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On the composition of forces

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The object of the present note will be the search for the formula for the composition of forces in PLÜCKER’s system of a *New Geometry of Space.*

1. First, let us present some metric relations that will be useful in what follows. Let (**a**, **b**, **c**, **d**) and (**A**, **B**, **C**, **D**) be the vertices and opposite faces of a tetrahedron, and let:

$$(\mathbf{f}, \mathbf{l}), (\mathbf{g}, \mathbf{m}), (\mathbf{h}, \mathbf{n}) \qquad \text{or else} \quad (\mathbf{L}, \mathbf{F}), (\mathbf{M}, \mathbf{G}), (\mathbf{N}, \mathbf{H})$$

denote the three pairs of its opposite edges, which one considers to be the joins of the pairs of points:

$$(\mathbf{b}, \mathbf{c}), (\mathbf{a}, \mathbf{d}); \quad (\mathbf{c}, \mathbf{a}), (\mathbf{b}, \mathbf{d}); \quad (\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d}),$$

or intersections of the pairs of planes:

$$(\mathbf{A}, \mathbf{D}), (\mathbf{B}, \mathbf{C}); \qquad (\mathbf{B}, \mathbf{D}), (\mathbf{C}, \mathbf{A}); \qquad (\mathbf{C}, \mathbf{D}), (\mathbf{A}, \mathbf{B}) .$$

For each of these pairs of lines, one forms the product of their minimum distance with the angle that they subtend, and denotes these three products by:

$$[\mathbf{f}, \mathbf{l}] = [\mathbf{F}, \mathbf{L}], \qquad [\mathbf{g}, \mathbf{m}] = [\mathbf{M}, \mathbf{G}], \qquad [\mathbf{h}, \mathbf{n}] = [\mathbf{N}, \mathbf{H}] .$$

One will have the relations:

$$\begin{aligned} \mathbf{abcd} &= \mathbf{bc} \cdot \mathbf{ad}\,[\mathbf{f}, \mathbf{l}] = \mathbf{ca} \cdot \mathbf{bd}\,[\mathbf{g}, \mathbf{m}] = \mathbf{ab} \cdot \mathbf{cd}\,[\mathbf{h}, \mathbf{n}] \\ &= \mathbf{aA} \cdot \mathbf{dcb} = \mathbf{dB} \cdot \mathbf{dac} = \mathbf{cC} \cdot \mathbf{dba} = \mathbf{dD} \cdot \mathbf{abc}, \end{aligned} \tag{1}$$

$$\begin{aligned} \sin \mathbf{ABCD} &= \sin \mathbf{BC} \sin \mathbf{AD}\,[\mathbf{L}, \mathbf{F}] = \sin \mathbf{CA} \sin \mathbf{BD}\,[\mathbf{M}, \mathbf{G}] = \sin \mathbf{AB} \sin \mathbf{CD}\,[\mathbf{N}, \mathbf{H}] \\ &= \mathbf{Aa} \sin \mathbf{DCB} = \mathbf{Bb} \sin \mathbf{DAC} = \mathbf{Cc} \sin \mathbf{DBA} = \mathbf{Dd} \sin \mathbf{ABC} . \end{aligned}$$

If **p** and **P** denote an arbitrary point and plane, resp., then one will have:

$$\mathbf{pP} = \frac{\mathbf{pA}\cdot\mathbf{Pa}}{\mathbf{Aa}} + \frac{\mathbf{pB}\cdot\mathbf{Pb}}{\mathbf{Bb}} + \frac{\mathbf{pC}\cdot\mathbf{Pc}}{\mathbf{Cc}} + \frac{\mathbf{pD}\cdot\mathbf{Pd}}{\mathbf{Dd}} \quad (2)$$

Refer **p** and **P** to the given tetrahedron as the fundamental tetrahedron, and take the coordinates of **p** and **P** to be the quantities:

$$a = \frac{\mathbf{pA}}{\mathbf{aA}}, \qquad b = \frac{\mathbf{pB}}{\mathbf{bB}}, \qquad c = \frac{\mathbf{pC}}{\mathbf{cC}}, \qquad d = \frac{\mathbf{pD}}{\mathbf{dD}},$$
$$A = \frac{\mathbf{Pa}}{\mathbf{Aa}}, \qquad B = \frac{\mathbf{Pb}}{\mathbf{Bb}}, \qquad C = \frac{\mathbf{Pc}}{\mathbf{Cc}}, \qquad D = \frac{\mathbf{Pd}}{\mathbf{Dd}},$$

respectively; a homogeneous, first-degree equation in (a, b, c, d) or (A, B, C, D) will then represent a plane or a point, respectively.

