"Sul movimento geometrico infinitesimo di un sistema rigido," Rend. dell'Accad. scienze fisiche Napoli **9** (1870), 89-100.

On the infinitesimal geometric motion of a rigid system

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As a continuation of the notes that were inserted in the Rendiconti (fascicules for February, May, and August 1869) and related to the *statics* of systems of invariable form, we now pass on to the treatment of the *kinematics* of those systems.

1. In order to evaluate an infinitesimal rotation around an axis that is impressed upon a rigid system by means of the space that is described at an arbitrary point of the system that is at a unit distance from the axis, let δL, δM, δN be infinitesimal rotations that are impressed in a rigid system around the edges **L**, **M**, **N** , resp., that belong to the face **D** of the fundamental tetrahedron. Set (*):

$$\delta R^2 = \delta L^2 + \delta M^2 + \delta N^2 + 2\,\delta M\,\delta N \cos \mathbf{MN} + 2\,\delta N\,\delta L \cos \mathbf{NL} + 2\,\delta L\,\delta M \cos \mathbf{LM},$$

so δR will be the resultant of the forces that one intends to represent by (δL, δM, δN), which act along the lines (**L**, **M**, **N**), resp., and the line of action of that resultant will be determined in the plane **D** by the equation:

$$\delta L a\,\frac{\mathbf{Aa}}{\sin \mathbf{AD}} + \delta M b\,\frac{\mathbf{Bb}}{\sin \mathbf{BD}} + \delta M c\,\frac{\mathbf{Cc}}{\sin \mathbf{CD}} = 0.$$

For any point **p** of **D**, one will then have the relation:

$$\delta R\ \mathbf{Rp} = \delta L\ \mathbf{Lp} + \delta L\ \mathbf{Lp} + \delta N\ \mathbf{Np},$$

so if one observes that the right-hand side of this equation denotes the infinitesimal displacement of the point **p** by the three simultaneous rotations δL, δM, δN around the axes **L**, **M**, **N** , resp., and that the left-hand side denotes the infinitesimal displacement of **p** by the single rotation δR around the axis **R**, then it will follow that the three rotations

(*) Note: "sulle compositione delle forze," Rend. dell'Accad., February 1869.

compose into just one in the same way that forces compose ([**]). They will compose into just one around an axis **R** that is the intersection of four planes:

$$(1)\qquad \begin{aligned}
&\delta H b \frac{\mathbf{bB}}{\sin \mathbf{AB}} - \delta G c \frac{\mathbf{cC}}{\sin \mathbf{CA}} + \delta L d \frac{\mathbf{dD}}{\sin \mathbf{AD}} = 0,\\
&\delta F c \frac{\mathbf{cC}}{\sin \mathbf{BC}} - \delta H a \frac{\mathbf{aA}}{\sin \mathbf{AB}} + \delta M d \frac{\mathbf{dD}}{\sin \mathbf{BD}} = 0,\\
&\delta G a \frac{\mathbf{aA}}{\sin \mathbf{CA}} - \delta F b \frac{\mathbf{bB}}{\sin \mathbf{BC}} + \delta N d \frac{\mathbf{dD}}{\sin \mathbf{CD}} = 0,\\
&\delta L a \frac{\mathbf{aA}}{\sin \mathbf{AD}} + \delta M b \frac{\mathbf{bB}}{\sin \mathbf{BD}} + \delta N c \frac{\mathbf{cC}}{\sin \mathbf{CD}} = 0,
\end{aligned}$$

if one satisfies the condition:

$$(2)\qquad \frac{\delta F\,\delta L}{\sin \mathbf{BC} \sin \mathbf{AD}} + \frac{\delta G\,\delta M}{\sin \mathbf{CA} \sin \mathbf{BD}} + \frac{\delta H\,\delta N}{\sin \mathbf{AB} \sin \mathbf{CD}} = 0.$$

The resultant rotation δR will then be given symbolically by:

$$(3)\qquad \delta R^2 = (\delta F \cos \mathbf{F} + \delta G \cos \mathbf{G} + \delta H \cos \mathbf{H} + \delta L \cos \mathbf{L} + \delta M \cos \mathbf{M} + \delta N \cos \mathbf{N})^2.$$

When the rotations (δF, …, δL, …) verify the conditions:

$$(4)\qquad \begin{gathered}
\frac{\delta N}{\mathbf{ab}} - \frac{\delta M}{\mathbf{ca}} + \frac{\delta F}{\mathbf{ad}} = 0, \quad \frac{\delta L}{\mathbf{bc}} - \frac{\delta N}{\mathbf{ad}} + \frac{\delta G}{\mathbf{bd}} = 0, \quad \frac{\delta M}{\mathbf{ca}} - \frac{\delta L}{\mathbf{bc}} + \frac{\delta H}{\mathbf{cd}} = 0,\\
\frac{\delta F}{\mathbf{ad}} + \frac{\delta G}{\mathbf{bd}} + \frac{\delta H}{\mathbf{cd}} = 0,
\end{gathered}$$

the line **R** will be at infinity and one will have $\delta R = 0$; in such a case, the given rotations will give rise to a *translation* of the system.

