“Über lineare Somenmanngifaltigkeiten,” Math. Ann. **81** (1920), 187-218.

On linear manifolds of somas

By

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1. Study introduced the *soma* ([1]), which is any of the ∞^6 positions of a rigid body (whose boundary will be ignored), as the basic concept in analytical mechanics. A trihedron that is fixed in the rigid body will serve as the intuitive carrier of a soma. The trihedron of the coordinate axes will be regarded as the *protosoma*. A *right-handed* soma will arise from it by a motion and a *left-handed* soma by a transfer. The protosoma is then itself regarded as right-handed.

It is customary to represent the figure that we now call a soma by twelve defining data. The soma has a *midpoint* (which is the coordinate origin for the protosoma). That will absorb three defining data – say, in ordinary rectangular point coordinates (x, y, z). Furthermore, a soma has three *axes*; i.e., *oriented* lines. We shall refer to the axis that emerges from the X-axis of the protosoma as the *first* axis. It will be represented by its three direction cosines (p_1, q_1, r_1); *their ratios do not suffice.* We arrive at the other two axes by establishing two more triples of defining data (p_2, q_2, r_2) and (p_3, q_3, r_3).

This representation of a soma indeed has a certain inherent symmetry to it. However, it is purchased at the expense of the fact that no less than *twelve* relations exist between the last nine of these defining data:

$$(1)\qquad \begin{aligned} &p_1^2 + q_1^2 + r_1^2 = 1, \text{ etc.} &\qquad& p_2\,p_3 + q_2\,q_3 + r_2\,r_3 = 0, \text{ etc.} \\ &p_1^2 + p_1^2 + p_1^2 = 1, \text{ etc.} &\qquad& q_1\,r_1 + q_2\,r_2 + q_3\,r_3 = 0, \text{ etc.} \end{aligned}$$

One more must be added to them:

$$(1\text{a})\qquad | \, p_1\,q_2\,r_3 \, | = \pm 1,$$

in which the upper sign is true for right-handed somas.

The consequence is that this analytical apparatus is already cumbersome to work with when one is dealing with only two or three somas.

However, one will diminish the number of relations in (1) by *one* when one appeals to a different representation of the soma. *Eight* ratios:

([1]) *Geometrie der Dynamen*, Appendix, pp. 556, 557, Sitzungsber. d. Berl. Math. Ges. **12** (1913), 36-90.

$$\mathfrak{X}_0 : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \qquad \mathfrak{X}_{123} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} ,$$

between which s single relation exists:

$$\mathfrak{X}_0\, \mathfrak{X}_{123} + \mathfrak{X}_{01}\, \mathfrak{X}_{23} + \mathfrak{X}_{02}\, \mathfrak{X}_{31} + \mathfrak{X}_{03}\, \mathfrak{X}_{12} = 0 \tag{2}$$

will be employed as the *coordinates* of a right-handed soma.

If we introduce:

$$(\mathfrak{X} \mid \mathfrak{X}) = \mathfrak{X}_0^2 + \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 , \tag{3}$$

for brevity, then we shall have:

$$\begin{aligned} (\mathfrak{X} \mid \mathfrak{X})\, p_1 &= \mathfrak{X}_0^2 + \mathfrak{X}_{01}^2 - \mathfrak{X}_{02}^2 - \mathfrak{X}_{03}^2 , \\ (\mathfrak{X} \mid \mathfrak{X})\, q_1 &= 2(\mathfrak{X}_{01}\, \mathfrak{X}_{02} - \mathfrak{X}_0\, \mathfrak{X}_{03}) , \\ (\mathfrak{X} \mid \mathfrak{X})\, r_1 &= 2(\mathfrak{X}_{01}\, \mathfrak{X}_{03} + \mathfrak{X}_0\, \mathfrak{X}_{02}) . \end{aligned} \tag{4}$$

One will get from this, by cyclic permutation of the indices 1, 2, 3: q_2 and r_3 from the first row, r_2 and p_3 from the second row, p_2 and q_3 from the third row.

With that, all of the relations (1), as well as (1a), will already be satisfied when one takes the plus sign in the latter.

Finally, let:

$$\begin{aligned} (\mathfrak{X} \mid \mathfrak{X})\, x &= 2(\mathfrak{X}_{02}\, \mathfrak{X}_{12} - \mathfrak{X}_{03}\, \mathfrak{X}_{31} - \mathfrak{X}_0\, \mathfrak{X}_{23} + \mathfrak{X}_{123}\, \mathfrak{X}_{01}), \\ (\mathfrak{X} \mid \mathfrak{X})\, y &= 2(\mathfrak{X}_{02}\, \mathfrak{X}_{23} - \mathfrak{X}_{01}\, \mathfrak{X}_{12} - \mathfrak{X}_0\, \mathfrak{X}_{31} + \mathfrak{X}_{123}\, \mathfrak{X}_{02}), \\ (\mathfrak{X} \mid \mathfrak{X})\, z &= 2(\mathfrak{X}_{01}\, \mathfrak{X}_{31} - \mathfrak{X}_{02}\, \mathfrak{X}_{23} - \mathfrak{X}_0\, \mathfrak{X}_{12} + \mathfrak{X}_{123}\, \mathfrak{X}_{03}), \end{aligned} \tag{5}$$

for the midpoint (x, y, z) of the soma.

Conversely, if the soma is given by the usual defining data $(x, y, z, p_1, \ldots, r_3)$ then one can ascertain its coordinates in two steps. First, one has:

$$\begin{array}{rcccccc} & \mathfrak{X}_0 & : & \mathfrak{X}_{01} & : & \mathfrak{X}_{02} & : & \mathfrak{X}_{03} \\ = & 1 + p_1 + q_2 + r_3 & : & q_3 - r_3 & : & r_1 - p_3 & : & p_3 - q_3 \\ & q_3 - r_3 & : & 1 + p_1 + q_2 + r_3 & : & p_2 + q_1 & : & r_1 + p_3 \\ & p_2 - q_1 & : & r_1 + p_3 & : & q_3 + r_2 & : & 1 - p_1 - q_2 + r_3 . \end{array} \tag{6a}$$

Each of these four systems of formulas can break down; however, one of them will always be usable. If one has found the ratios of the $\mathfrak{X}_0 : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03}$ then the remaining four coordinates of the soma will follow from the additional formulas:

$$2\, \mathfrak{X}_{123} = \quad * \; + x\mathfrak{X}_{01} + y\mathfrak{X}_{02} + z\mathfrak{X}_{03} ,$$

$$2\, \mathfrak{X}_{23} = -x\mathfrak{X}_0 + \; * \; - z\mathfrak{X}_{02} + y\mathfrak{X}_{03} ,$$

(6b)

$$2\,\mathfrak{X}_{31} = -y\mathfrak{X}_0 + z\mathfrak{X}_{01} + * - y\mathfrak{X}_{03}\,,$$

$$2\,\mathfrak{X}_{12} = -z\mathfrak{X}_0 - y\mathfrak{X}_{01} + x\mathfrak{X}_{02} + * \;.$$

In order for the formulas (4), (5) to make sense, one must require that $(\mathfrak{X} \mid \mathfrak{X}) \neq 0$. We will then speak of a *proper* (right-handed) soma. However, the concept of a soma shall also be extended to the case that was just excluded; i.e., since we are considering *only real* quantities here, to the case in which:

$$\mathfrak{X}_0 = \mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0.$$

We then speak of an *improper* (right-handed) soma. Hence, we shall now call *any system of eight real ratios* $\mathfrak{X}$ *that satisfy the relation* (2) a right-handed soma.

The improper somas have been up in the air, up to now; they were defined only formally, for the time being. In § 5, we will see that an improper soma can be associated with an *ordered triple of oriented directions.*

The basis for the introduction of improper somas lies in the following theorem, which is deduced from (2) immediately:

Theorem 1: *The totality of* ∞^6 *proper and* ∞^3 *improper real somas can be mapped, with no gaps in a single-valued and invertible way, to the totality of real points of a six-fold extended, singularity-free, quadratic manifold* M_6^2 *of signature zero that lives in a seven-dimensional space* R_7 .

That implies an approach to research; at the same time, however, kinematics is recognized to be the general counterpart to line geometry. In that field, as one knows, one maps the straight lines to the points of an M_4^2 whose equation will follow from (2) by "abbreviation" – i.e., when one drops all terms with $\mathfrak{X}_0$ and $\mathfrak{X}_{123}$. Meanwhile, that will also yield a wealth of things that have *no* analogue in line geometry, but will first occur when the number of dimensions is changed to *four,* and that case will attract special interest for us.