One has the relation:

$$a + b + c + d = 1$$

between (a, b, c, d), and the symbolic relation:

$$(A \cos \mathbf{A} + B \cos \mathbf{B} + C \cos \mathbf{C} + D \cos \mathbf{D})^2 = 1$$

between (A, B, C, D), in which it is intended that after developing this, one will set:

$$\cos^2 \mathbf{P}_i = \cos \mathbf{P}_i\,\mathbf{P}_i = 1, \qquad \cos \mathbf{P}_i \cos \mathbf{P}_j = \cos \mathbf{P}_i\,\mathbf{P}_j\,.$$

If $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ and $(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$ are the vertices and opposite faces of a tetrahedron then one will have:

$$\frac{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4}{\mathbf{abcd}} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \qquad \frac{\sin \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3\mathbf{P}_4}{\sin \mathbf{ABCD}} = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}. \quad (3)$$

Let the line $(\mathbf{r}, \mathbf{R})$ be the join of two points $(\mathbf{p}, \mathbf{P})$, as well as the intersection of the two planes $(\mathbf{P}_i, \mathbf{P}_j)$; take its coordinates to be the expressions:

$$\begin{aligned} f &= b_i\,c_j - c_i\,b_j\,, & g &= c_i\,a_j - a_i\,c_j\,, & h &= a_i\,b_j - b_i\,a_j\,, \\ l &= a_i\,d_j - d_i\,a_j\,, & m &= b_i\,d_j - d_i\,b_j\,, & n &= c_i\,d_j - d_i\,c_j\,, \\ F &= B_i\,C_j - C_i\,B_j\,, & G &= C_i\,A_j - A_i\,C_j\,, & H &= A_i\,B_j - B_i\,A_j\,, \\ L &= A_i\,D_j - D_i\,A_j\,, & M &= B_i\,A_j - D_i\,C_j\,, & N &= C_i\,D_j - D_i\,C_j\,, \end{aligned}$$

in which one will have:

$$fl + gm + hn = 0, \qquad FL + GM + HN = 0,$$

identically, and one will get:

$$(4)\quad \begin{aligned}
f &= \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{bc}}\frac{[\mathbf{r},\mathbf{l}]}{[\mathbf{f},\mathbf{l}]}, & g &= \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{ca}}\frac{[\mathbf{r},\mathbf{m}]}{[\mathbf{g},\mathbf{m}]}, & h &= \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{ab}}\frac{[\mathbf{r},\mathbf{n}]}{[\mathbf{h},\mathbf{n}]},\\
l &= \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{ad}}\frac{[\mathbf{r},\mathbf{f}]}{[\mathbf{l},\mathbf{f}]}, & m &= \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{bd}}\frac{[\mathbf{r},\mathbf{g}]}{[\mathbf{m},\mathbf{g}]}, & n &= \frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{cd}}\frac{[\mathbf{r},\mathbf{h}]}{[\mathbf{n},\mathbf{h}]},\\
F &= \frac{\sin\mathbf{P}_i\mathbf{P}_j}{\sin\mathbf{BC}}\frac{[\mathbf{R},\mathbf{L}]}{[\mathbf{F},\mathbf{L}]}, & G &= \frac{\sin\mathbf{P}_i\mathbf{P}_j}{\sin\mathbf{CA}}\frac{[\mathbf{R},\mathbf{M}]}{[\mathbf{G},\mathbf{M}]}, & H &= \frac{\sin\mathbf{P}_i\mathbf{P}_j}{\sin\mathbf{AB}}\frac{[\mathbf{R},\mathbf{N}]}{[\mathbf{H},\mathbf{N}]},\\
L &= \frac{\sin\mathbf{P}_i\mathbf{P}_j}{\sin\mathbf{AD}}\frac{[\mathbf{R},\mathbf{F}]}{[\mathbf{L},\mathbf{F}]}, & M &= \frac{\sin\mathbf{P}_i\mathbf{P}_j}{\sin\mathbf{BD}}\frac{[\mathbf{R},\mathbf{G}]}{[\mathbf{M},\mathbf{G}]}, & N &= \frac{\sin\mathbf{P}_i\mathbf{P}_j}{\sin\mathbf{CD}}\frac{[\mathbf{R},\mathbf{H}]}{[\mathbf{N},\mathbf{H}]}.
\end{aligned}$$

