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A new theory of gravitation

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Modern physics does not assume that forces propagate with infinite velocity. It does not believe that Newton's law is the true fundamental law of gravitation. Rather, it reduces that law of action at a distance to differential equations that assign a finite speed of propagation to gravitation.

One model of such a theory of immediate action is provided by Maxwell's theory of the electromagnetic field. Its fundamental laws are differential equations that couple the electric vector to the magnetic vector. The electromagnetic energy in that theory is distributed over the field. Whenever the field varies with time, there will be an energy current that is determined by the Poynting vector. For example, if an electron begins to vibrate then it will emit electromagnetic waves. Electromagnetic energy will propagate with the waves from the vicinity of the electron to the parts of space that were initially devoid of the field. That emission of energy will cause a decrement in the vibrations of the electrons.

The analogy between Coulomb's law and Newton's suggests a similar interpretation for the gravitational forces. It was Maxwell himself (¹) that developed such a theory of gravitation and pointed out the difficulty. The differing signs of the forces (viz., masses of equal signs attract, but charges of equal sign repel) causes the energy density of the gravitational field to become negative, although that assumes, of course, that where there is no field, there is no energy. It is necessary to renounce the last hypothesis and imagine that the ether that is devoid of a field is filled with a certain quantity of intrinsic energy that would be diminished by the gravitational field. However, such a hypothesis cannot escape all of the objections either.

For example, consider a material particle that is initially at rest and then put it into vibration. In theories of electromagnetic type, it will emit waves that are analogous to light waves, i.e., transverse waves that propagate with the speed of light. Along with the wave, the gravitational field will enter into regions of space in which the field was initially zero. The energy of the ether will then diminish in those regions; that is: The energy moves towards the vibrating mass which acquires energy at the expense of the intrinsic energy of the ether. That absorption of energy produces an increment in the vibrations of the particle; that will produce an instability of its equilibrium.

^{(&}lt;sup>1</sup>) J. C. Maxwell, *Scientific papers*, I, pp. 570.

Similar difficulties will oppose any gravitational theory of Maxwellian type. Among them, in particular, those of H. A. Lorentz (¹) and R. Gans (²) have found some proponents among the physicists. Starting from the hypothesis that matter is composed of positive and negative electrons and assuming that the attraction between electrons of opposite sign is slightly greater than the repulsion of electrons of equal sign, one will arrive at differential equations that are analogous to those of Maxwell, which couple the gravitational vector, which corresponds to the electric vector, to a second vector that corresponds to the magnetic vector. The gravitational energy current in that theory is expressed by a vector that is analogous to the Poynting vector, but with the opposite direction. The aforementioned difficulty results from that fact: Indeed, Gans found (³) that in that theory the reaction force of radiation has the opposite sign to the one that is valid in the dynamics of the electron. The acceleration of a neutral particle will increase by virtue of the radiation that is created, while that of the electron will decrease. The equilibrium of the neutral particle will not be stable then.

One must then renounce the strict analogy between gravitation and electromagnetism but retain the essential viewpoints of Maxwell's theory in the theory of gravitation, namely: The fundamental laws must be differential equations that describe the excitation and propagation of the gravitational field. One must assign a positive energy density to that field and an energy current.

The problem of the gravitational field is compelling, and all the more so because modern theoretical physics has established interesting relations between mass and energy. According the theory of relativity, one must have:

(1) $E = m c^2$ (*E* is energy, *m* is mass, *c* is the speed of light).

However, in the absence of a satisfactory theory of gravitation, that relation can refer to only the inertial mass. Does it also apply to gravitational mass?

Let a body displace in a gravitational field. Its potential energy will vary with the gravitational potential, and therefore the left-hand side of (1) will, as well. It will follow that one of the factors in the right-hand side or both of them will depend upon the gravitational potential. Take into consideration the hypothesis that the second factor – *i.e., the speed of light – depends upon the gravitational potential.* That hypothesis was stated by Einstein (⁴) in 1911. Starting from it, I propose to develop a theory of the gravitational field (⁵) that I can give a more satisfactory form (⁶) to when I avail myself of a contribution that comes from Einstein (⁷). In this lecture, I shall present the essential features of the new theory of gravitation.

