of relativistic invariance. The following relations exist between the Lagrange function L and the energy density U:

$$U = \sum_i E_i \frac{\partial L}{\partial E_i} - L . \tag{3}$$

In MAXWELLian electrodynamics, one has:

$$L = \frac{1}{8\pi}(E^2 - B^2), \qquad U = \frac{1}{8\pi}(E^2 + B^2) .$$

Anything that is added to this Lagrange function must, like the Lagrange function itself, be relativistically invariant. As long as we only restrict ourselves to slowly varying fields (condition (1)), these additional terms will only depend upon the values of the field strengths and not on their derivatives. They can therefore be functions of only the invariants $(E^2 - B^2)$ and $(EB)^2$. If we develop the additional terms in powers of the field strengths up to sixth order then we obtain:

$$L = \frac{1}{8\pi}(E^2 - B^2) + L',$$
$$L' = \alpha (E^2 - B^2)^2 + \beta (EB)^2 + \xi (E^2 - B^2)^3 + \zeta (E^2 - B^2)^2 (EB)^2 + \ldots$$

and therefore, from (3):

$$(4)\qquad \begin{cases} U = \dfrac{1}{8\pi}(E^2 + B^2) + U' \\ U' = \alpha(E^2 - B^2)(3E^2 + B^2) + \beta(EB)^2 + \\ \qquad + \xi(E^2 - B^2)^2(5E^2 + B^2) + \varsigma(EB)^2(3E^2 - B^2) + \cdots \end{cases}$$

The addition to the energy density is therefore completely determined by the invariance properties; in what follows, it will thus be necessary only for us to determine the constants α, β, ξ, ζ, …, that appear in it. These statements are already based upon the special assumption that U' includes no terms of second order in the field strengths, but only ones of higher order. This is equivalent to the statement that the vacuum possesses no polarizability independently of the fields.

The calculations of EULER and KOCKEL, as well as those of HEISENBERG and EULER, yield the following values for the constants:

$$\alpha = \frac{1}{360\pi^2}\frac{e^4 h}{m^4 c^7}, \qquad \beta = 7\alpha, \qquad \xi = \frac{1}{630\pi^2}\frac{e^6 h^3}{m^8 c^{12}}, \qquad \zeta = \frac{13}{2}\xi .$$

The dielectricity and permeability tensor that is given by (2) is obtained from the relations:

$$D_i = 4\pi\frac{\partial L}{\partial E_i}, \qquad H_i = -4\pi\frac{\partial L}{\partial B_i} .$$

In what follows, we shall derive these results in an essentially simply fashion.

The additional term U' to the MAXWELL vacuum energy density shall be determined from the additional term $\tilde{U}'$ that the vacuum electrons contribute. The energy density due to the presence of electrons in the states ψ_1, ψ_2, …, ψ_i … is given by:

$$U = \frac{1}{8\pi}(E^2 + B^2) + \tilde{U}'$$
$$\tilde{U}' = \sum_i \left\{ \psi_i^*, \left[\left(\vec{\alpha}, \frac{hc}{i}\,\mathrm{grad} + e\vec{A} \right) + \beta mc^2 \right] \psi_i \right\},$$

in which $\vec{\alpha}$, β are the DIRAC matrices and $\vec{A}$ is the vector potential. The additional term $\tilde{U}'$ to the MAXWELL density is therefore not equal to the total material energy density U_{mat} :*

* Here and in the sequel, when two eigenfunctions ψ and φ are placed between curly brackets: $\{\psi, \varphi\}$ means the inner product of the two spinors ψ and φ: $\{\psi, \varphi\} = \sum_k \psi^k \varphi^k$, in which k is the spin index.

(5) $$U_{\text{mat}} = ih\sum_i \left\{ \psi_i^*, \frac{\partial}{\partial t}\psi_i \right\},$$

but:

(6) $$\tilde{U}' = U_{\text{mat}} - \sum_i \{\psi_i^*, eV\psi_i\},$$

in which V is the scalar potential. One can identify $\tilde{U}'$ as the kinetic energy density. The total material energy density U_{mat} may, as we will see, be easily computed: the second term of (6) – the potential energy density – can be obtained from U_{mat} in the following way: When one thinks of the scalar potential as proportional to the constant factor λ then one has: [1]

(7) $$\lambda \int \sum_i \{\psi_i^*, eV\psi_i\} d\tau = \lambda \frac{\partial}{\partial \lambda} \int U_{\text{mat}} d\tau,$$

in which the integration is carried out over all of space. In the limiting case of constant fields, which, from the conditions (1), we shall consider here, we can regard the field strength E itself as the constant factor λ, and can, moreover, employ the relation (7) for the energy density. We then obtain the kinetic energy density as:

(7a) $$\tilde{U}' = U_{\text{mat}} - E\frac{\partial U_{\text{mat}}}{\partial E}.$$

If one equates this with (3) then one sees that the same relationship exists between the material and kinetic energy densities that exists between $-L$ and U. Thus, U_{mat} can be set equal to the aforementioned addition to the Lagrange function:

(8) $$\tilde{U}' = -\tilde{L}'.$$

Since the form of U' is completely determined by the requirement of relativistic invariance, it suffices to determine $\tilde{U}'$ for a particular field. We choose a homogeneous magnetic field $B = (B_x, 0, 0)$ and a spatially periodic electric field that is parallel to it, and whose potential is given by:

(9) $$V = V_0 e^{\frac{igx}{h}} + V_0^* e^{-\frac{igx}{h}}.$$

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[1] The proof proceeds as follows: When the energy operator H is independent of a parameter λ then the diagonal element H_{ii} of the energy operator changes by an infinitesimal adiabatic increment $d\lambda$ of λ according to:

$$dH_{ii} = \left(\frac{\partial H}{\partial \lambda}\right)_{ii} d\lambda.$$

When we now set:

$$H = H_0 + \lambda eV,$$

this yields:

$$\lambda (eV)_{ii} = \lambda \frac{\partial H_{ii}}{\partial \lambda}.$$

We thus equate this result with the general form (4) and will determine the coefficients of this form from that.

By contrast, HEISENBERG and EULER chose a constant electric field, which results in complications as a consequence of the KLEIN paradox: Any arbitrarily small homogeneous electric field creates electron pairs when it extends over all of space. The electron content of the energy density is therefore not precisely stationary. In the previous calculations, due to periodicity one can avoid the appearance of potential differences greater than $2mc^2$ so that no pair creation comes about.

