"Zur Elektrodynamik bewegten Körper," Rend. Circ. mat. Palermo (1) **28** (1909), 1-28.

On the electrodynamics of moving bodies

By **Max Abraham** (Göttingen)

Translated by D. H. Delphenich

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§ 1.

Introduction.

It is known that the basic equations of the electrodynamics of moving bodies that were presented by **H. HERTZ** ([1]), which can be regarded as the natural generalization of **MAXWELL**'s field equations for bodies at rest, prove to be inadequate: They contradict the experiments that **A. EICHENWALD** ([2]) and **H. A. WILSON** ([3]) performed on the behavior of moving dielectrics.

The results of those efforts are found to be in agreement with the electrodynamical theories of **H. A. LORENTZ** ([4]) and **E. COHN** ([5]). The heuristic ideas that guided the two researchers were completely different: Whereas **H. A. LORENTZ** started with hypotheses that related to the behavior of electrons and molecules, **E. COHN** sought the simplest description of electromagnetic processes in the **KIRCHHOFF** sense.

The theory of **E. COHN** explained the inconclusive nature of the attempts up to the time in a satisfactory way as the influence of the motion of the Earth on the electromagnetic processes that took place on the Earth's surface. By contrast, **LORENTZ**'s electron theory, which started with the electromagnetic field in the ether, was closely-related to the concept that the motion of a system through the ether might influence the perception of a co-moving observer. In that way, **H. A. LORENTZ** ([6])

([1]) H. HERTZ, "Über die Grundgleichungen der Elektrodynamik für bewegte Körper." [*Gesammelte Werke*¸Bd. II, pp. 256-285].

([2]) A. EICHENWALD, "Über die magnetische Wirkungen bewegter Körper im elektrostatischen Felde," Ann. Phys. (Leipzig) **11** (1903), 421-441.

([3]) H. A. WILSON, "On the Electric Effect of Rotating a Dielectric in a Magnetic Field," Phil. Trans. Roy. Soc. London A **204** (1905), 121-137.

([4]) H. A. LORENTZ, *Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpen*, Leiden, 1895.

([5]) E. COHN, "Zur Elektrodynamik bewegter Systeme II," Sitz. Kgl. Preuss. Akad. Wiss (Berlin), (1904), 1404-1416.

([6]) H. A. LORENTZ, "Electromagnetische verschijnselen in een stelsel dat sich met willekuerige snelheid, kleiner dan die van het licht, beweegt," Kon. Akad. Wet. Amsterdam **12** (1904), 2, 986-1009.

succeeded in adapting his theory to the relativity postulate by suitable hypotheses in the variations that the electrical and mechanical properties of matter should experience as a result of its motion through the ether. As is known, the fact that this is possible is explained by the property of the field equations in the ether that they go to themselves under certain transformations of coordinates and light rays, namely, the so-called ([7]) "**LORENTZ** transformations."

It is not my intention to discuss the entire complex of questions that are connected with the postulate of relativity in this paper. I have dealt with some of those questions elsewhere ([8]). Here, we shall be interested in that postulate only to the extent that it is connected with the electrodynamics of ponderable matter. A paper by **H. MINKOWSKI** ([9]) that appeared recently placed precisely that connection at its forefront. Here, the basic equation for moving bodies will be given in such a form that they will go to **MAXWELL**'s field equations for bodies at rest under a **LORENTZ** transformation.

MINKOWSKI's basic equations, like those of **E. COHN** and **H. A. LORENTZ**, explain the results of all experiments up to now. They have the symmetry of electric and magnetic quantities in common with **COHN**'s basic equations, which they coincide with when one neglects second-order quantities (in the quotient of the velocity of matter and that of light). By contrast, **LORENTZ**'s basic equations, in their original form, in which that symmetry was not present, already deviate from those of the other two theories by terms of first order. Therefore, that deviation, which was pointed out by **E. COHN** ([10]), would be relevant to only paramagnetic and diamagnetic insulators, and would be difficult to prove experimentally, due to its negligible magnitude.

However, it is not difficult to alter the relations between electric and magnetic vectors that **LORENTZ** assumed in such a way that the symmetry will still remain valid. Paragraphs (8) and (10) of the present investigation will address the form of **LORENTZ**'s theory, thus-modified. It will be shown that it overlaps **MINKOWSKI**'s theory completely, as far as actual content is concerned. The formal difference lies in the interpretation that will be given to the vectors that are denoted by $\mathfrak{E}$ and $\mathfrak{H}$. For **H. A. LORENTZ**, they represented the electric and magnetic excitation of the ether, while for **MINKOWSKI**, they lacked any intuitive meaning. In my opinion, it is in precisely that absence of an intuitive interpretation that **MINKOWSKI**'s theory lies. Now that the theory of the electron has borne such rich fruits, electrodynamics, in turn, seems to have entered a phenomenological phase of its development.

The method of the present examination is also phenomenological. Given the embarrassment that the increasing number of rival theories represents, it seems desirable to me to possess a system of the electrodynamics of moving bodies that is constructed upon **MAXWELLian** foundations and is free of the special Ansätzen of the individual theories from the outset. The assumptions of the system that will be presented here are contained in laws of impulse and energy (§ 3), and in addition, certain equations that we

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([7]) H. POINCARÉ, "Sur la dynamique de l'electron," Rend. Circ. Mat. Palermo **21** (1st semester 1906), 129-176.

([8]) M. ABRAHAM, *Theorie der Elektrizität*, Bd. II, 2nd ed., Leipzig, 1908, pp. 356-397.

([9]) H. MINKOWSKI, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern," Nachr. Kgl. Ges. Wiss. Göttingen (1908), 53-111.

([10]) E. COHN, "Zur Elektrodynamik bewegter Systeme. I," Sitz. Kgl. Preuss. Akad. Wiss. (Berlin) (1904), 1294-1303; pp. 1301.

shall refer to as the "main equations" (§ 4). Two of them, which are reasonable generalizations of the main equations of **MAXWELL**'s theory in the case of rest, couple the line integrals of the vectors $\mathfrak{E}'$ and $\mathfrak{H}'$ (viz., the forces on moving unit electric and magnetic poles) with the temporal variation of the surface integrals of the vectors $\mathfrak{B}$ and $\mathfrak{D}$ (viz., the magnetic and electric excitations). Together with the three closely-related main equations that expression the **JOULE** heat, the relative radiation, and the relative electromagnetic stresses in moving matter by means of vectors, they will define a mathematical framework into which the various pictures of electromagnetic processes can be inserted. Each of those pictures is characterized by two relations between the four vectors $\mathfrak{E}'$, $\mathfrak{H}'$, $\mathfrak{D}$, $\mathfrak{B}$. By appending these relations, the first two main equations will go to differential equations that represent the temporal variation of the electromagnetic field in the theory in question, while the other three main equations determine the energy processes and ponderomotor forces. Meanwhile, the extent to which one can pursue the consequence of the main equations without adding any special equations that would couple the theories in question is remarkable. In particular, the deviations that exist between the various theories in the expressions for the ponderomotor force (§ 12) are negligible. In the case of rest, the ponderomotor forces of **LORENTZ**'s, **COHN**'s, and **MINKOWSKI**'s theories will even be identical to each other.

While I shall organize the various theories of the electrodynamics of moving bodies into a general system, I shall ignore the topics in the individual pictures that are not required by the characteristic laws for coupling the electromagnetic vectors. Hopefully, one will excuse me for having introduced such alterations in some of the aforementioned theories, since the essential elements of the pictures in question will emerge all the more clearly with the manner of representation that will be given.

§ 2.

Useful mathematical formulas.

Time differentiation at a fixed point in space will be denoted by $\partial / \partial t$. The temporal change in a surface integral that is taken over a surface whose points move with a velocity of $\mathfrak{w}$:

$$\frac{d}{dt}\int df\,\mathfrak{A}_n = \int df \left\{\frac{\partial'\mathfrak{A}}{\partial t}\right\}_n$$

defines another type of time differentiation for vectors:

$$\frac{\partial'\mathfrak{A}}{\partial t} = \frac{\partial\mathfrak{A}}{\partial t} + \mathfrak{w}\,\mathrm{div}\,\mathfrak{A} + \mathrm{curl}\,[\mathfrak{A}\,\mathfrak{w}]. \tag{1}$$

Moreover, the differential quotient that refers to moving points will be:

(2) $$\dot{\mathfrak{A}} = \frac{\partial \mathfrak{A}}{\partial t} + (\mathfrak{w}\,\nabla)\,\mathfrak{A}.$$

It is connected with the temporal change in the volume integral of a vector by the relations:

$$\frac{d}{dt}\int dv\,\mathfrak{A} = \int dv\,\frac{\delta \mathfrak{A}}{\delta t},$$

(2_a) $$\frac{\delta \mathfrak{A}}{\delta t} = \dot{\mathfrak{A}} + \mathfrak{A}\,\text{div}\,\mathfrak{w}.$$

It follows from (2) and (2*a*) that:

(3) $$\frac{\delta \mathfrak{A}}{\delta t} = \frac{\partial \mathfrak{A}}{\partial t} + (\mathfrak{w}\,\nabla)\,\mathfrak{A} + \mathfrak{A}\,\text{div}\,\mathfrak{w}.$$

Correspondingly, for scalars, that will yield:

(3_a) $$\frac{\delta \psi}{\delta t} = \frac{\partial \psi}{\partial t} + \text{div}\,\psi\,\mathfrak{w}.$$

Finally, if one considers the general rule that:

$$\text{curl}\,[\mathfrak{A}\ \mathfrak{w}] = (\mathfrak{w}\,\nabla)\,\mathfrak{A} - (\mathfrak{A}\,\nabla)\,\mathfrak{w} + \mathfrak{A}\,\text{div}\,\mathfrak{w} - \mathfrak{w}\,\text{div}\,\mathfrak{A},$$

then the relation:

(4) $$\frac{\partial' \mathfrak{A}}{\partial t} = \frac{\delta \mathfrak{A}}{\delta t} - (\mathfrak{A}\,\nabla)\,\mathfrak{w}$$

will follow from (1) and (3).

Since the type of time differentiation that was introduced in (2) obeys the usual rules of calculus, one will have, if one recalls (2_a):

$$[\dot{\mathfrak{A}}\,\mathfrak{B}] + [\mathfrak{A}\,\dot{\mathfrak{B}}] = \frac{\delta}{\delta t}[\mathfrak{A}\ \mathfrak{B}] - [\mathfrak{A}\ \mathfrak{B}]\,\text{div}\,\mathfrak{w}.$$

From this equation, in conjunction with the ones that follow from (4) and (2*a*):

$$\frac{\partial' \mathfrak{A}}{\partial t} = \dot{\mathfrak{A}} + \mathfrak{A}\,\text{div}\,\mathfrak{w} - (\mathfrak{A}\,\nabla)\,\mathfrak{w},$$

$$\frac{\partial' \mathfrak{B}}{\partial t} = \dot{\mathfrak{B}} + \mathfrak{B}\,\text{div}\,\mathfrak{w} - (\mathfrak{B}\,\nabla)\,\mathfrak{w},$$

one will obtain:

$$\left[\frac{\partial' \mathfrak{A}}{\partial t}\mathfrak{B}\right] + \left[\mathfrak{A}\frac{\partial' \mathfrak{B}}{\partial t}\right] = \frac{\delta}{\delta t}[\mathfrak{A}\ \mathfrak{B}] + [\mathfrak{A}\ \mathfrak{B}]\,\text{div}\,\mathfrak{w} - [\mathfrak{A}\ (\mathfrak{B}\,\nabla)\,\mathfrak{w}] + [\mathfrak{B}\ (\mathfrak{A}\,\nabla)\,\mathfrak{w}].$$

On the basis of the easily-verified identity:

$$[\mathfrak{A}\,(\mathfrak{B}\,\nabla)\,\mathfrak{w}] - [\mathfrak{B}\,(\mathfrak{A}\,\nabla)\,\mathfrak{w}] = [\mathfrak{A}\,\mathfrak{B}]\,\mathrm{div}\,\mathfrak{w} - ([\mathfrak{A}\,\mathfrak{B}]\,\nabla)\,\mathfrak{w} - [[\mathfrak{A}\,\mathfrak{B}]\,\mathrm{curl}\,\mathfrak{w}],$$

one will get the relation:

(5) $$\left[\frac{\partial'\mathfrak{A}}{\partial t}\mathfrak{B}\right] + \left[\mathfrak{A}\frac{\partial'\mathfrak{B}}{\partial t}\right] = \frac{\delta}{\delta t}[\mathfrak{A}\,\mathfrak{B}] + ([\mathfrak{A}\,\mathfrak{B}]\,\nabla)\,\mathfrak{w} + [[\mathfrak{A}\,\mathfrak{B}]\,\mathrm{curl}\,\mathfrak{w}].$$

§ 3.

The energy equation and the impulse equations.