I then formulate the first postulate of the theory:

^{(&}lt;sup>1</sup>) H. A. Lorentz, Verslag. Akad. v. Wetenschapen te Amsterdam (1900), pp. 603.

^{(&}lt;sup>2</sup>) R. Gans, Phys. Zeit. (1905), pp. 803.

^{(&}lt;sup>3</sup>) R. Gans, *H. Weber Festschrift* (1912), pp. 75.

^{(&}lt;sup>4</sup>) A. Einstein, Ann. Phys. (Leipzig) **35** (1911), pp. 898.

^{(&}lt;sup>5</sup>) M. Abraham, Rend. d. R. Accad. dei Lincei XX² (1911), pp. 678, XXI¹ (1912), pp. 27 and 432.

^{(&}lt;sup>6</sup>) M. Abraham, Phys. Zeit. (1912), pp. 793. (Lecture that was presented to the International Congress of Mathematics at Cambridge in August 1912.)

^{(&}lt;sup>7</sup>) A. Einstein, Ann. Phys. (Leipzig) **38** (1912), pp. 355 and 443.

POSTULATE I:

The surfaces c = constant coincide with the equipotential surfaces of the gravitational field; that is: The negative gradient of c points in the direction of gravity.

If the speed of light varies in the gravitational field then according to Huyghens's principle, a light ray will be deflected in that field as it would be in an inhomogeneous medium. That consequence was announced by Einstein. He proved that a light ray that passes very closely to the surface of the Sun must be deflected as if it were attracted to the Sun. However, that deflection, which is observable only during solar eclipses, is quite small and exists at the limits of observation.

Apply the usual geometry to a body at rest in a gravitational field. Namely, assume that the unit of length (viz., the meter) does not depend upon the value of c and that it can serve to measure the length in any region of the field. Even better, consider two regions in which c has differing values c_1 and c_2 . Move an antenna of length one meter from the first region to the second one. Since the length remains invariant, obviously the period of electromagnetic vibration of the antenna will vary inversely to the velocity c:

$$\tau_1: \tau_2 = c_2: c_1.$$

If we can construct a clock whose ticking is independent of c then we can confirm that variation of the period by transporting it along with the antenna from region to the other. However, we shall exclude the existence of such a clock by asserting the following postulate:

POSTULATE II:

An observer that belongs to the observed material system cannot recognize whether the system has been transported into a region where c has a value that is different from its original one.

It follows from this postulate that the duration of any phenomenon will vary with c by the same ratio as the periods of proper vibration of the antenna, because otherwise the observer could verify the latter variation. The general theorem will then follow from the second postulate:

The duration of any phenomenon that is produced in a system will vary in inverse proportion to the c when the system changes position in the gravitational field,

or:

The times are of degree c^{-1} .

The unit of time then has only a local significance in this theory, while one attributes a universal validity to the unit of length, at least for the case of rest.

One must understand the second postulate to mean that the value of c itself does not influence the phenomena that are presented to an observer that belongs to the system. (However, the gradient of c – i.e., gravity – has an influence on those phenomena.) That postulate establishes a certain relativity. For example, imagine that the Earth is displaced into a region of space in which the gravitational field, and therefore c, has a different value. According to postulate II, it is not possible to confirm that fact with terrestrial observations. All of the measurements, and therefore all of the constants of physics, will remain immutable. Moreover, a measurement of the speed of light (for example, by Fizeau's method) will give the same result as the first one, because its variation will be compensated by the variation that is proportional to the angular velocity of the toothed wheel when it is referred to a terrestrial clock.

However, one should not rule out the possibility that an observer that does not belong to the system can perceive the influence of the gravitational potential on its period. For example, a terrestrial observer that compares the positions of the spectral line of the Sun with those of the corresponding terrestrial lines must find that their frequencies have the ratio:

$$\frac{V_2}{V_1} = \frac{\tau_1}{\tau_2} = \frac{c_2}{c_1} \ .$$

A relative displacement of the solar lines towards the red:

$$\frac{V_2 - V_1}{V_1} = \frac{C_2 - C_1}{C_1}$$

will then follow, because the gravitational potential on the Sun (and therefore c) will have a lower value that it has on Earth. Einstein found a value of 2×10^{-6} for that relative displacement of the solar lines, and according to Doppler's principle, that would correspond to a velocity of 0.6 kilometers per second. Now, the astrophysicists can measure displacements of that order to high precision and they have, in fact, observed that value, in the postulated sense of our theory. However, it can be explained in part by the Doppler effect for currents that descend in absorbent gases and in part as being something that is due to pressure, but perhaps all of the phenomena that take place on the surface of the Sun can be explained by taking the gravitational displacement into account.