The material energy density is, by the fact that it represents all negative energy states, given by:

(10) $$U_{\text{mat}} = \sum_i W_i \{\psi_i^*, \psi_i\}.$$

W_i is the energy that belongs to the eigenfunction ψ_i, and the summation is over all negative states; the sum is obviously infinite. Which finite piece of this sum has any physical meaning will be uniquely deducible from the explicit expression for U_{mat}.

The ψ_i obey the wave equation:

(11) $$\left\{\frac{ih}{c}\frac{\partial}{\partial t} - \frac{eV}{c} + \alpha_x ih\frac{\partial}{\partial x} + K\right\}\psi = 0$$

(12) $$K = \alpha_y ih\frac{\partial}{\partial y} + \alpha_x\left[ih\frac{\partial}{\partial x} - \frac{e}{c}|B|y\right] - \beta mc.$$

We temporarily follow the computations of HEISENBERG and EULER (loc. cit.), in which we only make inessential changes to the meanings of the variables.

For a solution, we start with:

(13) $$\psi_i = \frac{1}{\sqrt{2\pi h}} e^{\frac{i}{h}p_z z} \cdot u(y)X(x).$$

When the operator K is applied to ψ twice this yields:

$$K^2\psi = \left[-h^2\frac{\partial^2}{\partial y^2} - i\alpha_y\alpha_z\frac{eh}{c}|B| + \left(p_z + \frac{e}{c}|B|y\right)^2 + m^2c^2\right]\psi.$$

We now set:

$$\eta = \left(y + \frac{2p_z h}{b}\right)\sqrt{\frac{b}{2h^2}}, \qquad b = \frac{2eh}{c}|B|.$$

b is the magnitude of the magnetic field. By the introduction of η we arrive at the fact that K^2 has the form of the Hamilton function for an oscillator. We therefore set:

$$u(y) = \tilde{H}_n(\eta)\left(\frac{b}{2h^2}\right)^{1/4},$$

in which $\tilde{H}_n(\eta)$ is the n^{th} normalized eigenfunction. One then has $\int |\, u(y)\,|^2\, dy = 1$ and:

$$K^2\psi = \left\{ m^2c^2 + b\left(n + \frac{1-\sigma_x}{2} \right) \right\}\psi , \qquad \sigma_x = i\alpha_y\, \alpha_z . \tag{14}$$

It now remains to choose a representation for the four-component ψ in which σ_x is diagonal:

$$\sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

The first two components of ψ correspond to a positive spin in the *x*-direction and the other two, to a negative one. By this choice, the wave equation (11) decomposes into two separate systems of equations for the two components with the same spin, such that we obtain two wave equations for two-rowed matrices. The operator *K* may then be written in the form $K = \gamma |\, K\,|$, in which γ is a two-rowed matrix that satisfies the condition $\gamma^2 = 1$ and $|\, K\,|$ means the ordinary number:

$$|\, K\,| = \sqrt{m^2c^2 + b\left(n + \frac{1-\sigma_x}{2} \right)} ,$$

which depends upon the value σ_x of the spin. Likewise, the matrix α_x that enters into the wave equation is also two-rowed, and it anticommutes with γ: $\alpha_x\gamma + \gamma\alpha_x = 0$, since, from (12), α_x also anticommutes with *K*. The two wave equations may then be written in the form:

$$\left\{ \frac{ih}{c}\frac{\partial}{\partial t} + \alpha_x ih\frac{\partial}{\partial x} - \frac{e}{c}V + \gamma |\, K\,| \right\}\psi = 0 , \tag{16}$$

in which α_x and γ are two-rowed matrices that associate the same spin only to a pair of components. The difference between the wave equations for the two spin directions lies only in the differing values for $|\, K\,|$. Since the dependence of ψ on the variables *y* and *z* has already been established by (13), (16) represents a wave equation for just the function *X*(*x*). Up till now, the sequence of calculations has been essentially identical to those of HEISENBERG and EULER.

We now treat the case $V = 0$ to begin with. The eigenvalues and the normalized eigenfunctions for (16) read:

$$X_n^{(\pm)}(p_x) = a^{(\pm)}(p_x)\frac{1}{\sqrt{2\pi h}} e^{\frac{ip_x x}{h}} \cdot e^{\frac{iW_n^{\pm}(p_x)}{h} t} \tag{17}$$

(18) $$W_n^{(\pm)}(p_x) = \pm c\sqrt{p_x^2 + |K|^2} = \pm c\sqrt{p_x^2 + m^2c^2 + b\left(n + \frac{1-\sigma_x}{2}\right)}.$$

The upper index (+) or (–) distinguishes the positive and negative energy states. $a^{\pm}(p)$ is a normalized 2-component "spinor." The equation (16) and its solutions (17), (18) represent a one-dimensional analogue of the DIRAC equation in which one finds $\gamma | K | \psi$, in place of the mass term $\beta mc \cdot \psi$. One positive and one negative energy eigenvalue are associated with an impulse p_x. (The other two energy eigenvalues yield the wave equation for opposite spin.)

If we now substitute these quantities in the energy density (10) then we obtain:

$$U_{\text{mat}} = \sum_{\sigma=-1}^{+1}\sum_{n=0}^{\infty}\iint \frac{dp_x dp_z}{2\pi h} W_n^-(p_x)\,|\tilde{H}(\eta)|^2 \left(\frac{b}{2h^2}\right)^{\frac{1}{2}} |X_n^{(-)}(p_x)|^2 .$$

As a result of the fact that $dp_x = \sqrt{\frac{b}{2}}d\eta$, as well as $|X_n^{(-)}(p)|^2 = \frac{1}{2\pi h}$, the integration over p_x yields:

(19) $$U_{\text{mat}} = \frac{b}{8\pi^2 h^3}\sum_{\sigma=-1}^{+1}\sum_{n=0}^{\infty}\int_{+\infty}^{-\infty} dp\, W_n^-(p).$$

From now on, we shall write p, instead of p_x.

