"Sopra una classe particolare di deformazioni a spostamenti polidromi dei solidi cilindrici," Rend. Accad. dei Lincei (5) **16** (1907), 26-33.

On a particular class of deformations with polydromic displacements of cylindrical solids.

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Presented to the society by VITO VOLTERRA

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1. One considers an isotropic elastic solid that occupies a cylindrical space S. Let σ denote normal plane sections to the cylinder (along the barycenter of that section), and let σ' and σ'' denote the two extreme sections.

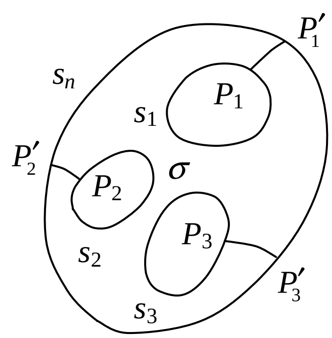

Figure 1.

Let the space S be multiply-connected. A section σ will be bounded by a certain number of closed lines that will be denoted by $s_1, s_2, \ldots, s_n$ (Fig. 1).

Refer the points of space to a triad of orthogonal axes, and take the z-axis to be the axis of the cylinder.

We propose to determine the most general state of deformation of the solid supposing that:

1. Its elements are subject to no volume force.
2. The elements of the external surface that are parallel to the axis (viz., the lateral surface) are not stressed.
3. The internal tension depends upon just the variables x, y.

We find that the most general deformation that satisfies these conditions can be decomposed into a deformation D_0, which relates to the cases that were examined by Saint-Venant, and which consequently corresponds to *monodromic* components of the displacement, and a deformation D for which that condition is not verified. The six fundamental internal tensions in the deformation D are expressed by the formula:

$$\tau_{11} = \frac{\partial^2\Phi}{\partial y^2}, \qquad \tau_{22} = \frac{\partial^2\Phi}{\partial x^2}, \qquad \tau_{12} = -\frac{\partial^2\Phi}{\partial x\,\partial y}, \quad \tau_{33} = \lambda\,\Delta^2\Phi, \tag{1}$$

$$\tau_{31} = \frac{\partial\varphi}{\partial y}, \qquad \tau_{32} = -\frac{\partial\varphi}{\partial x}, \tag{2}$$

where Φ is a biharmonic function ($\Delta^2\Delta^2\Phi = 0$) of the variables x, y, φ is a function of the same variables that verifies the equation $\Delta^2\varphi = k =$ const, and λ is the *contraction coefficient* ([1]).

2. The six internal tensions must verify the three equations:

$$\frac{\partial\tau_{11}}{\partial x} + \frac{\partial\tau_{12}}{\partial y} + \frac{\partial\tau_{13}}{\partial z} = 0, \qquad \dots \qquad (\tau_{12} = \tau_{21}\text{ , etc.})$$

and the other six:

$$\Delta^2\tau_{11} = -\frac{1}{1+\lambda}\frac{\partial^2 T}{\partial x^2}, \qquad \Delta^2\tau_{12} = -\frac{1}{1+\lambda}\frac{\partial^2 T}{\partial x\,\partial y}, \quad \dots$$

where

$$T = \tau_{11} + \tau_{22} + \tau_{33}\,.$$

From the last equation, one gets:

$$\Delta^2 T = 0.$$

Since the tensions must not contain the variable z, the nine preceding equations will become:

$$\frac{\partial\tau_{11}}{\partial x} + \frac{\partial\tau_{12}}{\partial y} = 0, \qquad \frac{\partial\tau_{21}}{\partial x} + \frac{\partial\tau_{22}}{\partial y} = 0, \tag{3}$$

$$\frac{\partial\tau_{31}}{\partial x} + \frac{\partial\tau_{32}}{\partial y} = 0, \tag{4}$$

$$\left\{\begin{aligned} \Delta^2\tau_{11} &= -\frac{1}{1+\lambda}\frac{\partial^2 T}{\partial x^2}, \\ \Delta^2\tau_{22} &= -\frac{1}{1+\lambda}\frac{\partial^2 T}{\partial y^2}, \\ \Delta^2\tau_{12} &= -\frac{1}{1+\lambda}\frac{\partial^2 T}{\partial x\,\partial y}, \end{aligned}\right. \tag{5}$$

$$\left\{\begin{aligned} \Delta^2\tau_{31} &= 0, \\ \Delta^2\tau_{32} &= 0, \\ \Delta^2\tau_{33} &= 0, \end{aligned}\right. \tag{6}$$

