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On the theory of moments

NOTE by ordinary member **G. Battaglini**

Translated by D. H. Delphenich

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As a continuation of the note "sulla composizione delle forze" (*), we will now treat the *theory of moments.*

1. Consider a system of forces with any sort of directions in space. The moment of a force r_i or R_i in the system that acts along the line $\mathbf{r}_i$ or $\mathbf{R}_i$, resp., with respect to an axis $\mathbf{r}$ or $\mathbf{R}$, resp. – namely, $r_i\,[\mathbf{r}_i, \mathbf{r}]$ or $R_i\,[\mathbf{R}_i, \mathbf{R}]$, resp. – is given by the formula:

$$\frac{r_i\, r\,[\mathbf{r}_i, \mathbf{r}]}{\mathbf{abcd}} = f_i\, l + g_i\, m + h_i\, n + l_i\, f + m_i\, g + n_i\, h,$$

or even:

$$\frac{R_i\, R\,[\mathbf{R}_i, \mathbf{R}]}{\sin \mathbf{ABCD}} = F_i\, L + G_i\, M + H_i\, N + L_i\, F + M_i\, G + N_i\, H\,.$$

If one lets $(\omega, \mathbf{R}) = \sum r_i\,[\mathbf{r}_i, \mathbf{r}]$ or $(\Omega, \mathbf{R}) = \sum R_i\,[\mathbf{R}_i, \mathbf{R}]$ denote *the moment of the system* with respect to the axis $\mathbf{r}$ or $\mathbf{R}$, resp., then one will have:

$$(1)\qquad \frac{r\,(\omega, \mathbf{r})}{\mathbf{abcd}} = (\textstyle\sum f_i)\, l + \ldots + (\sum l_i)\, f + \ldots; \quad \dfrac{R\,(\Omega, \mathbf{R})}{\sin \mathbf{ABCD}} = (\sum F_i)\, L + \ldots + (\sum L_i)\, F + \ldots$$

Therefore, if one knows the moments of the system with respect to the edges of the fundamental tetrahedron then the moment with respect to any other axis $\mathbf{r}$ or $\mathbf{R}$ will be given by one or the other of the equations:

$$(2)\qquad \begin{aligned} r(\omega, \mathbf{r}) &= f\,\mathbf{bc}\,(\omega, \mathbf{f}) + \ldots + l\,\mathbf{ad}\,(\omega, \boldsymbol{l}) + \ldots \\ R\,(\Omega, \mathbf{R}) &= F \sin \mathbf{BC}\,(\Omega, \mathbf{F}) + \ldots + L \sin \mathbf{AD}\,(\Omega, \mathbf{L}) + \ldots \end{aligned}$$

The coefficients of $(\omega, \mathbf{f}), \ldots, (\omega, \boldsymbol{l}), \ldots$ or $(\Omega, \mathbf{F}), \ldots, (\Omega, \mathbf{L}), \ldots$ in these formulas can be considered to be the forces that act along the edges $\mathbf{f}, \ldots, \boldsymbol{l}, \ldots$ or $\mathbf{F}, \ldots, \mathbf{L}, \ldots$, resp.,

(*) Rend. dell'Accad., Feb. 1869.

and that have the resultant force r or R that acts along the axis **r** or **R**, resp. As a result of that, if r', r'', …, $r^{(i)}$, … or R', R'', …, $R^{(i)}$, … represent forces in equilibrium that act along $\mathbf{r}'$, $\mathbf{r}''$, …, $\mathbf{r}^{(i)}$, … or $\mathbf{R}'$, $\mathbf{R}''$, …, $\mathbf{R}^{(i)}$, …, resp., then one will have (*):

$$\sum r^{(i)}\,(\omega, \mathbf{r}^{(i)}) = 0 \qquad \text{or even} \qquad \sum R^{(i)}\,(\Omega, \mathbf{R}^{(i)}) = 0, \text{ resp.},$$

with respect to those axes. Therefore:

If a system of forces is in equilibrium, and if one takes another system of forces arbitrarily then the sum of the products of each force in the first system times the moment of the of the second system with respect to the line along which it acts will be equal to zero.