In order to carry out the summation, we construct:

$$\sum_{\sigma=-1}^{+1}\sum_{n=0}^{\infty} W_n^- = W_0^- + 2\sum_{n=1}^{\infty} W_n^- .$$

We now apply the EULER summation formula for a function $F(x)$:

$$\frac{1}{2}F(a) + \sum_{r=1}^{N} F(a+rb) + \frac{1}{2}F(a+Nb) =$$
$$= \frac{1}{b}\left[\int_a^{n+Nb} F(x)dx - \sum_{m=1}^{\infty}(-)^m \frac{B_m}{(2m)!} b^{2m}\{F^{(2m-1)}(a+Nb) - F^{(2m-1)}(a)\right].$$

B_m is the m^{th} Bernoulli number. $F^{(m)}(x)$ is the m^{th} part of $F(x)$. When we apply this to (19), we obtain:

(20) $$U_{\text{mat}} = \frac{1}{4\pi^2 h^2}\int dp\left[\int_0^{\infty} F(x)\,dx + \sum_{m=1}^{\infty} b^{2m}\frac{B_m}{(2m)!}(-)^m F^{(2m-1)}(0)\right]$$
$$F(x) = -c\sqrt{p^2 + m^2c^2 + x}\,.$$

In the special case of a pure magnetic field, from (7a), one can set U_{mat} and $\tilde{U}'$ equal to each other. This expression already represents the energy density in a development in powers of the magnetic field strength *b*. Now it is very easy to determine any part of the contribution $\tilde{U}'$ of the vacuum electrons that shall be definitive for the real vacuum: The term that is independent of *b* represents the energy density of the field-free vacuum and is a divergent integral. Since the energy density must vanish for field-free vacuum, this expression can have no real meaning. Furthermore, the (likewise divergent) terms in b^2 must be omitted since the energy density shall possess no terms of second order in the field strengths. The omission of these terms is well-founded in the assumption that the polarizability of the vacuum tends to zero for a vanishing field. It must be emphasized that the subtraction described here is exclusively based on obvious assumptions about the field-free vacuum.

One then obtains the additional term to the MAXWELL energy density as:

$$U' = -\frac{c}{4\pi^2 h^2}\sum_{m=2}^{\infty}\frac{B_m(-)^m}{(2m)!}b^{2m}\,\frac{1\cdot 3\cdots(4m-5)}{2^{2m-1}}\int_{-\infty}^{+\infty}\frac{dp}{(p^2+m^2c^2)^{\frac{4m-3}{2}}}. \tag{21}$$

The power series may easily be represented by the power series development of the hyperbolic coth. One obtains:

$$U' = \frac{1}{8\pi^2}mc^2\left(\frac{mc}{h}\right)^2\int_0^{\infty}\frac{d\eta}{\eta^2}e^{-\eta}\left\{\eta\coth\eta\mathfrak{B}-1-\frac{\eta^2}{3}\mathfrak{B}^2\right\},$$

in which $\mathfrak{B}$ is the magnetic field strength measured in units of the critical field strength $\frac{m^2c^3}{eh}$:

$$\mathfrak{B} = \frac{eh}{m^2c^3}B.$$

$$U' = -\frac{1}{360\pi^2}\frac{e^4h}{m^4c^7}B^4 + \frac{1}{630\pi^2}\frac{e^6h^3}{m^8c^{13}}B^6 + \cdots$$

If we equate this with each term of (4) that includes fourth and sixth powers of the magnetic field then we obtain:

$$\alpha = \frac{1}{360\pi^2}\frac{e^4h}{m^4c^7}, \qquad \xi = \frac{1}{630\pi^2}\frac{e^6h^3}{m^8c^{13}}.$$

We will now consider an electric field, in addition. To this end, we solve the wave equation (16) for $X(x)$ with the method of the BORN approximation. We expect that the part of U_{mat} that depends on the potential *V* appears in the second approximation, proportional to V^2. When we develop U_{mat} in powers of *V*: $U_{\mathrm{mat}} = U_{\mathrm{mat}}^{(0)} + U_{\mathrm{mat}}^{(1)} + \ldots$ then, from (10), we obtain:

(22) $$U_{\mathrm{mat}}^{(2)} = \sum_i W_i^{-(2)} (|\psi_i|^2)^{(0)} + \sum_i W_i^{-(0)} (|\psi_i|^2)^{(2)} .$$

$W_i^{-(k)}, (|\psi_i|^2)^{(k)}$ are the k^{th} approximations in the corresponding developments of W_i^- and $|\psi_i|^2$. One observes that $W_i^{-(1)}$ vanishes in the given electric field. It may be easily shown that:

$$\int (|\psi|^2)^{(2)}\, dx\, dy\, dz = 0 ,$$

such that the spatial mean of U_{mat} is given by only the first term in (22):

$$\overline{U_{\mathrm{mat}}^{(2)}} = \sum_i W_i^{-(2)} (|\psi_i|^2)^{(0)} .$$

$(|\psi|^2)^{(0)}$ was already computed in the case of a pure magnetic field, and we thus obtain, in complete analogy to (19):

$$\overline{U_{\mathrm{mat}}^{(2)}} = \frac{b}{8\pi^2 h^3} \sum_{\sigma=-1}^{+1} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dp\, W_n^{-(2)}(p) .$$

The value of $W_i^{-(2)}$ may be computed by the method of the BORN approximation. With the eigenfunctions (17) one gets:

(23) $$\begin{cases} W_n^{-(2)} = e^2 |V_0|^2 \left[\dfrac{|\{a^{(+)*}(p+g), a^{(-)}(p)\}|^2}{W_n^-(p) - W_n^+(p+g)} + \dfrac{|\{a^{(-)*}(p+g), a^{(-)}(p)\}|^2}{W_n^-(p) - W_n^-(p+g)} \right] \\ \qquad + (\text{the same thing with } -g). \end{cases}$$

The expression between the { }-brackets represents a scalar product of two two-component spinors. By the integration of (23) over p, the second term in [ ]-bracket drops out if one carries out the integration of the terms in $-g$ over the variable $p' = p - g$:

(24) $$\begin{cases} \int dp\, W_n^{-(2)} = e^2 |V_0|^2 \int dp\, \dfrac{|\{a^{(+)*}(p+g), a^{(-)}(p)\}|^2}{W_n^-(p) - W_n^+(p+g)} + \\ \qquad + (\text{the same thing with } -g). \end{cases}$$

This approach is in no way unique since the integration of the second term in the { }-brackets of (23) leads to a divergent result, which, however, can be made finite by the addition of the corresponding terms in $-g$, or, as in (24), can be made to vanish no matter which manner by which one chooses the integration variables. However, this arbitrariness does not affect our computation since in performing the summation over n we employed only the terms that were proportional to b^2, b^4, etc., in which, on the basis of the EULER summation formula, only derivatives of $W_n^{-(2)}(p)$ up to n appear. As one

can easily convince oneself, these derivatives of second order in the [ ]-brackets no longer diverge under integration over *p*, such the result of this integration is independent of the choice of integration variables.