We understand *x*, *y*, *z*, *t* to mean coordinates and time, when measured in a system of reference in which the observer occupies a fixed position. The ponderomotor force that acts upon a unit volume of moving matter as a result of the electromagnetic process that such an observer will measure shall possess the components:

(6) $$\begin{cases} \mathfrak{K}_x = \dfrac{\partial X_x}{\partial x} + \dfrac{\partial X_y}{\partial y} + \dfrac{\partial X_z}{\partial z} - \dfrac{\partial \mathfrak{g}_x}{\partial t}, \\ \mathfrak{K}_y = \dfrac{\partial Y_x}{\partial x} + \dfrac{\partial Y_y}{\partial y} + \dfrac{\partial Y_z}{\partial z} - \dfrac{\partial \mathfrak{g}_y}{\partial t}, \\ \mathfrak{K}_z = \dfrac{\partial Z_x}{\partial x} + \dfrac{\partial Z_y}{\partial y} + \dfrac{\partial Z_z}{\partial z} - \dfrac{\partial \mathfrak{g}_z}{\partial t}. \end{cases}$$

We refer to the vector $\mathfrak{g}$ that appears here as the *electromagnetic quantity of motion density*, or more briefly, the *impulse density*. The system of *fictitious electromagnetic stresses* consists of six quantities, namely, the normal stresses X_x, Y_y, Z_z, and the pair-wise equal shear stresses:

(6*a*) $$X_y = Y_x, \qquad Y_z = Z_y, \qquad Z_x = X_z.$$

The *impulse equations* (6) replace the *energy equation:*

(7) $$\mathfrak{w}\mathfrak{K} + Q = -\,\mathrm{div}\,\mathfrak{S} - \frac{\partial\psi}{\partial t}.$$

In this, Q means the **JOULE** heat, ψ means the electromagnetic energy density, and $\mathfrak{S}$ means the energy current.

Like the impulse equations that determine the quantities of motion that are carried by the electromagnetic field, the energy equation, which yields the total energy per unit space and time, will be converted into a non-electromagnetic form (viz., work and heat).

If one introduces the temporal differential quotients that are defined by (3) and (3_a) into (6) and (7) then one will obtain another form for the law of impulse and energy:

$$(8) \qquad \begin{cases} \mathfrak{K}_x = \dfrac{\partial X'_x}{\partial x} + \dfrac{\partial X'_y}{\partial y} + \dfrac{\partial X'_z}{\partial z} - \dfrac{\delta \mathfrak{g}_x}{\delta t}, \\ \mathfrak{K}_y = \dfrac{\partial Y'_x}{\partial x} + \dfrac{\partial Y'_y}{\partial y} + \dfrac{\partial Y'_z}{\partial z} - \dfrac{\delta \mathfrak{g}_y}{\delta t}, \\ \mathfrak{K}_z = \dfrac{\partial Z'_x}{\partial x} + \dfrac{\partial Z'_y}{\partial y} + \dfrac{\partial Z'_z}{\partial z} - \dfrac{\delta \mathfrak{g}_z}{\delta t}, \end{cases}$$

$$(9) \qquad \mathfrak{w}\ \mathfrak{K} + Q = -\operatorname{div}\{\mathfrak{S} - \mathfrak{w}\ \psi\} - \frac{\delta \psi}{\delta t}.$$

Here, the vector:

$$\mathfrak{S} - \mathfrak{w}\ \psi$$

represents the *relative energy current.* The system of *relative stresses:*

$$(10) \qquad \begin{cases} X'_x = X_x + \mathfrak{w}_x\, \mathfrak{g}_x, & X'_y = X_y + \mathfrak{w}_y\, \mathfrak{g}_x, & X'_z = X_z + \mathfrak{w}_z\, \mathfrak{g}_x, \\ Y'_x = Y_x + \mathfrak{w}_x\, \mathfrak{g}_y, & Y'_y = Y_y + \mathfrak{w}_y\, \mathfrak{g}_y, & Y'_z = Y_z + \mathfrak{w}_z\, \mathfrak{g}_y, \\ Z'_x = Z_x + \mathfrak{w}_x\, \mathfrak{g}_z, & Z'_z = Z_z + \mathfrak{w}_y\, \mathfrak{g}_z, & Z'_z = Z_z + \mathfrak{w}_z\, \mathfrak{g}_z \end{cases}$$

is defined in such a way that (6) and (8) lead to the same values of the ponderomotor force.

The relations:

$$Y'_x - X'_y = \mathfrak{w}_x\, \mathfrak{g}_y - \mathfrak{w}_y\, \mathfrak{g}_x,$$
$$Z'_y - Y'_z = \mathfrak{w}_y\, \mathfrak{g}_z - \mathfrak{w}_z\, \mathfrak{g}_y,$$
$$X'_z - Z'_x = \mathfrak{w}_z\, \mathfrak{g}_x - \mathfrak{w}_x\, \mathfrak{g}_z$$

follow from (6_a) and (10), and they can be written vectorially as:

$$(11) \qquad \mathfrak{N}' = [\mathfrak{w}\ \mathfrak{g}].$$

$\mathfrak{N}'$ is the *rotational moment of the relative stresses* per unit volume. In ordinary mechanics, it will vanish, since the direction of the impulse vector coincides with that of the velocity vector here. In electromagnetic mechanics, it cannot be neglected in general, but it will be compensated by the rotational moment that originates in the co-moving quantities of motion when one refers them to a fixed moment point.

We can think of the relative energy current as being divided into two parts, one of which represents the energy transfer that is required by the relative stresses, while the other one represents the "relative radiation ([11])," which can be measured, e.g., in optics, by the production of heat on a black surface:

$$(12)\qquad \begin{cases} \mathfrak{S}_x - \mathfrak{w}_x\psi = \mathfrak{S}'_x - \{\mathfrak{w}_x X'_x + \mathfrak{w}_y Y'_x + \mathfrak{w}_z Z'_x\}, \\ \mathfrak{S}_y - \mathfrak{w}_y\psi = \mathfrak{S}'_y - \{\mathfrak{w}_x X'_y + \mathfrak{w}_y Y'_y + \mathfrak{w}_z Z'_y\}, \\ \mathfrak{S}_z - \mathfrak{w}_z\psi = \mathfrak{S}'_z - \{\mathfrak{w}_x X'_z + \mathfrak{w}_y Y'_z + \mathfrak{w}_z Z'_z\}. \end{cases}$$

We call the vector $\mathfrak{S}$ the *relative ray*.

We find the expression:

$$\begin{aligned} \mathfrak{w}\ \mathfrak{K} = -\ \mathfrak{w}\frac{\delta\mathfrak{g}}{\delta t} &+ \frac{\partial}{\partial x}(\mathfrak{w}_x X'_x + \mathfrak{w}_y Y'_x + \mathfrak{w}_z Z'_x) \\ &+ \frac{\partial}{\partial y}(\mathfrak{w}_x X'_y + \mathfrak{w}_y Y'_y + \mathfrak{w}_z Z'_y) \\ &+ \frac{\partial}{\partial z}(\mathfrak{w}_x X'_z + \mathfrak{w}_y Y'_z + \mathfrak{w}_z Z'_z) \end{aligned}$$

$$-\left\{X'_x\frac{\partial\mathfrak{w}_x}{\partial y} + Y'_x\frac{\partial\mathfrak{w}_y}{\partial x} + Z'_x\frac{\partial\mathfrak{w}_z}{\partial x} + X'_y\frac{\partial\mathfrak{w}_x}{\partial y} + Y'_y\frac{\partial\mathfrak{w}_y}{\partial y} + Z'_y\frac{\partial\mathfrak{w}_z}{\partial y} + X'_z\frac{\partial\mathfrak{w}_x}{\partial z} + Y'_z\frac{\partial\mathfrak{w}_y}{\partial z} + Z'_z\frac{\partial\mathfrak{w}_z}{\partial z}\right\}$$

for the work that is done by the ponderomotive force from the impulse equations (8). If we set:

$$(13)\qquad \begin{cases} P' = X'_x\dfrac{\partial\mathfrak{w}_x}{\partial x} + X'_y\dfrac{\partial\mathfrak{w}_y}{\partial y} + X'_z\dfrac{\partial\mathfrak{w}_z}{\partial z} \\ \qquad + Y'_x\dfrac{\partial\mathfrak{w}_y}{\partial x} + Y'_y\dfrac{\partial\mathfrak{w}_y}{\partial y} + Y'_z\dfrac{\partial\mathfrak{w}_y}{\partial z} \\ \qquad + Z'_x\dfrac{\partial\mathfrak{w}_z}{\partial x} + Z'_y\dfrac{\partial\mathfrak{w}_z}{\partial y} + Z'_z\dfrac{\partial\mathfrak{w}_z}{\partial z} \end{cases}$$

here, to abbreviate, then if we recall (12), the energy equation (9) will yield:

$$(14)\qquad Q + \operatorname{div}\mathfrak{S}' = -\frac{\delta\psi}{\delta t} + \mathfrak{w}\frac{\delta\mathfrak{g}}{\delta t} + P'.$$

The relation that follows from the laws of impulse and energy will prove to be important later on.

([11]) M. ABRAHAM, *loc. cit.* ([8]), pp. 324.

§ 4.

The main equations.

What all theories of the electrodynamics of moving bodies have in common is the form of the first two main equations:

$$c \text{ curl } \mathfrak{H}' = \frac{\partial' \mathfrak{D}}{\partial t} + \mathfrak{J}, \tag{I}$$

$$c \text{ curl } \mathfrak{E}' = -\frac{\partial' \mathfrak{B}}{\partial t}. \tag{II}$$

They are nothing by a general model that will first take on a physical meaning when one adds two relations between the vectors that appear in them. Two such relations are necessary in order to reduce the number of unknown vectors to two. The temporal variation of the fields of these two vectors will be described by the first two main equations.

We interpret the vectors $\mathfrak{E}'$, $\mathfrak{H}'$ as the *forces* that act upon *moving unit electric and magnetic poles.* We shall follow the terminology of the *Enzyklopädie der mathematischen Wissenschaften* and call the vectors $\mathfrak{D}$, $\mathfrak{B}$ the *electric and magnetic excitations*, resp.

The meaning of the vector $\mathfrak{E}'$ corresponds to the fact that we make the following Ansatz for the *heat* that is generated in the moving matter per unit time and space:

$$Q = \mathfrak{J}\,\mathfrak{E}'. \tag{III}$$

This third main equation is succeeded by a fourth one that couples the *relative ray* with the vectors $\mathfrak{E}'$, $\mathfrak{H}'$:

$$\mathfrak{S}' = c\,[\mathfrak{E}'\,\mathfrak{H}']. \tag{IV}$$

For the case of rest, that vector will go to **POYNTING**'s.

Finally, we shall need an Ansatz that expresses the quantity P' that is defined in equation (13), and therefore the relative stresses, in terms of the vectors $\mathfrak{E}'$, $\mathfrak{H}'$, $\mathfrak{D}$, $\mathfrak{B}$. We set:

$$P' = \mathfrak{E}'\,(\mathfrak{D}\,\nabla)\,\mathfrak{w} + \mathfrak{H}'\,(\mathfrak{B}\,\nabla)\,\mathfrak{w} - \tfrac{1}{2}\{\mathfrak{E}'\mathfrak{D} + \mathfrak{H}'\mathfrak{B}\}\text{ div }\mathfrak{w}, \tag{V}$$

and thus obtain the *relative stresses* as:

$$(\mathrm{V}_a) \quad \begin{cases} X'_x = \mathfrak{E}'_x \mathfrak{D}_x + \mathfrak{H}'_x \mathfrak{B}_x - \frac{1}{2}\{\mathfrak{E}'\mathfrak{D} + \mathfrak{H}'\mathfrak{B}\}, \\ X'_y = \mathfrak{E}'_x \mathfrak{D}_y + \mathfrak{H}'_x \mathfrak{B}_y, \\ X'_z = \mathfrak{E}'_x \mathfrak{D}_z + \mathfrak{H}'_x \mathfrak{B}_y, \\ Y'_x = \mathfrak{E}'_y \mathfrak{D}_y + \mathfrak{H}'_y \mathfrak{B}_y, \\ Y'_y = \mathfrak{E}'_y \mathfrak{D}_x + \mathfrak{H}'_y \mathfrak{B}_x - \frac{1}{2}\{\mathfrak{E}'\mathfrak{D} + \mathfrak{H}'\mathfrak{B}\}, \\ Y'_z = \mathfrak{E}'_y \mathfrak{D}_x + \mathfrak{H}'_y \mathfrak{B}_x, \\ Z'_x = \mathfrak{E}'_z \mathfrak{D}_x + \mathfrak{H}'_z \mathfrak{B}_x, \\ Z'_y = \mathfrak{E}'_z \mathfrak{D}_y + \mathfrak{H}'_z \mathfrak{B}_y, \\ Z'_z = \mathfrak{E}'_z \mathfrak{D}_z + \mathfrak{H}'_z \mathfrak{B}_z - \frac{1}{2}\{\mathfrak{E}'\mathfrak{D} + \mathfrak{H}'\mathfrak{B}\}. \end{cases}$$

The well-known formulas for the fictitious stresses follow from these in the case of rest.

On first glance, the choice of expressions (IV) and (V) seems to be completely arbitrary. However, it is the simplest generalization of the laws that are valid for bodies at rest, which only employ the four vectors that appear in the first two main equations.

Moreover, it follows from (V_a) that:

$$Y'_x - X'_y = \mathfrak{D}_x \mathfrak{E}'_y - \mathfrak{D}_y \mathfrak{E}'_x + \mathfrak{B}_x \mathfrak{H}'_y - \mathfrak{B}_y \mathfrak{H}'_x .$$

With that, the *rotational moment of the relative stresses* will be:

$$(\mathrm{V}_b) \qquad \mathfrak{N}' = [\mathfrak{D}\ \mathfrak{E}'] + [\mathfrak{B}\ \mathfrak{H}'].$$

The mechanical principles that were set forth in the previous paragraphs, and the five main equations, are the foundations upon which our system of the electrodynamics of moving bodies will rest.

§ 5.

Determining the impulse density and the energy density.