This proposition expresses the principal of *virtual velocity* in a different form.

If the axis **r** passes through the vertex **d** of the fundamental tetrahedron then one will have for the stated situation:

$$(3) \qquad (\omega, \mathbf{r}) = (\omega, 1)\frac{\sin \mathbf{rA}}{\sin \boldsymbol{l}\mathbf{A}} + (\omega, \mathbf{m})\,\frac{\sin \mathbf{rB}}{\sin \boldsymbol{l}\mathbf{B}} + (\omega, \mathbf{n})\,\frac{\sin \mathbf{rC}}{\sin \boldsymbol{l}\mathbf{C}},$$

so one will determine a plane **P** that goes through **d** by means of the relations:

$$\frac{\sin \boldsymbol{l}\mathbf{P}}{(\omega, \boldsymbol{l})} = \frac{\sin \mathbf{mP}}{(\omega, \mathbf{m})} = \frac{\sin \mathbf{nP}}{(\omega, \mathbf{n})},$$

and set:

$$(4) \qquad (\omega, \mathbf{r})^2 = \left[\frac{(\omega, \boldsymbol{l})}{\sin \boldsymbol{l}\mathbf{A}}\cos \mathbf{A} + \frac{(\omega, \mathbf{m})}{\sin \mathbf{mB}}\cos \mathbf{B} + \frac{(\omega, \mathbf{n})}{\sin \mathbf{nC}}\cos \mathbf{C}\right]^2,$$

so **d** in **P** will be an axis with *zero* moment, and all of the lines **r** that go through **d** and have equal inclination with respect to the plane **P** will be axes of equal moment.

The line **r** that goes through **d** normal to **P** is the axis of *maximum* moment relative to the point **d**. The value $(\omega, \mathbf{d})$ of this maximum moment will be called the *resultant* moment of the system with respect to the point **d.**

Analogously, if the axis **R** lies in the face **D** of the fundamental tetrahedron then one will have:

$$(3) \qquad (\Omega, \mathbf{R}) = (\Omega, \mathbf{L})\frac{\mathbf{Ra}}{\mathbf{La}} + (\Omega, \mathbf{M})\,\frac{\mathbf{Rb}}{\mathbf{Mb}} + (\Omega, \mathbf{N})\frac{\mathbf{Rc}}{\mathbf{Nc}},$$

so one will determine a point **p** of **D** by means of the relations:

$$\frac{\mathbf{Lp}}{(\Omega, \mathbf{L})} = \frac{\mathbf{Mp}}{(\Omega, \mathbf{M})} = \frac{\mathbf{Np}}{(\Omega, \mathbf{N})},$$

(*) MÖBIUS, *Lehrbuch der Statik,* vol. I, page 164.

and if one sets:

$$(\Omega,\,\mathbf{D}) = \frac{(\Omega,\mathbf{L})}{\mathbf{La}} + \frac{(\Omega,\mathbf{M})}{\mathbf{Mb}} + \frac{(\Omega,\mathbf{N})}{\mathbf{Nc}} \tag{4}$$

then one will get: $(\Omega,\,\mathbf{R}) = (\Omega,\,\mathbf{D})\,\mathbf{Rp}$. As a result of this, any line **R** that goes through **D** at **p** will be an axis of *zero* moment, and all of the lines **R** that go through **D** at equal distances from the point **p** will be axes of equal moment. The quantity $(\Omega,\,\mathbf{D})$ will be called the *resultant* moment of the system with respect to the plane **D**.

