"L'ordinaria teoria dell'Elasticità e la teoria delle deformazioni finite," Rend. R. Acc. dei Lincei (5) **26** (1917), 3-8.

Mechanics. – *The ordinary theory of elasticity and the theory of finite deformations*. Note by corresp. E. ALMANSI ([1]).

Translated by D. H. Delphenich

1. In some notes that were published in 1911 in these Rendiconti, I examined the finite deformations of elastic solids, and in particular, I proposed to obtain the formulas that one arrives at from those of the ordinary theory, as much as possible.

In the ordinary theory, one studies the *infinitesimal* deformations; in order to do that, one assumes, by way of approximation, that the formulas that one obtains are valid only for sufficiently small deformations, and are applicable to the examination of real phenomena only to that extent.

However, as is known, those formulas do not correspond to the results of observations with sufficient precision, from which, it results that, in particular, the components of the stresses are not representable by means of linear functions of the characteristics of the deformation without approximations for the major part of the materials, broadly speaking. It is then necessary to bring the *finite* deformations under examination: Namely, to study the deformations of elastic solids, while also taking into account the terms that are neglected in the usual theory. One can then obtain formulas that are more exact than the ones that the usual theory provides, and one can then appeal to experiment in order to determine the values of the constants that these formulas contain for various materials.

2. On the other hand, the theory of finite deformations also presents much that is of interest from a purely theoretical viewpoint, and in particular, the fact that the theorems that appear to be fundamental theorems in the theory of infinitesimal deformations do not persist in it.

Given an elastic body C, I call the passage from one initial state S_0 to a final state S_1 an *elastic deformation* when the projections u, v, w of the displacements at the points and their first and second derivatives are finite and *continuous* functions of the coordinates.

Assuming that a state S_0 of C is the reference state and denoting another state *of the same body* C by S_1, we intend that C can pass from the state S_0 to the state S_1 by means of a deformation that satisfies the indicated conditions.

Having said that, recall how in the ordinary theory of elasticity one proves that *if* S_0 *is an equilibrium state for a body* C *that is subject to no external forces then there exist no other equilibrium states for the body* C *in the absence of external forces* (provided that one considers the possible states *of the body* C to be only the ones that one arrives at by means of continuous displacements upon starting with S_0, according to the conventions that are made).

([1]) Received by the Academy on 22 June 1917.

Now, observation shows how the cases in which that theorem does not prove to be true will arise.

The simplest example is provided by the *inversion of a ring*. Consider the space that is generated by a planar surface σ that rotates around a line r that is situated in the plane of σ and that does not meet that surface. That space will be occupied by an isotropic, elastic body that is not subject to external forces and is in equilibrium. All of the particles that comprise C will be in the natural state. We call that state of the body S_0 .

It results from observation that there exists another equilibrium state S_1 for the body C in the absence of external forces. One passes from the state S_0 to the state S_1 by means of an elastic deformation that consists essentially of a rotation by 180° that is performed in each section σ of the ring around the barycentric line that is normal to the plane. The rotation is then accompanied by a small deformation that the section experiences in the plane in which it lies.

Therefore, two different states of equilibrium S_0 and S_1 exist for a ring that is not subject to external forces.

There is no reason why the theory of elasticity can exclude states of the type S_1 from its domain of research *a priori*. One also notes that they are not excluded as long as the form and dimensions of the body are not fixed, not even if one imposes a limit on the magnitude of the deformations of the individual material particles in some way. Suppose that one assigns an upper limit on the absolute value of the unitary elongations of the linear elements of the solid. In the case of the ring, the same intuition will make one expect that the limit will not be exceeded, even if the length of the axis of the ring (which goes through the barycenter of the section σ) is sufficiently large with respect to the linear dimensions of a section. It is obvious that if the length of the axis exceeds a certain limit then the state of deformation of the individual particles that is due to the inversion will not be very appreciable.

3. Merely the condition that the deformation is small for each material particle is therefore not enough to exclude that class of elastic deformations. They will be excluded when one supposes (as in the ordinary theory) that *all* of the first derivatives of u, v, w with respect to the coordinates are small. Meanwhile, in order for the deformation of each particle to be small, it is enough that the six quantities that characterize that deformation should be small.

In the study of finite deformations, one agrees to assume that the coordinates of the points of the solid in the *final* state S_1 are the independent variables, and they will be the variables with respect to which the derivatives of the components of stress refer in the indefinite equations of equilibrium. One can assume that the six quantities:

$$\begin{matrix} \varepsilon_{xx}, & \varepsilon_{yy}, & \varepsilon_{zz}, \\ \varepsilon_{yz}, & \varepsilon_{zx}, & \varepsilon_{xy}, \end{matrix}$$

which are given by the formula:

$$
(1) \qquad \begin{cases} \varepsilon_{xx} = \dfrac{\partial u}{\partial x} - \dfrac{1}{2}\left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2\right\}, \\ \dotfill \\ \varepsilon_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} - \dfrac{1}{2}\left\{\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\right\}, \end{cases}
$$

are the characteristics of the deformation at an arbitrary point P of the solid.

The necessary and sufficient condition for the deformation of the particle that is attached to the point P to be small is that these *six* quantities should be small at P, but that does not demand that the *nine* derivatives $\partial u / \partial x$, $\partial u / \partial y$, etc., should be small.