If one regards the line **R** as the intersection of two planes ($\mathbf{P}_i$, $\mathbf{P}_j$), and denote its coordinates by (F, …, L, …) then one will have:

$$(5)\qquad \begin{gathered}
\frac{\delta F}{F \sin \mathbf{BC}} = \frac{\delta G}{G \sin \mathbf{CA}} = \frac{\delta H}{H \sin \mathbf{AB}} = \frac{\delta L}{L \sin \mathbf{AD}} = \frac{\delta M}{M \sin \mathbf{BD}} = \frac{\delta N}{N \sin \mathbf{CD}}\\
= \frac{\delta R}{\sin \mathbf{P}_i \mathbf{P}_j} = \delta\tau.
\end{gathered}$$

([**]) POINSOT, *Théorie nouvelle de la rotation des corps.*

With this formula, an infinitesimal rotation $\delta R = \delta\tau \sin \mathbf{P}_i\,\mathbf{P}_j$ around the line **R** will *decompose* into six infinitesimal rotations:

$$\delta F = \delta\tau\, F \sin \mathbf{BC}, \ \ldots, \ \delta L = \delta\tau\, L \sin \mathbf{AD}, \ \ldots$$

around the edges (**F**, …, **L**, …), resp., of the fundamental tetrahedron.

This formula, when adapted to the case in which a translation $\delta r = \delta\tau\, \mathbf{p}_i\, \mathbf{p}_j$ is impressed upon the system along with a rotation, will express the infinitesimal displacement of an arbitrary point of the system by δr and the distances between two planes that are perpendicular to the direction of the translation by $\mathbf{p}_i\, \mathbf{p}_j$. One will then have the relations:

$$(4)\qquad \begin{aligned} &\frac{N \sin \mathbf{CD}}{\mathbf{ab}} - \frac{M \sin \mathbf{BD}}{\mathbf{ca}} + \frac{F \sin \mathbf{BC}}{\mathbf{ad}} = 0, \qquad && \frac{L \sin \mathbf{AD}}{\mathbf{bc}} - \frac{N \sin \mathbf{CD}}{\mathbf{ab}} + \frac{G \sin \mathbf{CA}}{\mathbf{bd}} = 0, \\ &\frac{M \sin \mathbf{BD}}{\mathbf{ca}} - \frac{L \sin \mathbf{AD}}{\mathbf{bc}} + \frac{H \sin \mathbf{AB}}{\mathbf{cd}} = 0, && \frac{F \sin \mathbf{BC}}{\mathbf{ad}} + \frac{G \sin \mathbf{CA}}{\mathbf{bd}} + \frac{H \sin \mathbf{AB}}{\mathbf{cd}} = 0, \end{aligned}$$

between the coordinates (F, …, L, …) of the line **R**, which is the intersection at infinity of the two planes above.

If (F, …, L, …) are the coordinates of a line $\mathbf{R}_k$ that is the intersection of two planes, and R_i is the sine of the angle between them, then one will have:

$$(6)\qquad \delta R\, R_k\, [\mathbf{R}, \mathbf{R}_i] = \delta\tau\, (F L_k + \ldots + L F_k + \ldots) \sin \mathbf{ABCD},$$

and it is easy to see that $\delta R\,[\mathbf{R}, \mathbf{R}_i]$ expresses the infinitesimal displacement, when *evaluated along the line* $\mathbf{R}_k$ *, (i.e., the virtual velocity)* that is *common* to all points of the line $\mathbf{R}_k$ of the system for the rotation δR of that system around the axis **R**. If one regards R_k as a force that acts along the line $\mathbf{R}_k$ then the quantity $\delta R\, R_k\, [\mathbf{R}, \mathbf{R}_i]$ will be what one calls the *virtual moment* of the force R_k relative to the infinitesimal rotation δR around the axis **R**.

Now, let several infinitesimal rotations δR_1, δR_2, …, δR_i, … around the axes $\mathbf{R}_1$, $\mathbf{R}_2$, …, $\mathbf{R}_i$, …, resp., be impressed on the system simultaneously. They will be equivalent to six simultaneous infinitesimal rotations around the edges of the fundamental tetrahedron that are expressed by:

$$(7)\qquad \delta F = \sum \delta F_i = \delta\tau \sin \mathbf{BC}\, (\sum \delta F_i), \ \ldots, \ \delta L = \sum \delta L_i = \delta\tau \sin \mathbf{AD}\, (\sum \delta L_i), \ \ldots$$

They will compose into just one infinitesimal rotation δR around the axis **R** that has the coordinates:

$$F = \sum F_i\,, \ \ldots, \ L = \sum L_i, \ \ldots,$$

which will verify the condition:

$$\frac{(\Sigma\delta F_i)(\Sigma\delta L_i)}{\sin \mathbf{BC} \sin \mathbf{AD}} + \frac{(\Sigma\delta G_i)(\Sigma\delta M_i)}{\sin \mathbf{AC} \sin \mathbf{BD}} + \frac{(\Sigma\delta H_i)(\Sigma\delta N_i)}{\sin \mathbf{AB} \sin \mathbf{CD}} = 0,$$

or also:

