"Sur l'équilibre d'un fil flexible et inextensible," Ann. fac. sci. Toulouse (1) **1** (1887), B1-B5.

On the equilibrium of a flexible, inextensible filament

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The goal of this note is to complete and prove directly a theorem that was only stated in a note that was presented to *l'Académie des Sciences* at the session on 12 March 1883.

Consider a flexible, inextensible filament that is entirely free whose element of length ds is subject to the force $F\,ds$ whose projections onto three rectangular axes are:

$$X\,ds, \quad Y\,ds, \quad Z\,ds,$$

in which X, Y, Z are functions of only the coordinates x, y, z of the point of application. Assume, moreover, that there exists a force function U; i.e., that:

$$X = \frac{\partial U}{\partial x}, \qquad Y = \frac{\partial U}{\partial y}, \qquad Z = \frac{\partial U}{\partial z}.$$

If one lets T denote the tension then the equations of equilibrium will be:

$$(1) \qquad \left\{ \begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right) + \frac{\partial U}{\partial x} = 0, \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) + \frac{\partial U}{\partial y} = 0, \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) + \frac{\partial U}{\partial z} = 0. \end{aligned} \right.$$

One then deduces immediately that:

$$(2) \qquad T = -(U + h),$$

in which h is an arbitrary constant. Having said that, in order to find the integrals of the equations of equilibrium (1), one can proceed in the following fashion:

Consider the partial differential equation:

$$(3) \qquad \left(\frac{\partial \Theta}{\partial x}\right)^2 + \left(\frac{\partial \Theta}{\partial y}\right)^2 + \left(\frac{\partial \Theta}{\partial z}\right)^2 = (U + h)^2,$$

which defines Θ *as a function of x, y, z, and suppose that one has found a complete integral:*

$$\Theta\,(x, y, z\,;\,\alpha, \beta, \gamma)$$

of that equation with two arbitrary constants α *and* β*, which are distinct from h and the constant that one can always add to* Θ*. The integrals of the equations of equilibrium* (1) *are then the following ones:*

$$\frac{\partial\Theta}{\partial x} = \alpha', \qquad \frac{\partial\Theta}{\partial y} = \beta', \qquad \frac{\partial\Theta}{\partial z} = s + h', \tag{4}$$

in which α', β', h', *are new constants, and s denotes the arc length of the equilibrium curve, when counted positively in a convenient sense.*

In order to prove that theorem, we shall see, upon following the method that Jacobi pointed out in his *Vorlesungen über Dynamik*, that the values of x, y, z as functions of s that one infers from equations (4) verify the differential equations (1) and the equation:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

Let us first calculate dx / ds, dy / ds, dz / ds. In order to do that, we must differentiate equations (4) while considering x, y, z to be implicit functions of s that are defined by those equations. We will then have:

$$\left\{\begin{aligned}
\frac{\partial^2\Theta}{\partial\alpha\,\partial x}\frac{dx}{ds} + \frac{\partial^2\Theta}{\partial\alpha\,\partial y}\frac{dy}{ds} + \frac{\partial^2\Theta}{\partial\alpha\,\partial z}\frac{dz}{ds} &= 0, \\
\frac{\partial^2\Theta}{\partial\beta\,\partial x}\frac{dx}{ds} + \frac{\partial^2\Theta}{\partial\beta\,\partial y}\frac{dy}{ds} + \frac{\partial^2\Theta}{\partial\beta\,\partial z}\frac{dz}{ds} &= 0, \\
\frac{\partial^2\Theta}{\partial h\,\partial x}\frac{dx}{ds} + \frac{\partial^2\Theta}{\partial h\,\partial y}\frac{dy}{ds} + \frac{\partial^2\Theta}{\partial h\,\partial z}\frac{dz}{ds} &= 1.
\end{aligned}\right. \tag{5}$$

On the other hand, if we replace Θ in the differential equation (3) with the function that was found:

$$\Theta\,(x, y, z\,;\,\alpha, \beta, h)$$

then the result of the substitution will be an identity in x, y, z ; α, β, h. If we take the partial derivatives of that identity (3) with respect to α, β, and h, in succession, then we will have:

(6)
$$\begin{cases} \dfrac{\partial^2\Theta}{\partial\alpha\,\partial x}\dfrac{\partial\Theta}{\partial x}+\dfrac{\partial^2\Theta}{\partial\alpha\,\partial y}\dfrac{\partial\Theta}{\partial y}+\dfrac{\partial^2\Theta}{\partial\alpha\,\partial z}\dfrac{\partial\Theta}{\partial z}=0, \\ \dfrac{\partial^2\Theta}{\partial\beta\,\partial x}\dfrac{\partial\Theta}{\partial x}+\dfrac{\partial^2\Theta}{\partial\beta\,\partial y}\dfrac{\partial\Theta}{\partial y}+\dfrac{\partial^2\Theta}{\partial\beta\,\partial z}\dfrac{\partial\Theta}{\partial z}=0, \\ \dfrac{\partial^2\Theta}{\partial h\,\partial x}\dfrac{\partial\Theta}{\partial x}+\dfrac{\partial^2\Theta}{\partial h\,\partial y}\dfrac{\partial\Theta}{\partial y}+\dfrac{\partial^2\Theta}{\partial h\,\partial z}\dfrac{\partial\Theta}{\partial z}=U+h. \end{cases}$$