The various theories of the electrodynamics of moving bodies differ from the relations that are chosen between the four vectors $\mathfrak{E}'$, $\mathfrak{H}'$, $\mathfrak{D}$, $\mathfrak{B}$ that appear in the main equations. However, before we go on to the discussion of the special theories, we would like to pursue the general development somewhat further. Therefore, only the following rather general assumption about the form of those relations shall be made: The vectors $\mathfrak{E}'$, $\mathfrak{H}'$, $\mathfrak{D}$, $\mathfrak{B}$ shall be coupled by equations that indeed contain the velocity vector $\mathfrak{w}$ itself, but not any sort of derivatives of them with respect to time or the coordinates.

The main equation (IV) yields:

$$\operatorname{div} \mathfrak{S}' = c\,\{\mathfrak{H}' \operatorname{curl} \mathfrak{E}' - \mathfrak{E}' \operatorname{curl} \mathfrak{H}'\}.$$

If one recalls the first two main equations then that will become:

$$\mathfrak{F}\mathfrak{E}' + \mathrm{div}\,\mathfrak{S}' = -\,\mathfrak{E}'\frac{\partial'\mathfrak{D}}{\partial t} - \mathfrak{H}'\frac{\partial'\mathfrak{B}}{\partial t}.$$

It follows from the main equation (III) and the relation (14) that:

$$(14_a)\qquad \frac{\delta\psi}{\delta t} - \mathfrak{w}\frac{\delta\mathfrak{g}}{\delta t} - P' = \mathfrak{E}'\frac{\partial'\mathfrak{D}}{\partial t} - \mathfrak{H}'\frac{\partial'\mathfrak{B}}{\partial t},$$

which is a condition that, from (4), one can also write as:

$$(14_b)\qquad \frac{\delta\psi}{\delta t} - \mathfrak{w}\frac{\delta\mathfrak{g}}{\delta t} - P' = \mathfrak{E}'\frac{\delta\mathfrak{D}}{\delta t} + \mathfrak{H}'\frac{\delta\mathfrak{B}}{\delta t} - \mathfrak{E}'\,(\mathfrak{D}\,\nabla)\,\mathfrak{w} - \mathfrak{H}'\,(\mathfrak{B}\,\nabla)\,\mathfrak{w},$$

and which will finally go to:

$$(15)\qquad \frac{\delta\psi}{\delta t} - \mathfrak{w}\frac{\delta\mathfrak{g}}{\delta t} = \mathfrak{E}'\frac{\delta\mathfrak{D}}{\delta t} + \mathfrak{H}'\frac{\delta\mathfrak{B}}{\delta t} - \tfrac{1}{2}\{\mathfrak{E}'\,\mathfrak{D} + \mathfrak{H}'\,\mathfrak{B}\}\,\mathrm{div}\,\mathfrak{w}$$

by appending the main equation (V). That relation serves to ascertain the densities of energy and quantity of motion from its dependency upon the electromagnetic vectors.

If one recalls (2_a) then it will read:

$$(15_a)\qquad \dot{\psi} - \mathfrak{w}\dot{\mathfrak{g}} + (\psi - \mathfrak{w}\,\mathfrak{g})\,\mathrm{div}\,\mathfrak{w} = \mathfrak{E}'\dot{\mathfrak{D}} + \mathfrak{H}'\dot{\mathfrak{B}} + \tfrac{1}{2}\{\mathfrak{E}'\,\mathfrak{D} + \mathfrak{H}'\,\mathfrak{B}\}\,\mathrm{div}\,\mathfrak{w}.$$

Since the type of time differentiation that is now employed satisfies the usual rules of calculus, when we set:

$$(16)\qquad \psi - \mathfrak{w}\,\mathfrak{g} = \varphi,$$

to abbreviate, it will then follow that:

$$(17)\qquad \dot{\psi} + \dot{\mathfrak{g}}\mathfrak{w} - \mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{H}'\dot{\mathfrak{B}} + \{\varphi - \tfrac{1}{2}\mathfrak{E}'\,\mathfrak{D} - \tfrac{1}{2}\mathfrak{H}'\,\mathfrak{B}\}\,\mathrm{div}\,\mathfrak{w}.$$

As was mentioned in the beginning of this paragraph, the relations that couple $\mathfrak{D}$, $\mathfrak{B}$ with $\mathfrak{E}'$, $\mathfrak{H}'$, will in fact contain the velocity vector $\mathfrak{w}$, but not its differential quotients with respect to time and position. The same thing will be demanded of the expressions that represent ψ and $\mathfrak{g}$ by the electromagnetic vectors, and our next goal shall be to find them. Accordingly, we can split off the terms in (17) that contain only differential quotients with respect to time, into which the divergence of $\mathfrak{w}$ enters as a factor. That will give the equations:

$$(17_a)\qquad \dot{\varphi} + \mathfrak{g}\dot{\mathfrak{w}} = \mathfrak{E}'\dot{\mathfrak{D}} + \mathfrak{H}'\dot{\mathfrak{B}},$$

$$(17_b)\qquad \varphi = \tfrac{1}{2}\mathfrak{E}'\mathfrak{D} + \tfrac{1}{2}\mathfrak{H}'\mathfrak{B}.$$

The elimination of φ yields:

$$(18)\qquad 2\,\mathfrak{g}\dot{\mathfrak{w}} = \mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' + \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{H}'\dot{\mathfrak{B}}\,.$$

That relation will serve to tell us the components of the impulse density, once the right-hand side is expressed as a linear function of the acceleration components on the basis of the relations between the electromagnetic vectors that characterize the theory in question.

From (V_b) and (11), one will get the condition:

$$(18_a)\qquad [\mathfrak{w}\mathfrak{g}] = [\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}'].$$

This must be fulfilled in every case, since otherwise our system would exhibit an internal contradiction.

(16) and (17_b) determine the energy density:

$$(19)\qquad \psi = \mathfrak{E}'\mathfrak{D} + \mathfrak{H}'\mathfrak{B} + \mathfrak{w}\mathfrak{g}\,.$$

From (V_a), the sum of the relative normal stresses will amount to:

$$X'_x + Y'_y + Z'_z = -\left\{\tfrac{1}{2}\mathfrak{E}'\mathfrak{D} + \tfrac{1}{2}\mathfrak{H}'\mathfrak{B}\right\},$$

so it will follow from (10) that:

$$X_x + Y_y + Z_z = -\left\{\tfrac{1}{2}\mathfrak{E}'\mathfrak{D} + \tfrac{1}{2}\mathfrak{H}'\mathfrak{B}\right\},$$

such that the remarkable relation will exist:

$$(19_a)\qquad X_x + Y_y + Z_z + \psi = 0.$$

If one substitutes the value (19) of ψ, as well as the expression (V_a) for the relative stress, in (12) then one will get:

$$(20)\qquad \mathfrak{S} = c\,[\mathfrak{E}'\mathfrak{H}'] + \mathfrak{w}\{\mathfrak{E}'\mathfrak{D} + \mathfrak{H}'\mathfrak{B}\} - \mathfrak{D}\,(\mathfrak{w}\mathfrak{E}') - \mathfrak{B}\,(\mathfrak{w}\mathfrak{H}') + \mathfrak{w}\,(\mathfrak{w}\mathfrak{g})$$

for the energy current, which is an expression that goes to:

$$\frac{\mathfrak{S}}{c} = [\mathfrak{E}'\mathfrak{H}'] + [\mathfrak{E}'[\mathfrak{q}\,\mathfrak{D}]] + [\mathfrak{H}'[\mathfrak{q}\,\mathfrak{B}]] + \mathfrak{q}\,(\mathfrak{q}\,c\,\mathfrak{g}),$$

on the basis of known rules of calculation, when one sets:

$$\mathfrak{q} = \frac{\mathfrak{w}}{c},$$

to abbreviate. One can also write that as:

$$(21) \qquad \frac{\mathfrak{S}}{c} = [\mathfrak{E}' - [\mathfrak{q}\,\mathfrak{B}],\, \mathfrak{H}' + [\mathfrak{q}\,\mathfrak{D}]] - \mathfrak{q}\,(\mathfrak{q}\,c\,\mathfrak{g}),$$

in which $\mathfrak{B}$ is understood to mean the vector:

$$(22) \qquad \mathfrak{B} = [\mathfrak{D}\;\mathfrak{B}] - c\,\mathfrak{g}.$$

We shall now go on to the discussion of the special theories, in which we restrict ourselves to isotropic bodies throughout.

§ 6.

Theory of H. Hertz.

The moving body of **HERTZian** electrodynamics sets the vectors $\mathfrak{D}$ and $\mathfrak{B}$ proportional to $\mathfrak{E}'$ and $\mathfrak{H}'$:

$$(23) \qquad \mathfrak{D} = \varepsilon\,\mathfrak{E}', \qquad \mathfrak{B} = \mu\,\mathfrak{H}'.$$

Correspondingly, as long as ε and μ can be considered to be constants for a particular material point in a moving body, one will have:

$$\mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' = 0, \qquad \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}' = 0.$$

It will then follow from (18) that:

$$(24) \qquad \mathfrak{g} = 0.$$

*The **HERTZian** theory does not include the electromagnetic quantities of motion.* It derives the ponderomotor force from the stresses alone, in which, from (10), it is irrelevant whether one refers the stresses to fixed or co-moving surfaces. A rotational moment of the relative stresses will not appear, since the two sides of (18_a) will also be equal to zero then.

From (19), the energy density will have the value:

$$(25) \qquad \psi = \tfrac{1}{2}\varepsilon\,\mathfrak{E}'^2 + \tfrac{1}{2}\mu\,\mathfrak{H}'^2 .$$

However, as was mentioned above, the simple Ansatz by which **HERTZ**'s theory couples the excitations $\mathfrak{D}$, $\mathfrak{B}$ to the electromagnetic forces $\mathfrak{E}'$, $\mathfrak{H}'$ has not been confirmed by experiment. All that remains is to choose between the theories that will be discussed in the following paragraphs.

§ 7.

Theory of E. Cohn.

E. COHN based the electrodynamics of moving bodies on the following coupling equations:

(26)
$$\begin{cases} \mathfrak{D} = \varepsilon \mathfrak{E}' - [\mathfrak{q}\mathfrak{H}'], \\ \mathfrak{B} = \mu \mathfrak{H}' + [\mathfrak{q}\mathfrak{E}']. \end{cases}$$

When $\dot{\varepsilon}$ and $\dot{\mu}$ are, in turn, set equal to zero, they will imply:

$$\mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' = \dot{\mathfrak{q}}[\mathfrak{E}'\mathfrak{H}'] + \mathfrak{q}[\mathfrak{E}'\dot{\mathfrak{H}}'] - \mathfrak{q}[\dot{\mathfrak{E}}'\mathfrak{H}'],$$
$$\mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}' = \dot{\mathfrak{q}}[\mathfrak{E}'\mathfrak{H}'] + \mathfrak{q}[\dot{\mathfrak{E}}'\mathfrak{H}'] - \mathfrak{q}[\mathfrak{E}'\dot{\mathfrak{H}}'].$$

Now, since the relation (18) demands that:

$$2\dot{\mathfrak{q}}c\,\mathfrak{g} = \mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' + \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}',$$

one can classify **COHN**'s theory within our system when one sets:

(27)
$$c\,\mathfrak{g} = [\mathfrak{E}'\mathfrak{H}'] = \frac{\mathfrak{S}'}{c}.$$

*In **COHN**'s electrodynamics, the impulse density must be set equal to the relative ray, divided by* c^2.

One easily confirms the fact that from (26) and (27), the relation (18_a) will also need to be satisfied when one observes that the following identity exists:

$$[\mathfrak{q}\,[\mathfrak{E}'\mathfrak{H}']] = [\mathfrak{E}'\,[\mathfrak{q}\,\mathfrak{H}']] - [\mathfrak{H}'[\mathfrak{q}\,\mathfrak{E}']].$$

It now follows from (19) that the electromagnetic *energy density* will be:

(28)
$$\psi = \tfrac{1}{2}\mathfrak{E}'\mathfrak{D} + \tfrac{1}{2}\mathfrak{H}'\mathfrak{B} + \mathfrak{q}\,[\mathfrak{E}'\mathfrak{H}'],$$

which is an expression that, according to (26), can also be written as:

(28*a*)
$$\psi = \tfrac{1}{2}\varepsilon\,\mathfrak{E}'^2 + \tfrac{1}{2}\mu\,\mathfrak{H}'^2 + 2\mathfrak{q}\,[\mathfrak{E}'\mathfrak{H}'];$$

this coincides with **E. COHN**'s Ansatz.

I shall come back to the calculation of the ponderomotive force later.

§ 8.

Theory of H. A. Lorentz.

When we change the coupling equations of the theory of **H. A. LORENTZ** in such a way that there is symmetry in the electric and magnetic vectors, we will arrive at the Ansatz:

$$(29)\qquad \begin{cases} \mathfrak{D} = \varepsilon\,\mathfrak{E}' - [\mathfrak{q}\,\mathfrak{H}], \\ \mathfrak{B} = \mu\,\mathfrak{H}' + [\mathfrak{q}\,\mathfrak{E}], \end{cases}$$

$$(30)\qquad \begin{cases} \mathfrak{E}' = \mathfrak{E} + [\mathfrak{q}\,\mathfrak{H}], \\ \mathfrak{H}' = \mathfrak{H} - [\mathfrak{q}\,\mathfrak{E}]. \end{cases}$$

Here, two new vectors $\mathfrak{E}$, $\mathfrak{H}$ appear, along with the four vectors that are contained in the main equations. This situation makes **LORENTZ**'s theory more complicated than **COHN**'s. The latter coupled the components of $\mathfrak{D}$, $\mathfrak{B}$ with those of $\mathfrak{E}'$, $\mathfrak{H}'$ directly by means of equations that were linear in the velocity components; by contrast, in this one, the coupling equations that are obtained eliminating $\mathfrak{E}$, $\mathfrak{H}$ [§ 10, eq. (37_b)] will no longer be linear in the velocity components.