Suppose that one has the relation $\sum r^{(i)}\,(\omega,\,\mathbf{r}_i) = 0$ or $\sum R^{(i)}\,(\Omega,\,\mathbf{R}_i) = 0$, resp., between the moments of the system of forces r_i or $\mathbf{R}_i$, resp., with respect to the axes $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$, resp. The forces that are represented by $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$ and act along those axes must be in equilibrium, so one will have one or the other system of six conditions:

$$\sum f^{(i)}\,\mathbf{bc} = \sum r^{(i)}\,\frac{[\mathbf{r}^{(i)},\boldsymbol{l}]}{[\mathbf{f},\boldsymbol{l}]} = 0,\ \ldots,\ \sum l^{(i)}\,\mathbf{ad} = \sum r^{(i)}\,\frac{[\mathbf{r}^{(i)},\mathbf{f}]}{[\boldsymbol{l},\mathbf{f}]} = 0,\ \ldots,$$

$$\sum F^{(i)}\sin\mathbf{BC} = \sum R^{(i)}\,\frac{[\mathbf{R}^{(i)},\mathbf{L}]}{[\mathbf{F},\mathbf{L}]} = 0,\ \ldots,\ \sum L^{(i)}\,\mathbf{AD} = \sum R^{(i)}\,\frac{[\mathbf{R}^{(i)},\mathbf{F}]}{[\mathbf{L},\mathbf{F}]} = 0,\ \ldots$$

If will follow from this that if there are seven axes $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$ that are supposed to be given then the preceding equations will generally give the values of the ratios of the seven quantities $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$. If there are six, five, or four axes $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$ then by taking five, four, or three, resp., of them arbitrarily, the coordinates of the remaining axis must verify one, two, or three linear conditions, resp. That axis must therefore belong to a first-degree system of the 3rd, 2nd, or 1st kind, resp., and one supposes that it satisfies conditions such that one will then have determined the ratios of the six, five, or four, resp., quantities $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$. It is ultimately clear that if there are three axes $\mathbf{r}^{(i)}$ or $\mathbf{R}^{(i)}$ then they must be concurrent at a point and lie in a plane, and two of them must coincide. It is easy to deduce the construction of the line that can be the line of action of the forces in equilibrium from this, or even such that the moments of an arbitrary system of forces with respect to it will have a linear relation between them (*).

2. We now examine the disposition in space of the axes of zero moment (**). The coordinates of these axes must verify one or the other equation:

$$(1)\quad \begin{aligned} &(\textstyle\sum f_i)\,l + (\sum g_i)\,m + (\sum h_i)\,l + (\sum l_i)\,f + (\sum m_i)\,g + (\sum n_i)\,h = 0,\\ &(\textstyle\sum F_i)\,L + (\sum G_i)\,M + (\sum H_i)\,N + (\sum L_i)\,F + (\sum M_i)\,G + (\sum N_i)\,H = 0,\end{aligned}$$

(*) MÖBIUS, *loc. cit.*, page 174, *et seq.* See "*intorno ai sistemi di rette di 1o grado*," Rend. dell'Accad., June 1866.
(**) MÖBIUS, *loc. cit.*, page 144, *et seq.*, CHASLES, *Comptes Rendus*, June 1843. POINSOT, *Élements de Statique.*

so they will form a system of lines of degree 1 and class 3. The lines of the system that pass through a point **p** with coordinates (a, b, c, d) will belong to the *corresponding* plane **P** with coordinates (A, B, C, D) that are determined by the equations:

$$\frac{(\Sigma n_i)b-(\Sigma m_i)c+(\Sigma f_i)d}{\mathbf{A}\cdot\mathbf{aA}} = \frac{(\Sigma l_i)c-(\Sigma n_i)a+(\Sigma g_i)d}{\mathbf{B}\cdot\mathbf{bB}} = \frac{(\Sigma m_i)a-(\Sigma l_i)b+(\Sigma h_i)d}{\mathbf{C}\cdot\mathbf{cC}}$$