2. The collineation (homogeneous point coordinates; $x = x_1 : x_0$, etc.):

(7)

$$x_0' = a_{00}\, x_0\,,$$

$$x_1' = a_{i0}\, x_0 + a_{i1}\, x_1 + a_{i2}\, x_2 + a_{i3}\, x_3 \qquad (i = 1, 2, 3),$$

will be a *motion* when the coefficients satisfy certain well-known relations. They will all be satisfied when one introduces eight homogeneous *parameters* ([2]):

$$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \qquad \beta_0 : \beta_1 : \beta_2 : \beta_3 ,$$

which satisfy only a single quadratic relation:

$$\alpha_0\, \beta_0 + \alpha_1\, \beta_1 + \alpha_2\, \beta_2 + \alpha_3\, \beta_3 = 0, \tag{8}$$

and are subject to the inequality:

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0, \tag{9}$$

moreover.

One must then set:

$$\begin{aligned} a_{00} &= \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 , \\ a_{11} &= \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 , \ \ldots \\ a_{23} &= 2\,(\alpha_2\alpha_3 + \alpha_0\alpha_1), \ \ldots \\ a_{32} &= 2\,(\alpha_2\alpha_3 - \alpha_0\alpha_1), \ \ldots \\ a_{10} &= 2\,(\alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_0\beta_1 + \alpha_1\beta_0), \ \ldots \end{aligned} \tag{10}$$

Here, *as well as henceforth*, the ellipses after the last four lines shall suggest that two sequences are missing that are defined by the cyclic permutation of the indices 1, 2, 3.

Any system of eight such parameters (α, β) that satisfy the requirements (8) and (9) will yield a motion, *and conversely*.

It is a *translation* for $\alpha_1 = \alpha_2 = \alpha_3 = \beta_0 = 0$. If one orients the direction of translation in such a way that one fixes its cosines (i.e., not merely their ratios) then the *step size* $2H_0^*$ of the translation will be determined uniquely:

$$\cos \lambda_1 : \cos \lambda_2 : \cos \lambda_3 : H_0^{*-1} = \beta_1 : \beta_1 : \beta_2 : - \alpha_0, \tag{11}$$

i.e.:

$$\cos \lambda_1 = \beta_1 : \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} , \ \ldots, \qquad H_0^* = - \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} : \alpha_0 ,$$

in which the square root is assigned an arbitrary value that is the same in both cases.

In the remaining cases, there is an *axis for the motion.* Its Plücker coordinates:

$$\mathfrak{P}_{01} : \mathfrak{P}_{02} : \mathfrak{P}_{03} : \mathfrak{P}_{23} : \mathfrak{P}_{31} : \mathfrak{P}_{12}$$

are deduced from the formulas:

([2]) Study, "Von den Bewegungen und Umlegungen," Math. Ann. **39** (1891), 527, 528. Unfortunately, some sign errors crept in *at crucial places* there that were corrected in *Geom. d. Dyn.*, but here it will be necessary for us to dwell upon them longer than would be necessary, especially for the transfers.

$$(12)\qquad \begin{aligned} \mathfrak{P}_{01} &= (\alpha_1^2+\alpha_2^2+\alpha_3^2)\,\alpha_1, \ldots \\ \mathfrak{P}_{23} &= (\alpha_1^2+\alpha_2^2+\alpha_3^2)\,\beta_1 + \alpha_0\beta_0\alpha_1, \ldots \end{aligned}$$

If one orients it by a convention on the sign of $\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}$ then the *step size* $2H_0$ will be determined uniquely, and the *rotation angle* $2\Theta_0$ will be determined up to a multiple of 2π:

$$(13)\qquad \cot\Theta_0 = -\alpha_0 : \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}\,, \qquad H_0 = +\beta_0 : \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}\,.$$

One must make a remark about this that was already true for (11), and shall not be repeated after this. If one endows the α, β with a proportionality factor $\rho \neq 0$ then $\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}$ must also be multiplied by $+\rho$, and not with $-\rho$.

If $\alpha_0 = 0$ then one will have $2\Theta_0 = \pi \pmod{2\pi}$. These ∞^5 screwing motions play an essential role in the further considerations, and are called *unscrewings* ([3]).

If $\beta_0 = 0$, without having $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then one is dealing with a *rotation* ($2H_0 = 0$). The intersection of the rotations with the unscrewings defined the ∞^4 involutory motions; they are called *reversals* ([4]) ($\alpha_0 = 0$, $\beta_0 = 0$).

If a *translation* is given by its oriented direction of translation ($\cos\lambda_1$, $\cos\lambda_2$, $\cos\lambda_3$) and step size $2H_0^*$ then, from (11), its parameters will be:

$$(14)\qquad -H_0^{*-1} : 0 : 0 : 0 : \qquad 0 : \cos\lambda_1 : \cos\lambda_2 : \cos\lambda_3\,.$$

The remaining motion will be characterized by the (*oriented*) axis $\mathfrak{P}$, the step size $2H_0$, and the angle of rotation $2\Theta_0$. Its parameters are then:

$$(15)\qquad \begin{aligned} \alpha_0 &= -\cot\Theta_0\,\sqrt{\mathfrak{P}_{01}^2+\mathfrak{P}_{02}^2+\mathfrak{P}_{03}^2}\,, &\qquad \alpha_1 &= \mathfrak{P}_{01}, \ldots \\ \beta_0 &= H_0\,\sqrt{\mathfrak{P}_{01}^2+\mathfrak{P}_{02}^2+\mathfrak{P}_{03}^2}\,, &\qquad \beta_1 &= \mathfrak{P}_{23} + H_0 \cot\Theta_0\,\mathfrak{P}_{01}, \ldots \end{aligned}$$

The soma $\mathfrak{X}$ arises from the protosoma by the motion with the parameters:

$$(16)\qquad \alpha_0 = \mathfrak{X}_0\,,\ \alpha_1 = \mathfrak{X}_{01}\,, \ldots, \qquad \beta_0 = \mathfrak{X}_{123}\,,\ \beta_1 = \mathfrak{X}_{23}\,, \ldots$$

Soma coordinates and parameters of motion are then essentially identical. We can manage with one of these systems of quantities. However, we believe that one's understanding of things is eased by their separation.

([3]) Math. Ann. **39** (1891), pp. 461.
([4]) H. Wiener, Sächs. Berichte, 1890.

3. A *left-handed* soma arises from the protosoma by a *transfer*, and therefore the most necessary ideas regarding those transformations will be stated now.

The collineation:

$$(17)\qquad \begin{aligned} x_0' &= -\,a_{00}\,x_0\,, \\ x_i' &= a_{i0}\,x_0 + a_{i1}\,x_1 + a_{i2}\,x_2 + a_{i3}\,x_3\,, \qquad (i = 1, 2, 3) \end{aligned}$$

is a transfer when the coefficients are replaced with (10), while giving consideration to (8) and (9). However, we prefer to avoid confusion by using a different notation. As Study recently did, we call the eight homogeneous parameters:

$$\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3 : \qquad \delta_0 : \delta_1 : \delta_2 : \delta_3$$

transfer parameters. They satisfy the equation:

$$(18)\qquad \gamma_0\,\delta_0 + \gamma_1\,\delta_1 + \gamma_2\,\delta_2 + \gamma_3\,\delta_3 = 0$$

and the inequality:

$$(19)\qquad \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \neq 0.$$

One shall then have [cf., (10)]:

$$(20)\qquad \begin{aligned} a_{00} &= \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2\,, \\ a_{11} &= \gamma_0^2 + \gamma_1^2 - \gamma_2^2 - \gamma_3^2\,, \\ a_{23} = 2\,(\gamma_2\gamma_3 + \gamma_0\gamma_1), \ldots, &\qquad a_{32} = 2\,(\gamma_2\gamma_3 - \gamma_0\gamma_1), \ldots, \\ a_{10} &= 2\,(\gamma_2\,\delta_3 - \gamma_{23}\delta_2 - \gamma_0\,\delta_1 + \gamma_1\,\delta_0), \ldots, \end{aligned}$$