Hence, the **LORENTZ** vectors $\mathfrak{E}$ and $\mathfrak{H}$ will take on an intuitive meaning. Namely, from eqs. (29), (30), the excitations $\mathfrak{D}$ and $\mathfrak{B}$ can be split into two parts:

$$(31)\qquad \begin{cases} \mathfrak{D} = \mathfrak{E} + \mathfrak{P}, \qquad \mathfrak{P} = (\varepsilon - 1)\,\mathfrak{E}', \\ \mathfrak{B} = \mathfrak{H} + \mathfrak{M}, \quad \mathfrak{M} = (\mu - 1)\,\mathfrak{H}'. \end{cases}$$

LORENTZ interpreted the first components of the electric and magnetic excitation, which the represented by $\mathfrak{E}$ and $\mathfrak{H}$, resp., as the electric and magnetic excitation of the *ether*, resp., and the second components, which is represented by the vectors $\mathfrak{P}$ and $\mathfrak{M}$ (viz., the electric and magnetic *polarization*), resp., was the electric and magnetic excitation of *matter*. The latter is set proportional to the electric and magnetic forces $\mathfrak{E}'$ and $\mathfrak{H}'$ that act upon the unit charges in the moving matter.

In this paragraph, we would like to consider ε and μ to be independent of velocity and time for a well-defined material point, although we shall lift those restrictions later on.

In order to ascertain the impulse density on the basis of the relation (18), we calculate the quantities:

$$(31_a)\qquad \begin{cases} \mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' = \mathfrak{E}'\dot{\mathfrak{E}} - \mathfrak{E}\,\dot{\mathfrak{E}}' + \mathfrak{E}'\dot{\mathfrak{P}} - \mathfrak{P}\,\dot{\mathfrak{E}}', \\ \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}' = \mathfrak{H}'\dot{\mathfrak{H}} - \mathfrak{H}\dot{\mathfrak{H}}' + \mathfrak{H}'\dot{\mathfrak{M}} - \mathfrak{M}\dot{\mathfrak{H}}'. \end{cases}$$

It follows from (30) that:

$$\mathfrak{E}'\dot{\mathfrak{E}} - \mathfrak{E}\,\dot{\mathfrak{E}}' = -\,\mathfrak{q}\,[\dot{\mathfrak{E}}\,\mathfrak{H}] + \mathfrak{q}\,[\mathfrak{E}\,\dot{\mathfrak{H}}] + \dot{\mathfrak{q}}\,[\mathfrak{E}\,\mathfrak{H}],$$

$$\mathfrak{H}'\dot{\mathfrak{H}} - \mathfrak{H}\dot{\mathfrak{H}}' = - \mathfrak{q}\,[\mathfrak{E}\,\dot{\mathfrak{H}}] + \mathfrak{q}\,[\dot{\mathfrak{E}}\,\mathfrak{H}] + \dot{\mathfrak{q}}\,[\mathfrak{E}\,\mathfrak{H}].$$

Now, according to (31), since the other two terms in (31_a) vanish, the relation (18) will imply that:

(32) $$c\,\mathfrak{g} = [\mathfrak{E}\,\mathfrak{H}]$$

is the value of the *electromagnetic impulse density*.

Now, the question arises whether that value likewise satisfies the condition (18_a):

$$[\mathfrak{q}\,c\,\mathfrak{g}] = [\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}'] .$$

From (29), one will have:

$$[\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}'] = [\mathfrak{E}'\,[\mathfrak{q}\,\mathfrak{H}]] - [\mathfrak{H}'\,[\mathfrak{q}\,\mathfrak{E}]].$$

It will further follow from (30) that:

$$[\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}'] = [\mathfrak{E}\,[\mathfrak{q}\,\mathfrak{H}]] - [\mathfrak{H}\,[\mathfrak{q}\,\mathfrak{E}]].$$

On the basis of the well-known identity:

$$[\mathfrak{q}\,[\mathfrak{E}\,\mathfrak{H}]] = [\mathfrak{E}\,[\mathfrak{q}\,\mathfrak{H}]] - [\mathfrak{H}\,[\mathfrak{q}\,\mathfrak{E}]],$$

one can prove that the expression (32) for the impulse density actually satisfies the condition (18_a).

Now, it follows from (19) that the *energy density* has the value:

(33) $$\psi = \tfrac{1}{2}\,\mathfrak{E}'\mathfrak{D} + \tfrac{1}{2}\,\mathfrak{H}'\mathfrak{B} + \mathfrak{q}\,[\mathfrak{E}\,\mathfrak{H}],$$

so one can also write:

$$\psi = \tfrac{1}{2}\,\mathfrak{E}^2 + \tfrac{1}{2}\,\mathfrak{H}^2 + \tfrac{1}{2}\,\mathfrak{E}'\mathfrak{H} + \tfrac{1}{2}\,\mathfrak{H}'\mathfrak{M}.$$

The first two terms are to be regarded, in the sense of **LORENTZ**'s theory, as contributions of the *ether* to the electromagnetic impulse density, while the last two are contributions from the polarized *matter*.

We now proceed to the calculation of the energy current. In **LORENTZ**'s theory, with consideration given to (32) and (31), we will have the expression:

(34) $$\mathfrak{W} = [\mathfrak{D}\,\mathfrak{B}] - [\mathfrak{E}\,\mathfrak{H}] = [\mathfrak{E}\,\mathfrak{M}] + [\mathfrak{P}\,\mathfrak{H}] + [\mathfrak{P}\,\mathfrak{M}]$$

for the vector $\mathfrak{W}$ that was introduced at the close of § 5. From (31) and (30), one has:

$$\mathfrak{E}' - [\mathfrak{q}\,\mathfrak{B}] = \mathfrak{E} - [\mathfrak{q}\,\mathfrak{M}],$$
$$\mathfrak{H}' + [\mathfrak{q}\,\mathfrak{D}] = \mathfrak{H} + [\mathfrak{q}\,\mathfrak{P}],$$

such that equation (21) will assume the form:

$$\frac{\mathfrak{S}}{c} = [\mathfrak{E} - [\mathfrak{q}\ \mathfrak{M}],\ \mathfrak{H} + [\mathfrak{q}\ \mathfrak{P}]] - \mathfrak{q}\ (\mathfrak{q}\ \mathfrak{W}).$$

Now since, from (34), one must set:

$$\mathfrak{q}\ (\mathfrak{q}\ \mathfrak{W}) = [[\mathfrak{q}\ \mathfrak{E}]\ [\mathfrak{q}\ \mathfrak{M}]] + [[\mathfrak{q}\ \mathfrak{P}]\ [\mathfrak{q}\ \mathfrak{H}]] + [[\mathfrak{q}\ \mathfrak{P}]\ [\mathfrak{q}\ \mathfrak{M}]],$$

it will finally follow that the *value of the energy current* is:

$$\frac{\mathfrak{S}}{c} = [\mathfrak{E}\ \mathfrak{H}] + [\mathfrak{E}'\ [\mathfrak{q}\ \mathfrak{P}]] + [\mathfrak{H}'\ [\mathfrak{q}\ \mathfrak{M}]]. \tag{35}$$

The first term can be regarded as the portion of the energy current that is due to the *ether*, while the second one is the portion that is due to electrically-*polarized matter*, as **G. NORDSTRÖM** ([12]) has explained in a work that appeared recently and is noteworthy other respects. The third term, which enters when magnetically-polarized matter is in motion, corresponds to the second one in such a way that it requires the symmetry of the electric and magnetic vectors that was assumed here.

§ 9.

Theory of H. Minkowski.

In that theory, the following relations between the electromagnetic vectors will be true:

$$\begin{cases} \mathfrak{D} = \varepsilon\,\mathfrak{E}' - [\mathfrak{q}\,\mathfrak{H}], \\ \mathfrak{B} = \mu\,\mathfrak{H}' + [\mathfrak{q}\,\mathfrak{E}], \end{cases} \tag{36}$$

$$\begin{cases} \mathfrak{E}' = \mathfrak{E} + [\mathfrak{q}\,\mathfrak{B}], \\ \mathfrak{H}' = \mathfrak{H} - [\mathfrak{q}\,\mathfrak{D}]. \end{cases} \tag{37}$$

Here, as well, a new vector-pair appears along with the two vector-pairs that are included in the main equations that mediates the relation between them.

From the standpoint of the system that we have used as our basis, the problem arises, in turn, of deriving the impulse density from the relation (18). It follows from (36) that:

$$\mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' = \dot{\mathfrak{q}}\ [\mathfrak{E}'\ \mathfrak{H}] + \mathfrak{q}\ [\mathfrak{E}'\ \dot{\mathfrak{H}}] + \mathfrak{q}\ [\dot{\mathfrak{E}}'\ \mathfrak{H}],$$
$$\mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}' = \dot{\mathfrak{q}}\ [\mathfrak{E}\ \mathfrak{H}'] + \mathfrak{q}\ [\dot{\mathfrak{E}}\ \mathfrak{H}'] + \mathfrak{q}\ [\mathfrak{E}\ \dot{\mathfrak{H}}'].$$

([12]) **G. NORDSTRÖM**, *Die Energiegleichung für das elektromagnetische Feld bewegter Körper*, Dissertation, Helsingfors, 1908.

Thus, the right-hand side of (18) will become:

(38) $$\mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' + \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}' = \dot{\mathfrak{q}}\{[\mathfrak{E}'\mathfrak{H}] + [\mathfrak{E}\mathfrak{H}']\} + \mathfrak{q}\{[\mathfrak{E}'\dot{\mathfrak{H}}] + [\dot{\mathfrak{E}}\mathfrak{H}'] - [\dot{\mathfrak{E}}'\mathfrak{H}] - [\mathfrak{E}\dot{\mathfrak{H}}']\}\,.$$

On the basis of (37), we express $\mathfrak{E}$, $\mathfrak{H}$, as well as $\dot{\mathfrak{E}}$, $\dot{\mathfrak{H}}$, in terms of the vectors that appear in the main equations and find that:

(38_a) $$[\mathfrak{E}'\ \mathfrak{H}] + [\mathfrak{E}\ \mathfrak{H}'] = 2\,[\mathfrak{E}'\ \mathfrak{H}'] + \mathfrak{q}\,(\mathfrak{E}'\ \mathfrak{D}) - \mathfrak{D}\,(\mathfrak{q}\ \mathfrak{E}') + \mathfrak{q}\,(\mathfrak{H}'\ \mathfrak{B}) - \mathfrak{B}\,(\mathfrak{q}\ \mathfrak{H}'),$$

(38_b) $$\left\{\begin{aligned} &[\mathfrak{E}'\dot{\mathfrak{H}}] + [\dot{\mathfrak{E}}\,\mathfrak{H}'] - [\dot{\mathfrak{E}}'\,\mathfrak{H}] - [\mathfrak{E}\,\dot{\mathfrak{H}}'] \\ &= \dot{\mathfrak{q}}\,(\mathfrak{E}'\mathfrak{D}) - \mathfrak{D}\,(\dot{\mathfrak{q}}\,\mathfrak{E}') + \dot{\mathfrak{q}}\,(\mathfrak{H}'\mathfrak{B}) - \mathfrak{B}\,(\dot{\mathfrak{q}}\,\mathfrak{H}') \\ &+ \mathfrak{q}\,\{\mathfrak{E}'\,\dot{\mathfrak{D}} - \mathfrak{D}\,\dot{\mathfrak{E}}' + \mathfrak{H}'\,\dot{\mathfrak{B}} - \mathfrak{B}\,\dot{\mathfrak{H}}'\} \\ &- \{\dot{\mathfrak{D}}\,(\mathfrak{q}\,\mathfrak{E}') - \mathfrak{D}\,(\mathfrak{q}\,\dot{\mathfrak{E}}') + \dot{\mathfrak{B}}\,(\mathfrak{q}\,\mathfrak{H}') - \mathfrak{B}\,(\mathfrak{q}\,\dot{\mathfrak{H}}')\}. \end{aligned}\right.$$

When we substitute ($38_{a,b}$) in (38), we will obtain:

($38c$) $$\left\{\begin{aligned} &\mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' - \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\,\dot{\mathfrak{H}}' \\ &= 2\,\dot{\mathfrak{q}}\,\{[\mathfrak{E}'\,\mathfrak{H}'] - \mathfrak{q}(\mathfrak{E}'\mathfrak{D}) + \mathfrak{q}\,(\mathfrak{H}'\mathfrak{B}) - \mathfrak{D}\,(\mathfrak{q}\,\mathfrak{E}') - \mathfrak{B}\,(\dot{\mathfrak{q}}\,\mathfrak{H}')\} \\ &+ (\dot{\mathfrak{q}}\,\mathfrak{D})(\mathfrak{q}\,\mathfrak{E}') - (\mathfrak{q}\mathfrak{D})(\dot{\mathfrak{q}}\,\mathfrak{E}') - (\mathfrak{q}\dot{\mathfrak{D}})(\mathfrak{q}\,\mathfrak{E}') + (\mathfrak{q}\mathfrak{D})(\mathfrak{q}\,\dot{\mathfrak{E}}') \\ &+ (\dot{\mathfrak{q}}\,\mathfrak{B})(\mathfrak{q}\,\mathfrak{H}') - (\mathfrak{q}\mathfrak{B})(\dot{\mathfrak{q}}\mathfrak{H}') - (\mathfrak{q}\dot{\mathfrak{B}})(\mathfrak{q}\,\mathfrak{H}') + (\mathfrak{q}\mathfrak{B})(\mathfrak{q}\,\dot{\mathfrak{H}}') \\ &+ \mathfrak{q}^2\{\mathfrak{E}'\,\dot{\mathfrak{D}} - \mathfrak{D}\,\dot{\mathfrak{E}}' + \mathfrak{H}'\,\dot{\mathfrak{B}} - \mathfrak{B}\,\dot{\mathfrak{H}}'\}. \end{aligned}\right.$$