One will have the relations:

$$(5)\quad \frac{f\cdot\mathbf{bc}}{L\sin\mathbf{AD}} = \frac{g\cdot\mathbf{ca}}{M\sin\mathbf{BD}} = \frac{h\cdot\mathbf{ab}}{N\sin\mathbf{CD}} = \frac{l\cdot\mathbf{ad}}{F\sin\mathbf{BC}} = \frac{m\cdot\mathbf{bd}}{G\sin\mathbf{CA}} = \frac{n\cdot\mathbf{cd}}{H\sin\mathbf{AB}} = \frac{\mathbf{p}_i\mathbf{p}_j}{\sin\mathbf{P}_i\mathbf{P}_j}$$

between the coordinates of **r** and those of **R**, and between (f, …, l, …), or even (F, …, L, …), one will have the symbolic relation:

$$(6)\quad \begin{aligned}
&(f\,\mathbf{bc}\cos\mathbf{f} + \ldots + l\,\mathbf{ad}\cos\boldsymbol{l} + \ldots)^2 = \mathbf{p}_i\,\mathbf{p}_j^2,\\
&(F\sin\mathbf{BC}\cos\mathbf{F} + \ldots + L\sin\mathbf{AD}\cos\mathbf{L} + \ldots)^2 = \sin^2\mathbf{P}_i\,\mathbf{P}_j,
\end{aligned}$$

in which it is intended that after developing this, one will set:

$$\begin{aligned}
\cos^2\mathbf{r}_i &= \cos\mathbf{r}_i\,\mathbf{r}_i = 1, &\quad \cos\mathbf{r}_i\cos\mathbf{r}_j &= \cos\mathbf{r}_i\,\mathbf{r}_j;\\
\cos^2\mathbf{R}_i &= \cos\mathbf{R}_i\,\mathbf{R}_i = 1, &\quad \cos\mathbf{R}_i\cos\mathbf{R}_j &= \cos\mathbf{R}_i\,\mathbf{R}_j.
\end{aligned}$$

Finally, if ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$) are two lines that are the joins of the pairs of points $(\mathbf{p}'_i,\mathbf{p}'_j)$, $(\mathbf{p}''_i,\mathbf{p}''_j)$, resp., and the intersections of the pairs of planes $(\mathbf{P}'_i,\mathbf{P}'_j)$, $(\mathbf{P}''_i,\mathbf{P}''_j)$, resp., then one will have:

$$(7)\quad \begin{aligned}
\frac{\mathbf{p}'_i\mathbf{p}'_j\mathbf{p}''_i\mathbf{p}''_j}{\mathbf{abcd}} &= f'l'' + g'm'' + h'n'' + l'f'' + m'g'' + n'h'',\\
\frac{\sin\mathbf{P}'_i\mathbf{P}'_j\,\mathbf{P}''_i\mathbf{P}''_j}{\mathbf{ABCD}} &= F'L'' + G'M'' + H'N'' + L'F'' + M'G'' + N'H''.
\end{aligned}$$

2. Assume, as is obviously true, that:

1. The resultant of two concurrent forces lies in their plane and bisects the angle between them when they are equal, in addition.
2. In order to find the resultant of more forces, one must distribute the given forces into groups, compose the forces in each group, and finally compose the partial resultants of the various groups.

Referring to the fundamental tetrahedron of the preceding number, consider the three concurrent lines $\boldsymbol{l}$, $\mathbf{m}$, $\mathbf{n}$ at the point $\mathbf{d}$, and let the symbol (l, m, n) denote the direction of the resultant of forces l, m, n that are equal to φ and directed along $\boldsymbol{l}$, $\mathbf{m}$, $\mathbf{n}$, respectively. Let (1, 0, 0), (0, 1, 0), (0, 0, 1) be the same lines as $\boldsymbol{l}$, $\mathbf{m}$, $\mathbf{n}$, resp., let (0, 1, 1), (1, 0, 1), (1, 1, 0) be the bisectors of the angles between ($\mathbf{m}$, $\mathbf{n}$), ($\mathbf{n}$, $\boldsymbol{l}$), ($\boldsymbol{l}$, $\mathbf{m}$), resp., and let (1, 1, 1) be the line along which the three planes intersect that are defined by the pairs of lines [(1, 0, 0), (0, 1, 1)], [(0, 1, 0), (1, 0, 1)], [(0, 0, 1), (1, 1, 0)].