(2)

$$= -\frac{(\Sigma f_i)a-(\Sigma g_i)b+(\Sigma h_i)c}{\mathbf{D}\cdot\mathbf{dD}} = \frac{(\omega,\mathbf{p})}{\mathbf{abcd}},$$

and the lines of the system that lie in the plane **P** with coordinates (A, B, C, D) will belong to the *corresponding* point **p** with coordinates (a, b, c, d) that is determined by the equations:

$$\frac{(\Sigma N_i)B-(\Sigma M_i)C+(\Sigma F_i)D}{a\cdot\mathbf{Aa}} = \frac{(\Sigma L_i)C-(\Sigma N_i)A+(\Sigma G_i)D}{b\cdot\mathbf{Bb}}$$

$$= \frac{(\Sigma M_i)A-(\Sigma L_i)B+(\Sigma H_i)D}{c\cdot\mathbf{Cc}}$$

(2)

$$= -\frac{(\Sigma F_i)A-(\Sigma G_i)B+(\Sigma H_i)C}{d\cdot\mathbf{Dd}} = \frac{(\Omega,\mathbf{P})}{\sin\mathbf{ABCD}}.$$

If one observes that one has the relations:

$$(A\cos\mathbf{A} + B\cos\mathbf{B} + C\cos\mathbf{C} + D\cos\mathbf{D})^2 = 1,$$
$$a + b + c + d = 1$$

then equations (2) will immediately give the resultant moments (ω, **p**) and (Ω, **P**) of the system relative to the point **p** and the plane **P**.

If $\mathbf{P}_i$ denotes the plane that goes through **p** and $\mathbf{r}_i$ then the moment of the force $\mathbf{r}_i$ with respect to an axis **r** that goes through **p** will be expressed by $r_i\,\mathbf{pr}_i\sin\mathbf{rP}_i$, so if **r** is an axis of zero moment then one must have $\sum r_i\,\mathbf{pr}_i\sin\mathbf{rP}_i = 0$. Similarly, if $\mathbf{p}_i$ denotes the point at which **P** and $\mathbf{R}_i$ meet then the moment of the force $\mathbf{R}_i$ with respect to an axis **R** that lies in **P** will be expressed by $R_i\,\mathbf{PR}_i\sin\mathbf{Rp}_i$, so if **R** is an axis of zero moment then one must have $\sum R_i\,\mathbf{PR}_i\sin\mathbf{Rp}_i = 0$. It follows from this that in order to construct the plane **P** that corresponds to the point **p**, it will suffice to pass lines through **p** that are perpendicular to the various planes $\mathbf{P}_i$ and proportional to $r_i\,\mathbf{pr}_i$, and these lines will be considered to be many forces from which one will find the resultant. That resultant will be proportional to the resultant moment (ω, **p**) of the system relative to the point and the plane that goes through **p** normal to the direction of that resultant will be the plane **P** that corresponds to **p**. Analogously, in order to construct the point **p** that corresponds to the plane **P**, it is sufficient to pass lines that are parallel and proportional to $R_i\sin\mathbf{PR}_i$ through the various points $\mathbf{p}_i$, and consider these lines to be many forces from which one finds the resultant. That resultant will be proportional to the resultant moment (Ω, **P**) of the system relative to the plane **P**, and the point at which **P** meets the direction of that resultant will be the point **p** that corresponds to **P**.

If the plane $\mathbf{P}'$ that corresponds to $\mathbf{p}'$ passes through the point $\mathbf{p}''$ then the plane $\mathbf{P}''$ that corresponds to $\mathbf{p}''$ will pass through $\mathbf{p}'$, or else if the point $\mathbf{p}'$ that corresponds to $\mathbf{P}'$ lies in the plane $\mathbf{P}''$ then the point $\mathbf{p}''$ that corresponds to $\mathbf{P}''$ will lie in $\mathbf{P}'$.

If the point $\mathbf{p}$ traverses a line $\mathbf{r}'$ then its corresponding plane $\mathbf{P}$ will pass through another line $\mathbf{r}''$. Similarly, if the plane $\mathbf{P}$ rotates around a line $\mathbf{P}'$ then its corresponding point $\mathbf{p}$ will traverse another line $\mathbf{R}''$. If $\mathbf{r}'$ coincides with $\mathbf{R}'$ then $\mathbf{r}''$ will coincide with $\mathbf{R}''$. The lines ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$) are said to be *conjugate* lines.

