"Das Gleichgewicht und die Bewegung einer unendlich dünnen, beliebig gekrümmten elastischen Schale," J. f. reine u. angew. Math. **78** (1874), 136-174.

The equilibrium and motion of an infinitely-thin, arbitrarily-curved, elastic shell

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Kirchhoff was the first to derive the true equations for small displacements and oscillations of infinitely-thin elastic plates from the general equations for elasticity in his treatise in volume **40** of this journal and calculated the sound figures on thin, circular discs. In volume **56** of this journal, **Kirchhoff** likewise developed the theory of the motion and equilibrium of infinitely-thin rods for which finite changes of form are possible, without the elements suffering more than small dilatations, which is the single assumption in the general theory of elasticity, and at the same time we remarked that one could obtain his results for infinitely-thin plates in the same way. Accordingly, in his Berlin dissertation that he submitted in 1860, **Gehring** treated the differential equations for the equilibrium and motion of infinitely-thin plates that suffer small deformations, and at the same time he allowed the plate to have a crystalline composition. In his book *Theorie der Elasticität fester Körper*, **Clebsch** also presented the equations for finite deformations of thin plates whose elements experience only infinitely-small dilatations by applying the principles that **Kirchhoff** had developed in his treatment of thin rods and derived the equations for infinitely-small displacements from those equations. Here, bolstered by the aforementioned work, I would like to present the general equations for the equilibrium and motion for arbitrarily-curved plates, and thus, for very thin elastic shells that can experience finite deformations, but for which the elements can enter into only infinitely-small dilatations, but then consider the case of infinitely-small displacements in particular.

I.

Let the coordinates of the middle surface of the shell be X, Y, Z, when referred to a rectilinear, right-angled, fixed coordinate system. Let those three coordinates be given as functions of two independent variables u, v. Following **Gauss**, one then sets:

$$\frac{\partial X}{\partial u} = a, \qquad \frac{\partial Y}{\partial u} = b, \qquad \frac{\partial Z}{\partial u} = c,$$

$$\frac{\partial X}{\partial v} = a', \qquad \frac{\partial Y}{\partial v} = b', \qquad \frac{\partial Z}{\partial v} = c' \, ;$$

furthermore:

$$\begin{aligned} a^2 + b^2 + c^2 &= E, \\ aa' + bb' + cc' &= F, \\ a'^2 + b'^2 + c'^2 &= G. \end{aligned}$$

I would now like to choose the u, v-coordinates in such a way that u = const., v = const. define two families of curves on the surface that cut at right angles, or in other words, that $F = 0$. Not only does such a coordinate system always exist on a surface, but an infinite number of them will exist on each surface, moreover. If the u, v have that property and ds^2 is the element of arc length on the surface then:

$$ds^2 = E\,du^2 + G\,dv^2.$$

s_1 shall now denote the length of the curve v = const. that goes through the point u, v, and indeed as measured from its point of intersection with the curve $u = 0$ to the point u, v. By contrast, s_2 shall denote the length of the curve $u = 0$ that goes through the point u, v, from its point of intersection with the curve $v = 0$ to the point u, v.

One then has:

$$ds_1^2 = E\,du^2, \quad ds_2^2 = G\,dv^2,$$

because u (v, resp.) are constant along those curves, so $du = 0$ ($dv = 0$, resp.).

I employ two such families of curves u = const., v = const. in order to decompose the surface into a doubly-infinite family of small rectangles whose dimensions in width and length have the same order as the thickness of the shell. The space coordinates of the center P of the rectangle that were X, Y, Z might be ξ, η, ζ after deformation. Such an element suffers only infinitely-small deformations while the shell experiences finite changes of form, and the work that corresponds to those deformations can then be found from the general equations of elasticity.

To that end, I imagine that each element has a second coordinate system. It is fixed in neither the element nor in space. Its position for each form of the shell shall now be established. Let its origin be at P, which is the center of the element considered, for the rest position of the shell, and let the x, y, and z axes have the directions of ds_1, ds_2, and n, resp., the last of which is the normal to the surface at the point in question. Let $x + \mathfrak{u}_0$, $y + \mathfrak{v}_0$, $z + \mathfrak{w}_0$ then be the coordinates of any point of the element when referred to the indicated coordinate system. x, y, z shall be the coordinates of that same point in the rest position of the element itself. However, for curved surfaces, that rest position of the element itself is different from the rest position of the same element when it is coupled to the surface. A pressure will then be exerted by one part on the other, and the element, which tries to remain in the tangent plane, will be first assigned its curvature in that way. Let the coordinates of the same material points relative to the coordinate system that was laid down be:

$$x + \mathfrak{u}_0 + \mathfrak{u}, \qquad y + \mathfrak{v}_0 + \mathfrak{v}, \qquad z + \mathfrak{w}_0 + \mathfrak{w},$$

after deformation.

The position of the coordinate system in the element itself and the element when coupled to the shell during or after the deformation shall now be determined in such a way that $\mathfrak{u}_0$, $\mathfrak{v}_0$, $\mathfrak{w}_0$, $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ will contain neither common displacements nor common rotations. Namely, if no common displacements are also considered then the origin of the coordinate system must be found at the point that was denoted by P, so for:

$$x = 0, \quad y = 0, \quad z = 0,$$

one must also have:

$$\mathfrak{u}_0 = 0, \quad \mathfrak{v}_0 = 0, \quad \mathfrak{w}_0 = 0,$$
$$\mathfrak{u} = 0, \quad \mathfrak{v} = 0, \quad \mathfrak{w} = 0.$$

However, common rotations shall also be omitted now. When the position of a point in a body is given then the position of the body will be determined by the position of a plane that goes through that point and the direction of a line that goes through that point. Now, let the xy-plane be chosen in both varied positions in such a way that it defines a tangent plane to the surface at the point P into which the original xy-plane will go in both cases. At the point P, one will then have:

$$\frac{\partial \mathfrak{w}_0}{\partial x} = 0, \qquad \frac{\partial \mathfrak{w}_0}{\partial y} = 0, \qquad \frac{\partial \mathfrak{w}}{\partial x} = 0, \qquad \frac{\partial \mathfrak{w}}{\partial y} = 0.$$

Furthermore, in both cases, the x-axis shall be the tangent to the curve that the original x-axis defines each time, and indeed, the positive sense shall be directed in the same way as the positive side of the original x-axis.

One also has $\frac{\partial \mathfrak{v}_0}{\partial x} = 0$, $\frac{\partial \mathfrak{v}}{\partial x} = 0$ at the point P then. However, since no displacement or rotation exists in the neighborhood of the point P then, there will be no common displacement or rotation *a fortiori*. We have arrived at this in such a way that the following six conventions were assumed for the position of the coordinates each time, namely, that for:

$$x = 0, \quad y = 0, \quad z = 0,$$

one must have:

$$\mathfrak{u}_0 = 0, \quad \mathfrak{v}_0 = 0, \quad \mathfrak{w}_0 = 0,$$
$$\frac{\partial \mathfrak{v}_0}{\partial x} = 0, \qquad \frac{\partial \mathfrak{w}_0}{\partial x} = 0, \qquad \frac{\partial \mathfrak{w}_0}{\partial y} = 0,$$

or

$$\mathfrak{u} = 0, \quad \mathfrak{v} = 0, \quad \mathfrak{w} = 0,$$
$$\frac{\partial \mathfrak{v}}{\partial x} = 0, \qquad \frac{\partial \mathfrak{w}}{\partial x} = 0, \qquad \frac{\partial \mathfrak{w}}{\partial y} = 0,$$

resp.

Let the direction cosines of the x, y, and z axes after deformation with respect to the three fixed coordinate axes be:

$$\alpha_0,\ \beta_0,\ \gamma_0, \qquad \alpha_1,\ \beta_1,\ \gamma_1, \qquad \alpha_2,\ \beta_2,\ \gamma_2,$$

resp., and then let ξ', η', ζ' be the coordinates of the original point in the element with respect to the fixed coordinate axes. Now, one can already see that the form of the shell will be determined completely when the position of the center point *P* of the individual element after deformation is given, and thus the coordinates ξ, η, ζ, as well. It must be also determined uniquely by the nine quantities:

$$\alpha_0,\ \beta_0,\ \gamma_0, \qquad \alpha_1,\ \beta_1,\ \gamma_1, \qquad \alpha_2,\ \beta_2,\ \gamma_2,$$

and furthermore, ξ', η', ζ' will already be determined for each point *x*, *y*, *z* in the element *P*, and also $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$, when the original form of the shell is known.

Six equations exist already for the nine direction cosines:

$$(1) \qquad \left\{ \begin{array}{ll} \alpha_0^2 + \beta_0^2 + \gamma_0^2 = 1, & \alpha_1\,\alpha_2 + \beta_1\,\beta_2 + \gamma_1\,\gamma_2 = 0, \\ \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1, & \alpha_2\,\alpha_0 + \beta_2\,\beta_0 + \gamma_2\,\gamma_0 = 0, \\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1, & \alpha_0\,\alpha_1 + \beta_0\,\beta_1 + \gamma_0\,\gamma_1 = 0. \end{array} \right.$$

We shall now address the problem of finding three more equations that relate ξ, η, ζ to each other.

The point *u*, *v* has the coordinates ξ, η, ζ after deformation. The point $u + du$, *v* then corresponds to the point:

$$\xi + \frac{\partial \xi}{\partial u}du, \quad \eta + \frac{\partial \eta}{\partial u}du, \quad \zeta + \frac{\partial \zeta}{\partial u}du$$

after deformation, or since:

$$du = \frac{ds_1}{\sqrt{E}},$$

$$\xi + \frac{\partial \xi}{\partial u}\frac{ds_1}{\sqrt{E}}, \quad \eta + \frac{\partial \eta}{\partial u}\frac{ds_1}{\sqrt{E}}, \quad \zeta + \frac{\partial \zeta}{\partial u}\frac{ds_1}{\sqrt{E}}.$$

Since ξ, η, ζ are the coordinates of the origin of the element ds_1 after deformation, these will be the final coordinates. Now, ds_1 goes to ds_1' after the deformation, so we will have:

$$\frac{\dfrac{\partial \xi}{\partial u}\dfrac{ds_1}{\sqrt{E}}}{ds_1'}, \qquad \frac{\dfrac{\partial \eta}{\partial u}\dfrac{ds_1}{\sqrt{E}}}{ds_1'}, \qquad \frac{\dfrac{\partial \zeta}{\partial u}\dfrac{ds_1}{\sqrt{E}}}{ds_1'}$$

as the direction cosines of the element after the deformation.

Now let ε_1 be the dilatation in the direction of the *x*-axis, so we will have:

$$\frac{\partial\xi}{\partial u}\frac{1}{(1+\varepsilon_1)\sqrt{E}}, \qquad \frac{\partial\eta}{\partial u}\frac{1}{(1+\varepsilon_1)\sqrt{E}}, \qquad \frac{\partial\zeta}{\partial u}\frac{1}{(1+\varepsilon_1)\sqrt{E}}$$

after deformation.

However, since the element ds_1' also defines the change of form of the x-axis, whose direction cosines are α_0, β_0, γ_0, we will have:

$$\frac{\partial\xi}{\partial u}\frac{1}{(1+\varepsilon_1)\sqrt{E}} = \alpha_0, \qquad \frac{\partial\eta}{\partial u}\frac{1}{(1+\varepsilon_1)\sqrt{E}} = \beta_0, \qquad \frac{\partial\zeta}{\partial u}\frac{1}{(1+\varepsilon_1)\sqrt{E}} = \gamma_0;$$

therefore:

$$(2) \qquad \frac{\partial\xi}{\partial u} = \alpha_0\sqrt{E}\,(1+\varepsilon_1), \qquad \frac{\partial\eta}{\partial u} = \beta_0\sqrt{E}\,(1+\varepsilon_1), \qquad \frac{\partial\zeta}{\partial u} = \gamma_0\sqrt{E}\,(1+\varepsilon_1).$$

When we multiply each of these equations by:

$$\alpha_0, \beta_0, \gamma_0, \qquad \alpha_1, \beta_1, \gamma_1, \qquad \alpha_2, \beta_2, \gamma_2,$$

resp., and add them three at a time, we will get another form for those three equations that we will need later, namely:

$$(2a) \qquad \left\{\begin{aligned} &\alpha_0\frac{\partial\xi}{\partial u} + \beta_0\frac{\partial\eta}{\partial u} + \gamma_0\frac{\partial\zeta}{\partial u} = \sqrt{E}(1+\varepsilon_1), \\ &\alpha_1\frac{\partial\xi}{\partial u} + \beta_1\frac{\partial\eta}{\partial u} + \gamma_1\frac{\partial\zeta}{\partial u} = 0, \\ &\alpha_2\frac{\partial\xi}{\partial u} + \beta_2\frac{\partial\eta}{\partial u} + \gamma_2\frac{\partial\zeta}{\partial u} = 0. \end{aligned}\right.$$

We will similarly find:

$$\frac{\partial\xi}{\partial u}\frac{1}{(1+\varepsilon_2)\sqrt{G}}, \qquad \frac{\partial\eta}{\partial u}\frac{1}{(1+\varepsilon_2)\sqrt{G}}, \qquad \frac{\partial\zeta}{\partial u}\frac{1}{(1+\varepsilon_2)\sqrt{G}}$$

for the direction cosines of the elements ds_2 after the deformation when we set its dilatation equal to ε_2 . However, the element ds_2 will no longer define the η-axis after deformation, but will deviate from it by a small angle τ, and as a result, we will have:

$$\alpha_1\frac{\partial\xi}{\partial u}\frac{1}{(1+\varepsilon_2)\sqrt{G}} + \beta_1\frac{\partial\eta}{\partial u}\frac{1}{(1+\varepsilon_2)\sqrt{G}} + \gamma_1\frac{\partial\zeta}{\partial u}\frac{1}{(1+\varepsilon_2)\sqrt{G}} = \cos\tau \quad \text{or} \qquad = 1$$

when we neglect second-order quantities. However, since the xy-plane should also contain the elements ds_1 and ds_2 after deformation, the x-axis should define an angle of $90^\circ - \tau$ with the ds_2 after deformation, so:

$$\alpha_0 \frac{\partial \xi}{\partial u} \frac{1}{(1+\varepsilon_2)\sqrt{G}} + \beta_0 \frac{\partial \eta}{\partial u} \frac{1}{(1+\varepsilon_2)\sqrt{G}} + \gamma_0 \frac{\partial \zeta}{\partial u} \frac{1}{(1+\varepsilon_2)\sqrt{G}} = \cos(90 - \tau) = \sin \tau = 1$$

when we neglect second-order quantities. By contrast, the z-axis will still define a right angle with ds_2 after deformation, and we will have:

$$\alpha_2 \frac{\partial \xi}{\partial u} \frac{1}{(1+\varepsilon_2)\sqrt{G}} + \beta_2 \frac{\partial \eta}{\partial u} \frac{1}{(1+\varepsilon_2)\sqrt{G}} + \gamma_1 \frac{\partial \zeta}{\partial u} \frac{1}{(1+\varepsilon_2)\sqrt{G}} = 0.$$

When these equations are converted, and the quantity $\tau \varepsilon$ is neglected as a second-order quantity, that will give:

$$(3a) \qquad \begin{cases} \alpha_0 \dfrac{\partial \xi}{\partial v} + \beta_0 \dfrac{\partial \eta}{\partial v} + \gamma_0 \dfrac{\partial \zeta}{\partial v} = \tau \sqrt{G}, \\ \alpha_1 \dfrac{\partial \xi}{\partial v} + \beta_1 \dfrac{\partial \eta}{\partial v} + \gamma_1 \dfrac{\partial \zeta}{\partial v} = \sqrt{G}(1+\varepsilon_1), \\ \alpha_2 \dfrac{\partial \xi}{\partial v} + \beta_2 \dfrac{\partial \eta}{\partial v} + \gamma_2 \dfrac{\partial \zeta}{\partial v} = 0. \end{cases}$$

After deformation, the x, y, z axes will define angles of inclination with the X, Y, Z axes whose cosines are:

$$\alpha_0, \beta_0, \gamma_0, \qquad \alpha_1, \beta_1, \gamma_1, \qquad \alpha_2, \beta_2, \gamma_2,$$

resp. Conversely, the X, Y, Z axes define angles of inclinations with the x, y, z axes whose cosines are:

$$\alpha_0, \alpha_1, \alpha_2, \qquad \beta_0, \beta_1, \beta_2, \qquad \gamma_0, \gamma_1, \gamma_2,$$

resp. Hence:

$$\begin{aligned} \alpha_0^2 + \alpha_1^2 + \alpha_2^2 &= 0, & \alpha_0 \beta_0 + \alpha_1 \beta_1 + \alpha_2 \beta_2 &= 0, \\ \beta_0^2 + \beta_1^2 + \beta_2^2 &= 0, & \beta_0 \gamma_0 + \beta_1 \gamma_1 + \beta_2 \gamma_2 &= 0, \\ \gamma_0^2 + \gamma_1^2 + \gamma_2^2 &= 0, & \gamma_0 \alpha_0 + \gamma_1 \alpha_1 + \gamma_2 \alpha_2 &= 0. \end{aligned}$$

If we multiply equations (3a) in succession by:

$$\alpha_0, \alpha_1, \alpha_2, \qquad \beta_0, \beta_1, \beta_2, \qquad \gamma_0, \gamma_1, \gamma_2,$$

resp., and add them three at a time then we will get:

$$(3) \qquad \begin{cases} \dfrac{\partial \xi}{\partial v} = \sqrt{G}(\alpha_1(1+\varepsilon_2)+\alpha_0\tau), \\ \dfrac{\partial \eta}{\partial v} = \sqrt{G}(\beta_1(1+\varepsilon_2)+\beta_0\tau), \\ \dfrac{\partial \zeta}{\partial v} = \sqrt{G}(\gamma_1(1+\varepsilon_2)+\gamma_0\tau). \end{cases}$$

In fact, since the six equations (2) and (3) contain three quantities ε_1, ε_2, τ, in addition to the desired ones, they collectively define three new equations. The quantities ε_1, ε_2, τ can be easily expressed in terms of ξ, η, ζ.