Moreover, from (24), this yields:

$$\int dp\, W_n^{-(2)} = -e^2 |V_0|^2 \frac{g^2}{4} \int dp \frac{|K|^2}{c(p^2+|K|^2)^{\frac{5}{2}}},$$

in which a series development in powers of *g* was already carried out and the terms of order higher than second were omitted. This means neglecting the derivatives of the field strength on the grounds of conditions (1). Likewise, as in the previous computation, $\overline{U_{\text{mat}}^{(2)}}$ is now given by (20) if we set:

$$F(x) = -e^2 |V_0|^2 \frac{g^2}{4} \frac{m^2c^2 + x}{c(p^2 + m^2c^2 + x)^{\frac{5}{2}}}.$$

One then obtains, if one first integrates over *p*:

$$\overline{U_{\text{mat}}^{(2)}} = -\frac{1}{4\pi^2 h^2 c} \frac{g^2}{3} e^2 |V_0|^2 \left[\int_0^\infty \frac{dx}{m^2c^2 + x} + \sum_{m=1}^{\infty} b^{2m} \frac{B_m (-)^m}{(2m)!} \left(\frac{d^{2m-1}}{dx^{2m-1}} \frac{1}{m^2c^2 + x} \right)_{x=0} \right].$$

Since this expression is quadratic in the electric field strengths, from (7a) we obtain for the kinetic energy density:

$$\overline{\tilde{U}^{(2)}} = -\overline{U_{\text{mat}}^{(2)}}.$$

From the previously discussed grounds, the terms of fourth and higher order in the field strengths can be regarded as physically meaningful for the vacuum, such that the divergent integral is omitted. We now replace V_0 with the electric field strength *E*:

$$\overline{E^2} = 2\frac{g^2}{h^2} |V_0|^2,$$

in which the overbar means the spatial mean, and we obtain for the first two terms:

$$U^{(2)} = \frac{5}{360\pi^2} \frac{e^4 h}{m^4 c^7} E^2 B^2 - \frac{7}{2} \frac{1}{360\pi^2} \frac{e^6 h^3}{m^8 c^{13}} E^2 B^4 + \cdots \tag{25}$$

in the limiting case of weakly varying fields whose spatial average can be neglected.

If one equates (25) with the terms in (4) that are proportional to $E^2 B^2$ and $E^2 B^4$ then one obtains the relations:

$$\beta - 2\alpha = \frac{5}{360\pi^2}\frac{e^4 h}{m^4 c^7}, \qquad 3\xi - \zeta = \frac{7}{2}\frac{1}{360\pi^2}\frac{e^6 h^3}{m^8 c^{13}},$$

and with the previously computed values for α and β:

$$\beta = 7\alpha, \qquad \zeta = \frac{13}{2}\xi.$$

The expression for $U^{(2)}$, which is exact in the magnetic field strength, is then given by:

$$U^{(2)} = \frac{1}{8\pi^2} mc^2 \left(\frac{mc}{h}\right)^2 \frac{1}{3}\mathfrak{E}^2 \int_0^\infty \frac{d\eta}{\eta} e^{-\eta}\{\eta\mathfrak{B}\coth\eta\,\mathfrak{B} - 1\}.$$

in which $\mathfrak{E} = \frac{eh}{m^2c^3}E$.

The higher approximations in E may be easily determined up to a constant factor. Let us think of the k^{th} approximation $W_n^{-(k)}(p)$ for the energy as being determined by the given p and n states: on the basis of the wave equation (16), it will have the following form:

$$W_n^{-(k)}(p) = g^k e^k \mid V_0 \mid^k \cdot G(e, h, \mid K \mid, p),$$

in which G is a function in which only the given quantities figure. As a result of the gauge invariance, $W^{(k)}$ must be of at least k^{th} order in g. The higher powers of g are disregarded. The energy density in k^{th} order then becomes:

$$(26) \qquad U_{\text{mat}}^{(k)} = \frac{-1}{4\pi^2 h^3} g^k e^k \mid V_0 \mid^k \left[\int_0^\infty dx \int_{-\infty}^{+\infty} G\,dp + \sum_{m=1}^{\infty} b^{2m} \frac{B_m}{(2m)!}(-)^m \left(\frac{d^{2m-1}}{dx^{2m-1}}\int_{-\infty}^{+\infty} G\,dp\right)_{x=0}\right].$$

The integral over G must have the dimension (energy)$^{-(k-1)}$ (impulse)$^{-(k-1)}$, and may longer depend upon the quantities c, h, $\mid K \mid$, which is possible only in the form:

$$\int_{-\infty}^{+\infty} G\,dp = f_k \frac{1}{c^{k-1}\mid K \mid^{2k-2}} = f_k \frac{1}{c^{k-1}(m^2c^2 + x)^{k-1}},$$

in which f_k is a numerical factor.

When one substitutes this in (26) $U_{\text{mat}}^{(k)}$ is given completely, up to a factor f_k.

However, the numerical factor f_k is easily determined by the fact that, from (8), U_{mat} must be relativistically invariant. Since U_{mat} must depend upon only $E^2 - B^2$ and $(EB)^2$ in this way, one must have, for instance, that the coefficient of E^4 differs from the coefficient of B^4 only by the factor $(-)^{k/2}$. The latter coefficient was already computed, and is given by (21). One then obtains:

$$f_{2m} = \frac{2^{3m-2} B^m}{m(2m-1)},$$

and can therefore calculate the representation that HEISENBERG and EULER have given for L'[1]:

$$L' = -\frac{1}{8\pi^2} mc^2 \left(\frac{mc}{h}\right) \int_0^\infty \frac{d\eta}{\eta^2} e^{-\eta} \left\{ \eta \mathfrak{B} \coth \eta \mathfrak{B} \cdot \cot \eta \mathfrak{B} - 1 + \frac{\eta^2}{3} (\mathfrak{E}^2 - \mathfrak{B}^2) \right\}$$

$$\mathfrak{E} = \frac{m^2 c^3}{eh} E, \qquad \mathfrak{B} = \frac{m^2 c^3}{eh} B.$$

This expression is computed for parallel fields. In order to generalize it to arbitrary fields, one must write is as a function of the two invariants $E^2 - B^2$ and $(EB)^2$. This is made possible by HEISENBERG and EULER in a simple way by the relation:

$$\cot \alpha \cot \beta = -i \frac{\cos \sqrt{\beta^2 - \alpha^2 + 2i\alpha\beta} + \text{conj.}}{\cos \sqrt{\beta^2 - \alpha^2 + 2i\alpha\beta} - \text{conj.}},$$

and one obtains:

$$L' = -\frac{1}{8\pi^2} \frac{e^2}{hc} \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ i\eta^2 (EB) \frac{\cos \eta \sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})} + \text{conj.}}{\cos \eta \sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})} - \text{conj.}} + \frac{m^4 c^6}{e^2 h^2} + \frac{\eta^2}{3} (B^2 - E^2) \right\}.$$

Due to the reality of the total expression this is actually dependent only on $E^2 - B^2$ and $(EB)^2$.