(8) $$(\sum F_i)(\sum L_i) + (\sum G_i)(\sum M_i) + (\sum H_i)(\sum N_i) = 0.$$

One will then have:

$$\delta R^2 = [(\sum \delta F_i) \cos \mathbf{F} + \ldots + (\sum \delta L_i) \cos \mathbf{L} + \ldots]^2 = (\sum \delta R_i \cos \mathbf{R}_i)^2,$$

or

(9) $$\delta R^2 = \delta\tau^2 \, [(\sum F_i) \sin \mathbf{BC} \cos \mathbf{F} + \ldots + (\sum L_i) \sin \mathbf{AD} \cos \mathbf{L} + \ldots]^2.$$

The resultant rotation will be converted into a translation that verifies the conditions that are obtained by taking $(\sum \delta F_i, \ldots, \sum \delta L_i, \ldots)$ or $(\sum F_i, \ldots, \sum L_i, \ldots)$, instead of $(\delta F, \ldots, \delta L, \ldots)$ or $(F, \ldots, L, \ldots)$, resp., in the first or second system of equations in (4).

In the general case, in which equation (7) is not satisfied, the infinitesimal rotation that are impressed upon the system simultaneously will be equivalent to two simultaneous infinitesimal rotations around two axes, one of which passes through a vertex of the fundamental tetrahedron and the other of which lies in the opposite face.

Finally, the rotations that are impressed upon the system will cancel each other when one has:

(10) $$\sum \delta F_i = 0, \ldots, \sum \delta L_i = 0, \ldots \quad \text{or} \quad \sum F_i = 0, \ldots, \sum L_i = 0, \ldots$$

If one applies equation (6) successively to each of the infinitesimal rotations that are impressed upon the system and takes the sum of the resultants then one will get:

(11) $$R_k \sum \delta R_i \, [\mathbf{R}_i, \mathbf{R}_k] = \delta\tau [(\sum F_i) L_k + \ldots + (\sum L_i) F_k + \ldots] \sin \mathbf{ABCD}.$$

$\sum \delta R_i \, [\mathbf{R}_i, \mathbf{R}_k]$ will be the infinitesimal displacement of the line $\mathbf{R}_k$ along the proper direction (i.e., the virtual velocity) that is due to the infinitesimal rotations that are impressed simultaneously on the system, and $R_k \sum \delta R_i \, [\mathbf{R}_i, \mathbf{R}_k]$ will be the virtual moment of the force R_k relative to those rotations.

Applying formula (11) to the various forces R_k of a system and summing the results will give:

(12) $$\sum R_k (\sum \delta R_i \, [\mathbf{R}_i, \mathbf{R}_k]) = \delta\tau [(\sum F_i)(\sum L_k) + \ldots + (\sum L_i)(\sum F_k) + \ldots] \sin \mathbf{ABCD},$$

so the forces R_k that will bring about equilibrium will then satisfy $\sum F_i = 0, \ldots, \sum L_i = 0, \ldots$, and one will get $\sum R_k (\sum \delta R_i \, [\mathbf{R}_i, \mathbf{R}_k]) = 0$, which is a formula that expresses the principle of *virtual velocity.*

2. Let **r** or **R** denote a line, which can be considered to be a locus of points or an intersection of planes, and let $(f, \ldots, l, \ldots)$ or $(F, \ldots, L, \ldots)$, resp., denote its coordinates, which depend upon the pair of points $(\mathbf{p}_i, \mathbf{p}_j)$ of **r** or the pair of planes $(\mathbf{P}_i, \mathbf{P}_j)$ that go through **R**, resp., so one will have:

$$\frac{f \, \mathbf{bc}}{L \sin \mathbf{AD}} = \ldots = \frac{l \, \mathbf{ad}}{F \sin \mathbf{BC}} = \ldots = \frac{\mathbf{p}_i \mathbf{p}_j}{\sin \mathbf{P}_i \mathbf{P}_j}.$$

For greater simplicity, we always suppose that $\mathbf{p}_i\,\mathbf{p}_j = \sin \mathbf{P}_i\,\mathbf{P}_j$, and if the system is to be impressed around the axis ($\mathbf{r}$, $\mathbf{R}$) of the infinitesimal rotation $\delta r = \delta\tau\, r = \delta R = \delta\tau\, R$ then we will put $\mathbf{p}_i\ \mathbf{p}_j = r = \sin\ \mathbf{P}_i\ \mathbf{P}_j = R$. That rotation will be equivalent to the simultaneous infinitesimal rotations:

$$\delta\tau\, f\, \mathbf{bc} = \delta\tau\, L \sin \mathbf{AD}, \ldots, \delta\tau\, l\, \mathbf{ad} = \delta\tau\, F \sin \mathbf{BC}, \ldots$$

around the edges ($\mathbf{f}$, $\mathbf{L}$), …, ($\boldsymbol{l}$, $\mathbf{F}$), … of the fundamental tetrahedron.