After squaring and adding (2), we will get an equation for ε_1 :

$$(4) \qquad \left(\frac{\partial \xi}{\partial u}\right)^2 + \left(\frac{\partial \eta}{\partial u}\right)^2 + \left(\frac{\partial \zeta}{\partial u}\right)^2 = E\,(1+\varepsilon_1)^2.$$

Upon squaring and adding (3) and neglecting second-order terms, it will follow that:

$$(5) \qquad \left(\frac{\partial \xi}{\partial v}\right)^2 + \left(\frac{\partial \eta}{\partial v}\right)^2 + \left(\frac{\partial \zeta}{\partial v}\right)^2 = G\,(1+\varepsilon_2)^2.$$

When we multiply each two equations of (2) and (3) that have the same order and add them, while neglecting higher-order quantities:

$$(6) \qquad \frac{\partial \xi}{\partial u}\frac{\partial \xi}{\partial v} + \frac{\partial \eta}{\partial u}\frac{\partial \eta}{\partial v} + \frac{\partial \zeta}{\partial u}\frac{\partial \zeta}{\partial v} = \sqrt{EG}\,\tau,$$

or if, following **Gauss**, I let E', F', G' denote the same thing for the surface ξ, η, ζ that E, G, F means for the surface X, Y, Z then it will follow that:

$$\sqrt{\frac{E'}{E}} = 1+\varepsilon_1, \qquad \sqrt{\frac{G'}{G}} = 1+\varepsilon_2, \qquad \frac{F'}{\sqrt{EG}} = \tau.$$

One can already infer a remarkable conclusion from these equations. If we neglect the small quantities ε_1, ε_2, τ in comparison to finite quantities then it will follow that:

$$\sqrt{\frac{E'}{E}} = 1, \qquad \sqrt{\frac{G'}{G}} = 1, \qquad \frac{F'}{\sqrt{EG}} = 0,$$

$$E = E', \qquad G = G', \qquad F = F' = 0;$$

i.e., the shells remain mutually developable under all changes of form. However, since have neglected small quantities in comparison to the finite ones, we can say only that the shells will remain mutually-developable surface under very small deviations.

We shall address the problem of obtaining ξ', η', ζ', as well as $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, as functions of u and v.

That initially yields three equations immediately.

Since ξ', η', ζ' is the point:

$$x + \mathfrak{u}_0 + \mathfrak{u}, \qquad y + \mathfrak{v}_0 + \mathfrak{v}, \qquad z + \mathfrak{w}_0 + \mathfrak{w}$$

in the element P whose coordinate origin is at ξ, η, ζ, and whose axes have the direction cosines:

$$\alpha_0, \beta_0, \gamma_0, \qquad \alpha_1, \beta_1, \gamma_1, \qquad \alpha_2, \beta_2, \gamma_2 ,$$

one will have:

$$(7) \qquad \left\{ \begin{aligned} \xi' &= \xi + \alpha_0(x + \mathfrak{u}_0 + \mathfrak{u}) + \alpha_1(y + \mathfrak{v}_0 + \mathfrak{v}) + \alpha_2(z + \mathfrak{w}_0 + \mathfrak{w}), \\ \eta' &= \eta + \beta_0(x + \mathfrak{u}_0 + \mathfrak{u}) + \beta_1(y + \mathfrak{v}_0 + \mathfrak{v}) + \beta_2(z + \mathfrak{w}_0 + \mathfrak{w}), \\ \zeta' &= \zeta + \gamma_0(x + \mathfrak{u}_0 + \mathfrak{u}) + \gamma_1(y + \mathfrak{v}_0 + \mathfrak{v}) + \gamma_2(z + \mathfrak{w}_0 + \mathfrak{w}). \end{aligned} \right.$$

In order to obtain three more equations, I shall appeal to the following considerations: A system of values u, v corresponds to a certain system of values s_1, s_2 . Hence, ξ, η, ζ, and the cosines of the angle of inclination of a point u, v can then be regarded as functions of and s_1 and s_2 . ξ', η', ζ' then correspondingly represent functions of:

$$s_1 + x + \mathfrak{u}_0 , \qquad s_1 + x + \mathfrak{u}_0 , \qquad z.$$

Therefore:

$$\frac{\partial \xi}{\partial s_1} = \frac{\partial \xi}{\partial (s_1 + x + \mathfrak{u}_0)} \cdot \frac{\partial (s_1 + x + \mathfrak{u}_0)}{\partial s_1} = \frac{\partial \xi}{\partial (s_1 + x + \mathfrak{u}_0)} \cdot \left(1 + \frac{\partial \mathfrak{u}_0}{\partial s_1}\right),$$

$$\frac{\partial \xi'}{\partial x} = \frac{\partial \xi'}{\partial (s_1 + x + \mathfrak{u}_0)} \cdot \frac{\partial (s_1 + x + \mathfrak{u}_0)}{\partial x} = \frac{\partial \xi'}{\partial (s_1 + x + \mathfrak{u}_0)} \cdot \left(1 + \frac{\partial \mathfrak{u}_0}{\partial x}\right),$$

since x is independent of s_1.

However, $\partial \mathfrak{u}_0 / \partial x$ has the same order as the dilatation, so it is infinitely small and can be neglected in comparison to 1, while $\partial \mathfrak{u}_0 / \partial s_1$ is infinite of an even higher order, which shall be shown immediately below. We can then set $\frac{\partial \xi'}{\partial s_1} = \frac{\partial \xi'}{\partial x}$, up to small quantities, or also:

$$\frac{1}{\sqrt{E}} \frac{\partial \xi'}{\partial u} = \frac{\partial \xi'}{\partial x},$$

and it will follow similarly that:

$$\frac{1}{\sqrt{E}}\frac{\partial \eta'}{\partial u} = \frac{\partial \eta'}{\partial x}, \qquad \frac{1}{\sqrt{E}}\frac{\partial \zeta'}{\partial u} = \frac{\partial \zeta'}{\partial x},$$

so it will likewise follow that:

$$\frac{1}{\sqrt{G}}\frac{\partial \xi'}{\partial v} = \frac{\partial \xi'}{\partial y}, \qquad \frac{1}{\sqrt{G}}\frac{\partial \eta'}{\partial v} = \frac{\partial \eta'}{\partial y}, \qquad \frac{1}{\sqrt{G}}\frac{\partial \zeta'}{\partial v} = \frac{\partial \zeta'}{\partial y}.$$

If we first differentiate the system of equations (7) with respect to x and then with respect to u and divide by $\sqrt{E}$, and likewise differentiate with respect to y and then v and divide by $\sqrt{G}$, then we will get twelve equations. The left-hand sides of any two of them are equal, so when we set the right-hand sides equal to each other, we will eliminate the six derivatives of ξ', η', ζ' and get six equations for the partial differential quotients of $\mathfrak{u}_0 + \mathfrak{u}$, $\mathfrak{v}_0 + \mathfrak{v}$, $\mathfrak{w}_0 + \mathfrak{w}$ with respect to x and y. The first of those equations reads:

$$\alpha_0\left(1+\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}\right)+\alpha_1\left(1+\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial x}\right)+\alpha_2\left(1+\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial x}\right)$$
$$= \frac{1}{\sqrt{E}}\left(\frac{\partial \xi}{\partial u}+\frac{\partial \alpha_0}{\partial u}(x+\mathfrak{u}_0+\mathfrak{u})+\alpha_0\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial u}\right.$$
$$+\frac{\partial \alpha_1}{\partial u}(y+\mathfrak{v}_0+\mathfrak{v})+\alpha_1\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial u}$$
$$\left.+\frac{\partial \alpha_2}{\partial u}(z+\mathfrak{w}_0+\mathfrak{w})+\alpha_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial u}\right).$$

We can now neglect $\mathfrak{u}_0 + \mathfrak{u}$, $\mathfrak{v}_0 + \mathfrak{v}$, $\mathfrak{w}_0 + \mathfrak{w}$ in comparison to x, y, and z. However, $\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial u}$, etc., can also be neglected in comparison to $\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}$, so one will have:

$$\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial u} = \sqrt{E}\,\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial s_1}.$$

In $\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial s_1}$, the numerator is the difference between the displacements *inside* of the element in the direction of the x-axis at two *corresponding* points of *neighboring* elements. The denominator ∂s_1 is the distance to the center of that element, when measured along the x-axis. In $\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}$, the numerator likewise means a difference between displacements inside of an element and in general at two *neighboring points of the same* element here, but it will still have the same order as above. By contrast, the denominator is an infinitely-small part of ds_1. Hence, $\dfrac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial s_1}$ is very small and can be neglected in comparison to $\dfrac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}$; the same thing will be true for the corresponding expressions, so it will follow that:

$$\alpha_0\left(1+\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}\right)+\alpha_1\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial x}+\alpha_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial x}=\frac{1}{\sqrt{E}}\left(\frac{\partial\xi}{\partial u}+\frac{\partial\alpha_0}{\partial u}x+\frac{\partial\alpha_1}{\partial u}y+\frac{\partial\alpha_2}{\partial u}z\right),$$

$$\beta_0\left(1+\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}\right)+\beta_1\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial x}+\beta_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial x}=\frac{1}{\sqrt{E}}\left(\frac{\partial\eta}{\partial u}+\frac{\partial\beta_0}{\partial u}x+\frac{\partial\beta_1}{\partial u}y+\frac{\partial\beta_2}{\partial u}z\right),$$

$$\gamma_0\left(1+\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}\right)+\gamma_1\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial x}+\gamma_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial x}=\frac{1}{\sqrt{E}}\left(\frac{\partial\zeta}{\partial u}+\frac{\partial\gamma_0}{\partial u}x+\frac{\partial\gamma_1}{\partial u}y+\frac{\partial\gamma_2}{\partial u}z\right).$$

Correspondingly, differentiating with respect to y and v will yield:

$$\alpha_0\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial y}+\alpha_1\left(1+\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial y}\right)+\alpha_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial y}=\frac{1}{\sqrt{G}}\left(\frac{\partial\xi}{\partial v}+\frac{\partial\alpha_0}{\partial v}x+\frac{\partial\alpha_1}{\partial v}y+\frac{\partial\alpha_2}{\partial v}z\right),$$

$$\beta_0\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial y}+\beta_1\left(1+\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial y}\right)+\beta_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial y}=\frac{1}{\sqrt{G}}\left(\frac{\partial\eta}{\partial v}+\frac{\partial\beta_0}{\partial v}x+\frac{\partial\beta_1}{\partial v}y+\frac{\partial\beta_2}{\partial v}z\right),$$

$$\gamma_0\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial y}+\gamma_1\left(1+\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial y}\right)+\gamma_2\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial y}=\frac{1}{\sqrt{G}}\left(\frac{\partial\zeta}{\partial v}+\frac{\partial\gamma_0}{\partial v}x+\frac{\partial\gamma_1}{\partial v}y+\frac{\partial\gamma_2}{\partial v}z\right).$$

When we multiply the first and last three equations in succession by:

$$\alpha_0,\ \beta_0,\ \gamma_0,\qquad \alpha_1,\ \beta_1,\ \gamma_1,\qquad \alpha_2,\ \beta_2,\ \gamma_2,$$

resp., and add them three at a time, while recalling (2*a*) and (3*a*), we will get:

$$(7a)\quad\begin{cases}\dfrac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial x}=\varepsilon_1+\dfrac{1}{\sqrt{E}}\left(\alpha_0\dfrac{\partial\alpha_1}{\partial u}+\beta_0\dfrac{\partial\beta_1}{\partial u}+\gamma_0\dfrac{\partial\gamma_1}{\partial u}\right)y+\dfrac{1}{\sqrt{E}}\left(\alpha_0\dfrac{\partial\alpha_2}{\partial u}+\beta_0\dfrac{\partial\beta_2}{\partial u}+\gamma_0\dfrac{\partial\gamma_2}{\partial u}\right)z,\\[2ex] \dfrac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial x}=\dfrac{1}{\sqrt{E}}\left(\alpha_1\dfrac{\partial\alpha_0}{\partial u}+\beta_1\dfrac{\partial\beta_0}{\partial u}+\gamma_1\dfrac{\partial\gamma_0}{\partial u}\right)x+\dfrac{1}{\sqrt{E}}\left(\alpha_1\dfrac{\partial\alpha_2}{\partial u}+\beta_1\dfrac{\partial\beta_2}{\partial u}+\gamma_1\dfrac{\partial\gamma_2}{\partial u}\right)z,\\[2ex] \dfrac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial x}=\dfrac{1}{\sqrt{E}}\left(\alpha_2\dfrac{\partial\alpha_0}{\partial u}+\beta_2\dfrac{\partial\beta_0}{\partial u}+\gamma_2\dfrac{\partial\gamma_0}{\partial u}\right)x+\dfrac{1}{\sqrt{E}}\left(\alpha_2\dfrac{\partial\alpha_1}{\partial u}+\beta_2\dfrac{\partial\beta_1}{\partial u}+\gamma_2\dfrac{\partial\gamma_1}{\partial u}\right)z;\end{cases}$$

furthermore:

$$(7b)\quad \begin{cases} \dfrac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial y} = \tau + \dfrac{1}{\sqrt{G}}\left(\alpha_0\dfrac{\partial\alpha_1}{\partial u}+\beta_0\dfrac{\partial\beta_1}{\partial u}+\gamma_0\dfrac{\partial\gamma_1}{\partial u}\right)y + \dfrac{1}{\sqrt{G}}\left(\alpha_0\dfrac{\partial\alpha_2}{\partial v}+\beta_0\dfrac{\partial\beta_2}{\partial v}+\gamma_0\dfrac{\partial\gamma_2}{\partial v}\right)z, \\ \dfrac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial y} = \varepsilon_2 + \dfrac{1}{\sqrt{G}}\left(\alpha_1\dfrac{\partial\alpha_0}{\partial v}+\beta_1\dfrac{\partial\beta_0}{\partial v}+\gamma_1\dfrac{\partial\gamma_0}{\partial v}\right)x + \dfrac{1}{\sqrt{G}}\left(\alpha_1\dfrac{\partial\alpha_2}{\partial v}+\beta_1\dfrac{\partial\beta_2}{\partial v}+\gamma_1\dfrac{\partial\gamma_2}{\partial v}\right)z, \\ \dfrac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial y} = \quad \dfrac{1}{\sqrt{G}}\left(\alpha_2\dfrac{\partial\alpha_0}{\partial v}+\beta_2\dfrac{\partial\beta_0}{\partial v}+\gamma_2\dfrac{\partial\gamma_0}{\partial v}\right)x + \dfrac{1}{\sqrt{G}}\left(\alpha_2\dfrac{\partial\alpha_1}{\partial v}+\beta_2\dfrac{\partial\beta_1}{\partial v}+\gamma_2\dfrac{\partial\gamma_1}{\partial v}\right)z. \end{cases}$$

Since x, y, z are themselves of order one, it will suffice to consider the coefficients x, y, z up to first-order quantities, since the error that will arise in that way will be of order two.