For the sake of brevity, set:

$$(\Sigma f_i)(\Sigma l_i) + (\Sigma g_i)(\Sigma m_i) + (\Sigma h_i)(\Sigma n_i) = \frac{k}{\mathbf{abcd}},$$

$$(\Sigma F_i)(\Sigma L_i) + (\Sigma G_i)(\Sigma M_i) + (\Sigma H_i)(\Sigma N_i) = \frac{K}{\sin \mathbf{ABCD}}.$$

One will have the following relations:

$$(3)\qquad \begin{aligned} &\frac{f'}{r'(\omega,\mathbf{r}')} + \frac{f''}{r''(\omega,\mathbf{r}'')} = \frac{\Sigma f_i}{k}, \ \ldots, \quad \frac{l'}{r'(\omega,\mathbf{r}')} + \frac{l''}{r''(\omega,\mathbf{r}'')} = \frac{\Sigma l_i}{k}, \ \ldots \\ &\frac{F'}{R'(\Omega,\mathbf{R}')} + \frac{F''}{R''(\Omega,\mathbf{R}'')} = \frac{\Sigma F_i}{K}, \ \ldots, \quad \frac{L'}{R'(\Omega,\mathbf{R}')} + \frac{L''}{R''(\Omega,\mathbf{R}'')} = \frac{\Sigma L_i}{K}, \ \ldots \end{aligned}$$

between the coordinate of two conjugate lines ($\mathbf{r}'$, $\mathbf{R}'$) and ($\mathbf{r}''$, $\mathbf{R}''$).

If two conjugate lines are such that one passes through a point $\mathbf{p}$ and the other one lies in the corresponding plane $\mathbf{P}$, and vice versa, then these two lines will be anharmonically dependent on each other. Given two pairs of conjugate lines, any line that intersects three of them will also intersect the fourth one. If a line coincides with its conjugate then it will be an axis of zero moment.

Among the conjugate lines, the ones for which the directions are orthogonal merit particular attention. If $\mathbf{r}'$ and $\mathbf{r}''$ are two such lines and $\mathbf{p}'$, $\mathbf{p}''$ are the points at which they meet the common perpendicular then $\mathbf{r}'$ will be the axis of maximum moment relative to the point $\mathbf{p}'$ and $\mathbf{r}''$ will be the axis of maximum moment relative to the point $\mathbf{p}''$.

If one of the two conjugate lines is at infinity then the other one will have a constant direction. One of these particular pairs of conjugate lines is orthogonal. The line of that pair that lies at infinity is called the *central axis* of the system. The common perpendicular to two arbitrary conjugate lines intersects the central axis and is perpendicular to it.

Recalling the relations (2), the equations of the central axis will be:

$$(A \cos \mathbf{A} + B \cos \mathbf{B} + \ldots)\frac{\cos \mathbf{A}}{(\Sigma h_i - \Sigma g_i + \Sigma l_i)\,\mathbf{aA}}$$

$$= -(A \cos \mathbf{A} + B \cos \mathbf{B} + \ldots)\ \frac{\cos \mathbf{B}}{(\Sigma f_i - \Sigma h_i + \Sigma m l_i)\,\mathbf{bB}}$$

(4)

$$= (A \cos \mathbf{A} + B \cos \mathbf{B} + \ldots)\ \frac{\cos \mathbf{C}}{(\Sigma g_i - \Sigma f_i + \Sigma n_i)\,\mathbf{cC}}$$

$$= (A \cos \mathbf{A} + B \cos \mathbf{B} + \ldots)\ \frac{\cos \mathbf{D}}{(\Sigma l_i + \Sigma m_i + \Sigma n_i)\,\mathbf{dD}}\,.$$

If one is given five axes of zero moment, which are considered four at a time, then two lines that intersect each of these tetrads of lines will be conjugate lines, and a line that passes through a point **p** and intersects that pair of conjugate lines will lie in the plane **P** of the axes of zero moment relative to **p,** since the lines lie in a plane **P** and that same couple of conjugate lines passes through the point **p** of concurrence of the axes of zero moment relative to **P**. The common perpendicular to all common perpendiculars relative to two pairs of conjugate lines will then be the central axis of the system.