Now, consider the infinitesimal motion of the system that is due to several rotations $\delta\tau\,(r_i\,,\,R_{\,i})$ around the axes ($\mathbf{r}_i\ \mathbf{R}_j$), resp. As we have already seen how these rotations compose among themselves in the same way that the forces that are expressed by (r_i , $R_{\,i}$) that act along the lines ($\mathbf{r}_i\ \mathbf{R}_j$), resp., compose. Now, let (ω, $\mathbf{r}$) = (Ω, $\mathbf{R}$) denote the moment of that system of forces with respect to a line ($\mathbf{r}$, $\mathbf{R}$), so the virtual velocity of that line, when provided with the given rotations, will be found to be expressed by:

$$\sum \delta r_i\, [\mathbf{r}_i, \mathbf{r}] = \delta\tau\,(\omega, \mathbf{r}), \qquad \text{or else} \qquad \sum \delta R_i\, [\mathbf{R}_i, \mathbf{R} = \delta\tau\,(\Omega, \mathbf{R}),$$

with which, the properties of the moments of the system of forces ($\mathbf{r}_i\ \mathbf{R}_j$) will translate immediately into properties of the virtual velocities of the corresponding system of rotations $\delta\tau\,(\mathbf{r}_i\ \mathbf{R}_j)$ (*).

The lines ($\mathbf{r}$, $\mathbf{R}$) of zero virtual velocity, or the ones that are normals to the trajectories of their points, constitute the first-degree complex that is represented by the equations:

(1) $(\sum f_i)\, l + \ldots + (\sum l_i)\, f + \ldots = 0, \qquad (\sum F_i)\, L + \ldots + (\sum L_i)\, F + \ldots = 0,$

and one will then have the following properties (**):

The line $\mathbf{r}$ that passes through the point $\mathbf{p}$ with coordinates (a, b, c, d) that belongs to the corresponding plane $\mathbf{P}$ with coordinates (A, B, C, D) is determined by the equations:

(2)
$$\frac{(\Sigma\, n_i)\, b - (\Sigma\, m_i)\, c + (\Sigma\, f_i)\, d}{A \cdot \mathbf{aA}} = \frac{(\Sigma\, l_i)\, c - (\Sigma\, n_i)\, a + (\Sigma\, g_i)\, d}{B \cdot \mathbf{bB}} = \frac{(\Sigma\, m_i)\, a - (\Sigma\, l_i)\, b + (\Sigma\, h_i)\, d}{C \cdot \mathbf{cC}}$$
$$= -\frac{(\Sigma\, f_i)\, a + (\Sigma\, g_i)\, b + (\Sigma\, h_i)\, c}{D \cdot \mathbf{dD}} = \frac{(\omega, \mathbf{p})}{\mathbf{abcd}},$$

and the lines $\mathbf{R}$ that lie in the plane $\mathbf{P}$ with coordinates (A, B, C, D) that belongs to the corresponding point $\mathbf{p}$ with coordinates (a, b, c, d) are determined by the equations:

(2)
$$\frac{(\Sigma N_i)\, B - (\Sigma M_i)\, C + (\Sigma F_i)\, D}{a \cdot \mathbf{Aa}} = \frac{(\Sigma L_i)\, C - (\Sigma N_i)\, A + (\Sigma G_i)\, D}{b \cdot \mathbf{Bb}} = \frac{(\Sigma M_i)\, A - (\Sigma L_i)\, B + (\Sigma H_i)\, D}{c \cdot \mathbf{Cc}}$$

(*) Note: "sulla teorica dei Momenti," Rend. dell'Accad., May 1869.
(**) CHASLES, *Mémoires de l'Institut,* 1843. – JONQUIÉRES, *Mélanges de Géométrie pure.*

$$= -\frac{(\Sigma F_i)A + (\Sigma G_i)B + (\Sigma H_i)C}{d \cdot \mathbf{Dd}} = \frac{(\Omega, \mathbf{P})}{\sin \mathbf{ABCD}}.$$

The quantities that are expressed by the symbols (ω, **p**) and (Ω, **P**) are deduced from equations (2), while taking into account the relations:

$$(A \cos \mathbf{A} + B \cos \mathbf{B} + C \cos \mathbf{C} + D \cos \mathbf{D})^2 = 1,$$

$$(a + b + c + d) = 1.$$

$\delta\tau(\omega, \mathbf{p})$ and $\delta\tau(\Omega, \mathbf{P})$ will be the *resultant virtual velocities* of the system relative to the point **p** and the plane **P**, resp. For a line **r** or **R** that goes through the point **p** or the plane **P**, resp., the virtual velocity will be expressed by:

$$\delta\tau(\omega, \mathbf{r}) = \delta\tau(\omega, \mathbf{p}) \sin \mathbf{rR} \quad \text{or} \quad \delta\tau(\Omega, \mathbf{R}) = \delta\tau(\Omega, \mathbf{P})\, \mathbf{Rp}, \text{ resp.}$$

If the plane **P′** that corresponds to **p′** passes through the point **p″** then the plane **P″** that corresponds to **p″** will pass through **p′**. If the point **p′** that corresponds to **P′** lies in the plane **P″** then the point **p′** that corresponds to **P″** will lie in **P′**. In other words, the pairs (**p**, **P**) of corresponding points and planes will describe correlated figures with the peculiarity that the plane **P** that corresponds to a point **p** will pass through **p** and that the point **p** that corresponds to a plane **P** will lie in **P**.