However, it follows from (36) that:

$$\begin{aligned} -(\mathfrak{q}\dot{\mathfrak{D}})(\mathfrak{q}\,\mathfrak{E}') + (\mathfrak{q}\mathfrak{D})(\mathfrak{q}\,\dot{\mathfrak{E}}') &= (\dot{\mathfrak{q}}\,\mathfrak{D})(\mathfrak{q}\,\mathfrak{E}') - (\mathfrak{q}\mathfrak{D})(\dot{\mathfrak{q}}\,\mathfrak{E}')\,, \\ -(\mathfrak{q}\dot{\mathfrak{B}})(\mathfrak{q}\,\mathfrak{H}') + (\mathfrak{q}\mathfrak{B})(\mathfrak{q}\,\dot{\mathfrak{H}}') &= (\dot{\mathfrak{q}}\,\mathfrak{B})(\mathfrak{q}\,\mathfrak{H}') - (\mathfrak{q}\mathfrak{B})(\dot{\mathfrak{q}}\mathfrak{H}')\,. \end{aligned}$$

The second and third rows on the right-hand side of (38_c) will then assume the values:

$$\begin{aligned} 2\{(\dot{\mathfrak{q}}\mathfrak{D})(\mathfrak{q}\,\mathfrak{E}') - (\mathfrak{q}\mathfrak{D})(\dot{\mathfrak{q}}\,\mathfrak{E}')\} &= 2([\dot{\mathfrak{q}}\mathfrak{q}][\mathfrak{D}\,\mathfrak{E}'])\,, \\ 2\{(\dot{\mathfrak{q}}\mathfrak{B})(\mathfrak{q}\,\mathfrak{H}') - (\mathfrak{q}\mathfrak{B})(\dot{\mathfrak{q}}\,\mathfrak{H}')\} &= 2([\dot{\mathfrak{q}}\mathfrak{q}][\mathfrak{B}\,\mathfrak{H}'])\,. \end{aligned}$$

Now, if one has, in fact:

(39) $$[\mathfrak{q}\ c\ \mathfrak{g}] = [\mathfrak{D}\ \ \mathfrak{E}] + [\mathfrak{B}\ \ \mathfrak{H}],$$

as (18_a) would demand, then the second and third row collectively will yield:

$$2([\dot{\mathfrak{q}}\mathfrak{q}]\,[\mathfrak{q}c\mathfrak{g}]) = 2((\dot{\mathfrak{q}}\mathfrak{q}))(\mathfrak{q}c\mathfrak{g}) - \mathfrak{q}^2\,(\dot{\mathfrak{q}}\,2c\,\mathfrak{q})\,.$$

Therefore, it will ultimately follow from (18) that:

(39$_a$) $$c\,\mathfrak{g} = [\mathfrak{E}'\,\mathfrak{H}'] + \mathfrak{q}\,(\mathfrak{E}'\,\mathfrak{D}) + \mathfrak{q}\,(\mathfrak{H}'\,\mathfrak{B}) - \mathfrak{D}\,(\mathfrak{q}\,\mathfrak{E}') - \mathfrak{B}\,(\mathfrak{q}\,\mathfrak{H}') + \mathfrak{q}\,(\mathfrak{q}\,c\,\mathfrak{g}).$$

A comparison with (20) will yield the important relation:

(40) $$\mathfrak{g} = \frac{\mathfrak{S}}{c^2}.$$

If we insert the ***Minkowski*** *coupling equations between the electromagnetic vectors into our system then the impulse density in the moving body will be equal to the energy current, divided by* c^2.

It follows from (40) and (21), when one recalls (37), that:

(40$_a$) $$c\,\mathfrak{g} = [\mathfrak{E}\mathfrak{H}] - \mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W}),$$

in which the vector:

(40$_b$) $$\mathfrak{W} = [\mathfrak{D}\mathfrak{B}] - c\,\mathfrak{g}$$

is determined from:

(40$_c$) $$\mathfrak{W} - \mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W}) = [\mathfrak{D}\mathfrak{B}] - [\mathfrak{E}\mathfrak{H}].$$

If we let the x-axis point in the direction of $\mathfrak{q}$ and set:

(40$_d$) $$k^2 = 1 - |\,\mathfrak{q}\,|^2$$

then the components of $\mathfrak{W}$ will become:

(41) $$\begin{cases} \mathfrak{W}_x = k^{-2}\{[\mathfrak{D}\mathfrak{B}]_x - [\mathfrak{E}\mathfrak{H}]_x\}, \\ \mathfrak{W}_y = [\mathfrak{D}\mathfrak{B}]_y - [\mathfrak{E}\mathfrak{H}]_y, \\ \mathfrak{W}_z = [\mathfrak{D}\mathfrak{B}]_z - [\mathfrak{E}\mathfrak{H}]_z, \end{cases}$$

and it follows from (40$_d$) that:

(42) $$\begin{cases} c\,\mathfrak{g}_x = \dfrac{\mathfrak{S}_x}{c} = k^{-2}[\mathfrak{E}\mathfrak{H}]_x - |\,\mathfrak{q}\,|^2\,k^{-2}[\mathfrak{D}\mathfrak{B}]_x, \\ c\,\mathfrak{g}_y = \dfrac{\mathfrak{S}_y}{c} = [\mathfrak{E}\mathfrak{H}]_y, \\ c\,\mathfrak{g}_z = \dfrac{\mathfrak{S}_z}{c} = [\mathfrak{E}\mathfrak{H}]_z. \end{cases}$$

The derivation above is missing something; viz., it lacks the proof that equation (39), which is assumed to be valid, is actually fulfilled. In order to show that, we calculate the vector:

$$\mathfrak{N}' = [\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}'] = [\mathfrak{E}'\,[\mathfrak{q}\mathfrak{H}]] - [\mathfrak{H}'\,[\mathfrak{q}\mathfrak{E}]]$$
$$= \mathfrak{q}\,(\mathfrak{E}'\mathfrak{H}) - \mathfrak{q}\,(\mathfrak{E}\mathfrak{H}') + \mathfrak{E}\,(\mathfrak{q}\mathfrak{H}') - \mathfrak{H}\,(\mathfrak{q}\mathfrak{E}').$$

Since one has:

$$\mathfrak{E}'\mathfrak{H} - \mathfrak{E}\mathfrak{H}' = \mathfrak{q}\ \{[\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}']\} = (\mathfrak{q}\ \mathfrak{N}'),$$
$$\mathfrak{E}\ (\mathfrak{q}\mathfrak{H}') - \mathfrak{H}\ (\mathfrak{q}\mathfrak{E}') = \mathfrak{E}\ (\mathfrak{q}\ \mathfrak{H}) - \mathfrak{H}\ (\mathfrak{q}\ \mathfrak{E}) = [\mathfrak{q}\ [\mathfrak{E}\ \mathfrak{H}]],$$

if one recalls (40_a) then one will have:

$$\mathfrak{N}' - \mathfrak{q}\ (\mathfrak{q}\ \mathfrak{N}') = [\mathfrak{q}\ c\ \mathfrak{g}].$$

Since the component of $\mathfrak{N}'$ in the direction of the vector $\mathfrak{q}$ is equal to zero by this, one can also write:

(43) $$\mathfrak{N}' = \{[\mathfrak{D}\mathfrak{E}'] + [\mathfrak{B}\mathfrak{H}'] = [\mathfrak{q}\ c\ \mathfrak{g}].$$

With that, the condition (18_a) is shown to be valid, and at the same time, the flaw in the derivation of the value of $\mathfrak{q}$ above is eliminated.

(19) implies the value of the *energy density:*

(44) $$\psi = \tfrac{1}{2}\,\mathfrak{E}'\ \mathfrak{D} + \tfrac{1}{2}\,\mathfrak{H}'\ \mathfrak{B} + \mathfrak{q}\ c\ \mathfrak{g},$$

which can be brought into the form:

(44*a*) $$\psi = \tfrac{1}{2}\,\mathfrak{E}\ \mathfrak{D} + \tfrac{1}{2}\,\mathfrak{H}\ \mathfrak{B} - \mathfrak{q}\ \mathfrak{W}$$

by using (37) and (40*b*).

In order to ease the comparison of our results with the **MINKOWSKI** Ansätze, we write:

$$c\ \mathfrak{g}_x = X_l\ , \qquad c\ \mathfrak{g}_y = Y_l\ , \qquad c\ \mathfrak{g}_z = Z_l\ ,$$
$$\mathfrak{S}_x = c\ T_x\ , \qquad \mathfrak{S}_y = c\ T_y\ , \qquad \mathfrak{S}_z = c\ T_z\ ,$$
$$ct = l, \qquad \mathfrak{m}\mathfrak{K} + Q = c\ \mathfrak{K}_t\ , \qquad \psi = T_l\ .$$

The impulse equations (6) and the energy equation (7) then read:

$$\mathfrak{K}_x = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} - \frac{\partial X_l}{\partial l},$$
$$\mathfrak{K}_y = \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} - \frac{\partial Y_l}{\partial l},$$
$$\mathfrak{K}_z = \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} - \frac{\partial Z_l}{\partial l},$$
$$\mathfrak{K}_t = -\frac{\partial T_x}{\partial x} - \frac{\partial T_y}{\partial y} - \frac{\partial T_z}{\partial z} - \frac{\partial T_l}{\partial l}.$$

From (19_a), one then has the relation:

$$X_x + Y_y + Z_z + T_l = 0.$$

Now, the relation (40) says that:

$$X_l = T_x\,, \qquad Y_l = T_y\,, \qquad Z_l = T_z\,.$$

In conjunction with (6_a), these relations include a remarkable symmetry property of that system of equations that is not found in **MINKOWSKI**'s Ansätzen. As the behavior under **LORENTZ** transformations would demand, the ten quantities:

$$\begin{array}{cccc} X_x\,, & Y_y\,, & Z_z\,, & -T_t\,, \\ X_y = Y_x\,, & Y_z = Z_y\,, & Z_x = X_z\,, & \\ -X_l = -T_x\,, & -Y_l = -T_y\,, & -Z_l = -T_z & \end{array}$$

transform like the squares and products of the coordinates x, y¸ z, and the light path length l. Correspondingly, this *space-time tensor* satisfies the *principle of relativity* in the **MINKOWSKI** sense. The same thing is true for the *space-time vector of the first kind* $\mathfrak{K}$ that is derived from it. The *ponderomotor force* that will be calculated in § 12 *also satisfies the principle of relativity then.*

§ 10.

Relationship between the theories of Lorentz and Minkowski.

We have emphasized that the intuitive meaning that the vectors $\mathfrak{E}$, $\mathfrak{H}$ take on in **LORENTZ**'s theory is that of the contribution of the ether to the electric and magnetic excitations. In **MINKOWSKI**'s theory, those vectors, by means of which, $\mathfrak{D}$, $\mathfrak{B}$ and $\mathfrak{E}'$, $\mathfrak{H}'$ are coupled to each other, lack any such intuitive explanation. When one takes the standpoint of the principle of relativity, there is also no basis for speaking of the ether and its electromagnetic properties. That principle considers only the motion of matter relative to an observer and the electromagnetic processes in that matter.

Meanwhile, for our system of the electrodynamics of moving bodies, the vectors $\mathfrak{E}$ and $\mathfrak{H}$ are defined more narrowly than the vectors $\mathfrak{D}$, $\mathfrak{B}$, $\mathfrak{E}'$, $\mathfrak{H}'$. If we couple those four vectors to each other directly by eliminating $\mathfrak{E}$ and $\mathfrak{H}$ then the relationship between the theories of **MINKOWSKI** and **LORENTZ** will become clearer.

A) Minkowski's theory.

If follows from equations (36) and (37) of § 9 that:

$$\left\{\begin{aligned} \mathfrak{D} + [\mathfrak{q}[\mathfrak{q}\mathfrak{D}]] &= \varepsilon\,\mathfrak{E}' - [\mathfrak{q}\mathfrak{H}'], \\ \mathfrak{B} + [\mathfrak{q}[\mathfrak{q}\mathfrak{B}]] &= \mu\,\mathfrak{H}' + [\mathfrak{q}\mathfrak{E}']. \end{aligned}\right. \tag{45}$$

If we lay the x-axis in the direction of $\mathfrak{q}$ then we will get:

(45$_a$)
$$\begin{cases} \mathfrak{D}_x = \varepsilon\, \mathfrak{E}'_x, \\ \mathfrak{B}_x = \mu\, \mathfrak{H}'_x \end{cases}$$

for the components in that direction. By contrast, we will have:

(45$_b$)
$$\begin{cases} k^2\mathfrak{D}_y = \varepsilon\, \mathfrak{E}'_y - [\mathfrak{q}\mathfrak{H}']_y, \\ k^2\mathfrak{B}_y = \mu\, \mathfrak{H}'_y + [\mathfrak{q}\mathfrak{E}']_y \end{cases}$$

for the components that are perpendicular to the direction of motion.

B) Lorentz's theory.