Now, from equation (2), one has, up to small quantities:

$$\alpha_0 = \frac{1}{\sqrt{E}}\frac{\partial\xi}{\partial u}, \qquad \beta_0 = \frac{1}{\sqrt{E}}\frac{\partial\eta}{\partial u}, \qquad \gamma_0 = \frac{1}{\sqrt{E}}\frac{\partial\zeta}{\partial u},$$

$$\alpha_1 = \frac{1}{\sqrt{G}}\frac{\partial\xi}{\partial v}, \qquad \beta_1 = \frac{1}{\sqrt{G}}\frac{\partial\eta}{\partial v}, \qquad \gamma_1 = \frac{1}{\sqrt{G}}\frac{\partial\zeta}{\partial v}.$$

In analogy with **Gauss**'s notation, except that I shall choose German symbols in place of the Latin ones in order to indicate that the surface is different from the one in the rest configuration, I shall now set:

$$\frac{\partial\xi}{\partial u} = \mathfrak{a}, \qquad \frac{\partial\eta}{\partial u} = \mathfrak{b}, \qquad \frac{\partial\zeta}{\partial u} = \mathfrak{c},$$

$$\frac{\partial\xi}{\partial v} = \mathfrak{a}', \qquad \frac{\partial\eta}{\partial v} = \mathfrak{b}', \qquad \frac{\partial\zeta}{\partial v} = \mathfrak{c}'.$$

It will then follow that:

$$\alpha_0\frac{\partial\alpha_1}{\partial u}+\beta_0\frac{\partial\beta_1}{\partial u}+\gamma_0\frac{\partial\gamma_1}{\partial u} = \frac{\mathfrak{a}}{\sqrt{E}}\frac{\partial\left(\dfrac{\mathfrak{a}'}{\sqrt{G}}\right)}{\partial u}+\frac{\mathfrak{b}}{\sqrt{E}}\frac{\partial\left(\dfrac{\mathfrak{b}'}{\sqrt{G}}\right)}{\partial u}+\frac{\mathfrak{c}}{\sqrt{E}}\frac{\partial\left(\dfrac{\mathfrak{c}'}{\sqrt{G}}\right)}{\partial u}$$

$$= \frac{1}{\sqrt{EG}}\left(\mathfrak{a}\frac{\partial\mathfrak{a}'}{\partial u}+\mathfrak{b}\frac{\partial\mathfrak{b}'}{\partial u}+\mathfrak{c}\frac{\partial\mathfrak{c}'}{\partial u}\right)+\frac{1}{\sqrt{E}}(\mathfrak{a}\mathfrak{a}'+\mathfrak{b}\mathfrak{b}'+\mathfrak{c}\mathfrak{c}')\frac{\partial\left(\dfrac{1}{\sqrt{G}}\right)}{\partial u}.$$

However, one has:

$$\mathfrak{a}\mathfrak{a}' + \mathfrak{b}\mathfrak{b}' + \mathfrak{c}\mathfrak{c}' = F' = 0,$$

up to small quantities, as was shown above; moreover, one has:

$$\frac{\partial^2 \xi}{\partial u\,\partial v} = \frac{\partial^2 \xi}{\partial v\,\partial u},$$

so

$$\frac{\partial \mathfrak{a}'}{\partial u} = \frac{\partial \mathfrak{a}}{\partial v},$$

hence:

$$\alpha_0 \frac{\partial \alpha_1}{\partial u} + \beta_0 \frac{\partial \beta_1}{\partial u} + \gamma_0 \frac{\partial \gamma_1}{\partial u} = \frac{1}{\sqrt{EG}}\left(\mathfrak{a}\frac{\partial \mathfrak{a}}{\partial u} + \mathfrak{b}\frac{\partial \mathfrak{b}}{\partial u} + \mathfrak{c}\frac{\partial \mathfrak{c}}{\partial u}\right) = \frac{1}{2\sqrt{EG}}\frac{\partial E'}{\partial v};$$

however, E' was equal to E (up to small quantities), so:

(8) $$\alpha_0 \frac{\partial \alpha_1}{\partial u} + \beta_0 \frac{\partial \beta_1}{\partial u} + \gamma_0 \frac{\partial \gamma_1}{\partial u} = \frac{1}{2\sqrt{EG}}\frac{\partial E}{\partial v}.$$

In a similar way, one finds that:

(9) $$\alpha_1 \frac{\partial \alpha_0}{\partial v} + \beta_1 \frac{\partial \beta_0}{\partial v} + \gamma_1 \frac{\partial \gamma_0}{\partial v} = \frac{1}{2\sqrt{EG}}\frac{\partial G}{\partial u}.$$

Now, one has:

$$\alpha_2\,\mathfrak{a} + \beta_2\,\mathfrak{b} + \gamma_2\,\mathfrak{g} = 0,$$
$$\alpha_2\,\mathfrak{a}' + \beta_2\,\mathfrak{b}' + \gamma_2\,\mathfrak{g}' = 0,$$

so:

$$\alpha_2 : \beta_2 : \gamma_2 = (\mathfrak{b}\mathfrak{c}' - \mathfrak{c}\mathfrak{b}') : (\mathfrak{c}\mathfrak{a}' - \mathfrak{a}\mathfrak{c}') : (\mathfrak{a}\mathfrak{b}' - \mathfrak{b}\mathfrak{a}'),$$

and when I introduce a notation here that also corresponds to **Gauss**'s notation, one will have:

$$\alpha_2 : \beta_2 : \gamma_2 = \mathfrak{A} : \mathfrak{B} : \mathfrak{C}.$$

Now, $\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1$, so it follows that:

$$\alpha_2 = \frac{\mathfrak{A}}{\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2}}, \quad \beta_2 = \frac{\mathfrak{B}}{\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2}}, \quad \gamma_2 = \frac{\mathfrak{C}}{\sqrt{\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2}}.$$

Now, $\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2 = E'G' - F'^2$, or (up to small quantities):

$$\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2 = EG,$$

so:

$$\alpha_2 = \frac{\mathfrak{A}}{\sqrt{EG}}, \quad \beta_2 = \frac{\mathfrak{B}}{\sqrt{EG}}, \quad \gamma_2 = \frac{\mathfrak{C}}{\sqrt{EG}},$$

and therefore:

$$\alpha_1 \frac{\partial \alpha_0}{\partial v} + \beta_1 \frac{\partial \beta_0}{\partial v} + \gamma_1 \frac{\partial \gamma_0}{\partial v} = \frac{1}{\sqrt{EG}} \left[\mathfrak{A} \frac{\partial}{\partial u}\left(\frac{\mathfrak{a}}{\sqrt{E}}\right) + \mathfrak{B} \frac{\partial}{\partial u}\left(\frac{\mathfrak{b}}{\sqrt{E}}\right) + \mathfrak{C} \frac{\partial}{\partial u}\left(\frac{\mathfrak{c}}{\sqrt{E}}\right) \right]$$

$$= \frac{1}{E\sqrt{G}} \left(\mathfrak{A} \frac{\partial \mathfrak{a}}{\partial u} + \mathfrak{B} \frac{\partial \mathfrak{b}}{\partial u} + \mathfrak{C} \frac{\partial \mathfrak{c}}{\partial u} \right) + \frac{1}{\sqrt{EG}} (\mathfrak{A}\mathfrak{a} + \mathfrak{B}\mathfrak{b} + \mathfrak{C}\mathfrak{c}) \frac{\partial \left(\frac{1}{\sqrt{E}} \right)}{\partial u} .$$

However, one has:

$$\mathfrak{A}\mathfrak{a} + \mathfrak{B}\mathfrak{b} + \mathfrak{C}\mathfrak{c} = \begin{vmatrix} \mathfrak{a} & \mathfrak{b} & \mathfrak{c} \\ \mathfrak{a} & \mathfrak{b} & \mathfrak{c} \\ \mathfrak{a}' & \mathfrak{b}' & \mathfrak{c}' \end{vmatrix} .$$

The determinant is zero, since two equal rows occur in it. However:

$$\mathfrak{A} \frac{\partial \mathfrak{a}}{\partial u} + \mathfrak{B} \frac{\partial \mathfrak{b}}{\partial u} + \mathfrak{C} \frac{\partial \mathfrak{c}}{\partial u}$$

is what **Gauss** called D, when it is written in Latin symbols. The **Gauss**ian notation shall remain for the surface in the rest configuration, while for the deformed surface Δ, Δ', Δ'' shall be used instead of D, D', D''. If we apply a similar process repeatedly then we will get:

(10)
$$\begin{cases} \alpha_2 \dfrac{\partial \alpha_0}{\partial u} + \beta_2 \dfrac{\partial \beta_0}{\partial u} + \gamma_2 \dfrac{\partial \gamma_0}{\partial u} = \dfrac{1}{E\sqrt{G}} \Delta, \\ \alpha_2 \dfrac{\partial \alpha_0}{\partial v} + \beta_2 \dfrac{\partial \beta_0}{\partial v} + \gamma_2 \dfrac{\partial \gamma_0}{\partial v} = \dfrac{1}{E\sqrt{G}} \Delta', \\ \alpha_2 \dfrac{\partial \alpha_1}{\partial u} + \beta_2 \dfrac{\partial \beta_1}{\partial u} + \gamma_2 \dfrac{\partial \gamma_1}{\partial u} = \dfrac{1}{G\sqrt{E}} \Delta', \\ \alpha_2 \dfrac{\partial \alpha_1}{\partial u} + \beta_2 \dfrac{\partial \beta_1}{\partial u} + \gamma_2 \dfrac{\partial \gamma_1}{\partial u} = \dfrac{1}{G\sqrt{E}} \Delta''. \end{cases}$$

Furthermore, since:

$$\alpha_1\, \alpha_0 + \beta_1\, \beta_0 + \gamma_1\, \gamma_0 = 0, \qquad \ldots,$$

one will have:

$$\alpha_1 \frac{\partial \alpha_0}{\partial u} + \beta_1 \frac{\partial \beta_0}{\partial u} + \gamma_1 \frac{\partial \gamma_0}{\partial u} = -\left(\alpha_0 \frac{\partial \alpha_1}{\partial u} + \beta_0 \frac{\partial \beta_1}{\partial u} + \gamma_0 \frac{\partial \gamma_1}{\partial u} \right) = -\frac{1}{2\sqrt{EG}} \frac{\partial E}{\partial v}, \quad \ldots$$

It will then follow that:

$$\frac{\partial (\mathfrak{u}_0 + \mathfrak{u})}{\partial x} = \frac{y}{2E\sqrt{G}} \cdot \frac{\partial E}{\partial v} - \frac{\Delta}{E\sqrt{EG}} z + \varepsilon_1 ,$$

$$\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial x} = -\frac{x}{2E\sqrt{G}}\cdot\frac{\partial E}{\partial v} - \frac{\Delta'}{EG}z\,,$$

$$\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial x} = \frac{\Delta}{E\sqrt{EG}}x + \frac{\Delta'}{EG}y\,,$$

$$\frac{\partial(\mathfrak{u}_0+\mathfrak{u})}{\partial y} = -\frac{y}{2G\sqrt{E}}\cdot\frac{\partial G}{\partial u} - \frac{\Delta'}{EG}z + \tau\,,$$

$$\frac{\partial(\mathfrak{v}_0+\mathfrak{v})}{\partial y} = -\frac{x}{2G\sqrt{E}}\cdot\frac{\partial G}{\partial u} - \frac{\Delta''}{G\sqrt{EG}}z + \varepsilon_2\,,$$

$$\frac{\partial(\mathfrak{w}_0+\mathfrak{w})}{\partial y} = \frac{\Delta'}{EG}x + \frac{\Delta''}{G\sqrt{EG}}y\,.$$

However, for the case of the equilibrium configuration of the shell, one will have:

$$\mathfrak{u} = 0, \qquad \mathfrak{v} = 0, \qquad \mathfrak{w} = 0,$$

$$\varepsilon_1 = 0, \qquad \varepsilon_2 = 0, \qquad \tau = 0,$$

$$\Delta = D, \qquad \Delta' = D', \qquad \Delta'' = D'',$$

and upon introducing these special values, we will get:

$$\frac{\partial u_0}{\partial x} = \frac{y}{2E\sqrt{G}}\cdot\frac{\partial E}{\partial v} - \frac{Dz}{E\sqrt{EG}}\,, \qquad \ldots,$$

When we substitutes these expressions in the ones above, we will get:

$$\frac{\partial\mathfrak{u}}{\partial x} = \frac{D-\Delta}{E\sqrt{EG}}z + \varepsilon_1\,, \qquad \frac{\partial\mathfrak{v}}{\partial x} = \frac{D'-\Delta'}{EG}z\,, \qquad \frac{\partial\mathfrak{w}}{\partial x} = -\frac{D-\Delta}{E\sqrt{EG}}x - \frac{D'-\Delta'}{EG}y\,,$$

$$\frac{\partial\mathfrak{u}}{\partial y} = \frac{D'-\Delta'}{EG}z + \tau, \qquad \frac{\partial\mathfrak{v}}{\partial y} = \frac{D''-\Delta''}{G\sqrt{EG}}z + \varepsilon_2\,, \quad \frac{\partial\mathfrak{w}}{\partial y} = -\frac{D'-\Delta'}{EG}x - \frac{D''-\Delta''}{G\sqrt{EG}}y\,,$$

moreover.

Although the expressions for $\mathfrak{u}_0 + \mathfrak{u}$, $\mathfrak{v}_0 + \mathfrak{v}$, $\mathfrak{w}_0 + \mathfrak{w}$ above were not integrable, the ones for $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ are integrable. That implies that:

$$\mathfrak{u} = \frac{D-\Delta}{E\sqrt{EG}}zx - \frac{D'-\Delta'}{EG}zy + \varepsilon_1 x + \tau y + \mathfrak{u}_1,$$

$$\mathfrak{v} = \frac{D'-\Delta'}{EG}zx - \frac{D''-\Delta''}{G\sqrt{EG}}zy + \varepsilon_2 y + \mathfrak{v}_1,$$

$$\mathfrak{w} = -\frac{D-\Delta}{E\sqrt{EG}}\frac{x^2}{2} - \frac{D'-\Delta'}{EG}xy - \frac{D''-\Delta''}{G\sqrt{EG}}\frac{y^2}{2} + \mathfrak{w}_1,$$

in which $\mathfrak{u}_1$, $\mathfrak{v}_1$, $\mathfrak{w}_1$ can be functions of z. In order to determine them, we appeal to the following considerations: That part of the external forces that acts upon the outer surface or the interior of an element, while the remaining ones keep the same distribution, is inessential for defining the form of the shell, and therefore for the change of form of the elements, as well. It influence is felt in the forces that act upon a part of the outer surface of the element that is not free – i.e., on its boundary. We must then demand that the displacements that one finds must satisfy the equations of elasticity for the interior and for the free outer surface of the elements when we assume that no external forces act upon the interior and the outer surface, and that will indeed suffice to determine the functions $\mathfrak{u}_1$, $\mathfrak{v}_1$, $\mathfrak{w}_1$. Namely, from the equations of elasticity for the interior, one will have:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0, \qquad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = 0, \qquad \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0,$$

and for the outer surface:

$$X_z = 0, \quad Y_z = 0, \quad Z_z = 0,$$

since the z-axis is the normal to the outer surface.