In the scalar theory, the computation of the energy density and Lagrange function of the vacuum is carried out with the same mathematical tools. An energy density of the vacuum comes about in this theory by way of the zero-point energy of the vacuum. The total energy is given by PAULI and WEISSKOPF (loc. cit., formula (29)) as:

$$E_{\text{mat}} = \sum_k W_k (N_k^+ + N_k^- + 1),$$

in which W_k is the energy of the k^{th} state and N_k^+ is the number of positrons, while N_k^- is the number of electrons that are associated with this state. In an empty vacuum, the sum is taken over all remaining energies W_k, in which the energy W_k, which characterizes the states of impulse p and the quantum number n in a magnetic field B, has the value:

[1] As for the question of convergence of this integral, we refer to the comments in the work of HEISENBERG and EULER on that matter (pp. 729).

$$W_n^{\mathrm{scal}}(p,B) = c\sqrt{p^2 + m^2c^2 + b\left(n+\frac{1}{2}\right)}\,.$$

The summation over all states and division by the total volume leads to the energy density, which, by analogy with (19), is easily given by:

$$U_{\mathrm{mat}} = \frac{b}{8\pi^2 h^3}\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} dp\, W_n^{\mathrm{scal}}(p,B)\,.$$

The only difference from the prior computation is in omitting the summation over the two spin directions. One now easily verifies the following relation between the energy $W_n^{\mathrm{scal}}(p,B)$ in the scalar theory of electrons and the energy $W_n^{-}(p,B)$ in DIRAC's theory:

$$2\sum_{n=0}^{N} W_n^{\mathrm{scal}}(p,B) = \sum_{\sigma=-1}^{+1}\sum_{n=0}^{N} W_n^{-}(p,B) - \sum_{\sigma=-1}^{+1}\sum_{n=0}^{2N} W_n^{-}(p,B/2)\,.$$

We can thus express the energy density $\tilde{U}'_{\mathrm{scal}}$ in the scalar theory in terms of the energy density $\tilde{U}'$ in the DIRAC theory of positrons in the following way:

$$2\tilde{U}'_{\mathrm{scal}}(B) = \tilde{U}'(B) - 2\tilde{U}'(B/2)\,.$$

One thus sees here, as well, that the quadratic part that is independent of the field strengths is infinite. The latter thus yields an infinite polarizability that is independent of the field strengths. In order for the field-free vacuum to produce a useful result, one must further delete these two parts, and as a result of the relation:

$$\coth\beta - 2\coth\frac{\beta}{2} = -\frac{1}{\sin\beta}$$

one obtains:

$$\tilde{U}'_{\mathrm{scal}} = -\frac{1}{16\pi^2}mc^2\left(\frac{mc}{h}\right)^3\int_{(0)}^{\infty}\frac{d\eta}{\eta^3}e^{-\eta}\left\{\eta\mathfrak{B}\frac{1}{\sin\eta\mathfrak{B}} - 1 + \frac{\eta^2}{2}\mathfrak{B}^2\right\}.$$

Carrying out an analogous perturbation calculation in an electric field leads in the same way to an additional term in the Lagrange function of the field that is achieved in the DIRAC theory of positrons and is very useful:

$$L'_{\mathrm{scal}} = -\frac{1}{16\pi^2}\frac{e^2}{hc}\int_{(0)}^{\infty}\frac{d\eta}{\eta^3}e^{-\eta}\left\{\frac{2i\eta^2(EB)}{\cos\eta\sqrt{(\mathfrak{E}^2-\mathfrak{B}^2)+2i(\mathfrak{E}\mathfrak{B})}-\mathrm{conj.}} + \frac{m^4c^6}{e^2h^2} - \frac{\eta^2}{6}(B^2-E^2)\right\}.$$

For the coefficients α, β that were defined in (4) one thus obtains:

$$\alpha = \frac{7}{16}\frac{1}{360\pi^2}\frac{e^4 h}{m^4 c^7}, \qquad \beta = \frac{4}{7}\alpha .$$

Now, we will prove the following property of the Lagrange function of the vacuum: For very large field strengths E or B the highest term in the addition L' to the MAXWELLian Lagrange function in the DIRAC theory of positrons has the form:

$$L' \approx -\frac{e^2}{24\pi^2 hc} E^2 \ln \mathfrak{E}, \qquad L' \approx \frac{e^2}{24\pi^2 hc} B^2 \ln \mathfrak{B}, \text{ resp.}$$

The ratio between the additional term L' and the MAXWELLian Lagrange function $L_0 = \frac{1}{8\pi}(E^2 - B^2)$ is thus logarithmic in the field strengths for higher values of them, and is, moreover, multiplied by the factor $\frac{e^2}{hc}$:

$$\frac{L'}{L_0} \approx \frac{-e^2}{3\pi hc} \ln \mathfrak{E}, \qquad \frac{L'}{L_0} \approx -\frac{e^2}{3\pi hc} \ln \mathfrak{B}, \text{ resp.}$$

The nonlinearity of the field equations thus represents only a small correction for field strengths that considerably higher than the critical field strength $\frac{m^2 c^3}{eh}$. The agreement that was observed in the note of EULER and KOCKEL (loc. cit.), as well as in the work of EULER (loc. cit.), between the nonlinearity in the field equations that follows from the theory of positrons and the nonlinear field theory of BORN and INFELD [1] is therefore superficial. In the latter theory the MAXWELL equations are already completely changed by the critical field strength $F_0 = \frac{m^2 c^4}{e^2}$ at the "surface of the electron," such that the finitude of the self-energy of a point charge is then achieved. By contrast, here the deviation from the MAXWELL field equations is still very small for fields of magnitude F_0 and drops off much too slowly to play a similar role in the self-energy problem. The extrapolation of the foregoing computations to the fields at the "surface of the electron" is generally not free of objections since the conditions (1) are not satisfied there. It is therefore not obvious that an exact analysis in this context would yield an essentially different result.

III.

In this section, the influence of arbitrary fields on the vacuum will be treated. We restrict ourselves first to the static fields. The stationary states of the electrons on the basis of the DIRAC wave equation and its energy eigenvalues will generally divided into two groups, which, when one turns on the static field adiabatically, originate in the

[1] M. BORN and L. INFELD, Proc. Roy. Soc. **143**, 410, 1933.

positive (negative, resp.) energy levels of the free electrons. This happens, for example, in the Coulomb field of an atomic nucleus and for all static fields that are found in Nature.