If a point **p** traverses a line **r′** then its corresponding plane **P** will pass along another line **R′**. Similarly, if the plane **P** turns around a line **R′** then its corresponding point **p** will traverse a line **r″**. If **r′** coincides with **R′** then **R″** will coincide with **r″**. The lines (**r′**, **R′**) and (**r″**, **R″**) are called *conjugate lines.* A line **r** that coincides with its conjugate **R** will be normal to the trajectories of its points.

Let (**r′**, **R′**) and (**r″**, **R″**) be two conjugate lines. Set:

$$(\Sigma f_i)(\Sigma l_i) + (\Sigma g_i)(\Sigma m_i) + (\Sigma h_i)(\Sigma n_i) = \frac{k}{\mathbf{abcd}},$$

$$(\Sigma F_i)(\Sigma L_i) + (\Sigma G_i)(\Sigma M_i) + (\Sigma H_i)(\Sigma N_i) = \frac{K}{\sin \mathbf{ABCD}},$$

so $k = K$, and furthermore:

(3) $$r'(\omega, \mathbf{r}') = r''(\omega, \mathbf{r}'') = k, \qquad R'(\Omega, \mathbf{R}') = R''(\Omega, \mathbf{R}'') = K,$$

and one will find that:

(4) $$f' + f'' = \Sigma f_i, \ldots, l' + l'' = \Sigma l_i, \ldots, \quad F' + F'' = \Sigma F_i, \ldots, L' + L'' = \Sigma L_i, \ldots$$

Therefore, (r', R') and (r'', R'') will be two forces that act along the lines (**r′**, **R′**) and (**r″**, **R″**), resp., and are equivalent to the system of forces (r_i, R_i) that act along the lines ($\mathbf{r}_i$,

$\mathbf{R}_j$). As a result of that, $\delta\tau(r', R')$ and $\delta\tau(r'', R'')$ will be two rotations around the axes ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$), resp., that are equivalent to the system of rotations $\delta\tau(r_i, R_i)$ that act along the lines ($\mathbf{r}_i$, $\mathbf{R}_j$). The lines ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$) are called *conjugate axes of rotation.*

The virtual velocities of the conjugate lines ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$), which are expressed by $\delta\tau(\omega, \mathbf{r}') = \delta\tau(\Omega, \mathbf{R}')$ and $\delta\tau(\omega, \mathbf{r}'') = \delta\tau(\Omega, \mathbf{R}'')$, are inversely proportional to the conjugate rotations relative to the axes ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$) that are expressed by $\delta\tau\ r' = \delta\tau\ R'$ and $\delta\tau\ r'' = \delta\tau\ R''$, resp.

Let one of two conjugate axes of rotation pass through a point **p** and the other one lie in the corresponding plane **P**, and vice versa. These two axes will be mutually anharmonically dependent upon each other. Given two pairs of conjugate axes of rotations, any line that crosses three of them will also cross the fourth one.

Let one of two conjugate axes be at infinity, and let the other one have a constant direction. One of these three particular pairs of conjugate axes will be orthogonal. In that pair of axes, one calls the one that lies at a finite distance the *axis of twisting rotation* of the system, since the infinitesimal motion of the system will obviously reduce to a rotation around that axes and a translation along the same axes.

Suppose one is given five lines that are normal to the trajectories of their points, or have zero virtual velocity, and consider them four at a time. The two lines that cross each of this tetrad of lines will be conjugate axes of rotation, and the lines that pass through a point **p** and cross that pair of conjugate axes will line in the plane **P** of the axes of zero virtual velocity relative to **p**, just as the lines that lie in a plane **P** and that cross that same pair of conjugate axes will pass through point **p** of concurrence of the axes of zero virtual velocity relative to **P**. The common perpendicular to the common perpendiculars relative to the two pairs of conjugate axes will then be the axis of twisting rotation of the system.

If six forces act along any lines at all and are required to be in equilibrium then one can take five of these lines arbitrarily (*) and the sixth one will then belong to the linear complex that they determine. Therefore, suppose that this complex is the one that is represented by equations (1). One will then find that in the infinitesimal motion that is impressed upon the system, six lines that are normal to the trajectories of their points will be the lines of action of six forces that can bring about equilibrium (**).

Among the conjugate axes, the ones whose directions are orthogonal merit special attention. Each of them will be tangent to the trajectory of its points at which it meets its common perpendicular, and will be the line of maximum virtual velocity relative to all of the lines that pass through that point or – what amounts to the same thing – the axis of minimum conjugate rotation.

Let ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$) be two of these conjugate axes. Their coordinates must verify equations (3), in addition to these other ones:

$$(4)\quad \begin{aligned} &(f'\,\mathbf{bc}\cos\mathbf{f} + \ldots + l'\,\mathbf{ad}\cos\boldsymbol{l} + \ldots)(f''\,\mathbf{bc}\cos\mathbf{f} + \ldots + l''\,\mathbf{ad}\cos\boldsymbol{l} + \ldots) = 0, \\ &(F'\,\mathbf{BC}\cos\mathbf{F} + \ldots + L'\,\mathbf{AD}\cos\mathbf{L} + \ldots)(F''\,\mathbf{BC}\cos\mathbf{F} + \ldots + L''\,\mathbf{AD}\cos\mathbf{L} + \ldots) = 0, \end{aligned}$$