Now, one has:

$$X_x = 2k\left[\frac{\partial \mathfrak{u}}{\partial x} + \Theta\left(\frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial \mathfrak{v}}{\partial y} + \frac{\partial \mathfrak{w}}{\partial z}\right)\right],$$

$$Y_y = 2k\left[\frac{\partial \mathfrak{v}}{\partial y} + \Theta\left(\frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial \mathfrak{v}}{\partial y} + \frac{\partial \mathfrak{w}}{\partial z}\right)\right],$$

$$Z_z = 2k\left[\frac{\partial \mathfrak{w}}{\partial z} + \Theta\left(\frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial \mathfrak{v}}{\partial y} + \frac{\partial \mathfrak{w}}{\partial z}\right)\right],$$

$$Z_y = Y_z = k\left(\frac{\partial \mathfrak{v}}{\partial z} + \frac{\partial \mathfrak{w}}{\partial y}\right),$$

$$Z_x = X_z = k\left(\frac{\partial \mathfrak{w}}{\partial x} + \frac{\partial \mathfrak{u}}{\partial z}\right),$$

$$X_y = Y_x = k\left(\frac{\partial \mathfrak{u}}{\partial y} + \frac{\partial \mathfrak{v}}{\partial x}\right).$$

Now, I set:

$$\frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial \mathfrak{v}}{\partial y} + \frac{\partial \mathfrak{w}}{\partial z} = J.$$

One will then have:

$$J = \frac{D-\Delta}{E\sqrt{EG}}\,z - \frac{D''-\Delta''}{G\sqrt{EG}}\,z + \varepsilon_1 + \varepsilon_2 + \frac{\partial \mathfrak{w}_1}{\partial z},$$

$$X_x = 2k\left(\frac{D-\Delta}{E\sqrt{EG}}\,z + \varepsilon_1 + \Theta J\right),$$

$$Y_y = 2k\left(\frac{D''-\Delta''}{G\sqrt{EG}}\,z + \varepsilon_2 + \Theta J\right),$$

$$Z_z = 2k\left(\frac{\partial \mathfrak{w}_1}{\partial z} + \Theta J\right),$$

$$Z_y = Y_z = k\frac{\partial \mathfrak{v}_1}{\partial z},$$

$$Z_x = X_z = k\frac{\partial \mathfrak{u}_1}{\partial z},$$

$$X_y = Y_x = k\left(\frac{2(D'-\Delta')}{EG}\,z + \tau\right),$$

so one must have:

$$\frac{\partial X_z}{\partial z} = 0, \qquad \frac{\partial Y_z}{\partial z} = 0, \qquad \frac{\partial Z_z}{\partial z} = 0$$

in the interior, or:

$$\frac{\partial^2 \mathfrak{u}_1}{\partial z^2} = 0, \qquad \frac{\partial^2 \mathfrak{v}_1}{\partial z^2} = 0,$$

$$\frac{\partial^2 \mathfrak{w}_1}{\partial z^2} + \Theta\left(\frac{D-\Delta}{E\sqrt{EG}} + \frac{D''-\Delta''}{G\sqrt{EG}} + \frac{\partial^2 \mathfrak{w}_1}{\partial z^2}\right) = 0,$$

and for the free outer surface:

$$\frac{\partial \mathfrak{u}_1}{\partial z} = 0, \qquad \frac{\partial \mathfrak{v}_1}{\partial z} = 0,$$

$$\frac{\partial \mathfrak{w}_1}{\partial z} \pm \Theta\left(\frac{(D-\Delta)h}{2E\sqrt{EG}} + \frac{(D''-\Delta'')h}{2G\sqrt{EG}} + \varepsilon_1 + \varepsilon_2 + \frac{\partial \mathfrak{w}_1}{\partial z}\right) = 0,$$

if h is the thickness of the shell, and the surface considered is the middle surface of the shell, such that one will have $z = \pm h / 2$ for the free outer surface.

It follows from the equations for the interior that, in general:

$$\frac{\partial \mathfrak{u}_1}{\partial z} = \text{const.}, \quad \frac{\partial \mathfrak{v}_1}{\partial z} = \text{const.},$$

and it follows from the equations for the outer surface that these constants are 0, so, in general:

$$\frac{\partial \mathfrak{u}_1}{\partial z} = 0, \qquad \frac{\partial \mathfrak{v}_1}{\partial z} = 0,$$

and therefore:

$$\mathfrak{u}_1 = \text{const.}, \qquad \mathfrak{v}_1 = \text{const.},$$

and since one must have $\mathfrak{u} = 0$, $\mathfrak{v} = 0$ for $x = y = z = 0$, it follows that:

$$\frac{\partial^2 \mathfrak{w}_1}{\partial z^2} = -\frac{\Theta}{1+\Theta}\left(\frac{D-\Delta}{E\sqrt{EG}} + \frac{D''-\Delta''}{G\sqrt{EG}}\right),$$

so:

$$\frac{\partial \mathfrak{w}_1}{\partial z} = -\frac{\Theta}{1+\Theta}\left(\frac{D-\Delta}{E\sqrt{EG}}\,z + \frac{D''-\Delta''}{G\sqrt{EG}}\,z\right) + \text{const.},$$

but for $z = \pm h / 2$:

$$\frac{\partial \mathfrak{w}_1}{\partial z} = \mp\frac{\Theta}{1+\Theta}\left(\frac{D-\Delta}{2E\sqrt{EG}}\,h + \frac{D''-\Delta''}{G\sqrt{EG}}\,h + \varepsilon_1 + \varepsilon_2\right),$$

so, in general:

$$\frac{\partial \mathfrak{w}_1}{\partial z} = -\frac{\Theta}{1+\Theta}\left(\frac{D-\Delta}{E\sqrt{EG}}\,z + \frac{D''-\Delta''}{G\sqrt{EG}}\,z + \varepsilon_1 + \varepsilon_2\right),$$

$$\mathfrak{w}_1 = -\frac{\Theta}{1+\Theta}\left(\frac{(D-\Delta)}{E\sqrt{EG}}\frac{z^2}{2} + \frac{(D''-\Delta'')}{G\sqrt{EG}}\frac{z^2}{2} + \varepsilon_1\,z + \varepsilon_2\,z\right) + \text{const.},$$

but since $\mathfrak{w} = 0$ for $x = y = z = 0$, one will also have $\mathfrak{w}_1 = 0$, and this constant will also be equal to zero, so we will get:

(11)
$$\left\{\begin{aligned} \mathfrak{u} &= \frac{D-\Delta}{E\sqrt{EG}}zx + \frac{D'-\Delta'}{EG}zy + \varepsilon_1 x + \tau y, \\ \mathfrak{v} &= \frac{D'-\Delta'}{EG}zx + \frac{D''-\Delta''}{G\sqrt{EG}}zy + \varepsilon_2 y, \\ \mathfrak{w} &= -\left(\frac{D-\Delta}{E\sqrt{EG}}\frac{x^2}{2} + \frac{D'-\Delta'}{EG}xy + \frac{D''-\Delta''}{G\sqrt{EG}}\frac{y^2}{2}\right) \\ &\quad -\frac{\Theta}{1+\Theta}\left(\frac{D-\Delta}{E\sqrt{EG}}\frac{z^2}{2} + \frac{D''-\Delta''}{G\sqrt{EG}}\frac{z^2}{2} + \varepsilon_2 z + \varepsilon_2 z\right). \end{aligned}\right.$$

Now, these displacements have the form that makes them satisfy the equations of elasticity for the interior and the free outer surface when we can neglect the external forces. As far as the outer surface that is not free is concerned – so the boundary of the elements – stresses will appear as a result of those displacements. However, they are always equal to the external forces that act in that case, so here they will be the forces that are exerted upon the rest of the shell, since action and reaction are equal. Therefore, the equations of elasticity are also satisfied on the boundary, and since no common displacements or rotations will exist for:

$$\begin{array}{lll} x = 0, & y = 0, & z = 0, \\ \mathfrak{u} = 0, & \mathfrak{v} = 0, & \mathfrak{w} = 0, \\ \dfrac{\partial \mathfrak{v}}{\partial x} = 0, & \dfrac{\partial \mathfrak{w}}{\partial x} = 0, & \dfrac{\partial \mathfrak{w}}{\partial y} = 0, \end{array}$$

as one sees, $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ will, in fact, be displacements that come about merely as a result of the elastic forces. The stresses that appear as a result of these displacements when one sets:

$$J = \frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial \mathfrak{v}}{\partial y} + \frac{\partial \mathfrak{w}}{\partial z} = \frac{1}{1+\Theta}\left(\frac{D-\Delta}{E\sqrt{EG}}z + \frac{D''-\Delta''}{G\sqrt{EG}}z + \varepsilon_1 + \varepsilon_2\right)$$

will be:

(12)
$$\left\{\begin{aligned} X_x &= 2k\left(\frac{D-\Delta}{E\sqrt{EG}}z + +\varepsilon_1 + \Theta J\right), \\ Y_y &= 2k\left(\frac{D''-\Delta''}{G\sqrt{EG}}z + +\varepsilon_2 + \Theta J\right), \end{aligned}\right.$$

$$\begin{aligned} &Z_z = 0, \\ &Z_y = Y_z = 0, \\ &Z_x = Y_z = 0, \\ &X_y = Y_x = k\left(\frac{2(D'-\Delta')}{EG}z + \tau\right). \end{aligned}$$

The work that must be done in order to deform an element $dx\ dy\ dz$ while the shell goes from its initial configuration to its second one will be represented by:

$$F = k\left[\left(\frac{\partial \mathfrak{u}}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{v}}{\partial y}\right)^2 + \left(\frac{\partial \mathfrak{w}}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial \mathfrak{v}}{\partial z} + \frac{\partial \mathfrak{w}}{\partial y}\right)^2 \right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial \mathfrak{u}}{\partial z} + \frac{\partial \mathfrak{w}}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \mathfrak{u}}{\partial y} + \frac{\partial \mathfrak{v}}{\partial x}\right)^2 + \Theta\left(\frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial \mathfrak{v}}{\partial y} + \frac{\partial \mathfrak{w}}{\partial z}\right)^2\right],$$

when this is multiplied by $dx\ dy\ dz$, so the total work that is necessary to deform the entire shell will be represented by:

$$\iiint F\, ds_1\, ds_2\, dz = \iiint F\sqrt{EG}\, du\, dv\, dz\,.$$

The fact that one can represent the element of each layer, and not just the middle layer, by $ds_1\ ds_2\ dz$ follows from the fact that the shell is assumed to be infinitely thin, so the element of the layer of the middle surface will be represented by $ds_1\ ds_2\ z$ if the thickness is z.

II.

We can further represent the general equations of motion with the help of **Hamilton**'s principle when we know the potential Ω of the external forces that act on the interior and the boundary, as well as the external conditions that the shell should be subjected to under its deformations (we have already developed the internal conditions that originate in the assumption of continuity):

$$0 = \delta\iiint\int(\tfrac{1}{2}\rho v^2 - \Omega - F + \lambda\varphi + \mu\psi + \cdots)\sqrt{EG}\, du\, dv\, dz\, dt\,,$$

in which v is the velocity of a point at time t, ρ is the density at that point, and $\varphi = 0$, $\psi = 0$, etc., are the external, as well as internal, conditions that the surface is subjected to, while λ and μ might mean undetermined coefficients.

However, since we have not assumed that the external forces have a potential, we shall introduce the virtual moment of the external forces that act in the interior and on the boundary in place of $-\delta\Omega$, but I would first like to convert the part that originates in the internal work:

$$\delta F = \frac{\partial F}{\partial \frac{\partial \mathfrak{u}}{\partial x}}\delta\frac{\partial \mathfrak{u}}{\partial x} + \frac{\partial F}{\partial \frac{\partial \mathfrak{u}}{\partial y}}\delta\frac{\partial \mathfrak{u}}{\partial y} + \frac{\partial F}{\partial \frac{\partial \mathfrak{u}}{\partial z}}\delta\frac{\partial \mathfrak{u}}{\partial z}$$

$$+\frac{\partial F}{\partial \frac{\partial \mathfrak{v}}{\partial x}}\delta\frac{\partial \mathfrak{v}}{\partial x}+\frac{\partial F}{\partial \frac{\partial \mathfrak{v}}{\partial y}}\delta\frac{\partial \mathfrak{v}}{\partial y}+\frac{\partial F}{\partial \frac{\partial \mathfrak{v}}{\partial z}}\delta\frac{\partial \mathfrak{v}}{\partial z}$$

$$+\frac{\partial F}{\partial \frac{\partial \mathfrak{w}}{\partial x}}\delta\frac{\partial \mathfrak{w}}{\partial x}+\frac{\partial F}{\partial \frac{\partial \mathfrak{w}}{\partial y}}\delta\frac{\partial \mathfrak{w}}{\partial y}+\frac{\partial F}{\partial \frac{\partial \mathfrak{w}}{\partial z}}\delta\frac{\partial \mathfrak{w}}{\partial z}.$$

Now:

$$\frac{\partial F}{\partial \frac{\partial \mathfrak{u}}{\partial x}}=X_x, \qquad \frac{\partial F}{\partial \frac{\partial \mathfrak{u}}{\partial y}}=X_y, \qquad \frac{\partial F}{\partial \frac{\partial \mathfrak{u}}{\partial z}}=X_z,$$

$$\frac{\partial F}{\partial \frac{\partial \mathfrak{v}}{\partial x}}=Y_x, \qquad \frac{\partial F}{\partial \frac{\partial \mathfrak{v}}{\partial y}}=Y_y, \qquad \frac{\partial F}{\partial \frac{\partial \mathfrak{v}}{\partial z}}=Y_z,$$

$$\frac{\partial F}{\partial \frac{\partial \mathfrak{w}}{\partial x}}=Z_x, \qquad \frac{\partial F}{\partial \frac{\partial \mathfrak{w}}{\partial y}}=Z_y, \qquad \frac{\partial F}{\partial \frac{\partial \mathfrak{w}}{\partial z}}=Z_z.$$

Now, from (12):

$$X_z = 0, \qquad Y_z = 0, \qquad Z_x = 0, \qquad Z_y = 0, \qquad Z_z = 0,$$

so what will remain when we set $X_y = Y_x = T_z$ will be:

$$\delta F = X_x\,\delta\frac{\partial \mathfrak{u}}{\partial x}+T_z\,\delta\left(\frac{\partial \mathfrak{u}}{\partial y}+\frac{\partial \mathfrak{v}}{\partial x}\right)+Y_y\,\delta\frac{\partial \mathfrak{v}}{\partial y},$$

so from (11):

$$\delta\iiint\int F\sqrt{EG}\,du\,dv\,dz\,dt=\iiint\int\left\{\begin{array}{l} -X_x\dfrac{z\,\delta\Delta}{E}+X_x\sqrt{EG}\,\delta\varepsilon_1 \\ -2T_z\dfrac{z\,\delta\Delta'}{\sqrt{EG}}+T_x\sqrt{EG}\,\delta\tau \\ -Y_y\dfrac{z\,\delta\Delta''}{G}+Y_y\sqrt{EG}\,\delta\varepsilon_2 \end{array}\right\}du\,dv\,dz\,dt.$$

The integration over z is easy to perform with the use of equations (12). I set:

$$(13)\qquad \left\{\begin{array}{ll} \int X_x\,dz = X_1, & \int X_x\,z\,dz = X_2, \\ \int Y_y\,dz = Y_1, & \int Y_y\,z\,dz = Y_2, \\ \int T_z\,dz = T_1, & \int T_z\,z\,dz = T_2, \end{array}\right.$$

and when we carry out the variations further, and thus introduce, from equation (10):

$$-\frac{\Delta}{E} = \sqrt{G}\left(\alpha_0 \frac{\partial \alpha_2}{\partial u} + \beta_0 \frac{\partial \beta_2}{\partial u} + \gamma_0 \frac{\partial \gamma_2}{\partial u}\right),$$

$$\frac{-2\Delta'}{\sqrt{EG}} = \sqrt{G}\left(\alpha_1 \frac{\partial \alpha_2}{\partial u} + \beta_1 \frac{\partial \beta_2}{\partial u} + \gamma_1 \frac{\partial \gamma_2}{\partial u}\right) + \sqrt{E}\left(\alpha_0 \frac{\partial \alpha_2}{\partial v} + \beta_0 \frac{\partial \beta_2}{\partial v} + \gamma_0 \frac{\partial \gamma_2}{\partial v}\right),$$

$$\frac{-\Delta''}{G} = \sqrt{E}\left(\alpha_1 \frac{\partial \alpha_2}{\partial v} + \beta_1 \frac{\partial \beta_2}{\partial v} + \gamma_1 \frac{\partial \gamma_2}{\partial v}\right).$$