Since lengths have degree c^0 and times have degree c^{-1} , velocity will have degree c; i.e., it will vary in proportion to the speed of light. If we then pass from kinematics to dynamics then the second postulate will imply that *all mechanical magnitudes of the same class must have the same degree in c*. For example, according to that postulate, a system of forces that is in equilibrium in one region where *c* has the value c_1 must remain in equilibrium when the system is transported to another region of the field where *c* is equal to c_2 , all forces must vary by the same ratio when *c* varies. Similarly, energy in any form must have the same degree in *c*, because if two forms of energy – for example, potential and kinetic energy – have different degrees then the transformation of energy from one form to the other would have to give rise to periodic phenomena whose frequencies would not have the proper degree (viz., *c*). Similar arguments apply to other dynamical quantities. Now, since all of those magnitudes are composed of mass, length, and time, in order to determine their degrees, it is enough to find *the degree of mass*.

Recently, I raised the question of whether gravitational mass might be proportional to energy in the same that inertial mass is. What would happen if the loss of energy (i.e., heat) that a radioactive element experiences under the transformation were to cause its inertial mass to decrease, but not its weight? Obviously, if uranium and its isotopes were suspended by strings of the same length then they would give pendula of different periods. The equilibrium positions of those pendula would also be different because the attraction of the Earth acts on the gravitational mass, while the centrifugal force acts on its inertial mass. That contradicts experiments. We then need to renounce any relationship between mass and energy or else assume that weight, like inertia, is proportional to energy. The former alternative would signify the bankruptcy of the new mechanics. We prefer the latter one and propose the following postulate:

POSTULATE III:

The forces that act on two bodies at the same place in the gravitational field behave like their energies.

One needs to take into account not only the potential and kinetic energy of the molar and molecular motions, but also the chemical and electromagnetic ones. For example, the electrons contained in a metal carry electromagnetic energy with them; they therefore gravitate (¹). The radiant energy that is contained in a vessel by virtue of the temperature of its walls must also be subject to gravitation. One can then interpret the third postulate as: *The laws of conservation of energy and mass will merge into just one law*.

Moving on to the mathematical representation, let us take the expression for the Lagrangian function that is valid in the dynamics of electrons (²):

(3)
$$L = -mc^2 f\left(\frac{v}{c}\right),$$

in which *v* signifies the velocity, and *m* is the inertial rest mass $(^3)$. One deduces the values of the impulse and energy from the Lagrangian function by means of the known formulas:

(3.a)
$$G = \frac{\partial L}{\partial y}$$

(3.b)
$$E = v \frac{\partial L}{\partial v} - L$$

When this is applied to the theory of relativity, one will have $f\left(\frac{v}{c}\right) = \sqrt{1 - \left(\frac{v}{c}\right)^2}$.

Otherwise, a numerical factor would have to enter into this.

^{(&}lt;sup>1</sup>) That was also shown to be probable in the observations of J. Königsberger, Verh. d. deutschen Physik. Ges. XIV (1911), pp. 185.

^{(&}lt;sup>2</sup>) Cf., "I progressi recenti della fisica," Lecture presented at the Royal University of Genoa (1909), pp. 152.

⁽³⁾ In order for *m* to have that significance, one must have $f^*(0) = -f(0)$.

while the equations of motion are:

(4)
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \text{etc.}$$

In discussing those equations, the new mechanics reverts to the case of constant c. In that case, the Lagrangian function will depend upon only the velocity v, but not position. Therefore, only the first terms will enter into the equations of motion, which are the ones that contain the derivatives of the components of impulse with respect to time:

(4.a)
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\frac{\dot{x}}{v}\right) = \frac{d}{dt}\left(G\frac{\dot{x}}{v}\right) = \frac{d\mathbf{G}_x}{dt}, \quad \text{etc.}$$

However, in the new theory of c, L also depends upon the coordinates. One then needs to retain the second terms in the Lagrange equations:

(4.b)
$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial c} \frac{\partial c}{\partial x}, \quad \text{etc}$$

They represent the components of a force that is proportional to the gradient of c. According to the first postulate, that force is gravity precisely. The equations of motion (4) can be written in vectorial form:

(5)
$$\frac{d\mathbf{G}}{dt} = \frac{\partial L}{\partial c} \operatorname{grad} c \,.$$

It will then result that *the Lagrange equations are nothing but the analytical expression of the first postulate*. (They are exact for the free motion of a material point in the gravitational field, but they also apply to a system whose dimensions are small enough that it can be equated to a material point.)