(*) *Loc. cit.*, May 1869.
(**) SYLVESTER and CHASLES, *Comptes rendus*, 1861.

which express their orthogonality. It will follow from this that the lines ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$) will both belong to the second-degree complex that is represented by the equations:

$$(5)\qquad \begin{gathered}(f\,\mathbf{bc}\cos\mathbf{f}+\ldots)[(\textstyle\sum f_i)\,\mathbf{bc}\cos\mathbf{f}+\ldots]\,[f(\textstyle\sum f_i)+\ldots]=\dfrac{k}{\mathbf{abcd}}(f\,\mathbf{bc}\cos\mathbf{f}+\ldots)^2\\ (F\sin\mathbf{BC}\cos\mathbf{F}+\ldots)[(\textstyle\sum F_i)\sin\mathbf{BC}\cos\mathbf{F}+\ldots][F(\textstyle\sum L_i)+\ldots]\\ =\dfrac{K}{\sin\mathbf{ABCD}}(F\sin\mathbf{BC}\cos\mathbf{F}+\ldots)^2.\end{gathered}$$

Any line ($\mathbf{r}'$, $\mathbf{R}'$) of this complex will correspond to another conjugate ($\mathbf{r}''$, $\mathbf{R}''$) that belongs to the same complex and will be coupled to the first one by the anharmonic dependence that was pointed out above.

The fundamental property of second-degree complexes is that all of the lines that are tangent to the trajectory of one of its points and that pass through a point $\mathbf{p}$ will belong to a conic surface S of order two, and their conjugates (which are also tangents to the trajectory of one of their points) will envelop a line s of class two that belongs to the plane $\mathbf{P}$ that corresponds to $\mathbf{p}$, and vice versa. In addition, from the form of equations (5), it will appear that $\mathbf{P}$ is one of the cyclic planes of S and that $\mathbf{p}$ is one of the foci of s, while the other focus is at infinity.

Since the rotations $\delta\tau R'$, $\delta\tau R''$ around the conjugate axes $\mathbf{R}'$, $\mathbf{R}''$ are equivalent to the rotations $\delta\tau R_i$ around the axes $\mathbf{R}_i$ that are impressed upon the system, let $\delta\tau R$ denote the rotation around the axis of twisting rotation $\mathbf{R}$, so $R(\Omega, \mathbf{R}) = K$, and one will have the relations:

$$R'^2 + R''^2 + 2R'R''\cos\mathbf{R}'\mathbf{R}'' = (\textstyle\sum R_i\cos\mathbf{R}_i)^2 = R^2,$$

$$(6)\qquad R' = R\,\frac{\sin\mathbf{RR}''}{\sin\mathbf{R}'\mathbf{R}''},\qquad R'' = R\,\frac{\sin\mathbf{RR}'}{\sin\mathbf{R}''\mathbf{R}'},$$

$$(7)\qquad R'[\mathbf{R}', \mathbf{R}_k] + R''[\mathbf{R}'', \mathbf{R}_k] = \textstyle\sum R_i\,[\mathbf{R}_i, \mathbf{R}_j] = (\Omega, \mathbf{R}_k),$$

$$(8)\qquad R'R''[\mathbf{R}', \mathbf{R}''] = \textstyle\sum R_i R_j\,[\mathbf{R}_i, \mathbf{R}_j] = K.$$

If ρ' and ρ'' are the minimum distances between $\mathbf{R}'$, $\mathbf{R}$ and $\mathbf{R}''$, $\mathbf{R}$ then one will again have:

$$(9)\qquad \rho'\tan\mathbf{RR}'' = \rho''\tan\mathbf{RR}' = \frac{(\Omega,\mathbf{R})}{R}.$$

If the directions of the conjugate axes $\mathbf{R}'$ and $\mathbf{R}''$ are orthogonal then one will have:

$$(10)\qquad \frac{\rho'}{\tan\mathbf{RR}'} = \frac{\rho''}{\tan\mathbf{RR}''} = \frac{(\Omega,\mathbf{R})}{R},\qquad \rho'\rho'' = \frac{(\Omega,\mathbf{R})^2}{R^2};$$

$$(11)\qquad (\Omega, \mathbf{R}) = (\Omega, \mathbf{R}')\cos\mathbf{RR}' = (\Omega, \mathbf{R}'')\cos\mathbf{RR}'',$$

in which the last equation will be deduced from the virtual velocity of the axis of twisting rotation, and will be the minimum of the maximum virtual velocities relative to the lines that go through the various points in space.

Equations (10) and (11) show clearly the disposition around the axis of twisting rotation of the lines of maximum virtual velocity that corresponding to the various points in space, as well as the way in which the values of that velocity vary.

What was said of the line of maximum virtual velocity must also be true for the axes of minimum conjugate rotations.

One can switch the upper-case and lower-case letters in formulas (6) and (11).