By the principles that were previously assumed, it is easy to see that for $(1, 1, n)$ (which is in the plane of the lines [(0, 0, 1), (1, 1, 0)]) and for (1, 0, 0) and (0, 1, 0), one is led to two planes that intersect the planes of the lines [(0, 1, 0), (0, 0, 1)], [(1, 0, 0), (0, 0, 1)] along $(0, 1, n)$ and $(1, 0, n)$, and the planes of the lines [$(1, 0, n)$, (0, 1, 1)], [$(0, 1, n)$, (1, 0, 1)] intersect along $(1, 1, n + 1)$; therefore, if one starts with (1, 1, 1) then one will successively get (1, 1, 2), (1, 1, 3), …, $(1, 1, n)$ by this construction. One will successively get (1, 2, 1), (1, 3, 1), …, $(1, m, 1)$ and (2, 1, 1), (3, 1, 1), …, $(l, 1, 1)$ in an analogous way. After that, the pairs of the planes of the lines [$(1, m, 1)$, (0, 0, 1)], [$(1, 1, n)$, (0, 1, 0)]; [$(1, 1, n)$, (1, 0, 0)], [$(l, 1, 1)$, (0, 0, 1)], [$(1, 1, n)$, (0, 1, 0)]; [$(1, 1, n)$, (1, 0, 0)], [$(l, 1, 1)$, (0, 0, 1)]; [$(l, 1, 1)$, (0, 1, 0)], [$(1, m, 1)$, (1, 0, 0)] will intersect along the lines $(1, m, n)$, $(l, 1, n$, $(l, m, 1)$, respectively, and the planes of the lines [$(1, m, n)$, (1, 0, 0)], [$(l, 1, n)$, (0, 1, 0)], [$(l, m, 1)$, (0, 0, 1)] will intersect along the line (l, m, n); one can thus obtain the direction (l, m, n) of the resultant of the system of forces $(l, m, n)\ \varphi$.

Now, determine a plane $\mathbf{P}$ that goes through $\mathbf{d}$ with respect to the triad of fundamental lines ($\boldsymbol{l}$, $\mathbf{m}$, $\mathbf{n}$) by means of the coordinates:

$$X = \sin \boldsymbol{l}\,\mathbf{P}, \qquad Y = \sin \mathbf{m}\,\mathbf{P}, \qquad Z = \sin \mathbf{n}\,\mathbf{P},$$

and one will have:

$$\sin \mathbf{r}\,\mathbf{P} = X\,\frac{\sin \mathbf{rA}}{\sin \boldsymbol{l}\mathbf{A}} + Y\,\frac{\sin \mathbf{rB}}{\sin \mathbf{mB}} + Z\,\frac{\sin \mathbf{rC}}{\sin \mathbf{nC}} \tag{1}$$

for an arbitrary line $\mathbf{r}$ through $\mathbf{d}$, and thus, the first-degree equation:

$$X\,\frac{\sin \mathbf{rA}}{\sin \boldsymbol{l}\mathbf{A}} + Y\,\frac{\sin \mathbf{rB}}{\sin \mathbf{mB}} + Z\,\frac{\sin \mathbf{rC}}{\sin \mathbf{nC}} = 0 \tag{2}$$

between the coordinates (X, Y, Z) of $\mathbf{P}$ will determine the line $\mathbf{r}$.

As a result of that, the equations of the line (1, 1, 1), and the lines (0, 1, 1), (1, 0, 1), (1, 1, 0) will be:

$$X + Y + Z = 0,$$
$$Y + Z = 0, \qquad Z + X = 0, \qquad X + Y = 0,$$

respectively.

In general, the equation of the line (l, m, n) will be:

$$xX + yY + zZ = 0,$$

and the equations of (0, m, n), (l, 0, n), (l, m, 0) will be:

$$yY + zZ = 0, \quad zZ + xX = 0, \quad xX + yY = 0.$$

Thus, if one starts with the equation for (1, 1, 1) and then forms the equations of the lines that were previously considered in order to arrive at the line (l, m, n) then one will successively find the equations:

$X + Y + nZ = 0$, …(1, 1, n); $X + mY + Z = 0$, …(1, m, 1); $lX + Y + Z = 0$, …(l, 1, 1);
$X + mY + nZ = 0$, …(1, m, n); $lX + Y + nZ = 0$, …(l, 1, m); $lX + mY + Z = 0$, …(l, m, 1);
$lX + mX + nZ= 0$… (l, m, n).

