"Réduction à la forme canonique des équations d'un fil flexible et inextensible," Comptes rendus de l'Académie des sciences **96** (1883), 688-691.

Reducing the equations of equilibrium for a flexible, inextensible filament to canonical form

By P. Appell

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Translated by D. H. Delphenich

Even though numerous analogies have been pointed out for some time between the equations of equilibrium for a filament and the equations of motion of a point ([1]), to my knowledge, no one has reduced those equations of equilibrium to a canonical form that would permit one to apply Jacobi's theorems.

I. – First, consider a flexible, inextensible filament that is entirely free whose element of length ds is subject to the force $F\,ds$ that has $X\,ds$, $Y\,ds$, $Z\,ds$ for its projections onto the coordinate axes, which are supposed to be rectangular, in which X, Y, Z are functions of only the coordinates x, y, z of the point of application. Furthermore, assume that the exists a force function U; i.e., that:

$$dU = X\,dx + Y\,dy + Z\,dz.$$

If one lets T denote the tension then the equations of equilibrium will be:

$$(1) \qquad \frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \qquad \frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0;$$

hence, one will deduce that:

$$(2) \qquad dT + dU = 0, \quad T = -(U + h),$$

in which h is an arbitrary constant.

Introduction an auxiliary independent variable σ into equations (1) that is coupled to s by the relation:

$$\frac{ds}{d\sigma} = T.$$

([1]) *See* a memoir of O. Bonnet, Journal de Mathématique **9** (1844), and the work by P. Serret, *Théorie géométrique et mécanique des lignes à double courbure.*

The equations will become:

$$\frac{d^2x}{d\sigma^2} + TX = 0, \qquad \frac{d^2y}{d\sigma^2} + TY = 0, \qquad \frac{d^2z}{d\sigma^2} + TZ = 0,$$

or, upon setting $V = \frac{1}{2}(U + h)^2$:

$$\text{(3)} \qquad \frac{d^2x}{d\sigma^2} = \frac{\partial V}{\partial x}, \qquad \frac{d^2y}{d\sigma^2} = \frac{\partial V}{\partial y}, \qquad \frac{d^2z}{d\sigma^2} = \frac{\partial V}{\partial z},$$

which are equations that are analogous to those of the motion of a point. We can now apply Jacobi's theorems to those equations. In order to do that, consider the partial differential equation:

$$\text{(4)} \qquad \left(\frac{\partial \Theta}{\partial x}\right)^2 + \left(\frac{\partial \Theta}{\partial y}\right)^2 + \left(\frac{\partial \Theta}{\partial z}\right)^2 = (U + h)^2,$$

and suppose that one has found an integral:

$$\Theta\,(x, y, z\,;\, \alpha, \beta, h)$$

of that equation, with two arbitrary constants α and β that are distinct from h and the additive constant that one can always add to Θ. The equations of the equilibrium curve are then:

$$\text{(5)} \qquad \frac{\partial \Theta}{\partial \alpha} = \alpha', \qquad \frac{\partial \Theta}{\partial \beta} = \beta',$$

in which α' and β' are two new constants.

II. –More generally, imagine that one employs an arbitrary coordinate system q_1, q_2, q_3 that is coupled with x, y, z by the equations:

$$\text{(6)} \qquad x = f(q_1, q_2, q_3), \qquad y = \varphi(q_1, q_2, q_3), \qquad z = \psi(q_1, q_2, q_3).$$

Let x', y', z', q_1', q_2', q_3' denote the derivatives of x, y, z, q_1, q_2, q_3 with respect to σ. The expression:

$$P = (x'^2 + y'^2 + z'^2)$$

will be a function of q_1, q_2, q_3, q_1', q_2', q_3', and upon setting:

$$p_1 = \frac{\partial P}{\partial q_1'}, \qquad p_2 = \frac{\partial P}{\partial q_2'}, \qquad p_3 = \frac{\partial P}{\partial q_3'},$$

one can express P as a function of q_1, q_2, q_3, p_1, p_2, p_3 . Finally, one will form the function:

$$H(q_1, q_2, q_3, p_1, p_2, p_3) = P - \tfrac{1}{2}(U + h)^2,$$

and the equations of equilibrium will come down to the canonical form:

$$\frac{dq_i}{d\sigma} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{d\sigma} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, 3). \tag{7}$$

However, in order to obtain the equations of the equilibrium curve, it is pointless to have the *general* integral [1] of equations (7). As before, it will suffice to consider the equation:

$$H\left(q_1, q_2, q_3; \frac{\partial\Theta}{\partial q_1}, \frac{\partial\Theta}{\partial q_2}, \frac{\partial\Theta}{\partial q_3}\right) = 0 \tag{8}$$

and find an integral $\Theta(q_1, q_2, q_3; \alpha, \beta, h)$ of that equation with two arbitrary constants α and β. The equations of the equilibrium curve will then be:

$$\frac{\partial\Theta}{\partial\alpha} = \alpha', \qquad \frac{\partial\Theta}{\partial\beta} = \beta'. \tag{9}$$

III. – Finally, suppose that one must seek the equilibrium position of a filament that is subject to the same form F and constrained to remain on a given surface. Since the coordinates of a point on that surface supposed to be expressed as functions of two parameters q_1 and q_2 , one will define the function:

$$P = \tfrac{1}{2}(x'^2 + y'^2 + z'^2),$$

and one will express it in terms of the parameters q_1, q_2, and some new variables p_1, p_2 defined by the equations:

$$p_1 = \frac{\partial P}{\partial q_1'}, \qquad p_2 = \frac{\partial P}{\partial q_2'}.$$

If one then lets $H(q_1, q_2 ; p_1, p_2)$ denote the function $P - \tfrac{1}{2}(U + h)^2$, and if one considers the partial differential equation:

$$H\left(q_1, q_2; \frac{\partial\Theta}{\partial q_1}, \frac{\partial\Theta}{\partial q_2}\right) = 0$$

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[1] Indeed, equations (7) give the first integral $H = C$. However, by virtue of the value (2) of T, one must attribute the particular value 0 to that constant C.

then it will suffice to find an integral $\Theta(q_1, q_2, \alpha, h)$ of that equation with an arbitrary constant α, and the equation of the equilibrium curve will be:

$$\frac{\partial \Theta}{\partial \alpha} = \alpha',$$

in which α' is a new arbitrary constant.