There are also static fields in which such a subdivision breaks down, in which transitions from negative to positive states take place as a result of the field. A well-known example of this is a step potential of height > $2mc^2$. These exceptional cases are not treatable as stationary fields and must be treated as time-independent fields that were turned on in a given time interval. This is all the more necessary since such fields can hardly be maintained as stationary, as a result of pair creation.

However, in the case in which the eigenvalue spectrum can be subdivided into two groups one can compute the energy density U and the current-charge density $\vec{i}$, ρ of the vacuum electrons by the formula:

$$\text{(29)} \qquad \begin{cases} U = ih\sum_i \left\{ \psi_i^*, \frac{\partial}{\partial t}\psi_i \right\} \\ \rho = e\sum_i \left\{ \psi_i^*, \psi_i \right\} \\ \vec{i} = e\sum_i \left\{ \psi_i^*, c\vec{\alpha}\psi_i \right\}, \end{cases}$$

in which the summation is over the states that correspond to negative energy states of the free electrons. The sum, as written, will diverge. However, when the physically meaningful part is broken off we will obtain a convergent expression.

In order to establish this part on the basis of assumptions (1), we develop the summands in expression (29) in powers of the external field strengths in such a way that we think of the latter as multiplied by a factor of λ and then develop in powers of this factor. This approach is identical with a successive perturbation calculation that starts with the free electrons as the zero order approximation.

Assumptions I_1, I_2 demand, above all, the vanishing of terms that are independent of λ, which arise from the contributions of the field that are independent of the free vacuum electrons. If we temporarily consider only those free vacuum electrons whose impulse satisfies $|p| < P$ then we obtain the following contribution for them: [1]

[1] This is really the current $\frac{ec\vec{p}}{\sqrt{p^2+m^2c^2}}$ that is associated with the impulse $\vec{p}$ and the number of states $\frac{d\vec{p}}{4\pi^3h^3}$ in the interval $d\vec{p}$.

$$(30)\qquad \begin{cases} U_0 = -\dfrac{1}{4\pi^3 h^3}\displaystyle\int_{|p|<P} d\vec{p}\, c\sqrt{p^2+m^2c^2}, \\ \rho_0 = \dfrac{e}{4\pi^3 h^3}\displaystyle\int_{|p|<P} d\vec{p}, \\ \vec{i}_0 = \dfrac{e}{4\pi^3 h^3}\displaystyle\int_{|p|<P} d\vec{p}\,\dfrac{c\vec{p}}{\sqrt{p^2+m^2c^2}}. \end{cases}$$

The total contribution of all electrons – i.e., $P \to \infty$ – naturally diverges.

However, by separating the terms that are independent of λ assumption I_2 is no longer completely satisfied. The charge and current density ρ_0, $\vec{i}_0$ of the field-free vacuum electrons also expresses itself by the fact that the presence of the potentials $V, \vec{A}$ yields additional terms $\rho_0 V$ and $(\vec{i}_0 \vec{A})$ in the energy density, which must likewise be separated out. These additional terms appear since the energy and impulse of the vacuum electrons, still unaffected by the fields, will also be changed by the presence of potentials by the amount eV ($\dfrac{e}{c}\vec{A}$, resp.).

Thus, assumptions I_1 and I_2 are not completely satisfied until one modifies the omitted contributions (30) in the following way:

$$(31)\qquad \begin{cases} U_0 = -\dfrac{1}{4\pi^3 h^3}\displaystyle\int_{|p|<P} d\vec{p}\left[c\sqrt{\left(p+\frac{e}{c}A\right)^2+m^2c^2}\right], \\ \rho_0 = \dfrac{e}{4\pi^3 h^3}\displaystyle\int_{|p|<P} d\vec{p}, \\ \vec{i}_0 = \dfrac{e}{4\pi^3 h^3}\displaystyle\int_{|p|<P} d\vec{p}\,\dfrac{c\left(\vec{p}+\frac{e}{c}\vec{A}\right)}{\sqrt{\left(p+\frac{e}{c}A\right)^2+m^2c^2}}, \end{cases}$$

in which, again, only the part that originates in the free vacuum electrons with impulse $|p| < P$ is written down. Now we still have to satisfy condition I_3. For that, we observe that a constant field-independent polarizability leads to terms on the energy density $U(x)$ that are proportional to the squares $E^2(x)$ and $B^2(x)$ of the field strengths at the point x. Likewise, they lead to a current and charge density that is proportional to the first derivative of the fields, on the grounds of the relations:

$$i = \text{rot } M + \frac{dP}{dt}$$

$$\rho = \operatorname{div} P,$$

in which M and P are the electric and magnetic polarizations, which are proportional to the fields in the case of a constant polarizability. In order to satisfy condition I_3 the terms in the energy density U of the vacuum electrons that are proportional to E^2 and B^2 must therefore vanish, and in the current-charge density the terms that are proportional to the first derivatives must be neglected. It is practical to not give the form of these terms explicitly, but to only recognize their properties in the course of computation.

By way of explanation, we calculate the charge and current density of the vacuum under the influence of an electric potential:

(32) $$V = V_0 e^{\frac{i(\vec{g}\vec{r})}{h}} + V_0^* e^{-\frac{i(\vec{g}\vec{r})}{h}}$$

and a magnetic potential:

(33) $$\vec{A} = \vec{A}_0 e^{\frac{i(\vec{g}\vec{r})}{h}} + \vec{A}_0^* e^{-\frac{i(\vec{g}\vec{r})}{h}}, \qquad (\vec{A}_0, \vec{g}) = 0,$$

with the help of perturbation theory. These computations have already been carried out by HEISENBERG [1], and more generally by SERBER [2] and PAULI and ROSE [3], and shall serve as only an illustration of our physical interpretation of the subtraction terms here. The charge density ρ is given, up to first order, by $\rho = \rho^{(0)} + \rho^{(1)}$, where:

$$\rho^{(1)} = e \sum_i \sum_k \frac{H_{ki}\{\psi_i^*, \psi_k\}}{W_i - W_k} + \text{conj.},$$

in which the i is summed over the occupied states and k is summed over the unoccupied states, and H_{ik} is the matrix element of the perturbation energy. If we substitute the potential (32) as the perturbing potential then we obtain ($W(p) = c\sqrt{p^2 + m^2c^2}$):

$$\rho^{(1)} = \frac{-e^2 V_0}{8\pi^3 h^3} \cdot \int d\vec{p} \left\{ \frac{W(p)W(p+g) - c^2(p, p+g) - m^2c^4}{W(p)W(p+g)[W(p) + W(p+g)]} + \right.$$
$$\left. + \begin{pmatrix} \text{the same thing} \\ \text{with} - g \end{pmatrix} \right\} e^{\frac{i(\vec{g}\vec{r})}{h}} + \text{conj.}$$