Comparing this equation for the line (l, m, n) – or **r** – with (2) will give:

$$\frac{l \sin \boldsymbol{l}\mathbf{A}}{\sin \mathbf{rA}} = \frac{m \sin \mathbf{mB}}{\sin \mathbf{rB}} = \frac{n \sin \mathbf{nC}}{\sin \mathbf{rC}} = r$$

and

(3)

$$r^2 = l^2 + m^2 + n^2 + 2mn \cos \mathbf{mn} + 2nl \cos \mathbf{n}\boldsymbol{l} + 2lm \cos \boldsymbol{l}\mathbf{m};$$

[By the known relation ([1]):

$$\frac{\sin^2 \mathbf{rA}}{\sin^2 \boldsymbol{l}\mathbf{A}} + \cdots + 2\frac{\sin \mathbf{rB}}{\sin \mathbf{mB}}\frac{\sin \mathbf{rC}}{\sin \mathbf{nC}} \cos \mathbf{mn} \ldots = 1,$$

so

$$\frac{l}{\sin \mathbf{rmn}} = \frac{m}{\sin \mathbf{rn}\boldsymbol{l}} = \frac{n}{\sin \mathbf{r}\boldsymbol{l}\mathbf{m}} = \frac{r}{\sin \boldsymbol{l}\mathbf{mn}}.]$$

Let $r'\varphi$ be the value of the resultant of the force (l, m, n) φ; if this force is rotated into the opposite sense then one will have the force $n\varphi$ that is equal and opposite to the resultant of the force (l, m, n') φ, so if one denotes the directions that are opposite to **r** and **n** by **r′** and **n′**, respectively, then one will have:

$$\frac{l}{\sin \mathbf{n'mr'}} = \frac{m}{\sin \mathbf{n'r'}\boldsymbol{l}} = \frac{r'}{\sin \mathbf{n'}\boldsymbol{l}\mathbf{m}},$$

so (paying attention to the signs in the expressions):

([1]) Note: "sulle forme geometriche di 2^a specie," Rendiconto dell'Accademia, February 1865.

$$\frac{l}{\sin \mathbf{rmn}} = \frac{m}{\sin \mathbf{rn}\boldsymbol{l}} = \frac{r'}{\sin \boldsymbol{l}\mathbf{mn}}.$$

Therefore $r' = r$, or $r\varphi$ will be the value of the resultant of the system $(l, m, n)\ \varphi$.

The preceding results, with just some considerations regarding the limits, can be extended to the general case in which l, m, n, are no longer integers, but the ratios of three arbitrary forces in the directions $\boldsymbol{l}$, $\mathbf{m}$, $\mathbf{n}$, respectively, and an arbitrary force φ that can be taken to be unity. Equations (3) will then lead to the composition of arbitrary forces that are concurrent at a point.

One should note that if (A, B, C, D) are the coordinates of an arbitrary plane **P** that were defined in the preceding number then the point at which the line (l, m, n) meets the plane **D** will be determined by the equation:

$$lA\frac{\mathbf{aA}}{\mathbf{ad}} + mB\frac{\mathbf{bB}}{\mathbf{bd}} + nC\frac{\mathbf{cC}}{\mathbf{cd}} = 0. \tag{4}$$

Observe that from equations (1) and (3), one has:

$$r \sin \mathbf{rP} = lX + mY + nZ = l \sin \boldsymbol{l}\mathbf{P} + m \sin \mathbf{mP} + n \sin \mathbf{nP}, \tag{5}$$

which is called the *moment* of a force with respect to a plane, and is the product of the force with the sine of its inclination with respect to the plane, so relation (5) will reduce to the property that *in a system of forces that are concurrent at a point, the moment of the resultant with respect to a plane is equal to the sum of the moments of the components.*

Now consider three lines **L**, **M**, **N** that lie in the plane **D**, and let the symbol (L, M, N) denote the direction of the result of L, M, N, which are forces that are equal to Φ and point in the directions **L**, **M**, **N**, respectively. By an argument that is similar to the preceding one, one will see how one can arrive at (L, M, N), since (1, 1, 1) can be defined by successive joins of some points. Consequently, if one determines a point **p** of **D** with the coordinates:

$$x = \mathbf{Lp}, \qquad y = \mathbf{Mp}, \qquad z = \mathbf{Np},$$

and as a consequence, if:

$$\mathbf{Rp} = x\frac{\mathbf{Ra}}{\mathbf{La}} + y\frac{\mathbf{Rb}}{\mathbf{Mb}} + z\frac{\mathbf{Rc}}{\mathbf{Nc}} \tag{1}$$

is a line **R** in **D** with the equation:

$$x\frac{\mathbf{Ra}}{\mathbf{La}} + y\frac{\mathbf{Rb}}{\mathbf{Mb}} + z\frac{\mathbf{Rc}}{\mathbf{Nc}} = 0 \tag{2}$$

then one will find that the equation of (L, M, N) – or **R** – is:

$$Lx + My + Nz = 0 \ldots (L, M, N).$$

One will then have the formula:

