“Sulle onde luminose e gravitazionali,” Nuov. Cim. (6) **3** (1912), 211-219.

# On luminous and gravitational waves ([1])

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The principle of relativity excludes propagation speeds that are greater than that of light ($c$), and it is probable that gravitation propagates with that speed.

Starting from those hypotheses, two possible theories of gravitation are possible that correspond to the two types of waves – viz., *transversal* and *longitudinal* – in an isotropic medium.

The first theory, which was proposed by H. A. Lorentz and R. Gans, assumes that the attraction of electricity of opposite signs is a little greater than the repulsion of charges of equal signs, in such a way that a neutral body will produce an electro-gravitational field. It is connected with a magneto-gravitational field by means of equations that are analogous to Maxwell’s. One is then dealing with the propagation of a transverse wave. That theory encounters two difficulties:

1. The static gravitational field has negative energy (Maxwell).

2. The second difficulty is related with the new development of the theory of relativity. As A. Einstein ([2]) observed, the speed of light ($c$) must depend upon the gravitational potential ($\Phi$). Now, such a relationship between $c$ and $\Phi$ would not agree with the theory that considers the gravitational field to be analogous to the electric field, because $c$ is a universal scalar – i.e., an invariant of the Lorentz group – but the electrostatic potential is not. Furthermore, that potential has no significance for fields that are not static.

However, the discrepancy will prove to be even greater when one seeks to apply Minkowski’s equations of motion ([3]):

$$\ddot{x} = F_x, \quad \ddot{y} = F_y, \quad \ddot{z} = F_z, \quad \ddot{u} = F_u \tag{1}$$

to a material point that moves in the gravitational field.

(The dot indicates derivation with respect to proper time. $F$ is the driving force that acts on a unit mass, and it is a universal vector of the first kind.

$$dx,\ dy,\ dz, \qquad \text{and} \qquad du = i\,c\,dt$$

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([1]) Conference presentation before the Società di Fisica in Rome on 17 February 1912.
([2]) A. Einstein, Ann. Phys. (Leipzig) **35** (1911), pp. 898.
([3]) H. Minkowski, “Spazio e tempo,” Nuov. Cim. **18** (1909).

are the components of an infinitesimal displacement in four-dimensional space.) Proper time is defined in such a way that the components of the "vector of motion" ([1]) satisfy the condition:

(2)
$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{u}^2 = -c^2.$$

If one differentiates this then one will find that:

(3)
$$\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \dot{z}\,\ddot{z} + \dot{u}\,\ddot{u} = -c\,\dot{c}.$$

Now, the electromagnetic force that acts upon charges that move in empty space satisfies the so-called "orthogonality condition":

(4)
$$\dot{x}\,F_x + \dot{y}\,F_y + \dot{z}\,F_z + \dot{u}\,F_u = 0.$$

Assume that the laws of the electromagnetic field are true for the gravitational field, so one attributes the property (4) to the gravitational driving force.

It follows from this, by virtue of (1) and (3), that:

$$\dot{c} = 0, \quad \text{i.e.,} \quad c = \text{constant.}$$

Therefore, in the first theory of gravitation, the speed of light does not vary in the gravitational field. However, as we shall see, the constancy of the speed of light, which is a postulate of Einstein's original theory, can no longer be assumed in the further development of his theory.

Now, let us discuss the second theory ([2]), which considers gravitation to be a force that propagates as a longitudinal wave. However, for a plane wave, one has two vectors that are normal to the ray and just one vector that is parallel to it.

We shall consider the gravitational potential $\Phi$ to be an invariant of the Lorentz group – i.e., a universal scalar. One derives the unit driving force from it as a negative gradient:

(5)
$$F_x = -\frac{\partial \Phi}{\partial x}, \quad F_y = -\frac{\partial \Phi}{\partial y}, \quad F_z = -\frac{\partial \Phi}{\partial z}, \quad F_u = -\frac{\partial \Phi}{\partial u}.$$

The equations of motion (1) will then become:

(6)
$$\ddot{x} = -\frac{\partial \Phi}{\partial x}, \quad \ddot{y} = -\frac{\partial \Phi}{\partial y}, \quad \ddot{z} = -\frac{\partial \Phi}{\partial z}, \quad \ddot{u} = -\frac{\partial \Phi}{\partial u}.$$

It will then follow from (3) that:

$$\frac{\partial \Phi}{\partial x}\,\dot{x} + \frac{\partial \Phi}{\partial y}\,\dot{y} + \frac{\partial \Phi}{\partial z}\,\dot{z} + \frac{\partial \Phi}{\partial u}\,\dot{u} = c\,\dot{c},$$

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([1]) H. Minkowski, *loc. cit.*

([2]) M. Abraham, Rend. R. Accad. dei Lincei XX, fasc. 12 (1911); XXI, 1st and 2nd fasc. (1912).

i.e.:

$$c\,\dot{c} = \dot{\Phi}\,,$$

and when one integrates that:

$$\tag{7} \tfrac{1}{2}c^2 - \tfrac{1}{2}c_0^2 = \Phi - \Phi_0\,.$$

That is to say:

*The increment in one-half the square of the speed of light is equal to the increment in the gravitational potential*.