3. We now seek the variations (δa, δb, δc, δd) of the coordinates (a, b, c, d) of an arbitrary point **p** of the system that come from the rotation $\delta R = \delta\tau\, R$ around the axis **R** with coordinates (F, …, L, …). Set:

$$(1)\qquad \begin{aligned} \frac{\delta R_a}{\delta\tau} &= bH\,\mathbf{bB} - cG\,\mathbf{cC} + dL\,\mathbf{dD}, & \frac{\delta R_b}{\delta\tau} &= cF\,\mathbf{cC} - aH\,\mathbf{aA} + dM\,\mathbf{dD}, \\ \frac{\delta R_c}{\delta\tau} &= aG\,\mathbf{aA} - bF\,\mathbf{bB} + dN\,\mathbf{dD}, & \frac{\delta R_d}{\delta\tau} &= -\,aL\,\mathbf{aA} - bM\,\mathbf{bB} - cN\,\mathbf{cC}, \end{aligned}$$

so $\frac{\delta R_a}{\delta\tau} = 0$, $\frac{\delta R_b}{\delta\tau} = 0$, $\frac{\delta R_c}{\delta\tau} = 0$, $\frac{\delta R_d}{\delta\tau} = 0$ will be the equations of four planes that pass through **R** and one of the vertices (**a**, **b**¸ **c**, **d**) of the fundamental tetrahedron.

Now, the variation δa, δb, δc, or δd will obviously be zero for an arbitrary point **p** of the plane that passes through **R** and is perpendicular to the face **A**, **B**, **C**, or **D**, resp., if one observes that one has the relations:

$$(2)\qquad \begin{aligned} &\mathbf{dbc} + \mathbf{dca}\cos\mathbf{BA} + \mathbf{dab}\cos\mathbf{CA} + \mathbf{cba}\cos\mathbf{DA} = 0, \\ &\mathbf{dbc}\cos\mathbf{AB} + \mathbf{dca} + \mathbf{dab}\cos\mathbf{CB} + \mathbf{cba}\cos\mathbf{DB} = 0, \\ &\mathbf{dbc}\cos\mathbf{AC} + \mathbf{dca}\cos\mathbf{BC} + \mathbf{dab} + \mathbf{cba}\cos\mathbf{DC} = 0, \\ &\mathbf{dbc}\cos\mathbf{AD} + \mathbf{dca}\cos\mathbf{BD} + \mathbf{dab}\cos\mathbf{CD} + \mathbf{cba} = 0, \end{aligned}$$

between the parts of that tetrahedron. From the relation $a + b + c + d = 1$ that exists between the coordinates of and arbitrary point, one must have $\delta a + \delta b + \delta c + \delta d = 0$, and one will easily find that:

$$(3)\qquad \begin{aligned} \delta a\;\mathbf{abcd} &= (\delta R_a + \delta R_b\cos\mathbf{AB} + \delta R_c\cos\mathbf{AC} + \delta R_d\cos\mathbf{AD})\,\mathbf{dbc}, \\ \delta b\;\mathbf{abcd} &= (\delta R_a\cos\mathbf{BA} + \delta R_b + \delta R_c\cos\mathbf{BC} + \delta R_d\cos\mathbf{BD})\,\mathbf{dca}, \\ \delta c\;\mathbf{abcd} &= (\delta R_a\cos\mathbf{CA} + \delta R_b\cos\mathbf{CB} + \delta R_c + \delta R_d\cos\mathbf{CD})\,\mathbf{dab}, \end{aligned}$$

$$\delta d\ \mathbf{abcd} = (\delta R_a \cos \mathbf{DA} + \delta R_b \cos \mathbf{DB} + \delta R_c \cos \mathbf{DC} + \delta R_d)\ \mathbf{cba}\ .$$

Set:

$$\frac{\delta R_a}{\mathbf{dbc}} = \frac{\delta R_b}{\mathbf{dca}} = \frac{\delta R_c}{\mathbf{dab}} = \frac{\delta R_d}{\mathbf{cba}} = \kappa;$$

so one will have, from equations (2):

$$\delta a = 0, \qquad \delta b = 0, \qquad \delta c = 0, \qquad \delta d = 0,$$

and vice versa. However, from equations (1), one will have:

$$a\,\delta R_a\ \mathbf{aA} + b\,\delta R_b\ \mathbf{bB} + c\,\delta R_c\ \mathbf{cC} + d\,\delta R_a\ \mathbf{dD} = 0,$$

and, on the other hand:

$$\mathbf{aA} \cdot \mathbf{dbc} = \mathbf{bB} \cdot \mathbf{dca} = \mathbf{cC} \cdot \mathbf{dab} = \mathbf{dD} \cdot \mathbf{cba} = \mathbf{abcd},$$

$$a + b + c + d = 1,$$

so one will have $\kappa = 0$, and the points of the system that will remain fixed during the infinitesimal motion that is provided by the impressed rotation will be the ones that belong to the line along which the four planes:

$$(4) \qquad \delta R_a = 0, \qquad \delta R_b = 0, \qquad \delta R_c = 0, \qquad \delta R_d = 0$$

intersect, namely (as is clear), the points of the axis of rotation.