([1]) On the theory of regular deformations with polydromic displacements, see the note of prof. Volterra in the Rendiconti della R. Accademia dei Lincei, 1904.

At the points of the lateral surface (which, by hypothesis, is unstressed), if cos α, cos β, 0 denote the cosines of the (external or internal) normal then one must have:

(7)
$$\left\{\begin{aligned} &\tau_{11}\cos\alpha + \tau_{12}\cos\beta = 0, \\ &\tau_{21}\cos\alpha + \tau_{22}\cos\beta = 0, \\ &\tau_{31}\cos\alpha + \tau_{22}\cos\beta = 0. \end{aligned}\right.$$

3. By virtue of equations (3), we can set:

$$\tau_{11} = \frac{\partial\Phi'}{\partial y}, \qquad \tau_{12} = -\frac{\partial\Phi'}{\partial x},$$
$$\tau_{21} = \frac{\partial\Phi''}{\partial y}, \qquad \tau_{22} = -\frac{\partial\Phi''}{\partial x},$$

where Φ' and Φ'' are functions of the variables x, y. Since $\tau_{12} = \tau_{12}$, one must have $\frac{\partial\Phi'}{\partial x} = \frac{\partial\Phi''}{\partial y}$, so $\Phi' = \frac{\partial\Phi}{\partial y}$, $\Phi'' = \frac{\partial\Phi}{\partial x}$, and Φ is a new function of the same variables. One thus has:

(8)
$$\tau_{11} = \frac{\partial^2\Phi}{\partial y^2}, \qquad \tau_{22} = \frac{\partial^2\Phi}{\partial x^2}, \qquad \tau_{12} = -\frac{\partial^2\Phi}{\partial x\,\partial y}.$$

Note that one has $\Delta^2 T = \Delta^2(\tau_{11} + \tau_{22} + \tau_{33}) = 0$, and from the last of (6), $\Delta^2\tau_{33} = 0$, so $\Delta^2(\tau_{11} + \tau_{22}) = 0$, and upon taking formula (8) into account:

$$\Delta^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = 0,$$

or, since the functions depend upon just the variables x, y:

(9)
$$\Delta^2\,\Delta^2\,\Phi = 0.$$

From equations (8), one gets:

$$\Delta^2\tau_{11} = \frac{\partial^2\Delta^2\Phi}{\partial y^2}, \qquad \Delta^2\tau_{22} = \frac{\partial^2\Delta^2\Phi}{\partial x^2}, \qquad \Delta^2\tau_{12} = -\frac{\partial^2\Delta^2\Phi}{\partial x\,\partial y},$$

and from (9):

$$\frac{\partial^2\Delta^2\Phi}{\partial y^2} = -\frac{\partial^2\Delta^2\Phi}{\partial x^2}.$$

Therefore, we can write:

$$\Delta^2 \tau_{11} = -\frac{\partial^2 \Delta^2 \Phi}{\partial x^2}, \qquad \Delta^2 \tau_{11} = -\frac{\partial^2 \Delta^2 \Phi}{\partial y^2}, \qquad \Delta^2 \tau_{12} = -\frac{\partial^2 \Delta^2 \Phi}{\partial x \, \partial y}.$$