If one develops this in powers of $\vec{g}$ then one obtains:

$$\rho^{(1)} = \frac{e^2 V_0}{8\pi^3 h^3} \cdot \int d\vec{p} \frac{c^2}{W^3(p)} e^{\frac{i(\vec{g}\vec{r})}{h}}$$

[1] HEISENBERG, Z. f. Phys., **90**, 209, 1934.
[2] R. SERBER, Phys. Rev. **48**, 49, 1935.
[3] W. PAULI and N. ROSE, Phys. Rev. **49**, 462, 1936.

$$\left\{\frac{g^2}{2}-\frac{c^2(pg)^2}{W^2(p)}-\frac{c^2g^4}{4W^2(p)}+\frac{25}{16}\frac{c^4(pg)^2g^2}{W^4(p)}-\frac{21}{8}\frac{c^6(pg)^4}{W^6(p)}+\cdots\right\}+\text{conj.}$$

Which part of this charge density is physically meaningful? Due to assumption I_2 , ρ_0 must drop out. In $\rho^{(1)}$, the terms in g^2 are proportional to second derivatives of V and thus to the first derivatives of field strengths and must be neglected as a result of I_3 . We also remark that only those terms lead to divergences. The remaining ones yield finite integrals and, from HEISENBERG, are written in the form:

(34) $$\rho^{(1)}=\frac{1}{60\pi^2}\frac{e^2}{hc}\left(\frac{h}{mc}\right)^2\Delta\Delta V + \text{higher derivatives of } V\,.$$

The exact computation in the first approximation was given by SERBER and PAULI and ROSE (loc. cit.).

As a further example, we consider the current density $\vec{i}$ of the field (33) in the first approximation:

$$\vec{i}^{(1)}=e\sum_{ik}\frac{H_{ki}\{\psi_i^*,\vec{\alpha}\psi_k\}}{W_i-W_k}+\text{conj.}$$

This yields:

$$\vec{i}^{(1)}=-e^2\frac{c\vec{A}_0}{8\pi^3h^3}\int d\vec{p}\,e^{\frac{i(\vec{g}\vec{r})}{h}}\left\{\frac{W(p)W(p+g)+E^2(p)+c^2(pg)-2c^2(n,p+g)(n,p)}{W(p+g)W(p)[W(p+g)+W(p)]}\right\}+$$
$$+\left(\begin{matrix}\text{the same thing}\\ \text{with}-g\end{matrix}\right)\Bigg\}+\text{conj.},$$

in which n is the unit vector in the direction $\vec{A}$. When this is developed in g it yields:

(35)
$$\begin{cases}\vec{i}^{(1)}=-\dfrac{e^2c\vec{A}}{4\pi^3h^3}\displaystyle\int d\vec{p}\,\frac{1}{W^3(p)}\\ \qquad\left\{W^2(p)-c^2(np)^2-\dfrac{c^2g^2}{2}+\dfrac{3}{4}\dfrac{c^4(pg)^2}{W^2(p)}-\dfrac{5}{2}\dfrac{c^6(np)^2(pg)^2}{W^4(p)}+\right.\\ \qquad\left.+\dfrac{3}{4}\dfrac{c^4(np)^2g^2}{W^2(p)}+\text{terms of order 4 and higher in } g\right\}.\end{cases}$$

Terms that are independent of g – hence, gauge invariant – also appear in this expression. However, they are identical with the omitted contribution $\vec{i}_0'$ from (31). Namely, if one develops $\vec{i}_0$ in $\vec{A}$ then one obtains:

$$\vec{i}_0' = \frac{e}{4\pi^3 h^3}\int d\vec{p}\left\{\frac{cp}{W(p)}+\frac{e\vec{A}}{W(p)}-\frac{c^2 p(e\vec{A}\cdot\vec{p})}{W^3(p)}+\cdots\right\}$$
$$= \vec{i}_0 + e^2\frac{c\vec{A}}{4\pi^3 h^3}\int d\vec{p}\,\frac{1}{W^3(p)}\left\{W^2(p)-c^2(np)^2+\cdots\right\}.$$

The terms of first order in $\vec{A}$ correspond with the terms in (35) that are independent of *g*. The terms that are proportional to g^2 in (35) will likewise drop out, and the remaining ones give the following convergent result that corresponds to (34):

$$\vec{i}^{(1)} = \frac{1}{60\pi^2}\frac{e^2}{hc}\left(\frac{h}{mc}\right)^2 \Delta\Delta\vec{A} + \text{higher derivatives.}$$

The two examples shall show that the omitted contributions are immediately recognizable under a perturbative calculation, and that the remaining ones, by assumption (1), do not create contributions from the vacuum electrons that would lead to divergences in the summation. Indeed, the cited example proves this only in the first approximation. However, the consequences can be extended to higher order approximations with no further assumptions.

The treatment of time-dependent fields is not essentially different from the above. It is necessary to let the time-dependent fields act from a time point t_0 onward, such that before t_0 the vacuum electrons were in a field-free state, or in stationary states that could be unambiguously subdivided into occupied and unoccupied states. The timelike variation of these states from the time point t_0 onward may be represented with the help of a perturbative calculation in powers of the external fields. The expression (31) and the terms that follow from condition (I_3) can then be separated in such a way that the remaining terms no longer lead to divergences. The computation of the charge and current density of the vacuum for arbitrary time-dependent fields in the first approximation can be found in SERBER (loc. cit.) and PAULI and ROSE (loc. cit.). The parts to be stripped off were formally extracted in the HEISENBERG work. They are, however, completely identical, as would follow from assumption I.

How does one now express the creation of pairs by time-dependent fields? The pairs are not immediately expressed by the energy, current, or charge densities. Pair creation shows itself only in a total energy that increases proportional to time and corresponds to the energy of the electrons that were created. The charge and current density is not immediately influenced by the pair creation since just as many positive and negative electrons will always be produced, which first affects the current-charge density, and the external fields can act on the electrons thus created only by way of the various charges. [1]

It is therefore practical to calculate the pair creation in an external field directly as the transition of a vacuum electron into a positive energy state. The creation probability of the electron pair is then identical with the increase in the intensity of the positive energy

[1] The current and charge density that was computed by SERBER for pair-creation fields is therefore attributed to the fluctuations of vacuum electrons and is perhaps not the "created current-charge density." The resonance condition that appears originates in the fact that these oscillations are particularly strong when the external frequency approaches an absorption frequency of the vacuum.

eigenfunction in question (the decrease in the intensity of the corresponding negative energy eigenfunction, resp.), as a result of the effect of the time-dependent field on the state that existed up to time t_0 . The calculation was carried out by BETHE and HEITLER [1], HULME and JAEGER [2], etc.