It follows from equations (30) of § 8 that:

(46)
$$\begin{cases} \mathfrak{E} + [\mathfrak{q}[\mathfrak{q}\mathfrak{E}]] = \varepsilon\, \mathfrak{E}' - [\mathfrak{q}\mathfrak{H}'], \\ \mathfrak{H} + [\mathfrak{q}[\mathfrak{q}\mathfrak{H}]] = \mu\, \mathfrak{H}' + [\mathfrak{q}\mathfrak{E}']. \end{cases}$$

The components of $\mathfrak{E}$ and $\mathfrak{H}$ that are parallel (perpendicular, resp.) to the direction of the velocity will then be:

(46$_a$)
$$\begin{cases} \mathfrak{E}_x = \mathfrak{E}'_x, \quad k^2\mathfrak{E}_y = \mathfrak{E}'_y - [\mathfrak{q}\mathfrak{H}']_y, \\ \mathfrak{H}_x = \mathfrak{H}'_x, \quad k^2\mathfrak{H}_y = \mathfrak{H}'_y + [\mathfrak{q}\mathfrak{E}']_y. \end{cases}$$

Whereas, for **MINKOWSKI**, ε and μ are independent of direction in isotropic bodies, for **LORENTZ**, it is permissible that different values of ε and μ can come under question for excitations that are parallel and perpendicular to $\mathfrak{q}$. Accordingly, from (29) and (46$_a$), one will get:

(47$_a$)
$$\begin{cases} \mathfrak{D}_x = \varepsilon_x\, \mathfrak{E}'_x, \\ \mathfrak{B}_x = \mu_x\, \mathfrak{H}'_x \end{cases}$$

for the longitudinal components of $\mathfrak{D}$ and $\mathfrak{B}$, and:

(47$_b$)
$$\begin{cases} k^2\mathfrak{D}_y = (k^2\varepsilon_y + |\mathfrak{q}|^2)\, \mathfrak{E}'_y - [\mathfrak{q}\mathfrak{H}']_y, \\ k^2\mathfrak{B}_y = (k^2\mu_y + |\mathfrak{q}|^2)\, \mathfrak{H}'_y + [\mathfrak{q}\mathfrak{E}']_y \end{cases}$$

for the transversal components.

If we compare (45$_a$) and (47$_a$), on the one hand, and (45$_b$) and (47$_b$), on the other, then we will recognize that the equations in both theories that couple $\mathfrak{D}$, $\mathfrak{B}$ and $\mathfrak{E}'$, $\mathfrak{H}'$ will coincide when one sets:

(48$_a$) $\varepsilon_x = \varepsilon, \qquad \mu_x = \mu$

(48$_b$) $\varepsilon_y - 1 = k^{-2}\,(\varepsilon - 1), \qquad \mu_y - 1 = k^{-2}\,(\mu - 1)$

in **LORENTZ**'s theory. From (31), the longitudinal and transversal components of the electric and magnetic polarization will be:

$$(48_c)\qquad \mathfrak{P}_x = (\varepsilon - 1)\mathfrak{E}'_x, \qquad \mathfrak{P}_y = k^{-2}(\varepsilon - 1)\mathfrak{E}'_y, \qquad \mathfrak{P}_z = k^{-2}(\varepsilon - 1)\,\mathfrak{E}'_z,$$

$$(48_d)\qquad \mathfrak{M}_x = (\mu - 1)\mathfrak{H}'_x, \qquad \mathfrak{M}_y = k^{-2}(\mu - 1)\mathfrak{H}'_y, \qquad \mathfrak{M}_z = k^{-2}(\mu - 1)\,\mathfrak{H}'_z.$$

H. A. LORENTZ (1904) has already spoken about the fact that if the relativity postulate is compatible with **LORENTZ**'s theory then the electric polarization of an isotropic body in the rest state must be influenced by its motion in the manner that is described by (48_c). If one assume the symmetry of the electric and magnetic vectors then they will imply the corresponding behavior for the magnetic polarization.

The assumption that was made in § 8 that ε and μ should be independent of the velocity is now obsolete. Hence, the values of impulse density, energy density, and energy current that were found there must also be corrected. One can no longer neglect the quantities:

$$(49)\qquad \begin{cases} \mathfrak{E}'\dot{\mathfrak{P}} - \mathfrak{P}\dot{\mathfrak{E}}' = 2\mathfrak{E}'\dot{\mathfrak{P}} - \dfrac{d}{dt}(\mathfrak{E}'\mathfrak{P}), \\[2ex] \mathfrak{H}'\dot{\mathfrak{M}} - \mathfrak{M}\dot{\mathfrak{H}}' = 2\mathfrak{H}'\dot{\mathfrak{M}} - \dfrac{d}{dt}(\mathfrak{H}'\mathfrak{M}) \end{cases}$$

that entered into (31_a). It follows from (48*c*) that:

$$(49_a)\qquad \mathfrak{E}'\mathfrak{P} = (\varepsilon - 1)\,\{\mathfrak{E}'^2_x + k^{-2}(\mathfrak{E}'^2_y + \mathfrak{E}'^2_z)\}.$$

Furthermore, if one considers the transversal acceleration and the rotation of the polarization ellipsoid that it demands then one will get the following expressions for the components of $\dot{\mathfrak{P}}$:

$$(49_b)\qquad \begin{cases} \dot{\mathfrak{P}}_x = (\varepsilon - 1)\dot{\mathfrak{E}}'_x - \dfrac{\dot{\mathfrak{q}}_y}{|\mathfrak{q}|}\mathfrak{P}_y - \dfrac{\dot{\mathfrak{q}}_z}{|\mathfrak{q}|}\mathfrak{P}_z, \\[2ex] \dot{\mathfrak{P}}_y = k^{-2}(\varepsilon - 1)\dot{\mathfrak{E}}'_y + 2\dot{\mathfrak{q}}_x\,|\mathfrak{q}|\,k^{-2}\mathfrak{P}_y + \dfrac{\dot{\mathfrak{q}}_z}{|\mathfrak{q}|}\mathfrak{P}_x, \\[2ex] \dot{\mathfrak{P}}_z = k^{-2}(\varepsilon - 1)\dot{\mathfrak{E}}'_z + 2\dot{\mathfrak{q}}_x\,|\mathfrak{q}|\,k^{-2}\mathfrak{P}_z + \dfrac{\dot{\mathfrak{q}}_z}{|\mathfrak{q}|}\mathfrak{P}_x. \end{cases}$$

This implies that:

$$(49_c)\qquad \begin{cases} 2\mathfrak{E}'\dot{\mathfrak{P}} = 2(\varepsilon - 1)\{\mathfrak{E}'_x\dot{\mathfrak{E}}'_x + k^{-2}\mathfrak{E}'_y\dot{\mathfrak{E}}'_y + k^{-2}\mathfrak{E}'_z\dot{\mathfrak{E}}'_z\} + 4\dot{\mathfrak{q}}_x\,|\mathfrak{q}|\,k^{-2}\{\mathfrak{E}'_y\mathfrak{P}_y + \mathfrak{E}'_z\mathfrak{P}_z\} \\[1ex] \qquad - 2\dfrac{\dot{\mathfrak{q}}_y}{|\mathfrak{q}|}\{\mathfrak{E}'_x\mathfrak{P}_y - \mathfrak{E}'_y\mathfrak{P}_x\} - 2\dfrac{\dot{\mathfrak{q}}_z}{|\mathfrak{q}|}\{\mathfrak{E}'_x\mathfrak{P}_z - \mathfrak{E}'_z\mathfrak{P}_x\}, \end{cases}$$

while (49_a) yields:

$$(49_d) \quad \frac{d}{dt}(\mathfrak{E}'\,\mathfrak{P}) = 2\,(\varepsilon - 1)\{\mathfrak{E}'_x\,\dot{\mathfrak{E}}'_x + k^{-2}\mathfrak{E}'_y\,\dot{\mathfrak{E}}'_y + k^{-2}\mathfrak{E}'_z\,\dot{\mathfrak{E}}'_z\} + 2\dot{\mathfrak{q}}_x\,|\,\mathfrak{q}\,|\,k^{-2}\{\mathfrak{E}'_y\mathfrak{P}_y + \mathfrak{E}'_z\mathfrak{P}_z\}.$$

Now, since one has, from (48_c), that:

$$\mathfrak{E}'_x\mathfrak{P}_y - \mathfrak{E}'_y\mathfrak{P} = |\,\mathfrak{q}\,|^2\,\mathfrak{E}'_x\mathfrak{P}_y\,,$$
$$\mathfrak{E}'_x\mathfrak{P}_z - \mathfrak{E}'_z\mathfrak{P} = |\,\mathfrak{q}\,|^2\,\mathfrak{E}'_x\mathfrak{P}_z\,,$$

that will imply:

$$(49_e) \qquad \mathfrak{E}'\dot{\mathfrak{P}} - \dot{\mathfrak{P}}\mathfrak{E}' = 2\dot{\mathfrak{q}}_x\,|\,\mathfrak{q}\,|\,k^{-2}\{\mathfrak{E}'_y\mathfrak{P}_y + \mathfrak{E}'_z\mathfrak{P}_z\} - 2\dot{\mathfrak{q}}_y\,|\,\mathfrak{q}\,|\,\mathfrak{E}'_x\mathfrak{P}_y - 2\dot{\mathfrak{q}}_z\,|\,\mathfrak{q}\,|\,\mathfrak{E}'_x\mathfrak{P}_z\,.$$

The introduction of this expression and the one that corresponds to it for the magnetic term in (31_c) will yield the corrected value for the impulse density:

$$(50) \qquad c\mathfrak{g} = [\mathfrak{E}\mathfrak{H}] + [\mathfrak{E}'[\mathfrak{q}\mathfrak{B}]] + [\mathfrak{H}'[\mathfrak{q}\mathfrak{M}]] + \dot{\mathfrak{q}}_x\,|\,\mathfrak{q}\,|\,k^{-2}\{\mathfrak{E}'_y\mathfrak{P}_y + \mathfrak{E}'_z\mathfrak{P}_z + \mathfrak{H}'_y\mathfrak{M}_y + \mathfrak{H}'_z\mathfrak{M}_z\}$$

in place of (32). It is easy to verify that the relation (18_a) is still fulfilled.

If the value (50) for $c\,\mathfrak{g}$ were substituted into the general formula (19) for the energy density then, instead of (33), one would have:

$$(51) \qquad \psi = \tfrac{1}{2}\mathfrak{E}^2 + \tfrac{1}{2}\mathfrak{H}^2 + \tfrac{1}{2}\mathfrak{E}'\mathfrak{P} + \tfrac{1}{2}\mathfrak{H}'\mathfrak{M} + |\,\mathfrak{q}\,|^2\,k^{-2}\{\mathfrak{E}'_y\mathfrak{P}_y + \mathfrak{E}'_z\mathfrak{P}_z + \mathfrak{H}'_y\mathfrak{M}_y + \mathfrak{H}'_z\mathfrak{M}_z\}.$$

On the basis of (20), one will also obtain the following corrected formula for the energy current:

$$(52) \qquad \frac{\mathfrak{S}}{c} = [\mathfrak{E}\mathfrak{H}] + [\mathfrak{E}'[\mathfrak{q}\mathfrak{P}]] + [\mathfrak{H}'[\mathfrak{q}\mathfrak{M}]] + \mathfrak{q}\,|\,\mathfrak{q}\,|^2\,k^{-2}\{\mathfrak{E}'_y\mathfrak{P}_y + \mathfrak{E}'_z\mathfrak{P}_z + \mathfrak{H}'_y\mathfrak{M}_y + \mathfrak{H}'_z\mathfrak{M}_z\}.$$

From (50) and (52), one sees that the relation between the energy current and impulse density that we encountered already in MINKOWSKI's theory, namely:

$$(53) \qquad \frac{\mathfrak{S}}{c} = c\,\mathfrak{g},$$

will exist in **LORENTZ**'s theory, as well, when one modifies it in the given way.

That result was to be expected. Once the equations that couple $\mathfrak{D}$ and $\mathfrak{B}$ with $\mathfrak{E}'$ and $\mathfrak{H}'$ are brought into agreement, no essential difference will exist between the two theories any longer from the standpoint of our system. Only the meaning of the vectors that are denoted by $\mathfrak{E}$, $\mathfrak{H}$ has changed. As would emerge from (50) and (51), the **LORENTZ** definition of these vectors also now allows the contribution from the ether to the electromagnetic energy and impulse and that of matter to differ from each other. Of course, formulas for the contribution of matter are now true that no longer admit a simple interpretation.

§ 11.

Consideration of the temporal change in ε and μ.

Up to now, we have considered the dielectric constant ε and the magnetic permeability μ to be quantities that possess constant values for a given material point, or at least (cf., § 10), vary with the velocity in a given way. Up to now, we have not contemplated the case in which those quantities depends upon the state of deformation of the body, and therefore upon time. How are the considerations to be modified when $\dot{\varepsilon}$ and $\dot{\mu}$ are not equal to zero?

A) Theories of H. Hertz and E. Cohn.

If we use the formulas (23) of **HERTZ**'s theory or formula (26) of **COHN**'s theory as a basis then we will find, in the case where ε and μ depend upon time, that the following relation will replace (18):

(54) $$\tfrac{1}{2}\{\mathfrak{E}'\dot{\mathfrak{D}} - \mathfrak{D}\dot{\mathfrak{E}}' + \mathfrak{H}'\dot{\mathfrak{B}} - \mathfrak{B}\dot{\mathfrak{H}}'\} = \mathfrak{g}\dot{\mathfrak{w}} + \xi\dot{\varepsilon} + \eta\dot{\mu},$$

in which we have set:

(54*a*) $$\zeta = \tfrac{1}{2}\mathfrak{E}'^2, \qquad \eta = \tfrac{1}{2}\mathfrak{H}'^2.$$

In this, we have assumed that the previous expression (24) [(27), resp.] is still true for the impulse density.