We have thus established the exact form of the relation between $c$ and $\Phi$. Since $c$ and $\Phi$ are universal scalars, it has an invariant character.

We shall now move on to the first difficulty, which relates to the value of the energy of the gravitational field. One has:

$$\tfrac{1}{2}\int dv\,\mu\,\Phi = -\int \frac{dv}{8\pi\,\gamma}(\text{grad}\,\Phi)^2$$

($\mu$ is density, $\gamma$ is the gravitational constant) for the total energy of a system of masses at rest.

It was precisely that negative value that Maxwell had to assign to the energy of the field. However, we shall set:

$$\tag{8} \tfrac{1}{2}\int dv\,\mu\,\Phi = \int dv\,\mu\,\Phi + \int \frac{dv}{8\pi\,\gamma}(\text{grad}\,\Phi)^2$$

and interpret $\mu\,\Phi$ as the *density of potential energy of the matter*, while:

$$\frac{1}{8\pi\,\gamma}(\text{grad}\,\Phi)^2 = \frac{1}{8\pi\,\gamma}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right\}$$

is the *energy density of the gravitational field*. That ultimate result is therefore essentially *positive*.

However, one might observe that the potential energy of matter is still negative, and that the introduction of such an energy in addition to that of the field might seem to be an *ad hoc* hypothesis. To respond to such objections, recall that the new mechanics assigns an enormous internal energy to atoms, only a small part of which manifests itself in the usual physical and chemical processes; it is the radioactive processes that reveal the presence of that atomic energy. That is why we say “the potential energy of matter,” and it is only one term in the expression for the total energy of matter.

As a matter of fact, the *law of conservation of energy*, which follows from the last of the equations of motion (6) for a material point that moves in a static gravitational field, will become:

$$\tag{9} c\,k^{-1} = \text{constant},$$

with

$$k = \left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad c = \{c_0^2 + 2(\Phi - \Phi_0)\}^{1/2}.$$

Obviously, the factor $c$ is a function of the potential $\Phi$, while the factor $k$ is a function of the velocity $v$.

When one sets $v = 0$, $c = c_0$ , one will get:

$$c^2 = c_0^2\, k^2; \quad \frac{c^2}{c_0^2} = k^2,$$

i.e.:

$$1 + 2\left(\frac{\Phi - \Phi_0}{c_0^2}\right) = 1 - \frac{v^2}{c^2},$$

or

(9.a)
$$\tfrac{1}{2} v^2 + \frac{c^2}{c_0^2}(\Phi - \Phi_0) = 0.$$

Except for the factor:

(9.b)
$$\frac{c^2}{c_0^2} = 1 + 2\frac{\Phi - \Phi_0}{c_0^2},$$

which differs slightly from unity, the last relation, which is valid for arbitrary velocities, corresponds to the usual form of the *law of conservation of energy.*

It will then result that assuming the existence of potential energy of matter whose density is $\mu\,\Phi$ in the rest state is not, in fact, an *ad hoc* hypothesis, but a necessary consequence of the general equations of motion. The energy of the gravitational field will then become positive.

Like the speed of light, the speed of gravitational waves will also depend upon the potential $\Phi$, which will complicate the exact solution of the problem of propagation. However, for the usual fields, it is always sufficient to consider the speed of propagation to be constant and to write the *differential equation of the field as:*

(10)
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 4\pi\,\gamma\,\nu$$

($\nu$ is the "rest density" of the attracting mass).

In the last of the notes that were published in Lincei, the elementary law of action between two material points was deduced from that equation. It resulted more simply from the elementary law of electrodynamics, because since there is no vector that is analogous to the magnetic one that acts upon the moving charge, the force will not depend upon the velocity of the *attracted* point. However, it will depend upon the velocity and acceleration of the *attracting* point.

The propagation of gravitation resembles that of the electric force, except for the longitudinal nature of gravitational waves. One knows that a vibrating charge $e$ exerts an *electric force:*

$$F^e = \frac{e\,e'}{c^2\,r}\cdot a \tag{11}$$

($a$ is the value of the acceleration) on another charge $e'$ that is located in a plane that passes through $e$ and is normal to the line of oscillation.

Analogously, a mass $m$ that vibrates along a line must exert a *gravitational force:*

$$F^m = \gamma\frac{m\,m'}{c^2\,r}\cdot a\,, \quad \text{with} \quad \gamma = 6.6\times 10^{-8}, \tag{12}$$

on another mass $m'$ that is located along the same line at a distance of $r$.