By comparing these equations with (5), set:

$$(10) \qquad T = (1 + \lambda)\, \Delta^2\Phi + G, \qquad G = G(x, y),$$

and one will have:

$$\frac{\partial^2 G}{\partial x^2} = 0, \qquad \frac{\partial^2 G}{\partial y^2} = 0, \qquad \frac{\partial^2 G}{\partial x \, \partial y} = 0.$$

One must therefore have $G = mx + ny + p$, where m, n, p denote constants, and from formula (10):

$$T = (1 + \lambda)\, \Delta^2\Phi + mx + ny + p.$$

However, $T = \tau_{11} + \tau_{22} + \tau_{33}$, and from (8), $\tau_{11} + \tau_{22} = \Delta^2\Phi$, so:

$$\tau_{33} = \lambda\, \Delta^2\Phi + mx + ny + p.$$

Note that the system of tensions $\tau_{33} = mx + ny + p$, $\tau_{11} = \tau_{22} = \tau_{12} = \tau_{31} = \tau_{32} = 0$ now corresponds to a deformation of the cylinder that reverts to the cases that were examined by Saint-Venant. Since our interest is only in considering the deformation D (cf., § 1), not D_0, we omit the term $mx + ny + p$ and get:

$$\tau_{33} = \lambda\, \Delta^2\Phi.$$

Finally, from formula (4), we can write:

$$\tau_{31} = \frac{\partial \varphi}{\partial y}, \qquad \tau_{32} = -\frac{\partial \varphi}{\partial x}, \qquad \varphi = \varphi(x, y).$$

The first and second equations in (6) become:

$$\frac{\partial \Delta^2 \varphi}{\partial y} = 0, \qquad \frac{\partial \Delta^2 \varphi}{\partial x} = 0,$$

namely, one must have:

$$\Delta^2 \varphi = k,$$

where k denotes a constant.

The nine equations (3), (4), (5), and (6) are thus verified. The internal tensions are expressed by formulas (1) and (2). Q. E. D.

4. Now, take the boundary conditions into account. When one substitutes the expressions found for the tensions, formula (7) will become:

$$\frac{\partial^2\Phi}{\partial y^2}\cos\alpha-\frac{\partial^2\Phi}{\partial x\,\partial y}\cos\beta = 0, \qquad -\frac{\partial^2\Phi}{\partial x\,\partial y}\cos\alpha+\frac{\partial^2\Phi}{\partial x^2}\cos\beta = 0,$$

$$\frac{\partial\varphi}{\partial y}\cos\alpha-\frac{\partial\varphi}{\partial x}\cos\beta = 0.$$

We must therefore have:

$$(11) \qquad \frac{\partial\left(\dfrac{\partial\Phi}{\partial y}\right)}{\partial s_i} = 0, \qquad \frac{\partial\left(\dfrac{\partial\Phi}{\partial x}\right)}{\partial s_i} = 0, \qquad \frac{\partial\varphi}{\partial s_i} = 0$$

at any point of an arbitrary s_i of the n closed lines that constitute the contour of a section σ, if s_i also denotes the arc length of that line when measured from a fixed point.

Let s_n be that one of the n lines s_i that encloses all of the other ones. Render the section σ simply-connected by means of $n - 1$ cuts that join the points $P_1, P_2, \ldots, P_{n-1}$ of the lines $s_1, s_2, \ldots, s_{p-1}$ to the points $P_1', P_2', \ldots, P_{n-1}'$ of the line s_n.

The tensions $\tau_{11}, \tau_{12}, \ldots$ (and thus, from formulas (1) and (2), the first derivatives of φ and second derivatives of Φ) must be functions of just one value at all of the points of σ; so φ, $\partial\Phi / \partial x$, $\partial\Phi / \partial y$ can have different values on the two parts of the cut.

Formula (11) expresses the condition that the functions φ, $\partial\Phi / \partial x$, $\partial\Phi / \partial y$ must have constant values at each of the two segments into which the n closed lines s_i are divided by the points P that bound the cuts.