The works of the celebrated founder of analytical mechanics then revealed an unpredicted and much more far-reaching property when the mathematicians of the entire world honored the centennial of his death.

Now, introduce the third postulate, which makes the gravitation proportional to the energy of the moving point for a given position. In order to satisfy it, one must write:

$$\frac{\partial L}{\partial c} = -\chi \cdot E,$$

in which χ depends upon just *c*, but *v*. The right-hand side of (5) – i.e., the gravitational force – will then become:

(6.a)
$$\mathbf{K} = -\chi(c) \cdot E \cdot \operatorname{grad} c \; .$$

If one sets:

(6.b)
$$\log \varphi(c) = \int dc \,\chi(c)$$

then one will have:

(6.c)
$$E = -\frac{1}{\chi(c)} \frac{\partial L}{\partial c} = -\varphi \frac{\partial L}{\partial \varphi}.$$

It follows from that relation and (3.b) that:

(7)
$$L = v \frac{\partial L}{\partial v} + \varphi \frac{\partial L}{\partial \varphi}$$

When one applies a known theorem of Euler that relates to homogeneous functions, one will conclude that: The Lagrangian function is a linear homogeneous function of v and φ . One writes it in the following form:

(7.a)
$$L = -M\varphi \cdot f\left(\frac{\nu}{\varphi}\right),$$

where *M* is a specific constant of the material point that is independent of φ and therefore of *c*. When one compares (3) and (7.a), one will find that:

(7.b)
$$\varphi = c, \quad M = m \ c \ .$$

It will then follow that the rest mass is:

(8)
$$m = \frac{M}{c},$$

so mass has degree c^{-1} . Now, if one knows the degrees of length (c^0) , time (c^{-1}) , and mass (c^{-1}) then one can deduce the degrees of all dynamical quantities: *Energy has degree c*, and force also has degree *c*. *However, action*, which is the product of energy and time, has degree c^0 ; i.e., *it is independent of c*.

When one takes into account the value of φ , which is c [eq. (7.b)], (6.b) will give:

$$\chi(c) = c^{-1}$$

in such a way that *the expression* (6.a) *for the gravitational force* will become:

(9)
$$\mathbf{K} = -\frac{E}{c} \operatorname{grad} c.$$

It will satisfy either the first postulate, because the force is proportional to the gradient of c, or the third postulate, because gravity is proportional to energy.

A *point at rest* in the gravitational field has a certain energy. It can be found from (3.b) and (3) by taking *v* equal to zero:

$$E = -L = m c^2,$$

i.e., according to (8):

$$(9.a) E = M \cdot c \; .$$

The force acting on a point at rest then follows from that expression for the energy at rest and (9):

$$\mathbf{K} = -M \operatorname{grad} c$$

The work done by that force is equal to the decrement in the rest energy (9.a), as it must be.

We are now in a position to determine the force that acts on a material point in a given gravitational field, i.e., for a given scalar field *c*. However, we do not also have the solution to the problem that was posed to begin with, namely, the problem of finding the gravitational field that corresponds to a given distribution of matter. Now, the reciprocity of action and reaction suggests that one should assume that the attracting mass, like the attracted one, should be proportional to the energy and to regard *energy* as *the source of the gravitational field*.

However, that raises the question: Does one need to take into account just the energy of the matter or also the energy of that gravitational field? If the expression for the energy of the field were known then at least the statics of the gravitational field would follow by applying the principle of virtual work. One then prefers to start from trusted expressions for the field energy and its current and to then append the relations that couple the gravitational vector with the density of matter and energy.