Equations (5) are three equations of first-degree in dx / ds, dy / ds, dz / ds, and equations (6) are of first degree in $\partial\Theta / \partial x$, $\partial\Theta / \partial y$, $\partial\Theta / \partial z$. Furthermore, equations (5) are deduced from equations (6) by substituting:

$$(U+h)\frac{dx}{ds}, \quad (U+h)\frac{dy}{ds}, \quad (U+h)\frac{dz}{ds},$$

for

$$\frac{\partial\Theta}{\partial x}, \frac{\partial\Theta}{\partial y}, \frac{\partial\Theta}{\partial z}.$$

One will then have:

$$(U+h)\frac{dx}{ds} = \frac{\partial\Theta}{\partial x}, \quad (U+h)\frac{dy}{ds} = \frac{\partial\Theta}{\partial y}, \quad (U+h)\frac{dz}{ds} = \frac{\partial\Theta}{\partial z}.$$

Upon taking the sum of the squares of those three equations and taking relation (3) into account, one will find the relation:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2 = 1,$$

which shows that s is indeed the arc length of the curve. If one replaces $U + h$ with $-T$ in the equations above then they will become:

(7)
$$T\frac{dx}{ds} = -\frac{\partial\Theta}{\partial x}, \qquad T\frac{dy}{ds} = -\frac{\partial\Theta}{\partial y}, \qquad T\frac{dz}{ds} = -\frac{\partial\Theta}{\partial z}.$$

Hence, upon differentiating these, one will get:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = -\frac{\partial^2\Theta}{\partial x^2}\frac{dx}{ds}-\frac{\partial^2\Theta}{\partial x\,\partial y}\frac{dy}{ds}-\frac{\partial^2\Theta}{\partial x\,\partial z}\frac{dz}{ds},$$
$$\dots\dots\dots\dots\dots\dots\dots,$$

or, since:

$$\frac{dx}{ds}=\frac{1}{U+h}\frac{\partial\Theta}{\partial x}, \qquad \frac{dy}{ds}=\frac{1}{U+h}\frac{\partial\Theta}{\partial y}, \qquad \frac{dz}{ds}=\frac{1}{U+h}\frac{\partial\Theta}{\partial z},$$

upon differentiating:

$$\frac{d}{ds}\left(T\,\frac{dx}{ds}\right) = -\,\frac{1}{U+h}\left(\frac{\partial^2\Theta}{\partial x^2}\frac{\partial\Theta}{\partial x} - \frac{\partial^2\Theta}{\partial x\,\partial y}\frac{\partial\Theta}{\partial y} - \frac{\partial^2\Theta}{\partial x\,\partial z}\frac{\partial\Theta}{\partial z}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

However, upon differentiating the identity (3) with respect to x, one will get:

$$\frac{\partial^2\Theta}{\partial x^2}\frac{\partial\Theta}{\partial x} - \frac{\partial^2\Theta}{\partial x\,\partial y}\frac{\partial\Theta}{\partial y} - \frac{\partial^2\Theta}{\partial x\,\partial z}\frac{\partial\Theta}{\partial z} = (U+h)\,\frac{\partial U}{\partial x}\,.$$

Hence, the preceding equation becomes:

$$\frac{d}{ds}\left(T\,\frac{dx}{ds}\right) = -\frac{\partial U}{\partial x}\,,$$

and one will similarly find that:

$$\frac{d}{ds}\left(T\,\frac{dy}{ds}\right) = -\frac{\partial U}{\partial y}\,,$$

$$\frac{d}{ds}\left(T\,\frac{dz}{ds}\right) = -\frac{\partial U}{\partial z}\,,$$

which are equations that are nothing but the equations of equilibrium (1). The theorem is thus proved.

Suppose that one would like to determine the constants that appear in the integrals (4) from the condition that the curve must pass through two points:

$$(x_0\,,\,y_0\,,\,z_0), \qquad (x_1\,,\,y_1\,,\,z_1)$$

and that it must have a length l between those two points. Upon setting:

$$\Theta_0 = \Theta\,(x_0\,,\,y_0\,,\,z_0\,;\,\alpha,\,\beta,\,h),$$

$$\Theta_1 = \Theta\,(x_1\,,\,y_1\,,\,z_1\,;\,\alpha,\,\beta,\,h),$$

One will then have to solve the three equations:

$$\frac{\partial\Theta_0}{\partial\alpha} - \frac{\partial\Theta_1}{\partial\alpha} = 0, \qquad \frac{\partial\Theta_0}{\partial\beta} - \frac{\partial\Theta_1}{\partial\beta} = 0, \qquad \frac{\partial\Theta_0}{\partial h} - \frac{\partial\Theta_1}{\partial h} = \pm\, l,$$

for α, β, h.

One can easily apply the method of integration that we just presented to each of the following special cases:

1. The function U depends upon only the distance from the point (x, y, z) to a fixed plane.

2. U depends upon only the distance from the point (x, y, z) to a fixed axis.

3. U depends upon only the distance from the point (x, y, z) to a fixed point.

However, the calculations that one must carry out in these three cases will only repeat those of Jacobi in his *Vorlesungen über Dynamik,* for example, Lecture XXIV; there would be no point in repeating them here.