However, the $n - 1$ lines $s_1, s_2, \ldots, s_{n-1}$ each contain just one point P. Therefore, the three functions φ, $\partial\Phi / \partial x$, $\partial\Phi / \partial y$ must reduce to constants for each of them.

Since the first derivatives of φ and the second derivatives of Φ are functions of just one value at all points of σ, the difference between the values of φ, $\partial\Phi / \partial x$, $\partial\Phi / \partial y$ on the two parts of a cut P_iP_i' must be the same at all points of the cut, and therefore zero, since it is zero at the point P_i that belongs to one of the $n - 1$ internal lines $s_1, s_2, \ldots, s_{n-1}$.

Therefore, φ, $\partial\Phi / \partial x$, $\partial\Phi / \partial y$ will also be functions of just one value at all points of σ, including the line s_n, and on any one s_i of these n lines $s_1, s_2, \ldots, s_n$ one must have:

$$(12) \qquad \varphi = q_i, \qquad \frac{\partial\Phi}{\partial x} = a_i, \qquad \frac{\partial\Phi}{\partial y} = b_i,$$

where q_i, a_i, b_i represent constants for the line s_i.

From formula (12), one gets (if one conveniently assigns a positive direction to each line s_i):

$$\frac{\partial\Phi}{\partial s_i} = b_i\cos\alpha - a_i\cos\beta = a_i\,\frac{\partial x}{\partial s_i} + b_i\frac{\partial y}{\partial s_i} = \frac{\partial(a_i x + b_i y)}{\partial s_i},$$

namely:

$$\frac{\partial(\Phi - a_i x - b_i y)}{\partial s_i} = 0.$$

By a line of reasoning that is analogous to the preceding one, we find that $\Phi - a_i x - b_i y$ must be a function of just one value at all points of σ, and constant on each of the n lines $s_1, s_2, \ldots, s_n$. We therefore have:

$$\Phi = a_i x + b_i y + c_i \qquad (c_i = \text{const.}) \tag{13}$$

for the line s_i.

From formula (12), if one lets ν_i denote the normal to σ that points inward at the points of s_i then one will get:

$$\frac{\partial \Phi}{\partial \nu_i} = \frac{\partial(a_i x + b_i y + c_i)}{\partial \nu_i}. \tag{14}$$

Conditions (13) and (14) can be substituted for (12).

5. Consider the constant:

$$M = \int_\sigma (x\tau_{32} - y\tau_{31})\, d\sigma \qquad (\textit{torsional moment}). \tag{15}$$

If we compose the deformation that we examined with a simple *torsion* of the cylinder (which would not modify our formula, since one should recall that it was presented relative to a simple torsion) then that can be done in such a way that:

$$M = 0.$$

Add this new condition, which, by virtue of formulas (15) and (2), can be written:

$$\int_\sigma \left(x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y} \right) d\sigma = 0. \tag{16}$$

This establishes a relation between the constants k and q_i, as one easily sees when one takes $\varphi = k\varphi' + \varphi''$, where φ' represents the function that verifies the equation $\Delta^2 \varphi' = 1$ in the area σ and is annulled on the contour, so φ'' is the harmonic function that assumes the values q_i along the line s_i.

In order to complete our study, we would like to prove that *the deformation that we considered* [when it also satisfies the condition (10)] *cannot be a deformation with a monodromic displacement, unless all of the tensions are zero*.

6. Hence, take into account the formulas:

$$(17) \qquad \frac{\partial u}{\partial x} = A\tau_{11} - BT, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = A\tau_{12}, \qquad \text{etc.} \qquad \left(A, B = \text{const.}, \ \frac{B}{A-B} = \lambda\right)$$

that link the components of the deformation to the tensions.

Introduce a harmonic function $\varphi_1(x, y)$ that satisfies the equation:

$$\frac{\partial^2 \varphi_1}{\partial x^2} = \Delta^2 \Phi.$$

It is always possible to determine such a function in an area σ_0 that is bounded by a rectilinear segment s' that is parallel to the y-axis (Fig. 2), two segments s'', s''' (possibly zero) that are also rectilinear and parallel to the x-axis, and a line s'''' that is met at just one point by any parallel to that axis. Moreover, one can always decompose the area σ into a certain number of areas like σ_0 .