The simplest position would be that the density of the energy of the static field is proportional to the square of the gradient of c, which is what gravity depends upon, as we saw [cf., (9.b)]. However, we know that energy, and therefore its density, as well, has degree c, while the square of the gradient of c has degree c^2 . That consideration suggests that we should introduce the auxiliary variable:

(10)
$$u = \sqrt{c}$$

and assign the value:

(11)
$$\varepsilon = \frac{1}{2\alpha} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{1}{c} \frac{\partial u}{\partial t} \right)^2 \right\}$$

to the *energy density of the gravitational field*, in which α signifies a universal constant of degree c^0 . That expression is also valid for the dynamical field in our theory. It is connected with the expression for the *gravitational energy current*:

(12)
$$\mathbf{S} = -\frac{1}{\alpha} \frac{\partial u}{\partial t} \operatorname{grad} u.$$

According to (11), the energy of the field is always positive. It will become zero only when the field vanishes. It follows that the energy always moves with the wave that propagates the gravitational perturbation. For example, imagine that such a perturbation, for which u is less than u_0 , enters a region in which u was initially equal to u_0 . The gradient of u will then point towards the unperturbed part of the field, while the derivative of u with respect to time as the wave passes will be negative. The expression (12) for the energy current will then have the desired sign. It will also retain it if u is greater than u_0 in the perturbed region because either the gradient of u or the derivative of u with respect to time will then have the opposite sign. Thus, in the present theory, the direction of the energy current will always correspond to an emission of energy from the perturbed region. One cannot raise the aforementioned objection against it then. The reaction of the radiation will always cause a decrease in the acceleration of a material particle, so its equilibrium would be unstable then.

Now consider a field in which one finds matter at rest. Treat a continuous distribution of mass over the volume *V*, and set:

(13)
$$M = \int \mu \, dV$$

Let:

(13.a)
$$\mu = \lim_{V=0} \left(\frac{M}{V}\right)$$

denote the "specific density" of matter. Since M does not depend upon c for an incompressible fluid (whose parts have constant volume), the specific density will be independent of position in the gravitational field.

According to (9.a), the energy density of matter in the rest state is:

$$\eta = \lim_{V=0} \left(\frac{M c}{V} \right) = \mu c,$$

or

(14) $\eta = \mu \, u^2 \, .$

It will then follow that:

(14.a)
$$\frac{\partial \eta}{\partial t} = 2\mu u \frac{\partial u}{\partial t}$$

is the increment in the rest energy density that is caused by the variation of the gravitational field.

The energy equation says that minus the divergence of the energy current is equal to the sum of the increments of the densities of the energy of the field (ε) and of matter (η) :

(15)
$$-\operatorname{div} \mathbf{S} = \frac{\partial \varepsilon}{\partial t} + \frac{\partial \eta}{\partial t}.$$

One derives from the expressions (11) and (12) that:

(15.a)
$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\alpha} \left\{ \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z \partial t} + \frac{1}{c} \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial u}{\partial t} \right) \right\},$$

(15.b)
$$-\operatorname{div} \mathbf{S} = \frac{1}{\alpha} \left\{ \frac{\partial^2 u}{\partial x \partial t} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y \partial t} \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial z \partial t} \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \Delta u \right\}$$

with

(15.c)
$$\Delta u \equiv \operatorname{div} \operatorname{grad} u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

In addition, set:

(16)
$$\Box u = \Delta u - \frac{1}{c} \frac{\partial u}{\partial t} \left(\frac{1}{c} \frac{\partial u}{\partial t} \right)$$

It will then follow from (15) that when one introduces the expressions (15.a,b) and (14.a):

(17)
$$\Box u = 2\alpha \ \mu \cdot u.$$

The right-hand side of (17) refers to matter at rest. However, if it is in motion then one must introduce its energy density (η) and set:

(17.a)
$$\Box u = 2\alpha \cdot \frac{\eta}{u}.$$

That is the fundamental equation of the gravitational field, which couples it with the energy of matter. It can be considered to be the analytical representation of the fourth postulate, which couples the attracting mass of a body to its energy.

The fundamental equation assumes the form:

(17.b)
$$\Delta u \equiv \operatorname{div} \operatorname{grad} u = 2\alpha \frac{\eta}{u} = 2\alpha \,\mu u$$

for the static field.

It results from this that *the divergence of the gradient of u is proportional to the energy density of matter* in a static field.