By contrast, if ε_1, ε_2, and τ are expressed as in equations (2*a*) and (3*a*) then we will get:

$$\iiint \left\{ \begin{aligned} & X_2\sqrt{G}\left(\frac{\partial \alpha_2}{\partial u}\delta\alpha_0 + \frac{\partial \beta_2}{\partial u}\delta\beta_0 + \frac{\partial \gamma_2}{\partial u}\delta\gamma_0 + \alpha_0\frac{\partial \delta\alpha_2}{\partial u} + \beta_0\frac{\partial \delta\beta_2}{\partial u} + \gamma_0\frac{\partial \delta\gamma_2}{\partial u}\right) \\ & + T_2\sqrt{G}\left(\frac{\partial \alpha_2}{\partial u}\delta\alpha_1 + \frac{\partial \beta_2}{\partial u}\delta\beta_1 + \frac{\partial \gamma_2}{\partial u}\delta\gamma_1 + \alpha_1\frac{\partial \delta\alpha_2}{\partial u} + \beta_1\frac{\partial \delta\beta_2}{\partial u} + \gamma_1\frac{\partial \delta\gamma_2}{\partial u}\right) \\ & + T_2\sqrt{E}\left(\frac{\partial \alpha_2}{\partial v}\delta\alpha_0 + \frac{\partial \beta_2}{\partial v}\delta\beta_0 + \frac{\partial \gamma_2}{\partial v}\delta\gamma_0 + \alpha_0\frac{\partial \delta\alpha_2}{\partial v} + \beta_0\frac{\partial \delta\beta_2}{\partial v} + \gamma_0\frac{\partial \delta\gamma_2}{\partial v}\right) \\ & + Y_2\sqrt{E}\left(\frac{\partial \alpha_2}{\partial v}\delta\alpha_1 + \frac{\partial \beta_2}{\partial v}\delta\beta_1 + \frac{\partial \gamma_2}{\partial v}\delta\gamma_1 + \alpha_1\frac{\partial \delta\alpha_2}{\partial v} + \beta_1\frac{\partial \delta\beta_2}{\partial v} + \gamma_1\frac{\partial \delta\gamma_2}{\partial v}\right) \\ & + X_1\sqrt{G}\left(\frac{\partial \xi}{\partial u}\delta\alpha_0 + \frac{\partial \eta}{\partial u}\delta\beta_0 + \frac{\partial \zeta}{\partial u}\delta\gamma_0 + \alpha_0\frac{\partial \delta\xi}{\partial u} + \beta_0\frac{\partial \delta\eta}{\partial u} + \gamma_0\frac{\partial \delta\zeta}{\partial u}\right) \\ & + T_1\sqrt{E}\left(\frac{\partial \xi}{\partial v}\delta\alpha_0 + \frac{\partial \eta}{\partial v}\delta\beta_0 + \frac{\partial \zeta}{\partial v}\delta\gamma_0 + \alpha_0\frac{\partial \delta\xi}{\partial v} + \beta_0\frac{\partial \delta\eta}{\partial v} + \gamma_0\frac{\partial \delta\zeta}{\partial v}\right) \\ & + Y_1\sqrt{E}\left(\frac{\partial \xi}{\partial v}\delta\alpha_1 + \frac{\partial \eta}{\partial v}\delta\beta_1 + \frac{\partial \zeta}{\partial v}\delta\gamma_1 + \alpha_1\frac{\partial \delta\xi}{\partial v} + \beta_1\frac{\partial \delta\eta}{\partial v} + \gamma_1\frac{\partial \delta\zeta}{\partial v}\right) \end{aligned} \right\} du\,dv\,dt.$$

We must next convert the last three summands in each row, which contain the variations of the differentials:

$$\iint X_2\sqrt{G}\frac{\partial \delta\alpha_2}{\partial u}du\,dv = \iint \frac{\partial(X_2\sqrt{G}\,\alpha_0\,\delta\alpha_2)}{\partial u}du\,dv - \iint \frac{\partial(X_2\sqrt{G}\,\alpha_0)}{\partial u}\delta\alpha_2\,du\,dv.$$

In order to be able to further convert the integral:

$$\iint \frac{\partial(X_2\sqrt{G}\,\alpha_0\,\delta\alpha_2)}{\partial u}du\,dv,$$

I must extend a theorem here that is well-known for the plane so that it will be true for arbitrary surfaces.

u and *v* shall now mean the rectangular coordinates of an arbitrary surface, and *f* shall be a function of *u* and *v*. We would now like to examine what:

$$\int_{u_0}^{u_1}\int_{v_0}^{v_1}\frac{\partial f}{\partial v}\,du\,dv$$

will mean when it is taken over a closed surface patch and *f* is single-valued, continuous, and finite on the entire surface. If I denote that integral by *J* then one will first have:

$$J = \int_{u_0}^{u_1}(f_1 - f_0)\,du\,,$$

in which I have used f_0 and f_1 to denote the values of *f* when *v* is replaced with v_0 (v_1, resp.), or when I multiply by the same quantities and divide:

$$J = \int_{u_0}^{u_1}\left(\frac{f_1}{\sqrt{E_1}}\sqrt{E_1}\,du - \frac{f_0}{\sqrt{E_0}}\sqrt{E_0}\,du\right).$$

Here, E_0 and E_1 once more mean the values of *E* when *v* is replaced with v_0 (v_1, resp.).

However, our coordinate system was a rectangular one, so:

$$ds^2 = E\,du^2 + G\,dv^2.$$

The arc length of the curve for v = const., which we have denoted by s_1, is then:

$$ds_1^2 = E\,du^2,$$

$$ds_1 = \sqrt{E}\,du.$$

The sign of the root is still arbitrary, because I have still not established the sense of the curve that will be reckoned as positive. Now, s_1 shall be reckoned to be positive on the side for which *u* increases. When *du* is positive, ds_1 shall be positive, so $\sqrt{E}$ must take the + sign, and therefore one must have:

$$J = \int_{u_0}^{u_1}\left[\left(\frac{f_1}{\sqrt{E_1}}\,ds_1\right)_1 - \left(\frac{f}{\sqrt{E}}\,ds_1\right)_0\right],$$

in which the positive sign is chosen, and the indices 0 and 1 shall mean that the corresponding expression shall be taken for the values $v = v_0$ and $v = v_1$, resp.

I would now like to introduce ds, which is the arc length element of the boundary curve of the surface over which one integrates. I must now establish the sense in which ds is reckoned to be positive.

I shall next give some definitions and theorems for the plane that still preserve their meaning for arbitrarily-curved surfaces, because the elements of any surface can be regarded as planar.

When a and b are two directions in a plane, I will say that b lies to the left of a when one needs to rotate a through less than two right angles to the left in order for it to coincide with the b direction, and otherwise it will lie to the right of b. It follows from this that when a lies to the right of b, b will lie to the right of a, and *vice versa*. It further follows that if a and b lie on the same side of c, and a defines a right angle with c, then a will define an acute angle with b. By contrast, they will define an obtuse angle when they lie on different sides of c. ds_1 and ds_2 (viz., the arc lengths of the curves v = const., u = const.) were defined as $\sqrt{E}\,du$ ($\sqrt{E}\,dv$, resp.), where the roots are taken to be positive; in that way, the directions of ds_1 and ds_2 will also be determined.

I would now like to prove that ds_2 lies on the same side of ds_1 over the entire surface. I would first like to prove it for the elements of a well-defined curve.

The curve $v = c + dc$, in which c is a constant and dc means a small quantity, cannot cut the curve $v = c$. The perpendicular distance between both curves is $ds_2 = \sqrt{G}\,dc$, so this must be zero at the point of intersection, and since dc is not equal to 0, one must have $G = 0$. However, since G is a sum of three squares, they must be zero individually; that would give three equations for the two variables u and v. Such points can therefore not exist on the surface, in general. However, if such a point were present on a surface then one would have to make any sort of slit through the point and then consider the surface patches that would arise in that way. Those exceptional points are the ones at which the surface has an edge or a vertex, so the curvature will be:

$$k = \frac{DD'' - D'^2}{(EG - F^2)^2},$$

in **Gauss**ian notation. $F = 0$ for us, so $E = 0$ or $G = 0$ would mean that the surface is infinitely curved at the points in question; i.e., it has an edge or a vertex. However, we shall not concern ourselves with those discontinuities.

However, if the curve $v = c + dc$ did not cut the curve $v = c$ then that would mean that ds_2 would always lie on the same side of ds_1 along the curve $v = c$. However, a curve $u = c_1$ does not cut the curve $u = c_1 + dc_1$ either (i.e., ds_1 lies on the same side of ds_2 along the curve $u = c_1$), so conversely, ds_2 would lie on the same side of ds_1 along those curves. However, if it were to lie on different sides of ds_1 along two curves $u = c_1$ and $u = C_1$ then it would also have to lie on different sides of ds_1 at the point of intersection of that curve with the curve $v = c$. Nonetheless, ds_2 must likewise lie on the same side of ds_1, so ds_2 must lie on the same side of ds_1 over the entire surface; I shall call that side the *plus side*. Now, I shall establish the sense in which the boundary curve is taken to be positive in such a way that if I think of lines piercing the surface through any point of the boundary curve then they should always lie on the plus side of ds (viz., the element of the boundary curve that is assumed to be positive) after they pierce it, or what amounts to the same thing, that the direction of the lines that pierce the surface should lie on the plus side of the direction ds. It will then be meaningful for me to speak of ds_1, ds_2, ds. I shall let (s, v) denote the angle that ds defines at any point of the boundary curve with ds_1, namely, the arc length element of the curve v = const. at that point.

ds and ds_1 define the hypotenuse and cathetus of a right triangle, which define either the angle (s, v) or $(s, v) - \pi / 2$, so:

$$ds_1 = \pm\, ds \cos(s, v).$$

One can show that the positive sign must be taken for pieces of the boundary curve for which v has smaller values than it does in the surface patches that are bounded, while the negative sign must be taken for pieces on which v has larger values than it does on the surface patches that are bounded. For the smaller values of v, the curve u = const. will increase in the surface, so ds_2 will have the direction of the lines that enter the surface, and it will then lie on the plus side of ds. However, ds_2 always lies on the plus side of ds_1, so conversely ds and ds_1 lie on the same side of ds_2. ds_1 defines a right angle with ds_2, so ds will define an acute angle with ds_1. Wherever:

$$v = v_0 ,$$

one will then have:

$$ds_1 = ds \cos(s, v),$$

while if the curve u = const. exists on the boundary pieces where v has larger values than it does in the surface that they bound then ds_2 will have the direction of the lies that exit the surface, so it will lie on the minus side of ds. However, ds_2 always lies on the plus side of ds_1, so ds and ds_1 will lie on different sides of ds_2. ds_1 and ds_2 define a right angle, so ds_1 and ds will define an obtuse angle, so at the location $v = v_1$:

$$ds_1 = -\, ds \cos(s, v),$$

so one will have:

$$\left(\frac{f}{\sqrt{E}}\, ds_1\right)_1 = -\left(\frac{f}{\sqrt{E}}\, ds \cos(s, v)\right)_1 ,$$

$$\left(\frac{f}{\sqrt{E}}\, ds_1\right)_0 = \left(\frac{f}{\sqrt{E}}\, ds \cos(s, v)\right)_0 ,$$

and therefore:

$$\int_{u_0}^{u_1}\left[\left(\frac{f_1}{\sqrt{E_1}}\, ds_1\right)_1 - \left(\frac{f}{\sqrt{E}}\, ds_1\right)_0\right] = -\int_{u_0}^{u_1}\left[\left(\frac{f}{\sqrt{E}}\, ds \cos(s, v)\right)_1 - \left(\frac{f}{\sqrt{E}}\, ds \cos(s, v)\right)_0\right]$$

over the entire boundary curve.

Ultimately, the integral over the surface is then:

$$\int_{u_0}^{u_1}\int_{v_0}^{v_1}\frac{\partial f}{\partial v}du\,dv = -\int\frac{f}{\sqrt{E}}ds\cos(s,v)\,,$$

in which the latter integral is taken over the entire boundary curve.

Furthermore, one has:

$$\int_{u_0}^{u_1}\int_{v_0}^{v_1}\frac{\partial f}{\partial v}du\,dv = \int_{v_0}^{v_1}\left[\left(\frac{f}{\sqrt{G}}ds_2\right)_1 - \left(\frac{f}{\sqrt{G}}ds_2\right)_0\right],$$

in which $\sqrt{G}$ must be taken to be positive.

Now, one has $ds_2 = \pm\, ds \cos(s, u)$, if (s, v) means the angle that ds (viz., the element of the boundary curve) makes with ds_2 (viz., the element of the curve u = const.). Here, one must likewise take the positive or negative sign according to whether (s, u) is an acute or obtuse angle, resp. For the smaller values of u, the curves v = const. enter into the surface increasing, so ds_1 will have the direction of the entering curves in the surface, so it will lie on the plus side of ds. Hence, ds will lie on the minus side of ds_1, but ds_2 will lie on the plus side of ds_1, and then ds and ds_2 will lie on different sides of ds ; ds_1 and ds_1 define a right angle, so ds and d_2 will define an obtuse angle. We have:

$$ds_2 = -\,ds\cos(s, u)$$

wherever $u = u_0$, while the curves v = const. will leave the surface increasing wherever $u = u_1$. ds_2 lies on the minus side of ds, so ds will lie on the plus side of ds_2 . However, ds_2 always lies on the plus side of ds_1, so ds and ds_2 will lie on the same side of ds_1 . ds_1 and ds_2 define a right angle, so ds and ds_2 will define an acute angle. Here, one has:

$$ds_2 = ds\cos(s, u),$$

and it will then follow that:

$$\left(\frac{f}{\sqrt{G}}ds_2\right)_1 = \left(\frac{f}{\sqrt{E}}ds\cos(s,u)\right)_1,$$

$$\left(\frac{f}{\sqrt{G}}ds_2\right)_0 = -\left(\frac{f}{\sqrt{G}}ds\cos(s,u)\right)_0,$$

so ultimately, one will have:

$$\int_{u_0}^{u_1}\int_{v_0}^{v_1}\frac{\partial f}{\partial u}du\,dv = \int\frac{f}{\sqrt{G}}ds\cos(s,u)\,.$$

We would now like to employ these theorems in order to convert the pieces that originate in the internal work that were pointed out already.

One has, e.g.:

$$\iint X_2\sqrt{G}\,\alpha_0\,\frac{\partial\delta\alpha_2}{\partial u}\,du\,dv = \iint \frac{\partial(X_2\sqrt{G}\,\alpha_0\,\delta\alpha_2)}{\partial u}\,du\,dv - \iint \frac{\partial(X_2\sqrt{G}\,\alpha_0)}{\partial u}\,\delta\alpha_2\,du\,dv\,,$$

so:

$$\iint X_2\sqrt{G}\,\alpha_0\,\frac{\partial\delta\alpha_2}{\partial u}\,du\,dv = \int X_2\,\alpha_0\,\delta\alpha_2\,ds\cos(s,u) - \iint \frac{\partial(X_2\sqrt{G}\,\alpha_0)}{\partial u}\,\delta\alpha_2\,du\,dv\,,$$

and thus the summands that follow from the internal work will be:

$$\text{(I)}\qquad -\iiint\sum\left\{\begin{array}{l}\left(X_2\sqrt{G}\,\dfrac{\partial\alpha_2}{\partial u}+T_2\sqrt{E}\,\dfrac{\partial\alpha_2}{\partial v}+X_1\sqrt{G}\,\dfrac{\partial\xi}{\partial u}+T_1\sqrt{E}\,\dfrac{\partial\xi}{\partial v}\right)\delta\alpha_0\\ +\left(T_2\sqrt{G}\,\dfrac{\partial\alpha_2}{\partial u}+Y_2\sqrt{E}\,\dfrac{\partial\alpha_2}{\partial v}+Y_1\sqrt{E}\,\dfrac{\partial\xi}{\partial v}\right)\delta\alpha_1\end{array}\right\}du\,dv\,dt,$$

in which the sign $\sum$ means, here and in what follows, that two more similar summands are added in which the symbols β and η or γ and ζ appear in place of α and ξ, resp.

$$\text{(II)}\quad +\iiint\sum\left\{\begin{array}{l}\left(\dfrac{\partial(X_2\sqrt{G}\,\alpha_2)}{\partial u}+\dfrac{\partial(T_2\sqrt{G}\,\alpha_2)}{\partial u}+\dfrac{\partial(T_2\sqrt{E}\,\alpha_0)}{\partial v}+\dfrac{\partial(Y_2\sqrt{E}\,\alpha_1)}{\partial v}\right)\delta\alpha_2\\ +\left(\dfrac{\partial(X_1\sqrt{G}\,\alpha_0)}{\partial u}+\dfrac{\partial(T_1\sqrt{E}\,\alpha_0)}{\partial v}+\dfrac{\partial(Y_1\sqrt{E}\,\alpha_1)}{\partial v}\right)\delta\xi\end{array}\right\}du\,dv\,dt,$$

$$\text{(III)}\ -\iint\sum\left\{\begin{array}{l}[X_2\,\alpha_0\cos(s,u)+T_2\,\alpha_1\cos(s,u)-T_2\,\alpha_0\cos(s,v)-Y_2\,\alpha_0\cos(s,u)]\,\delta\alpha_2\\ +[X_1\,\alpha_0\cos(s,u)-T_1\,\alpha_0\cos(s,v)-Y_1\,\alpha_1\cos(s,u)]\,\delta\xi\end{array}\right\}ds\,dt.$$

The work that originates in the external forces shall now be calculated, and first of all for the forces that act upon the points in all layers inside the shell whose coordinates are ξ', η', ζ'. Let A, B, C be the components of the force that act at a point ξ', η', ζ' when they are taken along the three axes. Hence, the work that they do will be:

$$\iiint\int (A\,\delta\xi' + B\,\delta\eta' + C\,\delta\zeta')\sqrt{EG}\;du\,dv\,dz\,dt\,.$$

A, *B*, *C* must be given as functions of ξ, η, ζ, z. Now, it will follow from equation (7) when one sets $x = 0$ and $y = 0$ (which one can do as long as one includes only all possible values of *u*, *v* in the integrals above when one neglects the quantities $\mathfrak{u}_1$, $\mathfrak{v}_1$, $\mathfrak{w}_1$, which are small in comparison to higher-order quantities) that:

$$\delta\xi' = \delta\xi + z\,\delta\alpha_2\,,\qquad \delta\eta' = \delta\eta + z\,\delta\beta_2\,,\qquad \delta\zeta' = \delta\zeta + z\,\delta\gamma_2\,.$$

In general, however, one can also neglect $z\,\delta\alpha_2$ in comparison to $\delta\xi$, etc., since z is, in fact, infinitely small; however, there is a case in which that is not true.