The transfer (17) leaves the *proper* plane $\delta_0 : \gamma_1 : \gamma_2 : \gamma_3$ fixed, along with the point $\gamma_0 : \delta_1 : \delta_2 : \delta_3$, which lies on it (viz., the *middle plane and midpoint*). Both of them can be undetermined (but not simultaneously!); one would then obtain the $2 \cdot \infty^3$ involutory transfers, namely, *reflections through (proper) planes u* (undetermined midpoint):

$$(21)\qquad 0 : u_1 : u_2 : u_3 : \qquad u_0 : 0 : 0 : 0,$$

and *reflections through (proper) points* ξ (undetermined midpoint):

$$(22)\qquad \xi_0 : 0 : 0 : 0 : \qquad 0 : \xi_1 : \xi_2 : \xi_3\,.$$

The midpoint ξ can be *improper* ($\gamma_0 = 0$). The transfers can then be generated by the reflection in the middle plane and a translation parallel to the middle plane that commutes with it, and whose direction runs perpendicular to its improper midpoint. After orientating the direction of translation, the step size of that translation $2H_0^*$ will be determined uniquely, and will then be called the step size *of the transfer*. Let that direction of translation be:

(23) $$\cos \lambda_1 = \frac{u_2\xi_3 - u_3\xi_2}{\sqrt{u_1^2 + u_2^2 + u_3^2}\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}}, \ldots$$

The parameters of the transfer can then be written:

(23) $$\gamma_0 = 0, \qquad \gamma_1 = u_1 \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, \ldots$$
$$\delta_0 = u_0\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, \qquad \delta_1 = -H_0^*\,\xi_1\sqrt{u_1^2 + u_2^2 + u_3^2}, \ldots$$

Conversely, one has:

$$H_0^* = -\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2} : \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2},$$
$$\cos \lambda_1 = \gamma_2\,\delta_3 - \gamma_3\,\delta_2 : \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2}, \ldots$$

Finally, one must still consider the "general" case ($\gamma_0 \neq 0$, $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 \neq 0$). The midpoint and middle plane are well-defined and proper, and determine the line whose Plücker coordinates are:

(24) $$\gamma_0\,\gamma_1 : \gamma_0\,\gamma_2 : \gamma_0\,\gamma_3 : \quad \gamma_3\,\delta_2 - \gamma_2\,\delta_3 : \gamma_1\,\delta_3 - \gamma_3\,\delta_1 : \gamma_2\,\delta_1 - \gamma_1\,\delta_2,$$

moreover, which is normal to the middle plane at the midpoint. It is called the *axis* of the transfer. It can now be generated by a rotation around the axis and a reflection that commutes with it. The angle of rotation will be different according to whether the latter is the reflection through the middle plane or the *midpoint*. (The difference amounts to π.) When we speak of the angle of rotation $2\Theta_0$ of the transfer, we choose the *second* possibility, and then, analogous to (13), we will have:

(25) $$\cot \Theta_0 = -\gamma_0 : \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2},$$

when we take the seventh coordinates of the transfer axis to be:

$$\gamma_0\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2},$$

in order to orient it.

Conversely, if the (oriented) middle plane u and the midpoint ξ are given then the transfer axis will already be oriented; If $2\Theta_0$ is the angle of rotation of the transfer then it will have the parameters:

(26) $$\gamma_0 = -\cot \Theta_0\,\xi_0\sqrt{u_1^2 + u_2^2 + u_3^2}, \qquad \gamma_1 = u_1\,\xi_0, \ldots$$
$$\delta_0 = u_0\,\xi_0, \qquad \delta_0 = -\cot \Theta_0\,\xi_1\sqrt{u_1^2 + u_2^2 + u_3^2}, \ldots$$

One can then employ the eight ratios:

$$(27) \qquad \mathfrak{X}_0^l = \gamma_0, \qquad \mathfrak{X}_{01}^l = \gamma_1, \ldots, \quad \mathfrak{X}_{123}^l = \delta_0, \qquad \mathfrak{X}_{23}^l = \delta_1, \ldots,$$

as the coordinates of the *left-hand soma* that emerges from the transfer (γ, δ) by the transfer, and which satisfy the quadratic relation:

$$(28) \qquad \mathfrak{X}_0^l\,\mathfrak{X}_{123}^l + \mathfrak{X}_{01}^l\,\mathfrak{X}_{23}^l + \mathfrak{X}_{02}^l\,\mathfrak{X}_{31}^l + \mathfrak{X}_{03}^l\,\mathfrak{X}_{12}^l = 0.$$

Therefore, this relation appears for a fourth time, but this time with a different meaning, and later on (§ 10) two (four) more interpretations will be addressed. The consistency of the basic analytic notions explains the altered notation!

4. We arrived at the proper, *right-handed soma* by a motion of the *protosoma*. Now, let $\mathfrak{X}$ and $\mathfrak{Y}$ be two proper, right-handed somas, where $\mathfrak{X}$ no longer needs to be the protosoma. We seek the motion that takes $\mathfrak{X}$ to $\mathfrak{Y}$. One finds its parameters to be:

$$(29) \qquad \begin{aligned} \alpha_0 = (\mathfrak{X}\,/\,\mathfrak{Y}) &= \mathfrak{X}_0\,\mathfrak{Y}_0 + \mathfrak{X}_{01}\,\mathfrak{Y}_{01} + \mathfrak{X}_{02}\,\mathfrak{Y}_{02} + \mathfrak{X}_{03}\,\mathfrak{Y}_{03}, \\ \alpha_1 = \quad &= \mathfrak{X}_0\,\mathfrak{Y}_{01} - \mathfrak{X}_{01}\,\mathfrak{Y}_0 - \mathfrak{X}_{02}\,\mathfrak{Y}_{03} + \mathfrak{X}_{03}\,\mathfrak{Y}_{02}, \ldots \\ \beta_0 = (\mathfrak{X}\mathfrak{Y}) &= \mathfrak{X}_0\,\mathfrak{Y}_{123} + \mathfrak{X}_{01}\,\mathfrak{Y}_{23} + \mathfrak{X}_{02}\,\mathfrak{Y}_{31} + \mathfrak{X}_{03}\,\mathfrak{Y}_{12}, \\ &\quad + \mathfrak{X}_{123}\,\mathfrak{Y}_0 + \mathfrak{X}_{23}\,\mathfrak{Y}_{01} + \mathfrak{X}_{31}\,\mathfrak{Y}_{02} + \mathfrak{X}_{12}\,\mathfrak{Y}_{03}, \\ \beta_1 = \quad &= \mathfrak{X}_0\,\mathfrak{Y}_{23} - \mathfrak{X}_{01}\,\mathfrak{Y}_{123} - \mathfrak{X}_{02}\,\mathfrak{Y}_{12} + \mathfrak{X}_{03}\,\mathfrak{Y}_{31}, \\ &\quad + \mathfrak{X}_{123}\,\mathfrak{Y}_{01} - \mathfrak{X}_{23}\,\mathfrak{Y}_0 - \mathfrak{X}_{31}\,\mathfrak{Y}_{03} + \mathfrak{X}_{12}\,\mathfrak{Y}_{02}, \ldots \end{aligned}$$

When these formulas are solved for $\mathfrak{Y}$, they will give merely the composition of two motions, when taken at their basis, and can be summarized very elegantly when one appeals to certain biquaternions ([5]). We deliberately refrain from employing that tool here. The solution of formulas (29) for $\mathfrak{Y}$ is then accomplished by means of formulas (36).

The abbreviated symbol ($\mathfrak{X}\,/\,\mathfrak{Y}$) (which is read as: $\mathfrak{X}$ into $\mathfrak{Y}$!) that appears in the first formula of (29) subsumes the one that appears in (3) as a special case and emerges from it by a process of polarization. A further-abbreviated symbol appears in the third row of (29). With the help of it, the quadratic relation (2) that exists between the coordinates of a soma can be written briefly as:

$$(2) \qquad \tfrac{1}{2}(\mathfrak{X}\mathfrak{X}) = 0.$$

It follows from (29) that:

([5]) Sitzungsber. d. Berl. Math. Ges. **12** (1913), pp. 40.

(30) $$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (\mathfrak{X} / \mathfrak{X})(\mathfrak{Y} / \mathfrak{Y}) - (\mathfrak{X} / \mathfrak{Y})^2,$$

and one will then obtain the expressions:

(31) $$\cot \Theta = -\frac{(\mathfrak{X} / \mathfrak{Y})}{\sqrt{(\mathfrak{X} / \mathfrak{X})(\mathfrak{Y} / \mathfrak{Y}) - (\mathfrak{X} / \mathfrak{Y})^2}}, \qquad H = +\frac{(\mathfrak{X}\mathfrak{Y})}{\sqrt{(\mathfrak{X} / \mathfrak{X})(\mathfrak{Y} / \mathfrak{Y}) - (\mathfrak{X} / \mathfrak{Y})^2}}$$

for the angle of rotation 2Θ and the step size $2H$ of that motion from (13) and (29). We now define the two quantities Θ and H to be the *angle* and *distance* between the two somas $\mathfrak{X}$ and $\mathfrak{Y}$. The process of abbreviation that was described in § 1 goes to the angle and distance between the two lines $\mathfrak{X}$ and $\mathfrak{Y}$ (which are assumed to not be parallel).