(17.b) does not involve the energy of the gravitational field. However, it is easy to transform it in a suitable way. One has:

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial^2 u^2}{\partial x^2} = 2u \frac{\partial u}{\partial x} + 2\left(\frac{\partial u}{\partial x}\right)^2, \quad \text{etc.},$$

so

$$\Delta c = 2u \ \Delta u + 2 \ (\text{grad } u)^2.$$

On the other hand, according to (11), the energy density in the static field is given by:

$$2\alpha \varepsilon = (\text{grad } u)^2$$

One can then write (17.b) in the form:

(18)
$$\Delta c = \operatorname{div} \operatorname{grad} c = 4\alpha \left(\eta + \varepsilon\right),$$

which says that *the divergence of the gradient of c is proportional to the total energy density in the static field.* In other words, if one integrates over a volume of the field that is bounded by the surface *f* and applies Gauss's theorem then:

(19)
$$\int df \, \frac{\partial c}{\partial n} = 4\alpha \int dV(\eta + \varepsilon) = 4\alpha \, E \, .$$

That is: The flux of the gradient of c across a closed surface in a static field is proportional to the total energy that is contained in the space that it bounds.

To better highlight the significance of those theorems, consider a special case: A *sphere* at rest, in which the distribution of mass is homogeneous within concentric spherical layers.

The fundamental equation (17.b) gives:

(20)
$$\Delta u = 2\alpha \mu \cdot u$$
 for $r < a$, i.e., inside of the sphere,

(20.a)
$$\Delta u = 0$$
 for $r > a$, i.e., outside of the sphere.

The last equation is the Laplace equation, whose integral that is symmetric about the center:

(21)
$$u = u_0 \left(1 - \frac{g}{r}\right), \quad \text{for } r > a$$

determines the gravitational field outside of the attracting sphere (u_0 denotes the value of $c^{1/2}$ for $r^{-1} = 0$, and \mathcal{G} is another constant). It follows from (21) that:

(21.a)
$$\frac{du}{dr} = u_0 \frac{g}{r^2} \,.$$

The radial gradient of c will then become:

(22)
$$\frac{dc}{dr} = 2u\frac{du}{dr} = 2c_0 \cdot \frac{9}{r^2} \cdot \left(1 - \frac{9}{r}\right)$$

Now, according to (9.b), gravity is proportional to that gradient. *It then follows that Newton's law is not exact in our theorem.* The center of the sphere (e.g., the Sun) attracts a material point (e.g., a planet) with a force into whose expression, a term in r^{-3} will enter, along with a term in r^{-2} .

According to the theorem that was established just recently [eq. (19)], the flux of the gradient of *c* across a sphere of radius *r* is proportional to the total energy that is contained in the sphere:

(23)
$$4\pi r^2 \frac{dc}{dr} = 4\alpha \cdot E \,.$$

A sphere of infinite radius encloses the *total energy* E_t of the attracting sphere and its gravitational field. It then follows from (22) and (23) that:

(23.a)
$$E_t = \frac{2\pi c_0}{\alpha} \cdot \mathcal{G}.$$

However, the sphere of radius a contains only the *internal energy* of the sphere:

(23.b)
$$E_t = \frac{2\pi c_0}{\alpha} \cdot \mathcal{G}\left(1 - \frac{\mathcal{G}}{a}\right).$$

The energy of the external field will then have the value:

(23.c)
$$E_e = E_t - E_i = \frac{g}{a} \cdot E_t.$$

Set:

(24)
$$\psi = \frac{g}{a} = \frac{E_e}{E_t}.$$

That quantity ψ (i.e., the ratio of the external energy to the total energy) enters into the expression (22) for the value of gravity outside of the sphere (e.g., the Sun):

(24.a)
$$\frac{dc}{dr} = 2c_0 \cdot \frac{a\psi}{r^2} \left(1 - \psi \cdot \frac{a}{r}\right) = \frac{\alpha E_t}{\pi r^2} \left(1 - \frac{E_e}{E_t} \cdot \frac{a}{r}\right)$$

One can then say that: The inexactness of Newton's law is due to the energy of the external field.

