"Sulla teoria della gravitazione," Rend. Reale Accad. Lincei, cl. sci. fis., mat. e nat. (5) **20** (1911), 678-682.

On the theory of gravitation

By MAX ABRAHAM

Presented to the Society by T. LEVI-CIVITA

Translated by D. H. Delphenich

In a recent article, A. Einstein ([1]) proposed the hypothesis that the speed of light depends upon the gravitational potential (Φ). In the present note, we propose a theory of gravitation that agrees with the principle of relativity and includes a relation between c and Φ that is equivalent to Einstein's in the first approximation. That theory assigns values to the energy density and energy current of the gravitational field that are different from the ones that have been assumed up to now.

Following Minkowski's representation ([2]), consider:

$$x, y, z, \qquad \text{and} \qquad u = i\, l = i\, c\, t$$

to be the coordinates of a four-dimensional space. Let the *rest density* ([2]) ν be a scalar in that space, and thus, the gravitational potential Φ, as well. They are coupled by the differential equation:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial u^2} = 4\pi\,\gamma\,\nu \tag{1}$$

(γ is the gravitational constant).

The *driving force* by which the gravity field acts upon a unit mass is equal and opposite to the gradient Φ :

$$F = -\operatorname{grad}\Phi\,, \tag{2}$$

i.e.:

$$F_x = -\frac{\partial\Phi}{\partial x}\,,\quad F_y = -\frac{\partial\Phi}{\partial y}\,,\quad F_z = -\frac{\partial\Phi}{\partial z}\,,\quad F_u = -\frac{\partial\Phi}{\partial u}\,. \tag{2.a}$$

([1]) A. Einstein, Ann. Phys. (Leipzig) **35** (1911), pp. 898.
([2]) H. Minkowski, Göttinger Nachrichten (1908), pp. 53.

According to (1) and (2), gravitation propagates with the speed of light, as is required by the principle of relativity. Hence, *gravitational waves are longitudinal*, while light waves are transverse.

Let $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{u}$ denote the first derivatives of the coordinates of a material point with respect to its *proper time* ([2]) τ, i.e., the components of the *velocity* vector, and let $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, $\ddot{u}$, viz., the second derivatives, denote the components of the *acceleration* vector. The *equations of motion* ([1]) are:

$$(3) \qquad \ddot{x} = F_x , \qquad \ddot{y} = F_x , \qquad \ddot{z} = F_x , \qquad \ddot{u} = F_x .$$

Now, the first derivatives satisfy the identity:

$$(4) \qquad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{u}^2 = - c^2 ,$$

i.e.:

$$\dot{l}^2 \left\{ \left(\frac{dx}{dl}\right)^2 + \left(\frac{dy}{dl}\right)^2 + \left(\frac{dz}{dl}\right)^2 - 1 \right\} = - c^2 ,$$

or, if one takes:

$$(4.a) \qquad \beta^2 = \left(\frac{dx}{dl}\right)^2 + \left(\frac{dy}{dl}\right)^2 + \left(\frac{dz}{dl}\right)^2 , \qquad k = \sqrt{1-\beta^2} ,$$

one will have:

$$(4.b) \qquad \dot{l} = \frac{c}{\sqrt{1-\beta^2}} = c\, k^{-1} .$$

By differentiating (4) with respect to proper time, Minkowski added the *orthogonality condition* between the velocity and acceleration vectors. However, that orthogonality will no longer be true when c is considered to be variable. Rather, in that case, differentiating (4) will give the relation:

$$(5) \qquad \dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \dot{z}\,\ddot{z} + \dot{u}\,\ddot{u} = - c \frac{dc}{d\tau} ,$$

which replaces the orthogonality condition. If one introduces the driving force (2) in place of acceleration into (5) then one will get:

([1]) If energy is transmitted to matter in a non-mechanical form then one must modify Minkowski’s equations of motion [see M. Abraham, Rend. Circ. mat. Palermo **30** (1910)]. However, in the present note, we shall consider only purely-mechanical actions.

$$\dot{x}\frac{\partial\Phi}{\partial x}+\dot{y}\frac{\partial\Phi}{\partial y}+\dot{z}\frac{\partial\Phi}{\partial z}+\dot{u}\frac{\partial\Phi}{\partial u} = c\frac{dc}{d\tau} ,$$

or

$$\frac{d\Phi}{d\tau} = c\frac{dc}{d\tau} .$$

Integrating that will give:

$$\frac{c^2}{2}-\frac{c_0^2}{2} = \Phi - \Phi_0 , \tag{6}$$

if c_0 is the speed of light at the origin, where the potential is Φ_0 . That is to say: *The increment in one-half the square of the speed of light is equal to the increment in the gravitational potential*.

When one neglects the square of the ratio of Φ to c^2, that relation, which exists rigorously in our theory, can substitute for Einstein's formula (*loc. cit.*, pp. 906):

$$c = c_0\left(1+\frac{\Phi-\Phi_0}{c^2}\right) .$$

Thus, (6) better exhibits the independence of the origin, which can be chosen arbitrarily.