This further yields the following terminology with no effort:

Two proper somas are called *parallel* to each other when each of them can be obtained from the other by a *translation.* The two somas $\mathfrak{X}$ and $\mathfrak{Y}$ are parallel to each other when:

(32) $$\mathfrak{X}_0 : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} = \mathfrak{Y}_0 : \mathfrak{Y}_{01} : \mathfrak{Y}_{02} : \mathfrak{Y}_{03},$$

and the parallelism of somas will be defined by this when improper somas come under consideration. All improper somas will then be parallel to each other, and also to any proper soma.

If each of the two proper somas $\mathfrak{X}$ and $\mathfrak{Y}$ can be obtained from the other one by a *rotation* then we will say that the two somas *intersect.* The necessary, but not sufficient condition for this, namely:

(33) $$(\mathfrak{X}\mathfrak{Y}) = 0,$$

once more serves to extend the concept to improper somas.

It is useful to add a few words that say that either the two somas $\mathfrak{X}$ and $\mathfrak{Y}$ are parallel to each other or they intersect. We then call them *incident.* (33) will then be necessary *and* sufficient for incident.

For $(\mathfrak{X}\mathfrak{Y}) \neq 0$, the two somas $\mathfrak{X}$ and $\mathfrak{Y}$ shall be called *skew* to each other. Each of them can then be obtained from the other one by a *screwing motion* that therefore does not reduce to a rotation or a translation.

In particular, if it is an *unscrewing* then we will call the two somas *orthogonal* to each other ([6]):

(34) $$(\mathfrak{X} / \mathfrak{Y}) = 0.$$

The concept is again extended to improper somas by that.

Finally, if every soma goes to the other one by a *reversal* then both of them are said to *intersect perpendicularly* ([7]):

([6]) "Hemi-symmetral," to Study.
([7]) "Symmetral," to Study.

$$(\mathfrak{X} \,/\, \mathfrak{Y}) = 0, \quad (\mathfrak{X}\mathfrak{Y}) = 0.$$

The expression on the right in:

$$H \tan \Theta = -(\mathfrak{X}\mathfrak{Y}) : (\mathfrak{X} \,/\, \mathfrak{Y}) \tag{35}$$

that is defined by (31) is *rational*, and is called the *moment* of the two somas $\mathfrak{X}$ and $\mathfrak{Y}$ ([8]), or also the "moment of the motion that takes $\mathfrak{X}$ to $\mathfrak{Y}$."

For the sake of completeness, we also have a measure H^* for the proper, *parallel* somas. The step size $2H^*$ of the translation that takes $\mathfrak{X}$ to $\mathfrak{Y}$ can be ascertained from the following formula [which is derived from (11) and (29)] by the use of:

$$\{\mathfrak{X} \,/\, \mathfrak{Y}\} = \mathfrak{X}_{123}\, \mathfrak{Y}_{123} + \mathfrak{X}_{23}\, \mathfrak{Y}_{23} + \mathfrak{X}_{31}\, \mathfrak{Y}_{31} + \mathfrak{X}_{12}\, \mathfrak{Y}_{12}$$

(which will not be used after this), namely:

$$H^* = -\sqrt{(\mathfrak{X}/\mathfrak{X})\{\mathfrak{Y}/\mathfrak{Y}\} - 2(\mathfrak{X}/\mathfrak{Y})\{\mathfrak{X}/\mathfrak{Y}\} + (\mathfrak{Y}/\mathfrak{Y})\{\mathfrak{X}/\mathfrak{X}\}} \;:\; (\mathfrak{X} \,/\, \mathfrak{Y}),$$

which can be written more simply if one dispenses with symmetry, and goes to the abbreviated expression for the distance between two parallel lines. H^* is then called *the distance between the parallel* somas $\mathfrak{X}$ and $\mathfrak{Y}$.

We cite two systems of formulas that we shall make much use of in what follows.

The *proper, right-handed* soma $\mathfrak{Y}$, which emerges from the *proper, right-handed* soma $\mathfrak{X}$ by the equation (α, β), is:

$$\begin{aligned}
\mathfrak{Y}_0 &= \alpha_0\, \mathfrak{X}_0 - \alpha_1\, \mathfrak{X}_{01} - \alpha_2\, \mathfrak{X}_{02} - \alpha_3\, \mathfrak{X}_{03}, \\
\mathfrak{Y}_{01} &= \alpha_1\, \mathfrak{X}_0 + \alpha_0\, \mathfrak{X}_{01} + \alpha_3\, \mathfrak{X}_{02} - \alpha_2\, \mathfrak{X}_{03}, \ldots \\
\mathfrak{Y}_{123} &= \beta_0\, \mathfrak{X}_0 - \beta_1\, \mathfrak{X}_{01} - \beta_2\, \mathfrak{X}_{02} - \beta_3\, \mathfrak{X}_{03} \\
&\quad + \alpha_0\, \mathfrak{X}_{123} - \alpha_1\, \mathfrak{X}_{23} - \alpha_2\, \mathfrak{X}_{31} - \alpha_3\, \mathfrak{X}_{12}, \\
\mathfrak{Y}_{23} &= \beta_1\, \mathfrak{X}_0 + \beta_1\, \mathfrak{X}_{01} + \beta_3\, \mathfrak{X}_{02} - \beta_2\, \mathfrak{X}_{03} \\
&\quad + \alpha_1\, \mathfrak{X}_{123} + \alpha_0\, \mathfrak{X}_{23} + \alpha_3\, \mathfrak{X}_{31} - \alpha_2\, \mathfrak{X}_{12}, \ldots
\end{aligned} \tag{36}$$

In other words, these are the formulas for the motion (α, β) in soma coordinates when one replaces $\mathfrak{Y}$ with $\mathfrak{X}'$, as usual. It is remarkable that the parameters of motion appear linearly here. These formulas are *considerably* simpler than the ones that fall out by the

([8]) Which deviates from Cayley, who called such a simultaneous invariant of two straight lines *irrational*.

use of the defining data (x, y, z; p_1, …, r_3) in § 1. Finally, they also tell one how the motion (α, β) will permute the *improper* somas; there is only a *three-parameter* family of them. The β then drop out of formulas (36), and in particular, any improper soma will be fixed by all translations.

If we call a *left-handed* soma $\mathfrak{X}^l$ *improper* when:

$$\mathfrak{X}_0^l = \mathfrak{X}_{01}^l = \mathfrak{X}_{02}^l = \mathfrak{X}_{03}^l = 0,$$

and otherwise *proper*, then we will further have:

The *proper, right-handed* soma $\mathfrak{Y}$ that emerges from the *proper, left-handed* soma $\mathfrak{X}^l$ by the *transfer* (γ, δ) reads:

$$(37)\quad \begin{aligned}
\mathfrak{Y}_0 &= \gamma_0 \mathfrak{X}_0^l - \gamma_1 \mathfrak{X}_{01}^l - \gamma_2 \mathfrak{X}_{02}^l - \gamma_3 \mathfrak{X}_{03}^l, \\
\mathfrak{Y}_{01} &= \gamma_1 \mathfrak{X}_0^l + \gamma_0 \mathfrak{X}_{01}^l + \gamma_3 \mathfrak{X}_{02}^l - \gamma_2 \mathfrak{X}_{03}^l, \ldots \\
\mathfrak{Y}_{123} &= -\delta_0 \mathfrak{X}_0^l + \delta_1 \mathfrak{X}_{01}^l + \delta_2 \mathfrak{X}_{02}^l + \delta_3 \mathfrak{X}_{03}^l \\
&\quad + \gamma_0 \mathfrak{X}_{123}^l - \gamma_1 \mathfrak{X}_{23}^l - \gamma_2 \mathfrak{X}_{31}^l - \gamma_3 \mathfrak{X}_{12}^l, \\
\mathfrak{Y}_{23} &= -\delta_1 \mathfrak{X}_0^l - \beta_1 \mathfrak{X}_{01}^l + \delta_3 \mathfrak{X}_{02}^l + \delta_2 \mathfrak{X}_{03}^l \\
&\quad + \gamma_1 \mathfrak{X}_{123}^l + \gamma_0 \mathfrak{X}_{23}^l + \gamma_3 \mathfrak{X}_{31}^l - \gamma_2 \mathfrak{X}_{12}^l, \ldots
\end{aligned}$$

These formulas can also be written more conveniently with the use of biquaternions; however, in every application, one will find the final formulas. There are two more similarly-constructed systems of formulas, which we shall not need; they are exhibited quite easily on the basis for the two remarks:

The inverse of the motion (α, β) has the parameters:

$$(38)\qquad \alpha_0 : -\alpha_1 : -\alpha_2 : -\alpha_3 : \qquad \beta_0 : -\beta_1 : -\beta_2 : -\beta_3 .$$

The inverse of the transfer (α, β) has the parameters:

$$(39)\qquad \gamma_0 : -\gamma_1 : -\gamma_2 : -\gamma_3 : \qquad -\delta_0 : \delta_1 : \delta_2 : \delta_3 .$$

Finally, one will get the midpoint and direction of the axis of a left-hand soma when one introduces a minus sign everywhere on the left in (4) and (5).