In order to determine the ratio ψ , one needs to integrate the equation (20), which is valid inside of the sphere. Compare it with the Poisson equation: In the latter, the right-hand side depends upon only the density of the attracting mass, while the right-hand side of (20) contains the potential uitself as a factor. It will then follow that in the new theory the contributions of the individual masses do not superpose. However, the contribution of the mass that is present in a region will decrease with the value of u in that region. Since the neighboring masses produce a reduction of u, it will result that strongly-concentrated matter would exert less of an attraction in the new theory that it would in the usual theory. However, one sees that the difference is not very appreciable for the Sun itself.

When one takes into account the symmetry with respect to the center, (20) will give:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=2\alpha\,\mu\cdot u\,,$$

or

(25)
$$\frac{d^2(ru)}{dr^2} = 2\alpha \,\mu \cdot ru, \quad \text{with } r < a$$

In order to be able to integrate (25), one must know the distribution of the specific density μ along a radius. We shall confine ourselves to treating the special case of a sphere that is filled with an incompressible fluid (μ = constant). Set:

(25.a)
$$k^2 = 2\alpha \,\mu\,,$$

so we will find that *u* is expressed in terms of the hyperbolic sine:

(26)
$$u = \frac{A}{r} \cdot \sinh(kr),$$

because that integral of (25) remains finite for r = 0.

The constant A is determined by the conditions that for r = a the values of u and:

(26.a)
$$\frac{du}{dr} = A \left\{ \frac{k}{r} \cosh(kr) - \frac{1}{r^2} \sinh(kr) \right\}$$

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will coincide with the ones that are valid outside of the sphere and are given by (21) and (21.a):

(26.b)
$$\begin{cases} \frac{A}{a}\sinh(ka) = u_0\left(1 - \frac{\vartheta}{a}\right), \\ \frac{A}{a}\{k \, a \cosh(k \, a) - \sinh(k \, a) = u_0 \cdot \frac{\vartheta}{a}. \end{cases}$$

Those two equations also determine the value of ψ [cf., (24)]; that is all that is of interest to us. We find that:

$$\frac{\psi}{1-\psi} = \frac{k a \cosh(k a) - \sinh(k a)}{\sinh(k a)} = \frac{k a - \tanh(k a)}{\tanh(k a)} = \xi,$$

and therefore:

(27)
$$\psi = \frac{\xi}{1-\xi} = 1 - \frac{\tanh(k\,a)}{k\,a}.$$

The gravitational force exerted by the incompressible sphere on an external point depends upon that quantity, which denotes the ratio of the external energy to the total energy of the sphere.

Let us first evaluate it for the practical case:

Case I: k a small:

$$\frac{\tanh{(k\,a)}}{k\,a} = \frac{1 + \frac{(k\,a)^3}{3!} + \dots}{1 + \frac{(k\,a)^2}{2!} + \dots} = 1 - \frac{(k\,a)^2}{3},$$

so it will follow that:

(27.a)
$$\psi = \frac{k^2 a^2}{3} = \frac{2\alpha \mu a^2}{3} = \frac{\alpha M}{2\pi a} \quad \left(\text{in which } M = \frac{4\pi a^2}{3} \cdot \mu \right).$$

Since ψ is small in this case, so is the ratio of the external energy to the total energy of the sphere. If one neglects it in (24.a) and (9.b) then one will find the force with which the sphere attracts the material point P':

$$M'\frac{dc}{dr} = \frac{\alpha}{\pi}c_0\frac{MM'}{r^2}.$$

Therefore, in this limiting case, the sphere acts as if all of its mass were concentrated at its center.

Let us now move on to the usual units, in which we have:

$$M = c m , \qquad M' = c m' .$$

If we neglect the difference between c, c', and c_0 (which has order ψ) then we will find that the attracting force is:

$$\frac{\alpha}{\pi}c_0^2\frac{mm'}{r^2} = \gamma\frac{mm'}{r^3}, \quad \text{with} \quad \alpha = \frac{\pi\gamma}{c_0^3},$$

in which γ denotes the usual gravitational constant. Therefore:

(27.b)
$$\psi = \frac{\gamma m}{2a c_0^2}.$$

For this case alone, we will find the value $\psi = 10^{-6}$. We are still in the case of k a and ψ small: Hence, at the astronomical scale, it is legitimate to substitute a material point for the Sun and to neglect the corrections to Newton's law that are due to the energy of the gravitational field (¹).

It then results that the opposite limiting case does not correspond to reality. However, one can briefly treat:

Case II: k a large, tanh (k a) = 1.