B) Theories of H. Minkowski and H. A. Lorentz.

The calculations that one makes when are starts from the coupling equations (36) and (37) of **MINKOWSKI**'s theory are somewhat more cumbersome. One must not only add the term:

$$\dot{\varepsilon}\,\mathfrak{E}'^2 + \dot{\mu}\,\mathfrak{H}'^2$$

to the right-hand side of (38), but one must also consider the variation of ε and μ in the calculation of the terms in (38_c) that include $\dot{\mathfrak{D}}$ and $\dot{\mathfrak{B}}$. As long as the value of $\mathfrak{g}$ does not change, one will also get a relation of the form (54) here. Hence, the quantities ζ, η will have a somewhat different meaning here:

(54_b) $$\begin{cases} \zeta = \tfrac{1}{2}\{\mathfrak{E}'^2_x + k^{-2}(\mathfrak{E}'^2_y + \mathfrak{E}'^2_z)\}, \\ \eta = \tfrac{1}{2}\{\mathfrak{H}'^2_x + k^{-2}(\mathfrak{H}'^2_y + \mathfrak{H}'^2_z)\}. \end{cases}$$

This result is also true for **LORENTZ**'s theory in the form that we gave it in § 10. All expressions in that theory that contain only the vectors $\mathfrak{E}'$, $\mathfrak{H}'$, $\mathfrak{D}$, $\mathfrak{B}$ will then be identical with the corresponding expressions in **MINKOWSKI**'s theory.

Now, since equation (54) contradicts the relation (18), and since we would not like to allow any change in the values of the impulse density and the energy density, we would regard it as necessary for us to correct the value for the quantity P' that was given in (V), and indeed give it the value:

$$-\zeta\,\dot{\varepsilon}-\eta\,\dot{\mu}\,.$$

The considerations of § 5 will lead directly to the relation (54), instead of the relation (18).

This way of looking at things finds some support in the theory of electrostriction ([13]). In the simplest-available case of fluids and gases, in which ε and μ depend upon only the density σ, one will have:

$$-\zeta\,\dot{\varepsilon}-\eta\,\dot{\mu} = -\dot{\sigma}\left\{\zeta\frac{d\varepsilon}{d\sigma}+\eta\frac{d\mu}{d\sigma}\right\}.$$

That will be a consequence of the continuity condition for matter:

$$-\zeta\,\dot{\varepsilon}-\eta\,\dot{\mu} = \operatorname{div}\mathfrak{w}\left\{\zeta\sigma\frac{d\varepsilon}{d\sigma}+\eta\sigma\frac{d\mu}{d\sigma}\right\}.$$

If one recalls the definition (13) of the quantity P' then one will see that this increase will correspond to an increase in the relative normal stresses by:

$$-p' = \zeta\sigma\frac{d\varepsilon}{d\sigma}+\eta\sigma\frac{d\mu}{d\sigma}\,. \tag{55}$$

In the case where ε and μ go up with increasing density, the additional pressure p' will be negative; i.e., the fluid will tend to contract in electric and magnetic fields. In the case of rest, (55), in conjunction with (54_a) or (54_b), will yield an Ansatz that is useful in the theory of electrostriction.

For solid bodies, some general considerations will be required if one is to represent the dependency of the electric and magnetic constants upon the state of deformation. **H. HERTZ** ([14]) has calculated the corresponding supplementary stresses in general from the standpoint of his theory. By contrast, **E. COHN**, as well as **H. MINKOWSKI**, passed over the introduction of such supplementary stresses. We will also allow that simplification from now on, since it is permitted by the negligible magnitude of the supplementary stress.

([13]) **F. POCKELS**, *Enzyklopädie der mathematischen Wissenschaften*, Bd. V, 2, article 16, no. 4.

([14]) **H. HERTZ**, "Über die Grundgleichungen der Elektrodynamik für bewegte Körper." [*Gesammelte Werke*, Bd. II, pp. 256-285; pp. 280.]

§ 12.

The ponderomotive force.

Since the dependency of the relative stresses, as well as the impulse density, upon the electromagnetic vectors is determined in each of the present theories from now on, equations (8) will then yield the components of the ponderomotor force. From (V_a), one has:

$$(56)\qquad \begin{cases} X'_x = \tfrac{1}{2}\{\mathfrak{E}'_x\mathfrak{D}_x - \mathfrak{E}'_y\mathfrak{D}_y - \mathfrak{E}'_z\mathfrak{D}_z\} + \tfrac{1}{2}\{\mathfrak{H}'_x\mathfrak{B}_x - \mathfrak{H}'_y\mathfrak{B}_y - \mathfrak{H}'_z\mathfrak{B}_z\}, \\ X'_y = \mathfrak{E}'_x\mathfrak{D}_y + \mathfrak{H}'_x\mathfrak{B}_y, \\ X'_z = \mathfrak{E}'_x\mathfrak{D}_z + \mathfrak{H}'_x\mathfrak{B}_z. \end{cases}$$

It follows from this that:

$$(57)\qquad \begin{cases} \dfrac{\partial X'_x}{\partial x} + \dfrac{\partial X'_y}{\partial y} + \dfrac{\partial X'_z}{\partial z} \\ \quad = \mathfrak{E}'_x \operatorname{div}\mathfrak{D} + \mathfrak{H}'_x \operatorname{div}\mathfrak{B} - \mathfrak{D}_y \operatorname{curl}_z\mathfrak{E}' + \mathfrak{D}_z \operatorname{curl}_y\mathfrak{E}' - \mathfrak{B}_y \operatorname{curl}_z\mathfrak{H}' + \mathfrak{B}_z \operatorname{curl}_y\mathfrak{H}' \\ \qquad -\dfrac{1}{2}\left\{\mathfrak{E}'\dfrac{\partial\mathfrak{D}}{\partial x} - \mathfrak{D}\dfrac{\partial\mathfrak{E}'}{\partial x} + \mathfrak{H}'\dfrac{\partial\mathfrak{B}}{\partial x} - \mathfrak{B}\dfrac{\partial\mathfrak{H}'}{\partial x}\right\}. \end{cases}$$

If we consider the last row then the analogy with the left-hand side of (54) will appear immediately. The expressions differ by only the fact that, there, the differentiation was with respect to time, while here, it is with respect to a coordinate.

Now, since the train of thought that led to the relation (54) was not based upon the meaning of the independent variables, one will have:

$$(57_a)\qquad \frac{1}{2}\left\{\mathfrak{E}'\frac{\partial\mathfrak{D}}{\partial x} - \mathfrak{D}\frac{\partial\mathfrak{E}'}{\partial x} + \mathfrak{H}'\frac{\partial\mathfrak{B}}{\partial x} - \mathfrak{B}\frac{\partial\mathfrak{H}'}{\partial x}\right\} = \mathfrak{g}\frac{\partial\mathfrak{w}}{\partial x} + \zeta\frac{\partial\varepsilon}{\partial x} + \eta\frac{\partial\mu}{\partial x}.$$

The vectorial generalization of (57) comes from the force contribution that originates in the relative stresses:

(58) $\mathfrak{K}_1 =$

$$= \mathfrak{E}' \operatorname{div}\mathfrak{D} + \mathfrak{H}' \operatorname{div}\mathfrak{B} - [\mathfrak{D} \operatorname{curl}\mathfrak{E}'] - [\mathfrak{B} \operatorname{curl}\mathfrak{E}'] - (\mathfrak{g}\nabla)\,\mathfrak{w} - [\mathfrak{g} \operatorname{curl}\mathfrak{w}] - \zeta\nabla\varepsilon - \eta\nabla\mu.$$

According to (8), the contribution that originates from the electromagnetic impulse will be:

$$(58_a)\qquad \mathfrak{K}_2 = -\frac{\delta\mathfrak{g}}{\delta t}.$$

We would like to convert the vector products that appear in (58) into:

$$-[\mathfrak{D}\ \mathrm{curl}\ \mathfrak{E}'] = \frac{1}{c}\left[\mathfrak{D}\frac{\partial'\mathfrak{B}}{\partial t}\right],$$

$$-[\mathfrak{B}\ \mathrm{curl}\ \mathfrak{H}'] = \frac{1}{c}[\mathfrak{J}\mathfrak{B}] + \frac{1}{c}\left[\frac{\partial'\mathfrak{D}}{\partial t}\mathfrak{B}\right],$$

with the help of the first two main equations of § 4.

If one recalls the rule (5) of § 1 then the sum of these two terms will be:

$$\frac{1}{c}[\mathfrak{J}\mathfrak{B}] + \frac{1}{c}\left\{\frac{\delta}{\delta t}[\mathfrak{D}\mathfrak{B}] + ([\mathfrak{D}\mathfrak{B}]\nabla)\mathfrak{w} + [[\mathfrak{D}\mathfrak{B}]\,\mathrm{curl}\,\mathfrak{w}]\right\}.$$

One gets the ponderomotor force by adding the forces $\mathfrak{K}_1$ and $\mathfrak{K}_2$; the expression that arises will simplify when one introduces the vector:

(59) $$\mathfrak{W} = [\mathfrak{D}\ \mathfrak{B}] - c\ \mathfrak{g}$$

that was defined in (22), and use the notations:

(59_a) $$\mathfrak{q} = \frac{\mathfrak{w}}{c}, \qquad l = ct.$$

One might further set:

(59_b) $$\mathrm{div}\ \mathfrak{D} = \rho$$

for the density of the true electricity and assume that the density of the true magnetism is zero:

(59_c) $$\mathrm{div}\ \mathfrak{B} = 0.$$

The electromagnetic measure of the current strength will also be introduced instead of the electrostatic one by:

(59_d) $$\mathfrak{J} = c\ i.$$

The expression for the *ponderomotor force* that acts per unit volume on the moving matter will then read:

(60) $$\mathfrak{K} = \mathfrak{E}'\rho + [i\ \mathfrak{B}] - \zeta\nabla\varepsilon - \eta\,\nabla\mu + \frac{\delta\mathfrak{g}}{\delta t} + (\mathfrak{W}\ \nabla)\ \mathfrak{q} + [\mathfrak{W}\ \mathrm{curl}\ \mathfrak{q}].$$

The first term represents the force that is applied to the moving electricity, while the second one represents the force that is applied to the electrical conduction current. The third and fourth terms are concerned with the influence of the inhomogeneity of the body. While those four terms already come under consideration for static or stationary fields in bodies at rest, the last terms, which contain the vector $\mathfrak{W}$, play a role only for non-stationary processes or in moving bodies.

In the expressions that were obtained for the ponderomotor force, when one ignores the extremely small deviations in the meanings of the quantities ζ and η [eq. ($54_{a,b}$)], the difference between the individual theories of electrodynamics of moving bodies will come about only by virtue of the fact that the vector $\mathfrak{W}$ assumes different values.

If $\mathfrak{K}$ yields the quantity of motion that given by the electromagnetic field then the energy that is converted into non-electromagnetic forms will be given by the sum of the **JOULE** heat and the work that is done by the ponderomotor force. According to the main equation (III) and (59*d*) one will have:

$$Q = cq = \mathfrak{J}\mathfrak{E}' = ci\ \mathfrak{E}'$$

for the **JOULE** heat, while (60) will provide the work that is done by the force $\mathfrak{K}$:

$$\mathfrak{q}\mathfrak{K} = \mathfrak{E}'\rho\,\mathfrak{q} - i\,[\mathfrak{q}\,\mathfrak{B}] - \zeta(\mathfrak{q}\,\nabla)\,\varepsilon - \eta\,(\mathfrak{q}\,\nabla)\,\mu + \mathfrak{q}\,\left\{\frac{\delta\mathfrak{W}}{\delta l} + (\mathfrak{W}\nabla)\,\mathfrak{q} + [\mathfrak{W}\operatorname{curl}\mathfrak{q}]\right\}.$$

If one ponders the fact that the calculation of the ponderomotor force is based upon the assumptions that:

$$\frac{1}{c}\dot{\varepsilon} = \frac{\partial\varepsilon}{\partial l} + (\mathfrak{q}\,\nabla)\,\varepsilon = 0,$$

$$\frac{1}{c}\dot{\mu} = \frac{\partial\mu}{\partial l} + (\mathfrak{q}\,\nabla)\,\mu = 0,$$

and the fact that from (3) and the known rules of calculation:

$$\nabla\,(\mathfrak{q}\,\mathfrak{W}) = (\mathfrak{q}\,\nabla)\,\mathfrak{W} + [\mathfrak{q}\operatorname{curl}\mathfrak{W}] + (\mathfrak{W}\,\nabla)\,\mathfrak{q} + [\mathfrak{W}\operatorname{curl}\mathfrak{q}],$$

then that will give:

$$\frac{\delta\mathfrak{W}}{\delta l} + (\mathfrak{W}\,\nabla)\,\mathfrak{q} + [\mathfrak{W}\operatorname{curl}\mathfrak{q}] = \frac{\partial\mathfrak{W}}{\partial l} + \mathfrak{W}\operatorname{div}\mathfrak{q} + \nabla\,(\mathfrak{q}\,\mathfrak{W}) - [\mathfrak{q}\operatorname{curl}\mathfrak{W}],$$

and furthermore, if one recalls (3_a) then it will follow that:

$$\mathfrak{q}\left\{\frac{\delta\mathfrak{W}}{\delta l} + (\mathfrak{W}\nabla)\mathfrak{q} + [\mathfrak{W}\operatorname{curl}\mathfrak{q}]\right\}$$
$$= -\,\mathfrak{W}\,\frac{\partial\mathfrak{q}}{\partial l} + \frac{\partial(\mathfrak{q}\mathfrak{W})}{\partial l} + \operatorname{div}\mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W}) = -\,\mathfrak{W}\frac{\partial\mathfrak{q}}{\partial l} + \frac{\delta(\mathfrak{q}\mathfrak{W})}{\delta l},$$

so one will ultimately get the following formula for the *energy delivered* to a unit volume per unit time:

$$(60_a) \qquad q + \mathfrak{q}\mathfrak{K} = \{i + \rho\,\mathfrak{q}\}\ \{\mathfrak{E}' - [\mathfrak{q}\ \mathfrak{B}]\} + \zeta \frac{\partial \varepsilon}{\partial l} + \eta \frac{\partial \mu}{\partial l} - \mathfrak{W} \frac{\partial \mathfrak{q}}{\partial l} + \frac{\delta(\mathfrak{q}\mathfrak{W})}{\delta l}.$$

Here as well, if one ignores the slight deviation in the meanings of ζ and η in quantities of second order then the various theories will differ merely by the value of the vector $\mathfrak{W}$ when one considers it from the standpoint of our system.