5. We would now like to make the concept of *improper* (right-handed) soma (which we defined only formally up to now) more intuitive. For that, we proceed in analogy to line geometry, in which one replaces an improper line with the totality of proper lines that are incident with it.

We then consider an improper soma:

$$(40) \qquad 0 : 0 : 0 : 0 : \qquad \mathfrak{A}_{123} : \mathfrak{A}_{23} : \mathfrak{A}_{31} : \mathfrak{A}_{12}$$

and look for all ∞^5 *proper* somas $\mathfrak{X}$ that are incident with them. The condition for that $(\mathfrak{A}\mathfrak{X}) = 0$ reduces to:

$$\mathfrak{A}_{123}\,\mathfrak{X}_0 + \mathfrak{A}_{23}\,\mathfrak{X}_{01} + \mathfrak{A}_{31}\,\mathfrak{X}_{02} + \mathfrak{A}_{12}\,\mathfrak{X}_{03} = 0.$$

By comparing this with (34) and the first of formulas (29), one finds that all of these somas $\mathfrak{X}$ are *orthogonal* to the *proper* somas:

$$(41) \qquad \mathfrak{A}_{123} : \mathfrak{A}_{23} : \mathfrak{A}_{31} : \mathfrak{A}_{12} : \qquad 0 : 0 : 0 : 0,$$

and likewise to the ∞^3 somas that are parallel to the latter soma (41).

The ∞^5 somas $\mathfrak{X}$ that are incident to the *improper* soma (40) emerge from these ∞^3 proper somas by way of the ∞^5 *unscrewings*.

The aforementioned somas that are parallel to the soma (41) are thus determined uniquely from the improper soma (40). On the other hand, they determine three *oriented* directions (which are syntactic to those of their axes). However, the direction cosines of the first axis are:

$$\cos\lambda_1 = \frac{\mathfrak{A}_{123}^2 + \mathfrak{A}_{23}^2 - \mathfrak{A}_{31}^2 - \mathfrak{A}_{12}^2}{\mathfrak{A}_{123}^2 + \mathfrak{A}_{23}^2 + \mathfrak{A}_{31}^2 + \mathfrak{A}_{12}^2},$$

$$(42) \qquad \cos\mu_1 = \frac{2(\mathfrak{A}_{23}\mathfrak{A}_{31} - \mathfrak{A}_{123}\mathfrak{A}_{12})}{\mathfrak{A}_{123}^2 + \mathfrak{A}_{23}^2 + \mathfrak{A}_{31}^2 + \mathfrak{A}_{12}^2},$$

$$\cos\nu_1 = \frac{2(\mathfrak{A}_{23}\mathfrak{A}_{12} + \mathfrak{A}_{123}\mathfrak{A}_{31})}{\mathfrak{A}_{123}^2 + \mathfrak{A}_{23}^2 + \mathfrak{A}_{31}^2 + \mathfrak{A}_{12}^2}.$$

When one agrees that $\mathfrak{X}_{123} = \mathfrak{X}_{231} = \mathfrak{X}_{312}$ then cyclic permutation of the indices 1, 2, 3 will produce $\cos\mu_2$ and $\cos\nu_2$ from $\cos\lambda_1$, $\cos\nu_2$ and $\cos\lambda_3$ from $\cos\mu_1$, and $\cos\lambda_2$ and $\cos\mu_3$ from $\cos\nu_1$.

The figure of these three oriented directions now gives us an intuitive picture of the improper somas. If one would like to proceed in complete analogy with line geometry then one would have to replace an improper line in that subject, not with the associated pencil of parallels, but one would have to go to the bundle of parallel normals.

Now that we have learned how to link the soma with a sufficiently clear picture in all cases, we return to Theorem 1, which showed us that the somas could be mapped to points of an M_6^2. We take from the algebra of quadratic forms the theorem that the points of the M_6^2 are permuted with each other in the most general way by the projectivities of a twenty-eight-parameter group of R_7. For the sake of brevity, we would now like to denote it by (G_{28}, H_{28}), and likewise for the group of soma transformations that is holomorphic to it. The fact that it consists of (*at least*) two separate families of transformations will be shown in § 10, where we will describe the structure of the group, moreover.

That yields a method of research in the spirit of Klein's Erlanger Programm. However, (G_{28}, H_{28}) plays the same role in kinematics that the group (G_{15}, H_{15}) of collineations and correlations does in line geometry.

The difference between proper and improper somas is inessential in the geometry of the group (G_{28}, H_{28}). Such a difference will first arise in the geometry of a twenty-two-parameter (proof in § 10) subgroup G_{22} of G_{28} that can also be described in parallel to the affinities.

It is now convenient to treat only the phenomena that have an invariant character under transformations of (G_{28}, H_{28}), so one will then also arrive at the simplest general laws. Meanwhile the difference between proper and improper somas is too profound for us too leave it completely unmentioned in a first introduction of the subject. We shall then study the *linear manifolds of somas*, which we shall now move on to, as well as their classification under G_{22}, which is closely connected with the kinematic generation of those structures and brings a greater degree of intuitiveness to the behavior that has seemed quite complicated up to now.

6. We define the *pencil of somas:*

$$\mathfrak{X} = \sigma_1\,\mathfrak{A} + \sigma_2\,\mathfrak{B} \tag{43}$$

from two *distinct, incident* (right-handed) somas $\mathfrak{A}$ and $\mathfrak{B}$. The two somas $\mathfrak{A}$ and $\mathfrak{B}$ must be incident, since otherwise the relation (2) would not be fulfilled for $\mathfrak{X}$. One shall then have $(\mathfrak{A}\mathfrak{A}) = (\mathfrak{A}\mathfrak{B}) = (\mathfrak{B}\mathfrak{B}) = 0$. However, one will then have:

Theorem 2: *Any soma of a pencil of somas is incident with any other such soma.*

In fact, let $\mathfrak{Y} = \tau_1\,\mathfrak{A} + \tau_2\,\mathfrak{B}$ be another soma of the pencil ($\sigma_1\tau_2 - \sigma_2\tau_1 \neq 0$), so one will have:

$$(\mathfrak{X}\mathfrak{Y}) = \sigma_1\tau_1\,(\mathfrak{A}\mathfrak{A}) + (\sigma_1\tau_2 + \sigma_2\tau_1)\,(\mathfrak{A}\mathfrak{B}) + \sigma_2\tau_2\,(\mathfrak{B}\mathfrak{B}) = 0.$$

Theorem 3: *The pencils of somas are associated with the generating R_1's (straight lines) in M_6^2.*

An easy count of the constants, which we would like to perform here, shows that there are ∞^9 pencils of somas. The point $\mathfrak{A}$ on M_6^2 can be chosen in ∞^6 ways, while the point $\mathfrak{B}$ can be chosen in only ∞^5 ways, since $(\mathfrak{A}\mathfrak{B}) = 0$. There are then ∞^{11} useful point-pairs on M_6^2. However, each generator of M_6^2 belongs to ∞^2 such pairs. Therefore, there will be only ∞^9 such generators, and thus, ∞^9 pencils of somas. One similarly shows that:

∞^4 pencils of somas run through every soma.

The ∞^9 pencils of somas define a single class under transformations of (G_{28}, H_{28}); i.e., every pencil of somas can be transformed into any other. The properties of the pencils of somas can then be studied in a particular pencil of somas, say, the pencil:

$$0 : \sigma_1 : \sigma_2 : 0 : \qquad 0 : 0 : 0 : 0 \,.$$

Things are different for the transformations of G_{22}, where the difference between proper and improper somas will become essential. We then have *three* classes to distinguish:

a) *Pencils of somas that consist of only proper somas.* No two somas in the pencil are parallel. We choose:

$$\sigma_1 : \sigma_2 : 0 : 0 : \qquad 0 : 0 : 0 : 0 \tag{44}$$

to be the "canonical" example.