One gets from (27) that:

^{(&}lt;sup>1</sup>) If one uses the principle of action and reaction as a basis then one can also deduce the limits of validity of the development that is based upon Lagrange's equations, in which one substitutes a material point for the body, as well as calculating the mass by taking into account only the energy of matter, but not that of the gravitational field. One will then see that this way of proceeding will also be legitimate for bodies that have order of magnitude of the fixed stars.

(27.c)
$$\psi = \frac{E_e}{E_t} = 1 - \frac{1}{ka}$$
 is slightly less than unity.

The energy is found to be almost entirely outside of the sphere. All that remains inside of it is the small fraction:

$$\frac{E_e}{E_t} = 1 - \psi = \frac{1}{k a}$$

It follows from (24.a) that for gravity:

$$\frac{dc}{dr} = \frac{2c_0 a}{r^2} \left(1 - \psi \frac{a}{r} \right).$$

At large distances, gravity is proportional to the radius (*a*) of the attracting sphere, instead of its volume. The aforementioned *shielding action of a highly-concentrated mass* then results (either for the magnitude of the radius or that of the density).

Let us return to the fundamental equation (17.a):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c} \frac{\partial u}{\partial t} \left(\frac{1}{c} \frac{\partial u}{\partial t} \right) = 2\alpha \frac{\eta}{u},$$

from which we deduced the perturbation of the gravitational field that is produced by the motion of matter. It describes a propagation of the speed of light (c) outside of matter. However, the exact treatment of the problem of propagation proves to be difficult because the perturbation itself modifies the value of u, and therefore the speed of propagation ($c = u^2$). That is also true for the propagation of sound, whose speed, which depends upon pressure, will vary with the passage of a sound wave. However, in all practical cases, the variation of u is small enough that one can still regard the speed of propagation of gravitation as a constant.

In the present theory, light and gravitation have equal propagation speeds. However, one of them propagates as a transverse wave, while the other one is longitudinal. Moreover, the problem of a vibrating particle can be treated in a similar manner to the problem of the vibrating electron. The intensity of the emitted gravitational waves will depend upon the product of the gravitating mass of the particle with its acceleration. Is it possible to detect such gravitational waves?

As discussed before on another occasion $(^1)$, that hope is illusory. In fact, in order to give a particle an appreciable acceleration, one would need another particle, which would be pushed in the opposite direction due to the law of action and reaction. Now, the intensity of the emitted gravitational wave will depend upon the sum of the products of the gravitating masses by the accelerations, while the law of action and reaction $(^2)$ makes the sum of the products of the two

^{(&}lt;sup>1</sup>) M. Abraham, Nuov. Cim. (1912), pp. 211.

^{(&}lt;sup>2</sup>) It is true that one does not take the impulse of the gravitational field into account in that law, but it is negligible in comparison to that of matter.

accelerations multiplied by the inertial masses disappear. In the new theory, which assumes the identity of the gravitational mass and the inertial mass, the intensity of the wave will prove to be zero. Therefore, according to the mechanism of the field, *the existence of gravitational waves is possible in our theory*, but the possibility of exciting them is excluded by the identity of the gravitational and inertial mass.

It results from this that the solar system does not lose its mechanical energy by radiation, while an analogous system that is composed of negative electrons that circulate around a positive center will lose its energy by radiation. Therefore, the survival of the solar system is not threatened by any such danger.

The present theory of gravitation, which is based upon the hypothesis that c is variable, contradicts the second postulate of the theory of relativity. However, the invariance with respect to Lorentz transformations is maintained in empty space, as is show by the fundamental equation:

$$\Box \psi = 0.$$

Outside of matter, the Lorentz group is also valid infinitesimally. *However, matter breaks the chain that leads to the Lorentz group*, because the left-hand side of the equation:

$$u \Box u = 2\alpha \eta$$

is an invariant of that group, but the right-hand side, which is proportional to the energy, is not. However, the hypothesis that the attracting mass is proportional to energy is so reliable that one is forced to abandon the Lorentz group, even infinitesimally.

Einstein's theory of relativity (1905) would then fade away. Would a newer, more general, principle of relativity arise like a phoenix from the ashes? Should we return to the idea of absolute space? Must we recall the much-despised theory of the ether in order to include the gravitational field along with the electromagnetic one?

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