Namely, one has:

$$\alpha_0\,\delta\alpha_0 + \alpha_1\,\delta\alpha_1 + \alpha_2\,\delta\alpha_2 = 0$$

Now, one has:

$$\alpha_0 = \frac{1}{\sqrt{E}}\frac{\partial\xi}{\partial u},$$

$$\alpha_1 = \frac{1}{\sqrt{G}}\frac{\partial\xi}{\partial v},$$

up to small quantities.

If I regard E and G as constant during the deformation, which is correct up to small quantities, then it will follow that:

$$\frac{1}{E}\frac{\partial\xi}{\partial u}\frac{\partial\delta\xi}{\partial u}+\frac{1}{G}\frac{\partial\xi}{\partial v}\frac{\partial\delta\xi}{\partial v} + \alpha_2\,\delta\alpha_2 = 0.$$

Now, $\dfrac{\partial\delta\xi}{\partial u}$ and $\dfrac{\partial\delta\xi}{\partial v}$ have the same order as $\delta\xi$, in general, so $\delta\alpha_2$ will have the same order as $\delta\xi$, and therefore $z\,\delta\alpha_2$ can be neglected in comparison to $\delta\xi$. That conclusion is not permissible in one case, namely, when $\alpha_2 = 0$; i.e., when the X-axis is parallel to the surface. The same thing will be true for $\delta\beta_2$ and $\delta\gamma_2$, in comparison to $\delta\eta$ and $\delta\zeta$, resp., when the Y or Z axis is parallel to the surface, so, e.g., for an elastic plate that is parallel to the XY-plane or when the elastic shell defines cylinder whose axis is parallel to the Z-axis. Meanwhile, later on, we will also have to convert the equations for the general case, as well, under which even that exceptional case will have to be brought under consideration for each point of an arbitrary shell. We would not like to neglect $z\,\delta\alpha_2$, etc., in comparison to $\delta\xi$ for the general case as well, and therefore the work that is done by the external forces under the deformation of the shell will be:

$$\iiint\!\!\int (A\,\delta\xi + B\,\delta\eta + C\,\delta\zeta + Az\,\delta\alpha_2 + Bz\,\delta\beta_2 + Cz\,\delta\gamma_2)\sqrt{EG}\;du\;dv\;dz\;dt\;.$$

The forces that act at the points of the outer surface of the shell, when one excludes the boundary, might have the components A', B', C' in the directions of the axes. Hence, the work that they do will be:

$$\iiint (A'\,\delta\xi + B'\,\delta\eta + C'\,\delta\zeta + A'z\,\delta\alpha_2 + B'z\,\delta\beta_2 + C'z\,\delta\gamma_2)\sqrt{EG}\;du\;dv\;dt\;.$$

I now set:

$$\int A\,dz + A' = A_1, \qquad \int B\,dz + B' = B_1, \qquad \int C\,dz + C' = C_1,$$

$$\int Az\,dz + A'z = A_2, \qquad \int Bz\,dz + B'z = B_2, \qquad \int Cz\,dz + C'z = C_2\,.$$

One can conclude from the type of summation that the forces that are applied to the outer surface will act like the forces that are distributed inside, and conversely.

The work that originates in those forces is then:

(IV) $$\iiint (A_1\,\delta\xi + B_1\,\delta\eta + C_1\,\delta\zeta + A_2\,\delta\alpha_2 + B_2\,\delta\beta_2 + C_2\,\delta\gamma_2)\sqrt{EG}\;du\;dv\;dt\;.$$

However, external forces can also act on the boundary. They might have the components U, V, W in the directions of the axes. The work that they do will then be:

$$\iiint (U\,\delta\xi' + V\,\delta\eta' + W\,\delta\zeta')\,ds\;dz\;dt$$

or

$$\iiint (U\,\delta\xi + V\,\delta\eta + W\,\delta\zeta + Uz\,\delta\alpha_2 + Vz\,\delta\beta_2 + Wz\,\delta\gamma_2)\;du\;dv\;dt\;.$$

If I set:

$$\int U\,dz = U_1, \quad \int V\,dz = V_1, \quad \int W\,dz = W_1,$$

$$\int Uz\,dz = U_2\,, \quad \int Vz\,dz = V_2\,, \quad \int Wz\,dz = W_2$$

then that work will be:

(V) $$\iiint (U_1\,\delta\xi + V_1\,\delta\eta + W_1\,\delta\zeta + U_2\,\delta\alpha_2 + V_2\,\delta\beta_2 + W_2\,\delta\gamma_2)\,ds\;dz\;dt\;.$$

We now come to the part that originates in the *vis viva*:

$$\tfrac{1}{2}\,\delta\iiiint\left[\left(\frac{\partial\xi'}{\partial t}\right)^2 + \left(\frac{\partial\eta'}{\partial t}\right)^2 + \left(\frac{\partial\zeta'}{\partial t}\right)^2\right]\rho\sqrt{EG}\;du\;dv\;dz\;dt,$$

in which ρ means the density.

Now, one has:

$$\tfrac{1}{2}\,\delta\int\left(\frac{\partial\xi'}{\partial t}\right)^2 dt = \int\frac{\partial\xi'}{\partial t}\,\partial\,\frac{\partial\xi'}{\partial t}\,dt = \int\left(\frac{\partial\xi'}{\partial t}\,\delta\xi'\right)_{t_0}^{t_1} - \int\delta\xi'\,\frac{\partial^2\xi'}{\partial t^2}\,dt\;.$$

The first summand on the right is zero, since $\delta\xi'$ is zero on the boundary because the path is varied, but not the starting and ending points.

Now:

$$\xi' = \xi + \alpha_2\,z,$$

so:

$$\frac{\partial^2\xi'}{\partial t^2} = \frac{\partial^2\xi}{\partial t^2} + \frac{z\,\partial^2\alpha_2}{\partial t^2}\,,$$

and furthermore:

$$\delta\xi' = \delta\xi + z\,\delta\alpha_2\,,$$

and therefore:

$$\frac{\partial^2\xi'}{\partial t^2}\,\delta\xi' = \frac{\partial^2\xi}{\partial t^2}\,\delta\xi + z\frac{\partial^2\xi}{\partial t^2}\,\delta\alpha_2 + z\frac{\partial^2\alpha_2}{\partial t^2}\,\delta\xi + z^2\frac{\partial^2\alpha_2}{\partial t^2}\,\delta\alpha_2\,.$$

We would now like to make use of the assumption that the surface ξ, η, ζ is the middle surface of the shell. If h is the thickness of the shell then we must integrate for z from $-h/2$ to $+h/2$, and after performing that integration, we will get:

$$\int_{-h/2}^{h/2}\frac{\partial^2\xi}{\partial t^2}\,dz = \frac{h\,\partial^2\xi}{\partial t^2}\,, \qquad \int_{-h/2}^{h/2} z\frac{\partial^2\xi}{\partial t^2}\,dz = 0,$$

$$\int_{-h/2}^{h/2} z\frac{\partial^2\alpha_2}{\partial t^2}\,dz = 0, \qquad \int_{-h/2}^{h/2} z\frac{\partial^2\alpha_2}{\partial t^2}\,dz = \frac{h^2}{12}\frac{\partial^2\alpha_2}{\partial t^2}\,.$$

The *vis viva* then yield the following summands:

(VI) $$-\iiint\sum\left(h\frac{\partial^2\xi}{\partial t^2}\,\delta\xi + \frac{h^2}{12}\frac{\partial^2\alpha_2}{\partial t^2}\,\delta\alpha_2\right)\rho\sqrt{EG}\;du\;dv\;dt.$$

We now have to calculate the summands that originate in the internal condition equations. These conditions, in terms of the varied quantities that exist, are:

$$\alpha_0^2 + \beta_0^2 + \gamma_0^2 - 1 = 0, \qquad \alpha_1\,\alpha_2 + \beta_1\,\beta_2 + \gamma_1\,\gamma_2 = 0,$$
$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 - 1 = 0, \qquad \alpha_2\,\alpha_0 + \beta_2\,\beta_0 + \gamma_2\,\gamma_0 = 0,$$
$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 - 1 = 0, \qquad \alpha_0\,\alpha_1 + \beta_0\,\beta_1 + \gamma_0\,\gamma_1 = 0,$$

$$\alpha_1\frac{\partial\xi}{\partial u} + \beta_1\frac{\partial\eta}{\partial u} + \gamma_1\frac{\partial\zeta}{\partial u} = 0,$$
$$\alpha_2\frac{\partial\xi}{\partial u} + \beta_2\frac{\partial\eta}{\partial u} + \gamma_2\frac{\partial\zeta}{\partial u} = 0,$$
$$\alpha_2\frac{\partial\xi}{\partial v} + \beta_2\frac{\partial\eta}{\partial v} + \gamma_2\frac{\partial\zeta}{\partial v} = 0.$$

These nine condition equations are supplied with undetermined coefficients. I multiply the first three by L_0, L_1, L_2, resp., the second three by M_0, M_1, M_2, resp., and the last three by N_0, N_1, N_2, resp.

The coefficients of the variations of the cosines of the inclination angles can be written down immediately; they define the summands:

$$\text{(VII)} \quad + \iiint \sum \left\{ \begin{array}{l} (2L_0\,\alpha_0 + M_1\,\alpha_2 + M_2\,\alpha_1)\,\delta\alpha_2 \\ + \left(2L_1\,\alpha_1 + M_2\,\alpha_0 + M_0\,\alpha_2 + N_0 \dfrac{\partial\xi}{\partial u} \right) \delta\alpha_1 \\ + \left(2L_2\,\alpha_2 + M_0\,\alpha_1 + M_1\,\alpha_1 + N_1 \dfrac{\partial\xi}{\partial u} + N_2 \dfrac{\partial\xi}{\partial v} \right) \delta\alpha_2 \end{array} \right\} \sqrt{EG}\, du\, dv\, dt.$$

By contrast, we convert the integral that includes the variations of the derivatives of ξ, η, ζ in the repeatedly-applied way into an integral over the surface and an integral over the boundary curve and obtain:

$$\text{(VIII)} \quad - \iiint \sum \left\{ \left(\frac{\partial(\sqrt{EG} N_0\,\alpha_1)}{\partial u} + \frac{\partial(\sqrt{EG} N_1\,\alpha_2)}{\partial u} + \frac{\partial(\sqrt{EG} N_2\,\alpha_2)}{\partial v} \right) \delta\xi \right\} du\, dv\, dt,$$

$$\text{(IX)} \quad + \iint \sum \left\{ \left(N_0\,\alpha_1 \sqrt{E} \cos(s,u) + N_1\,\alpha_2 \sqrt{E} \cos(s,u) - N_2\,\alpha_2 \sqrt{G} \cos(s,v) \right) \delta\xi \right\} ds\; st.$$

The internal condition equations will assume a different form for the boundary. Initially, only the variations $\delta\alpha_2$, $\delta\beta_2$, $\delta\gamma_2$, $\delta\xi$, $\delta\eta$, $\delta\zeta$ will come under consideration, so we will also require only the condition equations:

$$\alpha_0^2 + \beta_0^2 + \gamma_0^2 = 1,$$

$$\alpha_2 \frac{\partial\xi}{\partial u} + \beta_2 \frac{\partial\eta}{\partial u} + \gamma_2 \frac{\partial\zeta}{\partial u} = 0,$$
$$\alpha_2 \frac{\partial\xi}{\partial v} + \beta_2 \frac{\partial\eta}{\partial v} + \gamma_2 \frac{\partial\zeta}{\partial v} = 0.$$

However, the last two condition equations are only apparently two. They say that the z-axis of each element should be normal to the middle surface. However, for the boundary, that can only mean that the z-axis of the boundary element should remain normal to the boundary and that will give only the equation:

$$\alpha_2 \frac{\partial\xi}{\partial s} + \beta_2 \frac{\partial\eta}{\partial s} + \gamma_2 \frac{\partial\zeta}{\partial s} = 0.$$

I multiply the first condition equation by P and then by Q, so the coefficients of the variations $\delta\alpha_2$, $\delta\beta_2$, $\delta\gamma_2$ can be written down directly:

$$\text{(X)} \qquad \iint \sum \left[\left(2P\,\alpha_2 + Q\,\alpha_2 \frac{\partial\xi}{\partial s} \right) \delta\alpha_2 \right] ds\, dt\,.$$

By contrast, one has:

$$\int Q\,\alpha_2\,\partial\frac{\delta\xi}{\partial s}\,ds = (Q\ \alpha_2\ \delta\xi) - \int\frac{\partial(Q\,\alpha_2)}{\partial s}\,\delta\xi\,dt.$$

In the first part, one must subtract the value at the starting point of the boundary curve from the one at the end point. However, the curve is closed, so that part will be 0, and we ultimately get the part that is endowed with $\delta\xi$, $\delta\eta$, $\delta\zeta$:

$$\text{(XI)}\qquad -\int\sum\left(\frac{\partial(Q\alpha_2)}{\partial s}\delta\xi\right)ds\,dt.$$

III.

The sum of the integrals (I), (II), etc., up to (XI) must be 0. When one sets the coefficients of the individual variations that relate to the interior equal to zero, one will get twelve equations that refer to the interior of the middle surface and six that refer to the position of the boundary points. I shall set down only the first, fourth, etc. of them. The other ones will be obtained in such a way that one replaces the α, ξ, A, and U in them with the symbols β, η, B, and V or γ, ζ, C, and W, resp.

Main equations

$$\text{(1-3)}\qquad \left\{\begin{aligned} &\frac{\partial(X_1\sqrt{G}\,\alpha_0)}{\partial u}+\frac{\partial(T_1\sqrt{E}\,\alpha_0)}{\partial v}+\frac{\partial(Y_1\sqrt{E}\,\alpha_1)}{\partial v}+\left(A_1-h\rho\frac{\partial^2\xi}{\partial t^2}\right)\sqrt{EG}\\ &\quad-\frac{\partial(\sqrt{EG}\,N_0\,\alpha_1)}{\partial u}-\frac{\partial(\sqrt{EG}\,N_1\,\alpha_2)}{\partial u}-\frac{\partial(\sqrt{EG}\,N_2\,\alpha_2)}{\partial v}=0,\end{aligned}\right.$$

$$\text{(4-6)}\qquad \left\{\begin{aligned} &\frac{\partial(X_2\sqrt{G}\,\alpha_0)}{\partial u}+\frac{\partial(T_2\sqrt{G}\,\alpha_1)}{\partial u}+\frac{\partial(T_2\sqrt{E}\,\alpha_0)}{\partial v}+\frac{\partial(Y_2\sqrt{E}\,\alpha_1)}{\partial v}+\left(A_2-\frac{h^3\rho\,\partial^2\alpha_2}{12\,\partial t^2}\right)\sqrt{EG}\\ &\quad+\left(2L_2\,\alpha_2+M_0\,\alpha_1+M_1\,\alpha_0+N_1\frac{\partial\xi}{\partial u}+N_2\frac{\partial\xi}{\partial v}\right)\sqrt{EG}=0,\end{aligned}\right.$$

$$\text{(7-9)}\qquad \left\{\begin{aligned} &\left(Y_2\sqrt{E}\frac{\partial\alpha_2}{\partial v}+T_2\sqrt{G}\frac{\partial\alpha_2}{\partial u}+Y_1\sqrt{E}\frac{\partial\xi}{\partial v}\right)\\ &\quad+\left(2L_1\,\alpha_1+M_2\,\alpha_0+M_0\,\alpha_2+N_0\frac{\partial\xi}{\partial u}\right)\sqrt{EG}=0,\end{aligned}\right.$$

$$\text{(10-12)}\qquad \left\{\begin{aligned} &-\left(X_2\sqrt{G}\frac{\partial\alpha_2}{\partial u}+T_2\sqrt{E}\frac{\partial\alpha_2}{\partial v}+X_1\sqrt{G}\frac{\partial\xi}{\partial u}+T_1\sqrt{E}\frac{\partial\xi}{\partial v}\right)\\ &\quad+\left(2L_0\,\alpha_0+M_1\,\alpha_2+M_2\,\alpha_1\right)\sqrt{EG}=0.\end{aligned}\right.$$

<u>Boundary equations.</u>

$$(1\text{-}3)\quad \left\{\begin{aligned} &-X_1\,\alpha_0\cos(s,u)+T_1\,\alpha_0\cos(s,u)+Y_1\,\alpha_0\cos(s,v)+U_1\\ &+N_0\,\alpha_1\sqrt{E}\cos(s,u)+N_1\,\alpha_2\sqrt{E}\cos(s,u)-N_1\,\alpha_2\sqrt{G}\cos(s,v)-\frac{\partial(Q\,\alpha_2)}{\partial s}=0,\end{aligned}\right.$$

$$(4\text{-}6)\quad \left\{\begin{aligned} &-X_2\,\alpha_0\cos(s,u)-T_2\,\alpha_1\cos(s,u)+T_2\,\alpha_0\cos(s,v)\\ &\qquad+Y_0\,\alpha_1\cos(s,u)+U_2+2P\,\alpha_2+Q\,\alpha_2\,\frac{\partial\xi}{\partial s}=0.\end{aligned}\right.$$

The twelve main equations can be reduced to five.