(3)
$$L\,\frac{\mathbf{La}}{\mathbf{Ra}} = M\,\frac{\mathbf{Mb}}{\mathbf{Rb}} = N\,\frac{\mathbf{Nc}}{\mathbf{Rc}} = R,$$
$$R^2 = L^2 + M^2 + N^2 + 2MN\cos\mathbf{MN} + 2NL\cos\mathbf{NL} + 2LM\cos\mathbf{LM},$$

for the composition of forces in a plane.
[By the known relation ([1]):

$$\left(\frac{\mathbf{Ra}}{\mathbf{La}}\right)^2 + \ldots + 2\frac{\mathbf{Rb}}{\mathbf{Mb}}\frac{\mathbf{Rc}}{\mathbf{Nc}}\cos\mathbf{MN} + \ldots = 1,$$

so

$$\frac{L}{\sin\mathbf{RMN}} = \frac{M}{\sin\mathbf{RNL}} = \frac{N}{\sin\mathbf{RLM}} = \frac{R}{\sin\mathbf{LMN}},$$

and $R\Phi$ will be the value of the resultant of the system $(L, M, N)\,\Phi$.]

One will note that the equation of the plane that goes through **R** and **d** is:

(4)
$$La\frac{\mathbf{Aa}}{\sin\mathbf{AD}} + Mb\frac{\mathbf{Bb}}{\mathbf{BD}} + Nc\frac{\mathbf{Cc}}{\sin\mathbf{CD}} = 0$$

in terms of the coordinates (a, b, c, d) of a point **p** .

Finally, from the relation:

(5)
$$R\,\mathbf{Rp} = Lx + My + Nz = L\,\mathbf{Lp} + M\,\mathbf{Mp} + N\,\mathbf{Np},$$

one will deduce the property that *in a system of forces that lie in a plane, the moment of the resultant with respect to a point* (viz., the product of the force with the distance from its direction to the point) *is equal to the sum of the moments of the components.*

3. Now let $(f, g, h, l, m, n)\,\varphi$ be the forces in the directions of the same-named edges of the fundamental tetrahedron. If one composes the group of forces $(h, g, l)\,\varphi$, $(f, h, m)\,\varphi$, $(g, f, n)\,\varphi$, $(l, m, n)\,\varphi$ then the points at which their partial resultants will encounter the planes **A**, B**, C**, **D**, respectively, will be represented by the equations:

(1)
$$hB\frac{\mathbf{bB}}{\mathbf{ab}} - gC\frac{\mathbf{cC}}{\mathbf{ca}} + lD\frac{\mathbf{dD}}{\mathbf{ad}} = 0,$$
$$fC\frac{\mathbf{cC}}{\mathbf{bc}} - hA\frac{\mathbf{aA}}{\mathbf{ab}} + mD\frac{\mathbf{dD}}{\mathbf{bd}} = 0,$$

([1]) *Ibid.*

$$gA\frac{\mathbf{aA}}{\mathbf{ca}} - fB\frac{\mathbf{bB}}{\mathbf{bc}} + nD\frac{\mathbf{dD}}{\mathbf{cd}} = 0,$$

$$lA\frac{\mathbf{aA}}{\mathbf{ad}} + mB\frac{\mathbf{bB}}{\mathbf{bb}} + nC\frac{\mathbf{cC}}{\mathbf{cd}} = 0,$$

any one of which will be a consequence of the other three, and will be those four points and vertices of a tetrahedron that is, at the same time, inscribed and circumscribed by the fundamental tetrahedron.

The four points that are determined from (1) will belong to the same line **r** when one has the condition:

(2) $$\frac{fl}{\mathbf{bc}\cdot\mathbf{ad}} + \frac{gm}{\mathbf{ca}\cdot\mathbf{bd}} + \frac{hn}{\mathbf{ab}\cdot\mathbf{cd}} = 0.$$

In that case, the force in question will admit a resultant $r\varphi$ in the direction **r**, and one will have:

(3) $$f\frac{[\mathbf{f},\boldsymbol{l}]}{[\mathbf{r},\boldsymbol{l}]} = g\frac{[\mathbf{g},\mathbf{m}]}{[\mathbf{r},\mathbf{m}]} = h\frac{[\mathbf{h},\mathbf{n}]}{[\mathbf{r},\mathbf{n}]} = l\frac{[\boldsymbol{l},\mathbf{f}]}{[\mathbf{r},\mathbf{f}]} = m\frac{[\mathbf{m},\mathbf{g}]}{[\mathbf{n},\mathbf{g}]} = n\frac{[\mathbf{n},\mathbf{h}]}{[\mathbf{r},\mathbf{h}]} = r,$$
$$r^2 = (f\cos\mathbf{f} + g\cos\mathbf{g} + h\cos\mathbf{h} + l\cos\boldsymbol{l} + m\cos\mathbf{m} + n\cos\mathbf{n})^2,$$

from formula (6) of number 1.