If the system is impressed simultaneously with several infinitesimal rotations $\delta R_i = \delta\tau\ R_i$ around several axes $\mathbf{R}_i$ with coordinates $(F_i, \ldots, L_i, \ldots)$, resp., then for each of these rotations, one will have equations that are analogous to (1) and (3) for the determination of the *partial* variations $(\delta_i a, \delta_i b, \delta_i c, \delta_i d)$. If one sets:

$$F = \sum F_i\ , \qquad G = \sum G_i\ , \qquad H = \sum H_i\ , \qquad L = \sum L_i\ , \qquad M = \sum M_i\ , \qquad N = \sum N_i\ ,$$

$$(5) \qquad \delta R_a = \sum \delta_i R_a\ , \qquad \delta R_b = \sum \delta_i R_b\ , \qquad \delta R_c = \sum \delta_i R_c\ , \qquad \delta R_d = \sum \delta_i R_d\ ,$$

$$\delta a = \sum \delta_i a, \qquad \delta b = \sum \delta_i b, \qquad \delta c = \sum \delta_i c, \qquad \delta d = \sum \delta_i d$$

then one will know the *total* variations $(\delta a, \delta b, \delta c, \delta d)$ by means of equations (1) and (3).

When the condition:

$$(\sum F_i)\ (\sum L_i) + (\sum G_i)\ (\sum M_i) + (\sum H_i)\ (\sum N_i) = 0$$

is verified, or when the rotations that are impressed on the system reduce to just one of them $\delta R = \delta\tau\, R$ around the axis $\mathbf{R}$ with coordinates $(F, \ldots, L, \ldots)$, one will find, as above, that the planes (4) will intersect along that line, each point of it will thus remain fixed during the infinitesimal motion of the system that is provided by the given rotations. In the general case, for which that condition is not satisfied, or when the rotations that are impressed on the system cannot be reduced to two conjugate rotations, the planes (4) will be the faces of a tetrahedron that is, at the same time, inscribed in and circumscribed on the fundamental tetrahedron, and then it will not have any common point, nor will any point remain fixed during the infinitesimal motion of the system.

If one symbolically sets:

$$\delta R_a \cos \mathbf{A} + \delta R_b \cos \mathbf{B} + \delta R_c \cos \mathbf{C} + \delta R_d \cos \mathbf{D} = \delta\mathbf{P} \tag{6}$$

then equations (3) will take the form:

$$\begin{aligned} &\delta a\ \mathbf{abcd} = \delta\mathbf{Pdbc} \cos \mathbf{A}, && \delta b\ \mathbf{abcd} = \delta\mathbf{Pdca} \cos \mathbf{B}, \\ &\delta c\ \mathbf{abcd} = \delta\mathbf{Pdab} \cos \mathbf{C}, && \delta d\ \mathbf{abcd} = \delta\mathbf{Pcba} \cos \mathbf{D}\,. \end{aligned} \tag{7}$$

If one then sets:

$$\begin{aligned} &c\ \mathbf{cC} \cos \mathbf{B} - b\ \mathbf{bB} \cos \mathbf{C} = \Phi(\mathbf{F}), && d\ \mathbf{dD} \cos \mathbf{A} - a\ \mathbf{aA} \cos \mathbf{D} = \Phi(\mathbf{L}), \\ &a\ \mathbf{aA} \cos \mathbf{C} - c\ \mathbf{cC} \cos \mathbf{A} = \Phi(\mathbf{G}), && d\ \mathbf{dD} \cos \mathbf{B} - b\ \mathbf{bB} \cos \mathbf{D} = \Phi(\mathbf{M}), \\ &c\ \mathbf{cC} \cos \mathbf{B} - b\ \mathbf{bB} \cos \mathbf{C} = \Phi(\mathbf{F}), && d\ \mathbf{dD} \cos \mathbf{C} - c\ \mathbf{cC} \cos \mathbf{D} = \Phi(\mathbf{N}), \end{aligned} \tag{8}$$

$$F\ \Phi(\mathbf{F}) + \ldots + L\ \Phi(\mathbf{L}) + \ldots = R\ \Phi(\mathbf{R})$$

then those equations will become:

$$\begin{aligned} &\frac{\delta a}{\delta\tau}\mathbf{abcd} = R\ \Phi(\mathbf{R})\ \mathbf{dbc} \cos \mathbf{A}, && \frac{\delta b}{\delta\tau}\mathbf{abcd} = R\ \Phi(\mathbf{R})\ \mathbf{dca} \cos \mathbf{B}, \\ &\frac{\delta c}{\delta\tau}\mathbf{abcd} = R\ \Phi(\mathbf{R})\ \mathbf{dab} \cos \mathbf{C}, && \frac{\delta d}{\delta\tau}\mathbf{abcd} = R\ \Phi(\mathbf{R})\ \mathbf{cba} \cos \mathbf{D}\,. \end{aligned} \tag{9}$$

The quantities in these expressions that are multiplied by the *arbitrary constants* (F, …, L, …) are the values of the same expressions that correspond to infinitesimal rotations of the system around one of the edges (**F**, …, **L**, …) of the fundamental tetrahedron.

When the constants (F, …, L, …) do not verify the condition $FL + GM + HN = 0$, equations (9) will give the variations (δa, δb, δc, δd) that correspond to an *arbitrary infinitesimal motion* that is impressed upon the system. Therefore, without making recourse to long analytical developments, one will see by very simple geometric considerations how an arbitrary infinitesimal motion that is impressed on a system of invariable form can generally reduce to two infinitesimal rotations around two different axes.