I multiply equations (10-12) in sequence by α_2, β_2, γ_2, resp., and add them, and when one considers equations (2*a*) and (3*a*), it will then follow that: $M_1 = 0$.

If one multiplies those equations by and α_1, β_1, γ_1, resp., and adds them then when one considers equations (8), (9), (10) and neglects quantities of higher order, it will follow that:

$$\frac{X_2\,\Delta'}{\sqrt{EG}}+\frac{T_2\,\Delta''}{G}-T_1\sqrt{EG}+M_2\sqrt{EG}=0.$$

If one multiplies equations (7-9) by α_2, β_2, γ_2, resp., and adds them then it will follow that $M_0 = 0$.

If one multiplies the same equations by α_2, β_2, γ_2 , resp., and adds them then it will follow that:

$$\frac{Y_2\,\Delta'}{\sqrt{EG}}+\frac{T_2\,\Delta}{E}+M_2\sqrt{EG}+N_0\sqrt{E}\sqrt{EG}=0.$$

It then follows from both equations that:

$$N_0\sqrt{E}\sqrt{EG}=\frac{X_2\,\Delta'}{\sqrt{EG}}+\frac{T_2\,\Delta''}{G}-T_1\sqrt{EG}+\frac{Y_2\,\Delta'}{\sqrt{EG}}-\frac{T_2\,\Delta}{G}.$$

We shall not need equations (7)-(11), inclusive, from now on.

We now multiply equations (1)-(3), inclusive, by:

$$\alpha_0,\ \beta_0,\ \gamma_0,\qquad \alpha_1,\ \beta_1,\ \gamma_1,\qquad \alpha_2,\ \beta_2,\ \gamma_2,$$

resp., add them each time, and, at the same time, replace $\dfrac{\partial(X\sqrt{G}\alpha_0)}{\partial u}$, …:

$$X_1\sqrt{G}\,\frac{\partial\alpha_0}{\partial u}+\alpha_0\,\frac{\partial(X_1\sqrt{G})}{\partial u},\ \ \ldots,$$

then when we consider equations (8), (9), (10) and neglect quantities of higher order, it will follow that:

Main equations

$$(1)\quad \begin{cases} h\rho\sqrt{EG}\left(\dfrac{\partial^2\xi}{\partial t^2}\alpha_0+\dfrac{\partial^2\eta}{\partial t^2}\beta_0+\dfrac{\partial^2\zeta}{\partial t^2}\gamma_0\right) \\ \quad -\sqrt{EG}(A_1\,\alpha_0+B_1\,\beta_0+C_1\,\gamma_0)-\dfrac{\partial(X_1\sqrt{G})}{\partial u}-\dfrac{\partial(T_1\sqrt{E})}{\partial v} \\ \quad +\dfrac{Y_1}{2\sqrt{G}}\dfrac{\partial G}{\partial u}+\dfrac{N_0}{2}-\dfrac{\partial E}{\partial v}-N_1\dfrac{\Delta}{\sqrt{E}}-N_2\dfrac{\Delta'}{\sqrt{E}}=0, \end{cases}$$

$$(2)\quad \begin{cases} h\rho\sqrt{EG}\left(\dfrac{\partial^2\xi}{\partial t^2}\alpha_1+\dfrac{\partial^2\eta}{\partial t^2}\beta_1+\dfrac{\partial^2\zeta}{\partial t^2}\gamma_1\right) \\ \quad -\sqrt{EG}(A_1\,\alpha_1+B_1\,\beta_1+C_1\,\gamma_1)-\dfrac{\partial(Y_1\sqrt{E})}{\partial v}+\dfrac{X_1}{2\sqrt{E}}\dfrac{\partial E}{\partial v}-\dfrac{T_1}{2\sqrt{G}}\dfrac{\partial G}{\partial u} \\ \quad +\dfrac{\partial(N_0\sqrt{EG})}{\partial u}-N_1\dfrac{\Delta'}{\sqrt{E}}-N_2\dfrac{\Delta''}{\sqrt{E}}=0, \end{cases}$$

$$(3)\quad \begin{cases} h\rho\sqrt{EG}\left(\dfrac{\partial^2\xi}{\partial t^2}\alpha_2+\dfrac{\partial^2\eta}{\partial t^2}\beta_2+\dfrac{\partial^2\zeta}{\partial t^2}\gamma_2\right) \\ \quad -\sqrt{EG}(A_1\,\alpha_2+B_1\,\beta_2+C_1\,\gamma_2)-X_1\dfrac{\Delta}{E}-Y_1\dfrac{\Delta''}{\sqrt{G}} \\ \quad -T_1\dfrac{\Delta'}{\sqrt{EG}}+N_0\dfrac{\Delta'}{\sqrt{G}}+\dfrac{\partial(N_0\sqrt{EG})}{\partial u}+\dfrac{\partial(N_2\sqrt{EG})}{\partial v}=0. \end{cases}$$

If we multiply equations (4-6) successively by α_0, β_0, γ_0, α_1, β_1, γ_1, resp., and add them then it will follow that:

$$(4)\quad \begin{cases} \dfrac{h^2\rho}{12}\sqrt{EG}\left(\dfrac{\partial^2\alpha_2}{\partial t^2}\alpha_0+\dfrac{\partial^2\beta_2}{\partial t^2}\beta_0+\dfrac{\partial^2\gamma_2}{\partial t^2}\gamma_0\right) \\ \quad -\sqrt{EG}(A_2\,\alpha_0+B_2\,\beta_0+C_2\,\gamma_0)-\dfrac{\partial(X_2\sqrt{G})}{\partial u}-\dfrac{\partial(T_2\sqrt{E})}{\partial v} \\ \quad -\dfrac{T_2}{2\sqrt{E}}\dfrac{\partial E}{\partial v}+\dfrac{Y_2}{2\sqrt{G}}\dfrac{\partial G}{\partial u}-N_1E\sqrt{G}=0, \end{cases}$$

$$(5)\quad \left\{ \begin{aligned} &\frac{h^2\rho}{12}\sqrt{EG}\left(\frac{\partial^2\alpha_2}{\partial t^2}\alpha_1+\frac{\partial^2\beta_2}{\partial t^2}\beta_1+\frac{\partial^2\gamma_2}{\partial t^2}\gamma_1\right) \\ &\qquad -\sqrt{EG}(A_2\,\alpha_1+B_2\,\beta_1+C_2\,\gamma_1)-\frac{\partial(Y_2\sqrt{E})}{\partial v}-\frac{\partial(T_2\sqrt{G})}{\partial u} \\ &\qquad +\frac{X_2}{2\sqrt{E}}\frac{\partial E}{\partial v}-\frac{T_2}{2\sqrt{G}}\frac{\partial G}{\partial u}-N_2\,G\sqrt{E}=0. \end{aligned} \right.$$

<u>Boundary equations</u>

If we multiply equations (1-3) of the boundary equations by:

$$\alpha_0,\ \beta_0,\ \gamma_0,\qquad \alpha_1,\ \beta_1,\ \gamma_1,\qquad \alpha_2,\ \beta_2,\ \gamma_2,$$

resp., add them three at a time, and at the same time set:

$$\frac{\partial\alpha_2}{\partial z}=\frac{\partial\alpha_2}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial\alpha_2}{\partial v}\frac{\partial v}{\partial z}=\frac{\partial\alpha_2}{\partial u}\frac{1}{\sqrt{E}}\frac{\partial s_1}{\partial s}+\frac{\partial\alpha_2}{\partial v}\frac{1}{\sqrt{G}}\frac{\partial s_2}{\partial s}$$

$$=\frac{\partial\alpha_2}{\partial u}\frac{\cos(s,v)}{\sqrt{E}}+\frac{\partial\alpha_2}{\partial v}\frac{\cos(s,u)}{\sqrt{G}}$$

then we will set:

$$(1)\quad \left\{ \begin{aligned} &(U_1\,\alpha_0+V_1\,\beta_0+W_1\,\gamma_0)-X_1\cos(s,u)+T_1\cos(s,v) \\ &\qquad +\frac{Q\,\Delta}{E\sqrt{EG}}\cos(s,v)+\frac{Q\,\Delta'}{EG}\cos(s,u)=0, \end{aligned} \right.$$

$$(2)\quad \left\{ \begin{aligned} &(U_1\,\alpha_1+V_1\,\beta_1+W_1\,\gamma_1)+Y_1\cos(s,v)+N_0\sqrt{E}\cos(s,u) \\ &\qquad +\frac{Q\,\Delta'}{EG}\cos(s,v)+\frac{Q\,\Delta''}{G\sqrt{EG}}\cos(s,u)=0, \end{aligned} \right.$$

$$(3)\quad (U_1\,\alpha_2+V_1\,\beta_2+W_1\,\gamma_2)+N_1\sqrt{E}\cos(s,u)-N_2\sqrt{G}\cos(s,v)-\frac{\partial Q}{\partial s}=0.$$

We multiply equations (4-6) by α_0, β_0, γ_0, α_1, β_1, γ_1, resp., and observe that:

$$\frac{\partial\xi}{\partial s}=\frac{\partial\xi}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial\xi}{\partial v}\frac{\partial v}{\partial s}=\frac{\partial\xi}{\partial u}\frac{1}{\sqrt{E}}\frac{\partial s_1}{\partial s}+\frac{\partial\xi}{\partial v}\frac{1}{\sqrt{G}}\frac{\partial s_2}{\partial s}$$

$$=\frac{\partial\xi}{\partial u}\frac{\cos(s,v)}{\sqrt{E}}+\frac{\partial\xi}{\partial v}\frac{\cos(s,u)}{\sqrt{G}}.$$

It will follow that:

(4) $$U_2\,\alpha_0 + V_2\,\beta_0 + W_2\,\gamma_0 - X_2 \cos(s, u) + T_2 \cos(s, v) + Q \cos(s, v) = 0,$$

(5) $$U_2\,\alpha_1 + V_2\,\beta_1 + W_2\,\gamma_1 + Y_2 \cos(s, v) - T_2 \cos(s, u) + Q \cos(s, u) = 0.$$

Here:

$$X_1, Y_1, Z_1, \qquad X_2, Y_2, Z_2$$

are calculated from equations (12) on page 18 and substituted.

Since the integrations should be extended from $-h/2$ to $+h/2$, we should get:

$$X_1 = \frac{2kh}{1+\Theta}((1 + 2\Theta)\,\varepsilon_1 + \Theta\,\varepsilon_2),$$

$$Y_1 = \frac{2kh}{1+\Theta}(\Theta\,\varepsilon_1 + (1 + 2\Theta)\,\varepsilon_2),$$

$$T_1 = kh\,\tau,$$

$$X_2 = \frac{kh^3}{6(1+\Theta)}\left[(1+2\Theta)\frac{D-\Delta}{E\sqrt{EG}} + \Theta\frac{D''-\Delta''}{G\sqrt{EG}}\right],$$

$$Y_2 = \frac{kh^3}{6(1+\Theta)}\left[\Theta\frac{D-\Delta}{E\sqrt{EG}} + (1+2\Theta)\frac{D''-\Delta''}{G\sqrt{EG}}\right],$$

$$T_2 = \frac{kh^3}{6}\frac{D'-\Delta'}{EG}.$$

It follows from this, in turn, that:

$$N_0\sqrt{EG} = -\,kh\tau\sqrt{G} + \frac{kh^3}{6EG\sqrt{E}}\left(\frac{D\Delta' - D'\Delta}{E} + \frac{D''\Delta' - D'\Delta''}{G}\right).$$

If the shell is a plate, so the middle surface is a plane, then $D = D' = D'' = 0$. These will then include the derivatives of the cosines of the inclination angles of the normal with the axes with respect to u and v as factors, but that angle will be constant for the plane, so their derivatives are 0. If one further sets $E = 1$, $G = 1$ then the coordinate system that is assumed on the plate will be an orthogonal rectilinear one, so if we ignore the acceleration then our equations will then go over to the ones that **Clebsch** (*) gave for the equilibrium of an infinitely-thin elastic plate that suffers finite deformations.

(*) *Theorie der Elasticität*, §§ 69, 92, and 93.

Infinitely-small oscillations in the neighborhood of the equilibrium position

Whereas the coordinates ξ, η, ζ of each point were determined as functions of u, v, and t for finite changes of form by our equations, in the case where the shell executes only infinitely-small oscillations, one can convert the equations in such a way that the coordinates of the displacements of each points inside and normal to the surface will enter into them.

To begin with, the quantities that appear in those equations:

$$\varepsilon_1 , \qquad \varepsilon_2 , \qquad \tau ,$$

$$\Delta, \qquad \Delta', \qquad \Delta''$$

should be expressed in terms of the stated displacements. To that end, the things that were denoted by α_0 , β_0 , γ_0 , α_1 , β_1 , γ_1 , α_2 , β_2 , γ_2 during the deformation shall be denoted by a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2 for the surface in the equilibrium configuration.

I now set:

$$\begin{aligned}
\alpha_0 &= a_0 + \alpha_0' , & \beta_0 &= b_0 + \beta_0' , & \gamma_0 &= c_0 + \gamma_0' , \\
\alpha_1 &= a_1 + \alpha_1' , & \beta_1 &= b_1 + \beta_1' , & \gamma_1 &= c_1 + \gamma_1' , \\
\alpha_2 &= a_2 + \alpha_2' , & \beta_2 &= b_2 + \beta_2' , & \gamma_2 &= c_2 + \gamma_2'
\end{aligned}$$

for small deformations of the shell, in which the symbols that are expressed with a prime mean small quantities.