Formula (3) leads to the composition of forces that are directed in whatever way in space. Suppose, for the greatest simplicity, that the distance between the points ($\mathbf{p}_i$, $\mathbf{p}_j$) that is taken arbitrarily on the line **r** in order to determine the coordinates (f, g, h, l, m,n) is equal to the quantity r with which one can represent the force in the direction **r**, and now let (f), (g), (h), (l), (m), (n) denote the components of r along the edges (**f**, **g**, **h**, ***l***, **m**, **n**) of the fundamental tetrahedron. From equations (4) on number 1, formula (3) will give the very simple relations:

(4) $$(f) = f\,\mathbf{bc},\quad (g) = g\,\mathbf{ca},\quad (h) = h\,\mathbf{ab},\quad (l) = l\,\mathbf{ad},\quad (m) = m\,\mathbf{bd},\quad (n) = n\,\mathbf{cd}.$$

It follows from this that an arbitrary system of forces $r_1, r_2, \ldots, r_i, \ldots$ in the directions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots$ can reduce to six forces along the edges of the fundamental tetrahedron that are expressed by:

(5) $$(f) = \mathbf{bc}\sum f_i,\quad (g) = \mathbf{ca}\sum g_i,\quad (h) = \mathbf{ab}\sum h_i,\quad (l) = \mathbf{ad}\sum l_i,\quad (m) = \mathbf{bd}\sum m_i,\quad (n) = \mathbf{cd}\sum n_i.$$

This force will admit a resultant r along the line **r** that has the coordinates:

$$f = \sum f_i,\quad g = \sum g_i,\quad h = \sum h_i,\quad l = \sum l_i,\quad m = \sum m_i,\quad n = \sum n_i,$$

which verify the condition:

(5) $$(\sum f_i)(\sum l_i) + (\sum g_i)(\sum m_i) + (\sum h_i)(\sum n_i) = 0,$$

and one will then have:

$$r^2 = [(\sum f_i)\ \mathbf{bc} \cos \mathbf{f} + \ldots + (\sum l_i)\ \mathbf{ad} \cos \boldsymbol{l} + \ldots]^2,$$

so if one observes that:

$$(f_i\ \mathbf{bc} \cos \mathbf{f} + \ldots + l_i\ \mathbf{ad} \cos \boldsymbol{l} + \ldots)^2 = r_i^2$$

$$(f_i\ \mathbf{bc} \cos \mathbf{f} + \ldots + l_i\ \mathbf{ad} \cos \boldsymbol{l} + \ldots)(f_j\ \mathbf{bc} \cos \mathbf{f} + \ldots + l_j\ \mathbf{ad} \cos \boldsymbol{l} + \ldots) = \mathbf{r}_i\ \mathbf{r}_j \cos \mathbf{r}_i\ \mathbf{r}_j$$

then one will have:

$$r^2 = \sum r_i^2 + 2\sum r_i\, r_j \cos \mathbf{r}_i\, \mathbf{r}_j = \left(\sum r_i \cos \mathbf{r}_i\right)^2. \tag{6}$$

The line **r** will be at infinity, and will verify the conditions:

$$\begin{aligned}
&\sum h_i - \sum g_i + \sum l_i = h - g + l = 0,\\
&\sum f_i - \sum h_i + \sum m_i = f - h + m = 0,\\
&\sum g_i - \sum f_i + \sum n_i = g - f + n = 0,\\
&\sum l_i + \sum m_i + \sum n_i = l + m + n = 0,
\end{aligned} \tag{7}$$

and then, if one observes that in general one will have:

$$\begin{aligned}
- r^2 = {} & (f - h + m)(g - f + n)\ \mathbf{bc}^2 + (g - f + n)(h - g + l)\ \mathbf{ca}^2 + (h - g + l)(f - h + m)\ \mathbf{ab}^2\\
& - (h - g + l)(l + m + n)\ \mathbf{ad}^2 - (f - h + m)(l + m + n)\ \mathbf{bd}^2 - (g - f + m)(l + m + n)\ \mathbf{cd}^2,
\end{aligned}$$

then one will have $r = 0$ (i.e., the case of a resultant *couple*).

In the general case in which equation (5) is not satisfied, all of the forces in the system can reduce to two, one of which passes through the vertex of the fundamental tetrahedron, and the other of which points in the opposite direction.