We now imagine setting $\mathfrak{W}$ equal to those values that it takes on in each theory in question, and compare our expression (50) for the ponderomotor force with the ones that the other authors obtained.

The value for the ponderomotor force that was given by **E. COHN** exhibits a small deviation from our own. That arises, in part, from the fact that **E. COHN**'s Ansatz for the relative stresses is not completely identical with (V_a). Namely, he set $\varepsilon\,\mathfrak{E}'$ in place of $\mathfrak{D}$ there, probably in the hopes of making the rotational moment $\mathfrak{N}'$ of the relative stresses vanish. The difference in the values of the force contributions that originate in the relative stresses that is required for that is found to be equal to:

$$(\mathfrak{g}\ \nabla)\ \mathfrak{w} + \mathfrak{w}\ \text{div}\ \mathfrak{g}.$$

We have regarded it as necessary that $\mathfrak{N}'$ should vanish only when no electromagnetic impulse came into question, as in **HERTZ**'s theory. Meanwhile **E. COHN** likewise inserted a second part of the force into the calculation that was coupled with the vector $\mathfrak{g}$, namely:

$$\mathfrak{K}_2 = -\frac{\partial' \mathfrak{g}}{\partial t}.$$

From (4), this expression for the electromagnetic inertial force deviates from our own (58_a) by:

$$(\mathfrak{g}\ \nabla)\ \mathfrak{w}.$$

In total, the difference between **E. COHN**'s expression for force and the one that is obtained here amounts to:

$$2\ (\mathfrak{g}\ \nabla)\ \mathfrak{w} + \mathfrak{w}\ \text{div}\ \mathfrak{g},$$

in which $\mathfrak{g}$ is determined by (27). That is probably too small be experimentally provable.

We now go on to **MINKOWSKI**'s theory. It was already mentioned in § 9 that the close relationship that exists between impulse density and energy current because of the results of the present investigation was not assumed by **MINKOWSKI**. Correspondingly, our value (60) of the ponderomotor force also deviates from **MINKOWSKI**'s Ansatz. In particular, the term $\delta\mathfrak{W} / \delta l$, which already comes into question in the case of rest, is lacking from the latter. It was already proved by **A.**

EINSTEIN and **I. LAUB** ([15]) that the force that should act upon the polarization current in a magnetic field according **LORENTZ** is lacking with **MINKOWSKI**'s Ansatz. Now, an experimental proof of the existence of that force has not, in fact, emerged, so one bases one's confidence in its existence on the analogy that exists between conduction current and polarization current in the picture that is given by the theory of the electron. That analogy asserts itself in such a way that one would not like to reject that force without a compelling reason. As would emerge from eq. (63), our expression for the force contains that force. We have already remarked at the conclusion of § 9 that it does not contradict the principle of relativity.

In the case of rest, where one writes $\mathfrak{E}$, $\mathfrak{H}$, instead of $\mathfrak{E}'$, $\mathfrak{H}'$, the *ponderomotive force* will be:

$$\mathfrak{K} = \mathfrak{E}\,\rho + [i\,\mathfrak{B}] - \tfrac{1}{2}\mathfrak{E}^2\,\nabla\varepsilon - \tfrac{1}{2}\mathfrak{H}^2\,\nabla\mu + \frac{\partial\mathfrak{W}}{\partial l}\,. \tag{61}$$

In the various theories, the vector $\mathfrak{W}$ possesses the following values:

A) Theory of H. Hertz.

Here, it follows from (22) and (24):

$$\mathfrak{W} = [\mathfrak{D}\ \mathfrak{B}] = \varepsilon\mu\,[\mathfrak{E}\ \mathfrak{H}]. \tag{61_a}$$

B) Theories of E. Cohn, H. A. Lorentz, and H. Minkowski.

In all three theories, as would emerge from (27), (32), (40_a), one will have:

$$\begin{gathered} c\,\mathfrak{g} = [\mathfrak{E}\ \mathfrak{H}], \\ \mathfrak{W} = [\mathfrak{D}\mathfrak{B}] - [\mathfrak{E}\mathfrak{H}] = (\varepsilon\mu - 1)\,[\mathfrak{E}\mathfrak{H}] \end{gathered} \tag{61_b}$$

in the case of rest. The fact that the all three theories yield the same value for the ponderomotor force on bodies at rest is, in the sense of our system, based upon the fact that the equations that couple $\mathfrak{D}$ and $\mathfrak{B}$ with $\mathfrak{E}'$ and $\mathfrak{H}'$, with the inclusion of the terms that are linear in $\mathfrak{q}$, coincide. The notation of **LORENTZ**'s theory might be used in the discussion of the forces on bodies at rest.

If one sets $\mathfrak{W}$ equal to the value (61_b) then the ponderomotor force (61) can be decomposed into two parts:

$$\left\{ \begin{aligned} \mathfrak{K}_e &= \mathfrak{E}\rho - \tfrac{1}{2}\mathfrak{E}^2\nabla\varepsilon + (\varepsilon\mu - 1)\left[\mathfrak{E}\,\frac{\partial\mathfrak{H}}{\partial l}\right], \\ \mathfrak{K}_m &= [i\mathfrak{B}] - \tfrac{1}{2}\mathfrak{H}^2\nabla\mu + (\varepsilon\mu - 1)\left[\frac{\partial\mathfrak{E}}{\partial l}\,\mathfrak{H}\right], \end{aligned} \right. \tag{62}$$

([15]) **A. EINSTEIN** and **I. LAUB**, "Über die im elektromagnetischen Felde auf ruhende Körper ausgeübten ponderomotorischen Kräfte," Ann. Phys. (Leipzig) **26** (1908), 541-550.

which are to be interpreted as the electric and magnetic fields.

From the main equations for bodies at rest:

$$\text{curl}\ \mathfrak{H} = \frac{\partial \mathfrak{D}}{\partial l} + i,$$
$$\text{curl}\ \mathfrak{E} = -\frac{\partial \mathfrak{B}}{\partial l},$$

when one introduces the electric and magnetic polarizations:

$$\mathfrak{P} = \mathfrak{D} - \mathfrak{E} = (\varepsilon - 1)\ \mathfrak{E},$$
$$\mathfrak{M} = \mathfrak{B} - \mathfrak{H} = (\mu - 1)\ \mathfrak{H},$$

one can derive the following two relations:

$$0 = -\ [\mathfrak{P}\ \text{curl}\ \mathfrak{E}] - \mu\,(\varepsilon - 1)\left[\mathfrak{E}\frac{\partial \mathfrak{H}}{\partial l}\right],$$
$$[i\ \mathfrak{B}] = [i\ \mathfrak{H}] - [\mathfrak{M}\ \text{curl}\ \mathfrak{H}] - \varepsilon\,(\mu - 1)\left[\frac{\partial \mathfrak{E}}{\partial l}\mathfrak{H}\right].$$

If one takes them into account then the expressions (62) will go to:

$$(62_a)\qquad \left\{\begin{aligned} \mathfrak{K}_e &= \mathfrak{E}\rho - [\mathfrak{P}\ \text{curl}\ \mathfrak{E}] - \tfrac{1}{2}\mathfrak{E}^2\nabla(\varepsilon - 1)\left[\mathfrak{E}\frac{\partial \mathfrak{M}}{\partial l}\right], \\ \mathfrak{K}_m &= [i\ \mathfrak{H}] - [\mathfrak{M}\ \text{curl}\ \mathfrak{H}] - \tfrac{1}{2}\mathfrak{H}^2\nabla(\mu - 1)\left[\frac{\partial \mathfrak{P}}{\partial l}\mathfrak{H}\right]. \end{aligned}\right.$$

Since one further has:

$$\tfrac{1}{2}(\varepsilon - 1)\nabla\mathfrak{E}^2 + \tfrac{1}{2}\mathfrak{E}^2\ \nabla(\varepsilon - 1) = \tfrac{1}{2}\nabla(\varepsilon - 1)\ \mathfrak{E}^2 = \tfrac{1}{2}\nabla(\mathfrak{P}\mathfrak{E}),$$
$$\tfrac{1}{2}(\varepsilon - 1)\nabla\mathfrak{E}^2 = (\mathfrak{P}\nabla)\mathfrak{E} + [\mathfrak{P}\ \text{curl}\ \mathfrak{E}],$$
$$\tfrac{1}{2}(\mu - 1)\nabla\mathfrak{H}^2 + \tfrac{1}{2}\mathfrak{H}^2\ \nabla(\mu - 1) = \tfrac{1}{2}\nabla(\mu - 1)\ \mathfrak{H}^2 = \tfrac{1}{2}\nabla(\mathfrak{M}\mathfrak{H}),$$
$$\tfrac{1}{2}(\mu - 1)\nabla\mathfrak{H}^2 = (\mathfrak{M}\nabla)\mathfrak{H} + [\mathfrak{M}\ \text{curl}\ \mathfrak{H}],$$

one will ultimately have:

$$(63)\qquad \left\{\begin{aligned} \mathfrak{K}_e &= (\mathfrak{P}\nabla)\,\mathfrak{E} + \mathfrak{E}\rho + \left[\mathfrak{E}\frac{\partial \mathfrak{M}}{\partial l}\right] - \tfrac{1}{2}\nabla(\mathfrak{P}\mathfrak{E}), \\ \mathfrak{K}_m &= (\mathfrak{M}\nabla)\,\mathfrak{H} + [i\ \mathfrak{H}] + \left[\frac{\partial \mathfrak{P}}{\partial l}\mathfrak{H}\right] - \tfrac{1}{2}\nabla(\mathfrak{M}\mathfrak{H}). \end{aligned}\right.$$

The formulas for the electric and magnetic contributions to the force that one gets from (61) and (61_b) here might be compared with the Ansätzen that **A. EINSTEIN** and **I. LAUB** ([16]) made for the ponderomotor force on bodies at rest. The first three terms in $\mathfrak{K}_e$ and $\mathfrak{K}_m$ are found there, as well. The first term can be interpreted as the forces that the field exerts upon electrically and magnetically polarized volume elements, where the vector product of *i* and $\mathfrak{H}$ can be interpreted as the force of the magnetic field upon the electrical conduction current. The aforementioned force of the magnetic field on the electric polarization current is added to that, along with the force that corresponds to it when the electric field acts upon the magnetic polarization current. Nevertheless, the last two terms in the expression (63) are missing from the Ansatze of the aforementioned authors, which is connected with the fact that their values for the fictitious normal stresses deviate somewhat from the ones that are otherwise assumed. That term will drop away when one is dealing with the force on a region, upon whose boundary $\mathfrak{P}$ and $\mathfrak{M}$ are equal to zero. The surface integrals that they produce will then vanish.

In this not-infrequent present case, one might also appeal to the Ansatz of **EINSTEIN** and **LAUB**. However, the conclusion that those authors inferred – namely, that the vector $\mathfrak{B}$ was not definitive for the force on the conduction current – does not seem applicable to me. We saw in (61) that it is precisely the vector $\mathfrak{B}$ that determines the force on the current conductor. Meanwhile, it is the force that acts upon a current-carrying and simultaneously magnetized wire in a magnetic field, which is calculated, either approximately as the vector product of *i* and $\mathfrak{B}$ or as that of *i* and $\mathfrak{H}$. Moreover, that vector product is to be added to the force $-\frac{1}{2}\mathfrak{H}^2\nabla\mu$ that exists in the transition layer between the wire and the air, while this includes the force $(\mathfrak{M}\ \nabla)\ \mathfrak{H}$ that acts upon the magnetized wire in the event that the field itself is not, perchance, homogeneous. Except for those entirely-special cases, one must then set the force that is applied to magnetized volume elements of a homogeneous current-carrying wire equal to the vector product of *i* and $\mathfrak{B}$, but not that of *i* and $\mathfrak{H}$.

Moreover, do the formulas (63) do not, by any means, take on such a fundamental meaning as that of the original expressions (61) for the ponderomotor force. Whereas the latter were produced by a system of electrodynamics that also subsumes moving bodies, the former can hardly be generalized in such a way that they also determine the ponderomotor forces on moving bodies.

Ospedalletti, Liguria, December 1908.

MAX ABRAHAM

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([16]) **A. EINSTEIN** and **I. LAUB**, *loc. cit.* ([15]), pp. 549.