One should compare the relation that was obtained with the emission theory of light. Imagine that a luminous body can emit particles that move according to the laws of Galilean mechanics and are subject to gravity. That particle will experience an increment in kinetic energy that is equal to the *decrement* in the potential energy. However, according to (6), the increment of the *vis viva* of the particle is equal to the *increment* of its potential energy, i.e., it is equal in value, but opposite in sign, to the value that is calculated on the basis of the emission theory. Hence, the curvature ([1]) of light rays in the gravitational field, which follows from (6) by way of Huygens's principle, is identical to that of the trajectory of that particle. That is one of the many incomplete analogies between the modern theory of radiant energy and the emission theory.

Let us study the motion of a material point of mass m in the field of gravity. The first three of the equations of motion (3) give:

$$m\frac{d\dot{x}}{d\tau} = -m\frac{\partial\Phi}{\partial x}, \qquad m\frac{d\dot{y}}{d\tau} = -m\frac{\partial\Phi}{\partial y}, \qquad m\frac{d\dot{z}}{d\tau} = -m\frac{\partial\Phi}{\partial z} . \tag{7}$$

That expresses the impulse theorem. However, the last of (3), viz.:

$$m\frac{d\dot{l}}{d\tau} = -m\,i\,F_u = m\,i\frac{\partial\Phi}{\partial u} = m\frac{\partial\Phi}{\partial l} , \tag{8}$$

([1]) A. Einstein, *loc. cit.*, showed that a ray that passes by the surface of the Sun must deviate towards the center of the Sun, and drew the attention of astronomers to that consequence of the theory, since it was capable of being compared to observations.

expresses the *vis viva* theorem in Minkowskian mechanics. If the gravity field depends upon the only the coordinates x, y, z, and not on time, then when one multiplies (8) by c and takes (4.*b*) into account, one will get:

(9) $$m c \frac{d}{d\tau}(c k^{-1}) = 0 .$$

Now, Minkowski kept c constant and interpreted $m\,(c^2\,k^{-1} - 1)$ as the kinetic energy of the material point. However, in the present theory, which considers c to be variable, that procedure would no longer be valid, and it would seem impossible to assign a general expression to the energy of the material point whose decrement is exactly equal to the energy that is transmitted to the gravitational field.

However, one can show (at least for small velocities) that the theorem of the conservation of energy follows from (9), which is confirmed by the facts. One gets from (9) that:

(9.*a*) $$m\,c\,k^{-1} = \text{constant}.$$

If one neglects the squares and products of β^2 and $(\Phi - \Phi_0) / c^2$ then according to (4.*a*), one will have:

$$k^{-1} = (1-\beta^2)^{-1/2} = 1 + \tfrac{1}{2}\beta^2, \qquad \beta^2 = \frac{v^2}{c_0^2},$$

and according to (6):

$$c = [c_0^2 + 2\,(\Phi - \Phi_0)]^{1/2} = c_0 + \frac{\Phi - \Phi_2}{c_0}.$$

One will then get:

$$m\,c\,k^{-1} = m c_0 (1 + \tfrac{1}{2}\beta^2) + m \cdot \frac{\Phi - \Phi_0}{c_0} .$$

If one multiplies (9.*a*) by the constant c_0 then it will follow that:

(9.*b*) $$\tfrac{1}{2} m v^2 + m\,\Phi = \text{constant},$$

i.e., the conservation of energy theorem in its usual form. *The new mechanics agrees with the older one in the limiting case of small velocities*, and the following relation between c and Φ will result: *Potential energy* $m\,\Phi$, as well as *kinetic energy* $\tfrac{1}{2} m v^2$ *are transported away from that material point.*

Now consider two material points of masses m_0 and m that move with small velocities in a stationary gravity field. Either of the points will possess a potential energy that varies with the distance r between them:

$$-\gamma \frac{m_0\,m}{r} = -E .$$

Therefore, the variable part of the potential energy of the two points will be $-2E$. Hence, if the two points approach each other as a result their mutual attraction then the increment in their total kinetic energy will be equal to one-half the decrement in their total potential energy.

What happened to the other half? Obviously, it is still in the field of the gravitational force. Indeed, as one sees, our theory assigns the value E to the energy of the field in that case, which is equal and opposite to the one that was assumed up to now. The difficulty that Maxwell stated ([1]) will disappear, namely, that the density of the energy of the gravitational field (which is taken to be zero where the force is zero) will become negative elsewhere. *The expression* (13) *for the energy density in the gravity field*, which one must add to it, *is essentially positive.* However, the total energy of the system will contain the energy of matter, whose potential part is equal to $-2E$ in the stationary field, in addition to the energy of the field E.

Volterra ([2]) deserves the credit for having extended the concept of energy flux to the gravity field. However, his expression for flux was based upon the Maxwellian value for the energy of the field. Obviously, the present theory, which abandons the latter expression for energy, must assign some other expressions to the components of the energy currents.

The fictitious tensions, the energy current, and the densities of energy and impulse of the field depend upon a *four-dimensional tensor* ([3]), which will be determined in the following Note.

([1]) Clerk Maxwell, *Scientific Papers*, I, pp. 570.
([2]) V. Volterra, Nuovo Cimento (1899), pp. 3378.
([3]) On the subject of four-dimensional tensors, see M. Abraham, Rend. Circ. mat. Palermo (1910); A. Sommerfeld, Ann. Phys. (Leipzig) **32** (1910), pp. 749; M. Laue, *Das Relativitätsprincip*, Braunschweig, 1911, pp. 73.