One generates such a pencil of somas kinematically when one subjects any of its somas to *all rotations around a fixed axis* (in the example of the protosoma – say – all rotations around its first axis). Or:

One reflects a *left-handed* soma *through all planes of a pencil with proper axis* (in the example of – say – the left-handed soma:

$$0 : 0 : 1 : 0 : \quad 0 : 0 : 0 : 0,$$

through all planes through the X-axis). The proofs are by (36) and (37).

b) *Pencils of somas with a single improper soma.* All somas of the pencil will then be parallel to each other. There are ∞^7 such pencils of somas, through every proper soma there are ∞^2, and through every improper one, there are ∞^4 of them. Example:

$$\sigma_1 : 0 : 1 : 0 : \quad 0 : \sigma_2 : 0 : 0. \tag{45}$$

One will obtain the *proper* somas of such a pencil when one subjects a right-handed soma to *all translations along a fixed direction* [in the example, perhaps, the protosoma in the direction of the X-axis, cf., (14)] or when one *reflects* a left-handed soma *through the planes of a pencil of parallel planes* (in the example of the left-handed soma:

$$0 : 1 : 0 : 0 : \quad 0 : 0 : 0 : 0,$$

through all planes that are perpendicular to the X-axis).

One might be tempted to think that improper somas in such a pencil will be obtained from the common direction of the axes of its ∞^1 parallel, proper somas. Moreover, one has subjected that axis direction to a *reversal* around a line that is parallel to the direction of translation. With that, the improper soma in the pencil is also constructed.

c) *The pencil of somas consists of nothing but improper somas.* There are ∞^4 such pencils, and ∞^2 of them through any (improper) soma. One subjects a (right-handed) improper soma to, perhaps, *all rotations around a line* that can be chosen arbitrarily inside of the bundle of parallels that it determines. All pencils of somas can be obtained from the initial soma by changing that bundle of parallels, or one applies *all transfers with a fixed midpoint and fixed middle plane* to a left-handed, improper soma. For example:

(46) $$0 : 0 : 0 : 0 : \qquad 0 : \sigma_1 : \sigma_2 : 0\,.$$

7. The three right-handed somas $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ shall belong to no pencil of somas, but shall be incident to two of them. With that assumption, the system:

(47) $$\mathfrak{X} = \sigma_1\,\mathfrak{A} + \sigma_2\,\mathfrak{B} + \sigma_3\,\mathfrak{C}$$

will represent a manifold of ∞^2 somas, namely, a *bundle of somas.*

Theorem 4: *Any soma of a bundle of somas is incident with every other such soma.*

The proof is similar to that of Theorem 2.

Theorem 5: *The bundles of somas are associated with the generating R_2's (i.e., planes) on* M_6^2.

There are ∞^9 bundles of somas, so there are ∞^5 through any soma, and ∞^2 through every pencil of somas. A bundle of somas contains ∞^2 pencils of somas, of which, two distinct ones will always have a single common soma. Thus, two pencils of somas that have a coma in common do not have to belong to a bundle of somas. Any two distinct somas of the bundle can always be linked by a single pencil of somas that lies in the bundle completely. Two distinct bundles of comas can have one soma or a pencil of somas in common; however, it is also conceivable that they might be completely skew to each other. The ∞^9 bundles of somas define a single class under (G_{28}, H_{28}).

We now again turn our attention to the improper somas, and thus classify them under G_{22} and give kinematic generators for the individual types.

a) *All somas of the bundle of somas are proper*. No two somas in the bundle are parallel then; there are only pencils of somas of type *a*). One performs *all rotations around the straight lines of a pencil with a proper vertex* on a right-handed soma or reflects a left-handed soma *through all planes through a proper point.* Example:

(48) $$0 : \sigma_1 : \sigma_2 : \sigma_3 : \qquad 0 : 0 : 0 : 0\,.$$

b) *Bundle of somas with a single, improper soma.* There are ∞^8 of them, ∞^4 of them through any proper soma, and ∞^5 of them through any improper one. The somas of such a bundle can be divided into ∞^1 pencils of type *b*); all of the remaining pencils of somas in the figure belong to type *a*). One performs *all rotations around the straight lines of a pencil of parallels* on a right-handed soma or reflects a left-handed soma *through all* *planes of a bundle with an improper vertex.* One must then add the improper soma to this. [Cf., § 6, type *b*)]. Example:

(49) $$0 : 0 : \sigma_1 : \sigma_2 : \qquad 0 : \sigma_1 : 0 : 0 .$$

c) *The bundle contains a pencil of improper somas.* There are ∞^6 such bundles, ∞^2 of them through any proper soma, and ∞^4 through any improper one. All somas of the figure are parallel to each other; pencils of somas of type *a*) no longer appear. One subjects a right-handed soma to *all translations that are parallel to a plane* or reflects a left-handed soma through *all points of a plane.* Any one-parameter group of translations determines an improper soma. Example:

(50) $$0 : \sigma_1 : 0 : 0 : \qquad 0 : 0 : \sigma_2 : \sigma_3 .$$

d) *The bundle of somas contains only imaginary somas.* There are ∞^3 bundles of this kind, and ∞^2 of them go through any (improper) soma. One performs *all rotations around the lines of a pencil with a proper vertex* on an improper, right-handed soma or performs the transfers that were described in *a*) on a left-handed, improper soma. Example:

(51) $$0 : 0 : 0 : 0 : \qquad 0 : \sigma_1 : \sigma_2 : \sigma_3 .$$

8. The four right-handed somas $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ shall not belong to any bundle of somas, but shall be incident to two of them. The system:

(52) $$\mathfrak{X} = \sigma_0\,\mathfrak{A} + \sigma_1\,\mathfrak{B} + \sigma_2\,\mathfrak{C} + \sigma_3\,\mathfrak{D}$$

then represents a manifold of ∞^3 somas, namely, a *bush of somas.*

Theorem 6: *Any soma of a bush of somas is incident with every other one.*

Theorem 7: *The bushes of somas are associated with the generating three-dimensional space R_3 on M_6^2.*

There are then ∞^3 bundles of somas in a bush of somas, two of which split into a pencil of somas, and therefore ∞^4 bundles of somas. If two distinct somas among them have a soma in common then they can also be coupled by a bundle of somas that is contained completely in the bush. Three bundles of somas in the bush have at least one soma in common. There are ∞^2 pencils of somas through a soma in the bush, and just as many bundles of somas that run through the bush completely. One can lay ∞^1 bundles of somas through a pencil of somas of the bush that belong to the bush.

a) *Bush of somas with no improper soma.* There are ∞^6 of them, and ∞^2 of them through any soma. No two somas in the bush are parallel to each other. One subjects a right-handed soma to *all rotations around a fixed point* or a left-handed soma to *all transfers with a fixed proper midpoint.*

(53) $$\sigma_0 : \sigma_1 : \sigma_2 : \sigma_3 : \qquad 0 : 0 : 0 : 0.$$

b) *A single improper soma.* There are likewise ∞^6 such bushes of somas, and ∞^3 of them through any soma. The somas of the figure can be divided into ∞^2 pencils of parallel somas. One performs *all rotations around the lines of a plane* on a right-handed soma or performs the reflections *through all planes in space* to a left-handed soma:

(54) $$\sigma_0 : 0 : \sigma_2 : \sigma_3 : \qquad 0 : \sigma_1 : 0 : 0.$$

c) *A pencil of improper somas.* There are ∞^5 such bushes of somas, ∞^2 of them through any proper soma, and ∞^3 of them through any improper one. The proper somas of the figure can be divided into ∞^1 bundles of parallel somas. A right-handed soma will be subjected to *all rotations around the lines of a bundle of parallels;* a left-handed soma will be subjected to *all transfers with fixed, improper midpoints.* In that way, this case proves to be a degenerate case of type *a*):

(55) $$\sigma_0 : \sigma_1 : 0 : 0 : \qquad 0 : 0 : \sigma_2 : \sigma_3 .$$

d) *A bundle of improper somas.* There are ∞^3 such bushes of somas, a single one of them through any proper soma, and ∞^2 of them through any improper one. All somas of the figure are parallel to each other. One subjects a right-handed soma to *all translations* [degeneracy of type *b*)] or a left-handed one to the reflections *through all points in space.*

(56) $$\sigma_0 : 0 : 0 : 0 : \qquad 0 : \sigma_1 : \sigma_2 : \sigma_3 .$$

e) *The bush of all improper somas.*

(57) $$0 : 0 : 0 : 0 : \qquad \sigma_0 : \sigma_1 : \sigma_2 : \sigma_3 .$$

All of the linear manifolds of somas are enumerated with that. Namely, if there were a generating R_4 on M_6^2 then, as a count of the constants would give directly, there must be a discrete number of them present; however, M_6^2 is free of singularities.

