“Aperçu sur l’emploi possible de l’énergie d’accélération dans les équations de l’électrodynamique,” C. R. Acad. Sci. Paris **154** (1912), 1037-1040.

Outline of the possible use of the energy of acceleration in the equations of electrodynamics

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1. – In a volume of the collection *Scientia* that was published in 1902 under the title of “L’électricité déduite de l’expérience et ramenée au principe des travaux virtuels,” Carvallo, following Maxwell’s theory, studied the application of the Lagrange to electrodynamical phenomena ([1]), and notably to the cases of two or three-dimensional conductors. He explained (pp. 81) the inadequacy of the Lagrange equations by remarking that the three parameters θ, q_1, q_2 whose arbitrary infinitely small variations define the most general displacement of the system in Barlow’s experiment with the wheel are not true coordinates, and that the system can be composed of a hoop in an analogous way, to which the Lagrange equations do not apply, as is known from a remark by Ferrers (Quarterly Journal of Mathematics 1871-1873). In other words, in Hertz’s terminology, the system is not *holonomic*.

2. – Under those conditions, if one can hope to attach the equations of electrodynamics to those of analytical mechanics then one must seek to attach them to a general form of equations that is applicable to all systems, whether holonomic or not.

In order to form such equations, as I showed in the *Comptes rendus* (session on 7 August 1899), one can proceed as follows ([2]): Imagine a material system whose virtual displacement (which is compatible with the constraints at the time t) is defined by the arbitrary variations δq_1, δq_2, …, δq_k of the parameters q_1, q_2, …, q_k. For that displacement, the sum of the elementary forces done by the given forces Q is:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k\,.$$

On the other hand, let the energy of acceleration be:

$$S = \tfrac{1}{2}\sum m\,J^2\,, \tag{1}$$

([1]) Likewise, see a book by Carvallo that is entitied *Leçons d’Électricité*, Béranger, editor, 1904.
([2]) See also my *Traité de Mécanique rationelle*, t. II, Chap. XXIV, § **6**.

which is equal to one-half the sum of the products that are obtained by multiplying the mass m of each point by the square of its acceleration J. That expression S is a function of degree two in the second derivatives q_1'', q_2'', …, q_k'' of the parameters q_1, q_2, …, q_k, resp., with respect to time. The equations of motion are then:

$$\frac{\partial S}{\partial q_\nu''} = Q_\nu \qquad (n = 1, 2, \ldots, k) ; \tag{2}$$

they express the idea that the accelerations at each instant t minimize the function:

$$R = S - \sum Q\, J \cos Q\, J . \tag{3}$$

That is the form of the equations that one would like to be able to extend to electrodynamic phenomena that depend upon a finite number of parameters. The difficulty will obviously be in calculating the energy of acceleration S. From a purely formal standpoint, one can calculate it for all phenomena for which the Lagrange equations apply, because when one calls the energy of velocity or kinetic energy T, one will then have:

$$\frac{\partial S}{\partial q_\nu''} = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} \qquad (\nu = 1, 2, \ldots, k),$$

which are equations that give S, up to a term that is independent of q''. However, when the Lagrange equations do not apply, the initial condition for equations (2) to be capable of accounting for the phenomenon is that when the equations of motion have been put into the form:

$$f_\nu(q_1'', q_2'', \ldots, q_k'') = Q_\nu \qquad (\nu = 1, 2, \ldots, k),$$

the f_ν must be partial derivatives of the same function S with respect to the q_ν''.

3. – For the example of the Barlow wheel, when one uses the notations of Carvallo (*loc. cit.*, pp 78-80), the parameters will be θ, q_1, q_2, while the right-hand sides Q_ν of equations will be Q, $E_1 - r_1 q_1'$, $E_2 - r_2 q_2'$; the equations themselves will be:

$$\begin{aligned} I\,\theta'' - K\, q_1'\, q_2' &= Q, \\ L_1\, q_1'' + K\,\theta''\, q_2' &= E_1 - r_1\, q_1', \\ L_2\, q_2'' \qquad\qquad &= E_2 - r_2\, q_2'. \end{aligned}$$

Now, the left-hand side are the partial derivatives with respect to θ, q_1'', and q_2'' of the function:

$$S = \tfrac{1}{2}[I\,\theta''^2 + L_1\, q_1''^2 + L_2\, q_2''^2 + 2K\, q_2'(\theta'\, q_1'' - \theta''\, q_1') + \cdots], \tag{4}$$

in which the unwritten terms do not contain second derivatives. The equations of motion are then indeed of the form (2). They express the idea that the accelerations at each instant minimize the function:

$$R = S - Q\,\theta'' - (E_1 - r_1\,q_1')\,q_1'' - (E_2 - r_2\,q_2')\,q_2''.$$

By analogy, the function *S* that is given by equation (4) must then be regarded as the energy of acceleration of the system. However, the truly important point will be to know if that function *S*, which is thus-formed analytically, can be obtained directly by physical considerations that are attached to the defining formula (1).

4. – As an example of a case with an infinite number of parameters, in the next volume, which will be published in Italy, in honor of Lagrange, I will show how one can deduce the general equations of hydrodynamics from the principle of the minimum of the function *R* that is defined by equation (3).

I recently indicated (C. R. Acad. Sci. Paris, session on 8 May 1911; Rendiconti del Circolo matematico di Palermo, t. XXXII, session on 14 May 1911, and t. XXXIII, session on 25 February 1912) how that same principle can be applied to the motion of systems that are subject to constraints that are nonlinear with respect to the velocities. That question was examined more deeply in a very general and complete way by Delassus in various notes that were included in *Comptes rendus* ([1]).

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([1]) E. Delassus, "Sur la réalisation matérielle des liaisons," C. R. Acad. Sci. Paris **152**, session on 19 June 1911, pp. 1739-1743; "Sur les liaisons non linéaires," *ibid.*, **153**, session on 2 October 1911, pp. 626-628; "Sur les liaisons non linéaires et les mouvements étudiés par M. Appell," *ibid.*, **153**, session on 16 October 1911, pp. 707-710; "Sur les liaisons d'ordre quelconque des systèmes matériels," *ibid.*, **154**, session on 15 April 1912, pp. 964-967.