Pair annihilation and the resulting radiation of light, like any other spontaneous radiation process, can only be treated by the quantization of the wave fields, or by a corresponding inversion of the light absorption process.

In the representation used up to now, the parts of the vacuum electrons that were taken away were not explicitly given, but only their form and their dependence on the external field was determined. In order to represent them explicitly, one must choose one of several options since these parts certainly include divergent expressions. For this, the density matrix that was introduced by DIRAC is suitable, and which DIRAC and particularly HEISENBERG (loc. cit.) have applied to this problem. The density matrix *R* is given by the following expression:

$$(x', k' \mid R \mid x'', k'') = \sum_i \psi_i^*(x', k')\psi_i(x'', k'') ,$$

in which x' and x'' are two spacetime points, and k' and k'' are two spin indices. The sum shall extend over all occupied states. From this matrix, one can then easily construct the current and charge densities $\vec{i}$, r and the energy-impulse tensor [3] U_ν^μ on the basis of the relations:

$$\vec{i} = \lim_{x' \to x''} e \sum_{k'k''} (\vec{\alpha})_{k'k''} (x'k' \mid R \mid x''k'')$$

$$\rho = \lim_{x' \to x''} e \sum_{k'k''} (x'k' \mid R \mid x''k'')$$

$$U_\nu^\mu = \lim_{x' \to x''} \frac{1}{2} \left\{ ich \left[\frac{\partial}{\partial x'_\mu} - \frac{\partial}{\partial x''_\mu} \right] - e[A^\mu(x') + A^\mu(x'')] \right\} \sum_{k'k''} (\alpha^\nu)_{k'k''} (x'k' \mid R \mid x''k'')$$

α^4 = identity matrix.

The density matrix has the advantage that for $x' \neq x''$ the summation over the vacuum electrons does not diverge, although an expression that is given for $x' = x''$ will be singular.

From the assumptions (I), one can now say uniquely which part of the density matrix of the vacuum electrons turns into the omitted part when $x' = x''$, and one then arrives at an explicit representation for these terms.

The physically meaningless part must then consist of those terms that are independent of the field strengths, those that lead to terms in the current density, those that are proportional to the derivatives of the fields, and lead to terms in the energy density that

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[1] H. BETHE and W. HEITLER, Proc. Roy. Soc. **146**, 34, 1934.

[2] H. H. HULME and J. C. JAEGER, Proc. Roy. Soc.

[3] The complete energy-impulse tensor is composed of the sum of U_ν^μ and the MAXWELLian energy-impulse tensor of the field. The component U_4^4 is therefore not the entire material energy density, but only the kinetic part.

are proportional to the square of the field strengths. Moreover, the part of the density matrix that was removed must still be multiplied by the factor:

$$u' = \exp\left[\frac{ie}{hc}\int_{x'}^{x''}\left(\sum_{i=1}^{3} A_i dx_i - V dt\right)\right],$$

in which the integral inside the exponent is taken along the straight line from the point x' to the point x". This factor adds a contribution to the removed energy-impulse tensor that is precisely the one that originates in the fact that the unperturbed vacuum electrons in the field acquire an additional energy eV and an additional impulse $\frac{e}{c}\vec{A}$, and shall be omitted, as a consequence of assumption I_2.

Since the part of the density matrix that was removed is, up to the factor u', of at most second order in the field strengths, it may be obtained from a perturbative calculation from the density matrix of the free electrons. This, in principle simple, but in practice very complicated, calculation laid the foundations for the determination of this matrix by HEISENBERG (loc. cit.). The result may be formulated in a mathematically simple way when one always takes the mean of any quantity that is calculated with the help of the foregoing theory and those that are calculated with the help of a theory in which the electron charge is positive and the negative electron represents a "hole." The result is indeed the same in both cases. The density matrix R will then be replaced with R':

$$(x'\,k' \mid R' \mid x''\,k'') = \frac{1}{2}\left\{\sum_i \psi_i^*(x'k')\psi_i(x''k'') - \sum_k \psi_k^*(x'k')\psi_k(x''k'')\right\},$$

in which the first sum is taken over the occupied states and the second one is taken over the unoccupied states.

The omitted part $(x'\,k' \mid S \mid x''\,k'')$ then has the form:

$$(x'\,k' \mid S \mid x''\,k'') = u'S_0 + \frac{\bar{a}}{|x'-x''|^2} + \bar{b}\lg\frac{|x'-x''|^2}{C}.$$

In this, S_0 is the matrix R' for a vanishing potential, $\bar{a}$ and $\bar{b}$ are functions of the field strength and their derivatives; C is a constant. These quantities are given explicitly by HEISENBERG (loc. cit.) and by HEISENBERG and EULER (loc. cit.).

For the sake of doing particular calculations it is more practical to understand the structure of the omitted terms than to fall back on the explicit expression of HEISENBERG. This is, above all, simpler since the remaining expressions will no longer be singular for $x' = x''$, such that one does not need the formal assistance of the density matrix for that calculation at all. The summation over all vacuum electrons thus no longer leads to divergent expressions. In general, the explicit HEISENBERG representation is suitable for showing the relativistic invariance and the validity of the conservation laws in the process.

It is therefore apparent that the determination of the physical properties of the vacuum electrons that was described here involves no essential arbitrariness since only those effects were omitted that must be omitted on the grounds of the assumptions of positron theory: the energy and the charge of the vacuum electrons that were unperturbed by the field and the physically absurd field-independent constant polarizability of the vacuum. All physically meaningful effects of the vacuum electrons will be considered and lead to convergent integrals. One may very well reach the conclusion that the hole theory of positrons leads to no difficulties for the electron theory as long as one restricts oneself to the treatment of unquantized wave fields.

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This paper deals with the modifications introduced into the electrodynamics of the vacuum by Dirac's theory of the positron. The behavior of the vacuum can be described unambiguously by assuming the existence of an infinite number of electrons occupying the negative energy states, provided that certain well-defined effects of these electrons are omitted, but only those to which it is obvious that no physical meaning can be ascribed.

The results are identical with those of Heisenberg's and Dirac's mathematical method of obtaining finite expressions in positron theory. A simple method is given of calculating the polarizability of the vacuum for slowly varying fields.