The criterion for the classification under G_{22} will be obtained from the ranks of the three matrices:

$$\left\| \begin{matrix} \mathfrak{A}_0 & \mathfrak{A}_{01} & \mathfrak{A}_{02} & \mathfrak{A}_{03} \\ \mathfrak{B}_0 & \mathfrak{B}_{01} & \mathfrak{B}_{02} & \mathfrak{B}_{03} \\ \mathfrak{C}_0 & \mathfrak{C}_{01} & \mathfrak{C}_{02} & \mathfrak{C}_{03} \\ \mathfrak{D}_0 & \mathfrak{D}_{01} & \mathfrak{D}_{02} & \mathfrak{D}_{03} \end{matrix} \right\|,$$

in which, for a bundle of somas, one suppresses the last row, and for a pencil of somas, one suppresses the last two rows.

In each case, for the construction of the individual figures by *motions*, we have employed only one soma that already belongs to the manifold. In that way, the families of motions in question, which consist of only rotations and translations here (i.e., not screws), define a *continuous group* in each case. Otherwise, one would be able to give other constructions, as well; for the pencils of somas and bundles of somas, one can manage with *reversals*. The axes of reversals fill up a pencil of lines with a proper vertex [viz., a pencil of somas of type *a*)], a pencils of parallels [type *b*)], a bundle of lines with a proper vertex [bundle of somas, type *a*)], a planar field of lines [type *b*)], or finally, a bundle of parallels [type *c*)]. The bush of somas cannot be obtained in that way, since there are no somas that simultaneously cut four linearly-independent somas perpendicularly.

The canonical examples are chosen for all pencils and bushes of somas that do not consist of only improper ones in such a way that they will contain protosomas. Naturally, that is not necessary, which is why we have intentionally given other examples for the bundles of somas.

The fact that there are two types of bushes of somas of the same number of constants – namely, six – demands further investigation, for which, we will require other tools ([9]).

9. The figure of the soma, and thus of three perpendicularly-intersecting spears, is already quite complicated, and it proves to be desirable to arrive at a clear picture of the manifolds of somas that are considered, whose visualization brings certain difficulties with it, in another way.

A simpler figure that likewise presents ∞^6 exemplars is *the complex point in three-dimensional space.* Let it be proper and have the inhomogeneous coordinates X_1, X_2, X_3; the conjugate imaginary points will be denoted by $\bar{X}_1, \bar{X}_2, \bar{X}_3$. We then set:

$$(58) \qquad \mathfrak{X}_0 = 1, \qquad \mathfrak{X}_{01} = \tfrac{1}{2}(\bar{X}_1 + X_1), \ldots$$

$$\mathfrak{X}_{123} = \tfrac{1}{4} i\{X_1^2 + X_2^2 + X_3^2 - \bar{X}_1^2 - \bar{X}_2^2 - \bar{X}_3^2\}, \quad \mathfrak{X}_{23} = \tfrac{1}{2} i(\bar{X}_1 - X_1), \ldots$$

The expressions for $\mathfrak{X}_{01}$, …, $\mathfrak{X}_{23}$, … are closely related ([10]), while the one for $\mathfrak{X}_{123}$ follows from (2).

Thus, every *proper, complex* point is associated with a *(real), proper soma*, and the converse is also true *when one does not have* $\mathfrak{X}_0 = 0$. If the somas $\mathfrak{X}$ and $\mathfrak{Y}$ belong to the points X and Y then one will have:

([9]) Part of the results that were demonstrated up to now are due to de Saussure ("Exposé résumé de la géométrie des feuillets," Geneva, 1910). A "feuillet" is essentially identical to a *proper* soma; the difference between right-handed and left-handed somas, which both occur, is not clear. De Saussure called the pencil of somas of type *a*) a *couronne*, the bundle of somas of type *a*), a *couronoïde*, and the bush of somas of type *b*), a *hyper couronoïde*. The remaining nine types of linear manifold of somas have eluded him, especially the *most import* bush of somas of type *a*) (left bush, cf., § 10).

([10]) Cf., say, Study, *Ebene analytische Kurven*, Leipzig, 1911, pp. 21.

(59) $$(\mathfrak{X}\mathfrak{Y}) = \tfrac{1}{4} i \{(X_1 - Y_1)^2 + \ldots - (\bar{X}_1 - \bar{Y}_1)^2 - \ldots\}.$$

It follows directly from this that:

Theorem 8: *Incident somas map to complex points whose distance-squared is real.*

Thus, the image of a pencil of somas proves to be the real progression of a real line, the image of a bundle of somas is the real progression of a real plane, and finally, we have the totality of all real points in space as the image of a bush of somas. However, those are the images of only ∞^4 pencils, ∞^3 bundles, and a single bush, resp. The remaining linear manifolds of somas map to *imaginary* point-structures here, and likewise by two other associations of real somas with complex points.

However, investigations are already available that replace complex points in space with a real figure, such as a real, oriented circle (Laguerre) or *two ordered real points* ([11]), and the latter proves to be useful for us. The pair of proper, real points $(x_1, x_2, x_3) \rightarrow (x_1', x_2', x_3')$ (inhomogeneous coordinates) shall be placed in the following relationship with the complex point X:

(60) $$x_1 + x_1' = \bar{X}_1 + X_1, \ldots, x_1 - x_1' = i(\bar{X}_1 - X_1), \ldots$$

(deviating from Graustein). As a result of (58), one will have:

(61) $$\mathfrak{X}_0 = 1, \qquad \mathfrak{X}_{01} = \tfrac{1}{2}(x_1 + x_1'), \ldots$$
$$\mathfrak{X}_{123} = \tfrac{1}{4}(x_1'^2 + x_2'^2 + x_3'^2 - x_1^2 - x_2^2 - x_3^2\}, \qquad \mathfrak{X}_{23} = \tfrac{1}{2}(x_1 - x_1'), \ldots$$

The inverse formulas read:

(62) $$x_1 = \mathfrak{X}_{01} + \mathfrak{X}_{23} : \mathfrak{X}_0, \ldots, \qquad x_1' = \mathfrak{X}_{01} - \mathfrak{X}_{23} : \mathfrak{X}_0, \ldots$$

Thus, every ordered pair of real, proper points of space will be assigned to a unique soma, and conversely, as long as $\mathfrak{X}_0$ does not vanish.

The last restriction can be lifted (§ 11); we shall not go into that at this point, in order to not disrupt the train of thought.

If the soma $\mathfrak{Y}$ belongs to the point-pair $y \rightarrow y'$ then:

(63) $$(\mathfrak{X}\mathfrak{Y}) = \tfrac{1}{4}\{(y_1' - x_1')^2 + \ldots - (y_1 - x_1)^2 - \ldots\}.$$

If the two somas $\mathfrak{X}$ and $\mathfrak{Y}$ are incident then the two "starting points" x and y will have the same separation-squared as the two "endpoints" x' and y': the two point-pairs will then be called *isometric:*

([11]) W. C. Graustein, "Eine reelle Abbildung analytischer komplexer Raumkurven," Diss. Bonn 1913.

Theorem 9: *Incident somas can be associated with isometric point-pairs.*

Thus, the map proves to be useful in examining the geometry of (G_{28}, H_{28}).

10. The linear manifolds of somas now correspond to the most intuitive figures, as long as one does not have $\mathfrak{X}_0 = 0$ for all somas. The ∞^1 point-pairs that belong to a *pencil of somas* have their starting points on a line, and likewise their endpoints on a line (which can coincide with the first one); both lines are related to each other isometrically (i.e., congruently) by the ∞^1 point-pairs.

The image of a *pencil of somas* is just as simple. The starting points of the ∞^2 point-pairs fill up a plane, as do the endpoints. Both planes, which can coincide, are mapped to each other isometrically (i.e., congruently) by the point-pairs.

One can read off the numbers of constants and a whole series of properties of the figures from this with no further analysis.