3. In equations (3) of the preceding number, the quantities $\mathbf{r}'$, $\mathbf{r}''$ were arbitrary. Set $\mathbf{r}'(\omega, \mathbf{r}') = \mathbf{r}''(\omega, \mathbf{r}'') = k$, for simplicity. One will then have:

(1) $$f' + f'' = \sum f_i\,, \quad \ldots, \qquad l' + l'' = \sum l_i\,, \quad \ldots,$$

and then the values of r' and r'' [which are inversely proportional to the moments (ω', $\mathbf{r}'$) and (ω'', $\mathbf{r}''$), resp.], which represent forces that act along the conjugate lines $\mathbf{r}'$ and $\mathbf{r}''$, will be equivalent to the given system of forces r_i . Let **r** denote the central axis, and once more take $r(\omega, \mathbf{r}) = k$. From the invariant property of the equivalent system of forces, one will have the relations:

(2)
$$r'^2 + r''^2 + 2r'\,r'' \cos \mathbf{r}'\,\mathbf{r}'' = (\sum r_i \cos \mathbf{r}_i)^2 = r^2,$$

$$r' = r\frac{\sin \mathbf{rr}''}{\sin \mathbf{r}'\mathbf{r}''}\,, \qquad r'' = r\frac{\sin \mathbf{rr}'}{\sin \mathbf{r}''\mathbf{r}'}\,,$$

(3) $$r'\,[\mathbf{r}', \mathbf{r}_k] + r''\,[\mathbf{r}'', \mathbf{r}_k] = \sum r_i\,[\mathbf{r}_i, \mathbf{r}_j] = k,$$

which is the theorem of CHASLES on the constant volume of the tetrahedron whose opposite sides are two arbitrary ones of the forces that can substitute for the given system of forces.

If, in equation (3), one successively makes the axis $\mathbf{r}_k$ coincide with the central axis and the axis that, together with the central axis and the common perpendicular to two conjugate lines $\mathbf{r}'$ and $\mathbf{r}''$, completes a system of three orthogonal axes, and lets ρ' and ρ'' denote the minimum distances between $\mathbf{r}'$, $\mathbf{r}$ and $\mathbf{r}''$, $\mathbf{r}$, respectively, then one will find that:

(5) $$\rho' \tan \mathbf{rr}'' = \rho'' \tan \mathbf{rr}' = \frac{(\omega, \mathbf{r})}{r}\,.$$

If the directions of the conjugate lines $\mathbf{r}'$ and $\mathbf{r}''$ are orthogonal then one will have:

$$(6)\qquad \frac{\rho'}{\tan \mathbf{rr}'} = \frac{\rho''}{\tan \mathbf{rr}''} = \frac{(\omega, \mathbf{r})}{r}, \quad \rho'\rho'' = \frac{(\omega, \mathbf{r})^2}{r^2};$$

in addition, one will then have $r' = r \cos \mathbf{rr}'$, $r'' = r \cos \mathbf{rr}''$, so:

$$(7)\qquad (\omega, \mathbf{r}) = (\omega, \mathbf{r}') \cos \mathbf{rr}' = (\omega, \mathbf{r}'') \cos \mathbf{rr}'',$$

from which, one will deduce that the moment of the system with respect to the central axis is the minimum of the maximum moments relative to the various points of space.

Equations (6) and (7) show clearly the disposition around the central axis of the axes of maximum moments for the various points of space, and also the way in which the values of those moments vary.