(11) and (12) are valid for distances $r$ that are large compared to the wavelength and velocities that are small compared to $c$.

Now, it is known that radioactive substances emit high-velocity material particles (e.g., $\alpha$ rays). One might hope to find gravitational waves that are emitted by the $\alpha$ at the moment of acceleration among the various types of radiation that are emitted by those substances. However, that hope seems illusory; compare the gravitational force (12) and the electric one (11). One has:

$$\frac{e}{c\,m} = 10^4, \quad \frac{e}{m} = 3\times 10^{14}$$

for the $\alpha$ particle, so one will find that the ratio of the forces that act between two such $\alpha$ particles is:

$$\frac{F^m}{F^e} = \gamma\left(\frac{m}{e}\right)^2 = \frac{6.6\times 10^{-8}}{9\times 10^{28}} < 10^{-36}.$$

It then results that the gravitational force is, in fact, insignificance compared to the electric one.

However, things are worse than that: In order to emit an $\alpha$ particle, one needs another particle that will give it the desired velocity, either by an impact or an electrical or gravitational force. Now, the law of action and reaction:

$$m_1\,a_1 + m_2\,a_2 = 0 \tag{13}$$

is true for the reciprocal acceleration of the two masses, at least approximately, if one ignores the impulse of the field.

Now, since the amplitude of the gravitational waves will depend upon precisely that value (13), the intensity of the gravitational wave will again be much less than the one that was calculated above.

Moreover, it will also vary for the electromagnetic wave if the ratio of the charge and mass is the same for all particles. It is precisely the existence of particles whose specific charges differ in value and sign that makes the emission of electromagnetic waves possible.

To summarize: One cannot hope to detect gravitational waves. *The motion of gravitating masses can always be considered to be quasi-stationary.* That is, one can neglect the forces that decrease with the reciprocal distance and take into account only the forces that decrease with the square of the reciprocal distance. That explains why Newton's law is in good agreement with observation.

Now let us address light waves that propagate in the gravitational field. If the speed of light is a function of position then the propagation will take place as it would in an inhomogeneous medium. A. Einstein has already pointed out that according to Huygens's principle, a ray that passes the surface of the Sun must be deflected as if it were attracted to the solar center.

In order to find the *trajectory of light* in a given gravitational field, one agrees to apply Fermat's principle: *It is the curve that makes the time that is employed in order to connect the source O to a generic point P a minimum:*

(14) $$\delta t = \delta \int_O^P \frac{ds}{c} = 0, \quad \text{with} \quad c = \sqrt{c_0^2 + 2(\Phi - \Phi_0)}\,.$$

For example, consider a homogeneous field. The force is parallel to the $z$-axis, so:

$$\Phi - \Phi_0 = -g\,z, \quad c^2 = c_0^2 - 2g\,z.$$

The speed of light is then equal to that of a material point that moves according to Galilean mechanics in a *field with the opposite direction*. The curves that make the elapsed time a minimum are identical to the light rays that are deduced from Fermat's principle for a field with the opposite direction. Since the time of G. Bernoulli (1696), it has been known that those brachistochrones are *cycloids*, and it was only G. Bernoulli, in his fundamental work on the calculus of variations, that could show that the cycloids represent the trajectories of light in an inhomogeneous medium in which their velocities varies with the square root of $z$. Hence, the new ideas are connected with those of two hundred years ago. Moreover, the theory of the brachistochrone can also serve to determine the trajectories of light in other fields.

Finally, consider the consequence that one derives in the general theory of relativity for the relation between the speed of light $c$ and the gravitational potential $\Phi$. Space remains *isotropic* around any isolated point, but not homogeneous. *The Lorentz group is true only infinitesimally*, and the element of a universal curve will become:

$$dS^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2,$$

where $c$ varies with position. The aforementioned postulate of the constancy of the speed of light that Einstein assumed in his original theory will then fade away. As Prof. Castelnuovo observed, that postulate introduces something absolute into the theory of relativity. In order to make relativity more perfect, one must assume that $c$ varies, but in a sense that is different from ours. In the theory that he developed, $c$ lost all of its absolute character and became a relative concept (viz., relative to the gravitational field), and the last remnant of the absolute then disappeared. On the other hand, one sees that the various reference systems will be distinct from the ones in which the gravitational field is static or differs slightly from static, because $c$ will depend upon only position in them, but not time, while $c$ will depend upon time, as well, for all other reference systems that are in a state of translation of rotation with respect the latter. The quasi-static field that is produced by a fixed star will realize the "$\alpha$ body" that was postulated by C. Neumann in order to fix the absolute reference system. One can then call motion relative to a system in which the gravitational field is static "absolute," and the relative will then become absolute. Even better: The concepts of absolute and relative will merge into a greater whole.