The function φ_1 and its derivatives can present discontinuities on the separation lines around the area σ_0 .

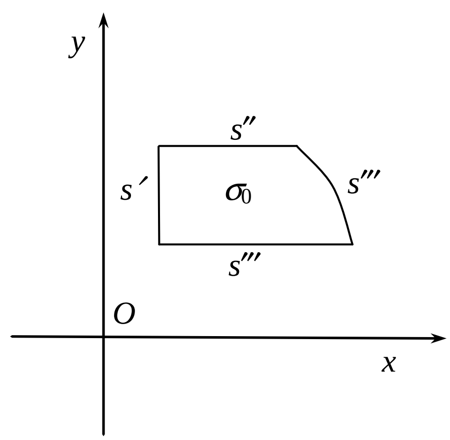

Figure 2.

We also introduce the function $\varphi_2(x, y)$ that is linked to $\varphi(x, y)$ by the equations:

$$\frac{\partial \varphi_2}{\partial x} = \frac{\partial \varphi}{\partial y} - \frac{k}{2} y, \qquad \frac{\partial \varphi_2}{\partial y} = -\frac{\partial \varphi}{\partial x} + \frac{k}{2} x,$$

which are mutually compatible, since $\Delta^2 \varphi = k$.

It is easy to verify that the formula that results from (17) when one sets:

$$(18)\qquad \begin{cases} u = -A\dfrac{\partial \Phi}{\partial x} + C\dfrac{\partial \varphi_1}{\partial x} + Akyz, \\ v = -A\dfrac{\partial \Phi}{\partial y} - C\dfrac{\partial \varphi_1}{\partial y} - Akxz, \\ w = 2A\varphi_2, \end{cases} \qquad C = \frac{A(A-2B)}{A-B}$$

is verified.

Suppose that the space S that is occupied by the cylinder has been rendered simply-connected by means of n – 1 cuts T_i that form a surface Σ_i that is parallel to the axis and passes through the line $P_i P_i'$ (§ 4) that is traced in σ. Since the internal tensions, and therefore the components of the deformations, are monodromic, we can dispose of the arbitrariness that remains in the functions φ_1 and φ_2 in such a way that the displacements u, v, w present no discontinuities across the surface Σ_i.

In order to prove that u, v, w cannot be continuous in all of the area s unless all of the tensions are non-zero, consider the quantity:

$$Q = \int_\sigma \left\{ \tau_{11}\frac{\partial u}{\partial x} + \tau_{22}\frac{\partial v}{\partial y} + \tau_{33}\frac{\partial w}{\partial z} + \tau_{23}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \tau_{31}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{12}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \right\} d\sigma,$$

which is essentially positive or zero, and which will reduce to zero only when all of the tensions are zero. I say that if u, v, w are continuous then one must have $Q = 0$.

In fact, under that hypothesis, one will have:

$$\int_\sigma \left\{ \tau_{11}\frac{\partial u}{\partial x} + \tau_{22}\frac{\partial v}{\partial y} + \tau_{12}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \right\} d\sigma = 0,$$

$$\int_\sigma \left\{ \tau_{31}\frac{\partial w}{\partial x} + \tau_{32}\frac{\partial w}{\partial y} \right\} d\sigma = 0$$

as one recognizes upon integrating by parts and taking equations (3) and (4) into account, along with the boundary conditions (7).

Furthermore, from formula (18), $\dfrac{\partial u}{\partial z} = Aky$, $\dfrac{\partial v}{\partial z} = -Akx$, $\dfrac{\partial w}{\partial z} = 0$; therefore:

$$Q = Ak \int_\sigma (y\tau_{31} - x\tau_{32})\, d\sigma = -AkM = 0 \qquad \text{(cf., § 5).}$$

Q. E. D.