Finally, the force in question can be in equilibrium when:

$$\sum f_i = 0, \quad \sum g_i = 0, \quad \sum h_i = 0, \quad \sum l_i = 0, \quad \sum m_i = 0, \quad \sum n_i = 0. \tag{8}$$

If one applies the first of equations (7) in number 1 to the two lines $\mathbf{r}_i$, $\mathbf{r}_k$ then this can be put into the form:

$$[\mathbf{r}_i, \mathbf{r}_k] = \frac{[\mathbf{r}_i, \boldsymbol{l}]}{[\mathbf{f}, \boldsymbol{l}]}\ [\mathbf{r}_k, \mathbf{f}] + \ldots + \frac{[\mathbf{r}_i, \mathbf{f}]}{[\boldsymbol{l}, \mathbf{f}]}\ [\mathbf{r}_k, \boldsymbol{l}] + \ldots,$$

so

$$\begin{aligned}
r_i\, [\mathbf{r}_i, \mathbf{r}_k] &= f_i\ \mathbf{bc}\ [\mathbf{r}_k, \mathbf{f}] + \ldots + l_i\ \mathbf{ad}\ [\mathbf{r}_k, \boldsymbol{l}] + \ldots,\\
\sum r_i\, [\mathbf{r}_i, \mathbf{r}_k] &= (\sum f_i)\ \mathbf{bc}\ [\mathbf{r}_k, \mathbf{f}] + \ldots + (\sum l_i)\ \mathbf{ad}\ [\mathbf{r}_k, \boldsymbol{l}] + \ldots,\\
&= f\ \mathbf{bc}\ [\mathbf{r}_k, \mathbf{f}] + \ldots + l\ \mathbf{ad}\ [\mathbf{r}_k, \boldsymbol{l}] + \ldots = r\, [\mathbf{r}, \mathbf{r}_k]\ ,
\end{aligned} \tag{9}$$

which is called the *moment* of a force with respect to an axis, and is the product of the force with the minimum distance from its direction to the axis and the sine of the angle that is subtended between its direction and the axis, one will have the property that *in an arbitrary system of forces that admits a resultant, the moment of that resultant with respect to an axis will be equal to the sum of the moments of its components.*

Suppose that the two systems of forces $(r_1', r_2', \ldots)$, $(-r_1'', -r_2'', \ldots)$ are in equilibrium, or in other words, that the systems $(r_1', r_2', \ldots)$, $(-r_1'', -r_2'', \ldots)$ are *equivalent*, so one will have:

$$\sum f_i' = \sum f_i'', \ldots, \sum l_i' = \sum l_i'', \ldots$$

and therefore:

$$\left(\sum r_i' \cos \mathbf{r}_i'\right)^2 = \left(\sum r_i'' \cos \mathbf{r}_i''\right)^2 \qquad \text{and} \qquad \sum r_i' [\mathbf{r}_i', \mathbf{r}_k] = \sum r_i'' [\mathbf{r}_i'', \mathbf{r}_k]$$

so for all equivalent systems of forces (whether the forces can or cannot reduce to just *one* resultant) the quantities $(\sum r_i \cos \mathbf{r}_i)^2$ and $\sum r_i [\mathbf{r}_i, \mathbf{r}_k]$ will always preserve the same value. Therefore, one can say that such a quantity is an *invariant* of the system of forces, where the invariant properties refer to the transformations of that system of forces into another one.

Similarly, if $(F, G, H, L, M, N)\ \Phi$ are forces that are directed along the edges of the same name of the fundamental tetrahedron then by an argument that is similar to the preceding one, one will get formulas that are analogous to (1), (2), …, (8), (9) [with the exception of (7)] that can be deduced from the latter by changing the lower-case letters to upper-case and then taking the sines of the angles between the faces that pass through the opposite edges, instead of the lengths of the edges of the fundamental tetrahedron. One supposes, in addition, that the two planes $(\mathbf{P}_i, \mathbf{P}_j)$ pass through the line $\mathbf{R}$ arbitrarily, and thus determines the coordinate by which one represents the force in the direction of $\mathbf{R}$, which takes the form of an angle whose sine (for a conveniently-chosen unit of force Φ) can be assumed to be equal to the quantity R.

The dual to the process of the composition of forces that was presented above relates to the mechanical principal of duality that has its origins in the double way of considering forces as producing either translations of points or rotations of planes. The passage from a system of forces of the first category to an equivalent one that belongs to the second category, as it is developed from the theory of moments, will form the object of another note.