If the six characteristics prove to be small, *and in addition, one makes the convention that one should neglect quantities whose order of magnitude is quadratic* then one can assert that the first three of them – viz., ε_{xx}, ε_{yy}, ε_{zz} – represent the unitary elongations that relate to the directions of the coordinates axes, while the remaining three are the shears that relate to couples whose directions are y and z, z and x, and x and y, respectively.

If the nine derivatives are that small then one can set:

$$
\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \ldots, \qquad \varepsilon_{xy} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}
$$

and substitute these expressions in the formulas of the ordinary theory.

4. These considerations draw attention to a noteworthy fact: Namely, it might be possible to extend the scope of research in the theory of elasticity *while still remaining within the order of approximation of the ordinary theory*.

Let ζ denote a quantity that can be considered to be small of first order. The six characteristics ε_{xx} , …, ε_{xy} that are given by formulas (1) are small of order ζ, and we agree to ignore small quantities of higher order in their expression. We can then adjoin terms δ_{xx} , …, δ_{xy} that are small of order higher than ζ, along with their first derivatives, but otherwise arbitrary, to the right-hand side of (1), that is, we can set:

$$
(2) \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} + \delta_{xx}, \qquad \text{etc.,}
$$

and assume that these six quantities are the expressions for the elongations and shears. As for the expressions for the components of the stress τ_{xx} , …, τ_{xy} , we can preserve the ones in the ordinary theory. In particular, if the solid is isotropic then one can assume that:

$$
\begin{aligned} &\tau_{xx} = A\,\{\varepsilon_{xx} + k\,(\varepsilon_{yy} + \varepsilon_{zz})\}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\tau_{xy} = B\,\varepsilon_{xy}, \end{aligned}
$$

in which A, B, k denote constants. The equations of equilibrium will then provide the components X, Y, Z and L, M, N of the volume forces and the stresses that act upon the surface, resp.

If we know how to determine the functions u, v, w (which are finite, continuous, etc.) and the additive terms δ_{xx} , …, δ_{xy} (which are small of some desired order) in such a way that the state that is assumed by the solid with the displacements (u, v, w), which is a state that we believe to be due to the external forces (X, Y, Z) and (L, M, N), which correspond to the givens in the problem, then we can consider that problem to have been resolved.

The foundation of the ordinary theory consists of setting:

$$\delta_{xx} = \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} + \delta_{xx}, \qquad \text{etc.,}$$

so, from formula (2), $\varepsilon_{xx} = \partial u / \partial x$, etc., this presupposes that the first derivatives of the functions u, v, w are all small of first order (or higher). The general theory of elastic deformations then acquires great simplicity and elegance. However, one then excludes a class of problems that can be solved to the same approximation, provided that one assigns various expressions to the terms δ_{xx}, …, δ_{xy} .

In the case of a circular ring, for which, $2\pi R$ denotes the length of the axis, and a denotes the width of its section σ – viz., the maximum distance between two lines that are tangent to its contour and parallel to the line r (§ 2) – then the maximum absolute value of the unitary elongations that originate in the inversion (assuming that a is much smaller than R) is sensibly equal to the ratio $\zeta = a / R$. We then propose to determine the state of deformation that the ring assumes as a result of the inversion when we ignore quantities that are small of order higher than ζ in the expressions for its six characteristics.

However, this does not exclude the possibility that the problem can be studied and solved with a higher approximation.

5. The inversion of a ring provides a first example of a state of equilibrium S_1 that is different from the natural state S_0 that an elastic solid can assume in the absence of external forces.

Another example is provided by an elastic lamina of small thickness that is not perfectly planar. A state of equilibrium S_1 exists in many cases that is different from the natural state that one obtains by requiring that the gibbousness that is presented by the lamina to pass to the opposite side with respect to which it was found initially.

The important research that given rise to the theory of *distortions* in recent years is well-known. It also exhibits states of equilibrium for elastic solids that are not subject to external forces that correspond to non-zero externals stresses.

There obviously exists no analogy between this class of phenomena and the one that gave us our examples above.

The equilibrium states S_1 that we consider to have been obtained, in fact, belong to a state S_0 in which the stresses are all zero by means of a deformation *with continuous displacements*.

Moreover, in the theory of distortions, one examines the equilibrium states, which one calls S_0', that cannot be considered to have been obtained by starting with an initial state S_0 in which all of the stresses are zero *if one does not assume that relative displacements of the points present a discontinuity surface.* The passage from the state S_0 to a state S_0' cannot therefore be a true and proper elastic deformation, according to the adopted criteria.

Allow me to note, in regard to this statement, that in the study of elastic equilibrium, one might perhaps agree to return the concept of *deformation* to its original significance, and not include the possibility that one does not deform with continuous displacements in that term. Apart from any consideration of an analytical character, the hypothesis of continuity of the displacements seems justified by the fact that if the displacements are discontinuous then the *physical* phenomena that might present are considerably diverse.

We would then like to limit the consideration to a well-defined body C whose only states are obtained by means of elastic deformations upon starting with a reference state, or as one would say in the theory of distortions, we do not consider equilibrium states of the body that admit a state S_0 in which no particle is subject to any stress. However, it teaches us how to construct, by special procedures, bodies in which one always has non-zero internal stress, even when the external forces are zero, and to examine the equilibrium states that these bodies assume in the absence of external forces.