Now, since the α_0 , ... satisfy equations (1) on page 4, but the a_0, … must satisfy similar equations, when one ponders the fact that α_0', … mean small quantities and that one would like to neglect quantities of higher order, it will follow that:

$$a_0\,\alpha_0' + b_0\,\beta_0' + c_0\,\gamma_0' = 0, \qquad a_1\,\alpha_1' + b_1\,\beta_1' + c_1\,\gamma_1' = 0, \qquad a_2\,\alpha_2' + b_2\,\beta_2' + c_2\,\gamma_2' = 0,$$

$$\begin{aligned}
a_0\,\alpha_1' + b_0\,\beta_1' + c_0\,\gamma_1' + a_1\,\alpha_0' + b_1\,\beta_0' + c_1\,\gamma_0' &= 0, \\
a_1\,\alpha_2' + b_1\,\beta_2' + c_1\,\gamma_2' + a_2\,\alpha_1' + b_2\,\beta_1' + c_2\,\gamma_1' &= 0, \\
a_2\,\alpha_0' + b_2\,\beta_0' + c_2\,\gamma_0' + a_0\,\alpha_2' + b_0\,\beta_2' + c_0\,\gamma_2' &= 0.
\end{aligned}$$

One will satisfy those equations when one sets:

$$\begin{aligned}
\alpha_0' &= p_1\,a_2 - p_2\,a_1 , & \beta_0' &= p_1\,b_2 - p_2\,b_1 , & \gamma_0' &= p_1\,c_2 - p_2\,c_1 , \\
\alpha_1' &= p_2\,a_0 - p_0\,a_2 , & \beta_1' &= p_2\,b_0 - p_0\,b_2 , & \gamma_1' &= p_2\,c_0 - p_0\,c_2 , \\
\alpha_2' &= p_0\,a_1 - p_1\,a_0 , & \beta_2' &= p_0\,b_1 - p_1\,b_0 , & \gamma_2' &= p_0\,c_1 - p_1\,c_0 ,
\end{aligned}$$

in which p_0 , p_1 , p_2 mean small quantities. Now, from equation (10) on page 12, one will have:

$$\frac{\Delta}{E\sqrt{G}} = (a_2 + \alpha_2')\frac{\partial(a_0 + \alpha_0')}{\partial u} + (b_2 + \beta_2')\frac{\partial(b_0 + \beta_0')}{\partial u} + (c_2 + \gamma_2')\frac{\partial(c_0 + \gamma_0')}{\partial u} ,$$

$$\frac{\Delta}{E\sqrt{G}} = a_2\frac{\partial a_0}{\partial u} + b_2\frac{\partial b_0}{\partial u} + c_2\frac{\partial c_0}{\partial u} + a_2\frac{\partial \alpha_2'}{\partial u} + b_2\frac{\partial \beta_2'}{\partial u} + c_2\frac{\partial \gamma_2'}{\partial u} + \alpha_2'\frac{\partial a_0}{\partial u} + \beta_2'\frac{\partial b_0}{\partial u} + \gamma_2'\frac{\partial c_0}{\partial u} ,$$

so it will follow, with the help of equations (8), (9), and (10), that:

$$\frac{\Delta}{E\sqrt{G}} = \frac{D}{E\sqrt{G}} + \frac{\partial p_1}{\partial u} - \frac{p_2\,D'}{G\sqrt{E}} - \frac{p_0}{2\sqrt{EG}}\frac{\partial E}{\partial v} .$$

It will follow similarly that:

$$\frac{\Delta'}{E\sqrt{G}} = \frac{D'}{E\sqrt{G}} + \frac{\partial p_1}{\partial v} - \frac{p_2\,D''}{G\sqrt{E}} + \frac{p_0}{2\sqrt{EG}}\frac{\partial G}{\partial u} ,$$

$$\frac{\Delta'}{G\sqrt{E}} = \frac{D'}{E\sqrt{G}} + \frac{\partial p_1}{\partial v} - \frac{p_2\,D''}{G\sqrt{E}} + \frac{p_0}{2\sqrt{EG}}\frac{\partial G}{\partial u} ,$$

$$\frac{\Delta'}{E\sqrt{G}} = \frac{D'}{G\sqrt{E}} - \frac{\partial p_1}{\partial u} + \frac{p_2\,D}{E\sqrt{G}} - \frac{p_1}{2\sqrt{EG}}\frac{\partial E}{\partial v} ,$$

$$\frac{\Delta''}{G\sqrt{E}} = \frac{D''}{G\sqrt{E}} - \frac{\partial p_0}{\partial v} + \frac{p_2\,D'}{E\sqrt{G}} + \frac{p_1}{2\sqrt{EG}}\frac{\partial G}{\partial u} .$$

Now, from equation (2) on page 5:

$$\frac{\partial \xi}{\partial u} = (a_0 + \alpha_0')\sqrt{E}\,(1 + \varepsilon_1) , \ldots$$

I set $\xi = X + \xi'$, $\eta = Y + \eta'$, $\zeta = Z + \zeta'$, in which ξ', η', ζ' mean the small changes in the coordinates under deformation. Hence:

$$\frac{\partial X}{\partial u} + \frac{\partial \xi'}{\partial u} = (p_1\,a_2 - p_2\,a_1)\sqrt{E}\,(1 + \varepsilon_1) + a_0\sqrt{E}\,(1 + \varepsilon_1) .$$

Now:

$$\frac{\partial X}{\partial u} = a_0\sqrt{E} ,$$

so it will follow that when one neglects second-order quantities:

$$\frac{1}{\sqrt{E}}\frac{\partial\xi'}{\partial u} = p_1\, a_2 - p_2\, a_1 + a_0\, \varepsilon_1\ ;$$

it likewise follows that:

$$\frac{1}{\sqrt{E}}\frac{\partial\eta'}{\partial u} = p_1\, b_2 - p_2\, b_1 + b_0\, \varepsilon_1\ ;$$

$$\frac{1}{\sqrt{E}}\frac{\partial\zeta'}{\partial u} = p_1\, c_2 - p_2\, c_1 + c_0\, \varepsilon_1\ .$$

In a similar way, it follows from equations (3) on page 6 that:

$$\frac{1}{\sqrt{G}}\frac{\partial\xi'}{\partial v} = p_2\, a_0 - p_0\, a_2 + a_1\, \varepsilon_2 + a_0\, \tau,$$

$$\frac{1}{\sqrt{G}}\frac{\partial\eta'}{\partial v} = p_2\, b_0 - p_0\, b_2 + b_1\, \varepsilon_2 + b_0\, \tau,$$

$$\frac{1}{\sqrt{G}}\frac{\partial\zeta'}{\partial v} = p_2\, c_0 - p_0\, c_2 + c_1\, \varepsilon_2 + c_0\, \tau\,.$$

If one multiplies the equations above by a_0, b_0, c_0 in succession and adds them then it will follow that:

$$\sqrt{E}\varepsilon_1 = a_0\frac{\partial\xi'}{\partial u} + b_0\frac{\partial\eta'}{\partial u} + c_0\frac{\partial\zeta'}{\partial u}\,.$$

It will follow similarly that:

$$\sqrt{G}\varepsilon_2 = a_1\frac{\partial\xi'}{\partial v} + b_1\frac{\partial\eta'}{\partial v} + c_1\frac{\partial\zeta'}{\partial v}\,,$$

$$\tau = \frac{1}{\sqrt{G}}\left(a_0\frac{\partial\xi'}{\partial v} + b_0\frac{\partial\eta'}{\partial v} + c_0\frac{\partial\zeta'}{\partial v}\right) - p_2\,,$$

$$p_0 = -\frac{1}{\sqrt{G}}\left(a_2\frac{\partial\xi'}{\partial v} + b_2\frac{\partial\eta'}{\partial v} + c_2\frac{\partial\zeta'}{\partial v}\right),$$

$$p_1 = \frac{1}{\sqrt{E}}\left(a_2\frac{\partial\xi'}{\partial u} + b_2\frac{\partial\eta'}{\partial u} + c_2\frac{\partial\zeta'}{\partial u}\right),$$

$$p_2 = -\frac{1}{\sqrt{E}}\left(a_1\frac{\partial\xi'}{\partial u} + b_1\frac{\partial\eta'}{\partial u} + c_1\frac{\partial\zeta'}{\partial u}\right).$$

It is essential to simplify these expressions. I imagine that the geometric locus of the middle surface of the shell is fixed in its rest configuration. The actual middle surface oscillates about it, as the equilibrium configuration. Let u, v be any point of that middle surface in the equilibrium configuration. After deformation, I imagine that an altitude has been dropped from that point to the fixed middle surface. Let $u + u'$, $v + v'$ be the surface coordinates of the foot of that altitude, and let n be its magnitude. I now imagine that the fixed coordinate system is arranged in such a way that its origin falls upon the point u, v of the middle surface, and the axes rotate in such a way that the X-axis assumes the direction of ds_1, the Y-axis assumes the direction of ds_2, and the Z-axis assumes the direction of the normal. The cosines of the inclination angles of ds_1 with respect to the three axes are a_0, b_0, c_0 (1, 0, 0, resp.). However, one can set $d\xi' = \sqrt{E}\,du'$ in the neighborhood of the origin, and it will then follow that:

$$a_0 \frac{\partial \xi'}{\partial u} + b_0 \frac{\partial \eta'}{\partial u} + c_0 \frac{\partial \zeta'}{\partial u} = \sqrt{E}\frac{\partial u'}{\partial u},$$

and one will then find that:

$$\varepsilon_1 = \frac{\partial u'}{\partial u}.$$

It follows in a similar way that:

$$\varepsilon_1 = \frac{\partial v'}{\partial v}, \qquad \tau = \sqrt{\frac{E}{G}}\frac{\partial u'}{\partial v} - p_2,$$

$$p_0 = -\frac{1}{\sqrt{G}}\frac{\partial n}{\partial u}, \qquad p_1 = \frac{1}{\sqrt{E}}\frac{\partial n}{\partial u}, \qquad p_2 = -\sqrt{\frac{G}{E}}\frac{\partial v'}{\partial u},$$

so one ultimately has:

$$\tau = \sqrt{\frac{E}{G}}\frac{\partial u'}{\partial v} + \sqrt{\frac{G}{E}}\frac{\partial v'}{\partial u}.$$

With the use of the values for p_0, p_1, p_2 that were found above, Δ, Δ', Δ'' can also be expressed in terms of u', v', n:

$$\frac{\Delta}{E\sqrt{G}} = \frac{D}{E\sqrt{G}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial n}{\partial u}\right) + \frac{D'}{E\sqrt{G}}\frac{\partial v'}{\partial u} + \frac{1}{2G\sqrt{E}}\frac{\partial E}{\partial v}\frac{\partial n}{\partial v},$$

$$\frac{\Delta'}{E\sqrt{G}} = \frac{D'}{E\sqrt{G}} + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial n}{\partial u}\right) + \frac{D''}{E\sqrt{G}}\frac{\partial v'}{\partial u} - \frac{1}{2G\sqrt{E}}\frac{\partial G}{\partial u}\frac{\partial n}{\partial v},$$

$$\frac{\Delta'}{G\sqrt{E}} = \frac{D'}{G\sqrt{E}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial n}{\partial v}\right) - \frac{D}{E\sqrt{E}}\frac{\partial v'}{\partial u} - \frac{1}{2E\sqrt{G}}\frac{\partial E}{\partial v}\frac{\partial n}{\partial u},$$

$$\frac{\Delta''}{G\sqrt{E}} = \frac{D''}{G\sqrt{E}} + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial n}{\partial v}\right) - \frac{D'}{E\sqrt{E}}\frac{\partial v'}{\partial u} + \frac{1}{2E\sqrt{G}}\frac{\partial G}{\partial u}\frac{\partial n}{\partial u}.$$

We can express $\Delta' / (EG)$ in two different ways with these equations:

$$\frac{\Delta'}{EG} = \frac{D'}{EG} + \frac{1}{\sqrt{EG}}\frac{\partial^2 n}{\partial u\,\partial v} - \frac{1}{2E\sqrt{EG}}\frac{\partial E}{\partial v}\frac{\partial n}{\partial u} + \frac{D''}{EG}\frac{\partial v'}{\partial u} - \frac{1}{2G\sqrt{EG}}\frac{\partial G}{\partial u}\frac{\partial n}{\partial v},$$

$$\frac{\Delta'}{EG} = \frac{D'}{EG} + \frac{1}{\sqrt{EG}}\frac{\partial^2 n}{\partial u\,\partial v} - \frac{1}{2G\sqrt{EG}}\frac{\partial G}{\partial u}\frac{\partial n}{\partial v} - \frac{D}{E^2}\frac{\partial v'}{\partial u} - \frac{1}{2E\sqrt{EG}}\frac{\partial E}{\partial v}\frac{\partial n}{\partial u}.$$

However, these two expression differ by quantities that include $\partial v' / \partial u$. We conclude from this that, in general, $\partial v' / \partial u$ is small and can be neglected in comparison to the quantities that come under consideration, and therefore the derivatives of *n* with respect to *u* and *v*. We then ultimately get:

$$\frac{\Delta}{E\sqrt{EG}} = \frac{D}{E\sqrt{EG}} + \frac{1}{G}\frac{\partial^2 n}{\partial u^2} - \frac{1}{2E^2}\frac{\partial E}{\partial u}\frac{\partial n}{\partial u} + \frac{1}{2EG}\frac{\partial E}{\partial v}\frac{\partial n}{\partial v},$$

$$\frac{\Delta'}{EG} = \frac{D'}{EG} + \frac{1}{\sqrt{EG}}\frac{\partial^2 n}{\partial u\,\partial v} - \frac{1}{2E\sqrt{EG}}\frac{\partial E}{\partial v}\frac{\partial n}{\partial u} - \frac{1}{2G\sqrt{EG}}\frac{\partial E}{\partial u}\frac{\partial n}{\partial v},$$

$$\frac{\Delta''}{G\sqrt{EG}} = \frac{D''}{G\sqrt{EG}} + \frac{1}{\sqrt{E}}\frac{\partial^2 n}{\partial v^2} - \frac{1}{2G^2}\frac{\partial G}{\partial v}\frac{\partial n}{\partial v} + \frac{1}{2EG}\frac{\partial G}{\partial u}\frac{\partial n}{\partial u}.$$

The stresses that were given on page 35 can be expressed in terms of only the time-dependent quantities u', v', n with the help of these expressions for ε_1, ε_2, and τ:

$$\varepsilon_1 = \frac{\partial u'}{\partial u}, \qquad \varepsilon_2 = \frac{\partial v'}{\partial v}, \qquad \tau = \sqrt{\frac{E}{G}}\frac{\partial u'}{\partial v} + \sqrt{\frac{G}{E}}\frac{\partial v'}{\partial u}.$$

However, one remarks that these stresses are independent of D, D', D'', and depend upon only E and G. The theorem follows from this that:

If two surfaces are mutually developable and the derivatives of n, u′, v′ with respect to u and v (i.e., the relative displacements) are equal at corresponding locations then the stresses that appear at those places will also be equal.

It still remains for us to convert the summands in the equations of motion that originate in accelerations, if we ignore the external forces. We have:

$$\alpha_0 \frac{\partial^2 \xi}{\partial t^2} + \beta_0 \frac{\partial^2 \eta}{\partial t^2} + \gamma_0 \frac{\partial^2 \zeta}{\partial t^2} = \sqrt{E} \frac{\partial^2 u'}{\partial t^2},$$

$$\alpha_1 \frac{\partial^2 \xi}{\partial t^2} + \beta_1 \frac{\partial^2 \eta}{\partial t^2} + \gamma_1 \frac{\partial^2 \zeta}{\partial t^2} = \sqrt{G} \frac{\partial^2 v'}{\partial t^2},$$

$$\alpha_2 \frac{\partial^2 \xi}{\partial t^2} + \beta_2 \frac{\partial^2 \eta}{\partial t^2} + \gamma_2 \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 n}{\partial t^2},$$

$$\alpha_0 \frac{\partial^2 \alpha_2}{\partial t^2} + \beta_0 \frac{\partial^2 \beta_2}{\partial t^2} + \gamma_0 \frac{\partial^2 \gamma_2}{\partial t^2} = -\frac{\partial^2 p_1}{\partial t^2} = -\frac{1}{\sqrt{E}} \frac{\partial^3 n}{\partial t^2 \, \partial u},$$

$$\alpha_1 \frac{\partial^2 \alpha_2}{\partial t^2} + \beta_1 \frac{\partial^2 \beta_2}{\partial t^2} + \gamma_1 \frac{\partial^2 \gamma_2}{\partial t^2} = \quad \frac{\partial^2 p_1}{\partial t^2} = -\frac{1}{\sqrt{G}} \frac{\partial^3 n}{\partial t^2 \, \partial v}.$$

The last two summands are not to be considered for the case of a plane or a cylinder that is parallel to an axis, but only for any surface at all. When we apply our latter considerations to that point, that point will become a starting point, as we mentioned on page 27.

With the conversions that were made, our equations for the equilibrium and motion of an elastic shell that performs infinitely-small oscillations in the neighborhood of the equilibrium configuration will go to the equations for u', v', n, viz., the displacements inside and normal to the surface.

These equations simplify essentially for the plane, for which $D = D' = D'' = 0$. Namely, one sees that the first two main equations and boundary equations will contain only u' and v' then, while n will appear in the remaining ones, in addition to them. Those four equations for the oscillations inside of the plane will be satisfied when one sets $u' = 0$, $v' = 0$. If one also substitutes those values for u' and v' in the remaining equations then one will get equations for the transversal oscillations of the plate that correspond to the ones that **Kirchhoff** first applied to the calculation of the sound figures of circular plated. In order to obtain the equations that are appropriate to a circular plate from our equations in polar coordinates, one chooses $u = r$, $v = \vartheta$, where r is the radius vector, and ϑ means the amplitude. Since:

$$ds^2 = dr^2 + r^2 \, d\vartheta^2,$$

one must set $E = 1$, $G = r^2$. One will then get the same equations that **Clebsch** (*) had obtained by transforming the equations into orthogonal rectilinear coordinates.

Berlin, 1873

(*) *Theorie der Elasticität*, §§ 78, 129.