The images of *bushes of somas* behave somewhat differently. One of them will indeed once more correspond to an isometric association, but this time, the space itself can be produced in two essentially different ways, namely, through a *motion* or through a *transfer*. Accordingly, there are two different types of bushes of somas, which we would like to distinguish as *left-bushes* and *right-bushes*. Of the types *a*) to *d*) that we presented in § 8, *a*) and *c*) are left-bushes, while *b*) and *d*) right-bush. For example, as a result of (62), it will follow from (55) that:

$$x_1 = \sigma_1 : \sigma_0\,, \quad x_2 = \sigma_2 : \sigma_0\,, \quad x_3 = \sigma_3 : \sigma_0\,,$$
$$x_1' = \sigma_1 : \sigma_0\,, \quad x_2' = -\,\sigma_2 : \sigma_0\,, \quad x_3' = -\,\sigma_3 : \sigma_0\,.$$

One is then dealing with the reversal around the X-axis ($x_1' = x_1$, $x_2' = -x_2$, $x_3' = -x_3$), and therefore a motion.

In order to have a criterion for when the bush of somas (52) is a left-bush, we compare the volumes of associated tetrahedra and find:

Left-bush (motions):

(64a) $| \mathfrak{A}_0\, \mathfrak{B}_{23}\, \mathfrak{C}_{31}\, \mathfrak{D}_{12} | + | \mathfrak{A}_0\, \mathfrak{B}_{02}\, \mathfrak{C}_{03}\, \mathfrak{D}_{23} | + | \mathfrak{A}_0\, \mathfrak{B}_{03}\, \mathfrak{C}_{01}\, \mathfrak{D}_{31} | + | \mathfrak{A}_0\, \mathfrak{B}_{01}\, \mathfrak{C}_{02}\, \mathfrak{D}_{12} | =$

Right-bush (transfers):

(64b) $| \mathfrak{A}_0\, \mathfrak{B}_{01}\, \mathfrak{C}_{02}\, \mathfrak{D}_{03} | + | \mathfrak{A}_0\, \mathfrak{B}_{31}\, \mathfrak{C}_{12}\, \mathfrak{D}_{01} | + | \mathfrak{A}_0\, \mathfrak{B}_{12}\, \mathfrak{C}_{23}\, \mathfrak{D}_{02} | + | \mathfrak{A}_0\, \mathfrak{B}_{23}\, \mathfrak{C}_{31}\, \mathfrak{D}_{03} | =$

It follows directly from this that the soma transformation:

(65) $\mathfrak{X}_0' = \mathfrak{X}_0,\ \mathfrak{X}_{01}' = \mathfrak{X}_{23},\ \ldots,\ \mathfrak{X}_{123}' = \mathfrak{X}_{123},\ \mathfrak{X}_{23}' = \mathfrak{X}_{01},\ \ldots$

converts any left-bush into a right-bush, and conversely. With that, we have arrived at a first inkling of the structure of the twenty-eight-parameter group of soma transformations (§ 5). It divides into two separate families of transformations, the first of which (“G_{28}”) always takes left-bushes into other left-bushes, and right-bushes to other right-bushes. These transformations define a group, in their own right. The transformations of the other family (“H_{28}”) switch both families of bushes of somas. Neither of the two families defines a continuum, moreover, as we will remark here in passing, but each of them again decomposes into four separate continua.

The analogue to the transformations of G_{28} in line space are the *collineations*, while the analogue to those of H_{28} are the *correlations.* Namely, (cf., 66*a*) the left-bushes correspond to the bundles of lines there (i.e., left-handed generators R_2 on M_4^2), and the right-bushes correspond to the line-fields (right-handed generators R_2 on M_4^2). We then extend Theorem 7 to:

Theorem 10: *The left-bushes are associated with the “left-handed” generators* R_3 on M_6^2, *and the right-bushes are the “right-handed” generators* R_3 .

However, the analogy is not rigorous, since the group (G_{15}, H_{15}) of projectivities in line space encompasses only *four* separate continua of transformations. One will again obtain *eight* separate families when one once more returns to two dimensions in the form of the group of a real, rectilinear, singularity-free second-order surface ([12]). In regard to that, one recalls the remark at the conclusion of § 1.

A point x can be taken to a point x' by ∞^3 motions and ∞^3 transfers. In fact, one can vary the α in equations (7) to (10) arbitrarily [cf., however, (9)!], and determine the β uniquely from them:

$$
(66)\qquad
\begin{aligned}
2\beta_0 &= \quad * \qquad - (x_1 - x_1')\,\alpha_1 - (x_2 - x_2')\,\alpha_2 - (x_3 - x_3')\,\alpha_3,\\
2\beta_1 &= (x_1 - x_1')\,\alpha_0 \quad * \quad - (x_3 + x_3')\,\alpha_2 + (x_2 + x_2')\,\alpha_3,\\
2\beta_2 &= (x_2 - x_2')\,\alpha_0 + (x_3 + x_3')\,\alpha_1 \quad * \quad - (x_1 + x_1')\,\alpha_3,\\
2\beta_3 &= (x_3 - x_3')\,\alpha_0 - (x_2 + x_2')\,\alpha_1 + (x_1 + x_1')\,\alpha_2 \quad * \quad .
\end{aligned}
$$

One finds the ∞^3 transfers in a corresponding way:

$$
(67)\qquad
\begin{aligned}
2\delta_0 &= \quad * \qquad - (x_1 + x_1')\,\gamma_1 - (x_2 + x_2')\,\gamma_2 - (x_3 + x_3')\,\gamma_3,\\
2\delta_1 &= (x_1 + x_1')\,\gamma_0 \quad * \quad - (x_3 - x_3')\,\gamma_2 + (x_2 - x_2')\,\gamma_3,\\
2\delta_2 &= (x_2 + x_2')\,\gamma_0 - (x_3 - x_3')\,\gamma_1 \quad * \quad - (x_1 - x_1')\,\gamma_3,
\end{aligned}
$$

([12]) Cf., an article by the author in Amer. Trans. **11** (1910), 418-420, 424-426.

$$2\delta_3 = (x_3 + x_3')\,\gamma_0 - (x_2 - x_2')\,\gamma_1 + (x_1 - x_1')\,\gamma_2 \qquad * \quad .$$

It follows from this by our map that:

Theorem 11: ∞^3 *left-bushes and* ∞^3 *right-bushes run through a soma.*

The proof breaks down for somas with the property that $\mathfrak{X}_0 = 0$. However, one then has only to show that these ∞^5 somas do not define an invariant submanifold, and one sees that already in the transformation:

$$\mathfrak{X}_0' = \mathfrak{X}_{123}, \qquad \mathfrak{X}_{01}' = \mathfrak{X}_{01}, \ldots, \quad \mathfrak{X}_{123}' = \mathfrak{X}_0, \qquad \mathfrak{X}_{23}' = \mathfrak{X}_{23}, \ldots$$

that belongs to H_{28} .

Should a motion (66) or a transfer (67) also take the point y to y' then the two point-pairs $x \to x'$ and $y \to y'$ would have to be isometric. For a fixed y, that will give ∞^2 additional positions for y', or:

Theorem 12: ∞^1 *left-bushes and* ∞^1 *right-bushes run through a pencil of somas.*

A single motion and a single transfer is determined by three pairs $x \to x'$, $y \to y'$, $z \to z'$ that are pair-wise isometric.

Theorem 13: *A single left-bush and a single right-bush run through a bundle of somas.*

The analogue in line geometry is: A single bundle of lines and a single line-field run through every pencil of lines.

Theorems 11 to 13 show that there are none of the differences between pencils of somas and bundles of somas that one finds between left-bushes and right-bushes.

There are skew left-bushes [*right-bushes, resp.*]. One example will suffice:

$$\begin{array}{ll} \sigma_0 : \sigma_1 : \sigma_2 : \sigma_3 : & 0 \;:\; 0 \;:\; 0 \;:\; 0, \\ \sigma_0 : \sigma_1 : \;0 : \;0 : & \sigma_1 : -\sigma_1 : \sigma_2 : \sigma_3 . \end{array}$$

We take this opportunity to replace the cumbersome criteria (64*a*, *b*) with simpler ones. The consideration of improper somas (§ 8) yields:

The bush of somas:

(52) $$\mathfrak{X} = \sigma_0\,\mathfrak{A} + \sigma_1\,\mathfrak{B} + \sigma_2\,\mathfrak{C} + \sigma_3\,\mathfrak{D}$$

is a right-somas when the rank of the matrix:

$$\| \mathfrak{A}_0\ \mathfrak{B}_{01}\ \mathfrak{C}_{02},\ \mathfrak{D}_{03} \|$$

is an odd number.

That would lead us to define *the bush of all improper somas* to be a *left-bush*, and in fact, the following two theorems are necessary for that. In them, the analogy with line geometry will once more be undermined, in which the totality of all improper lines is regarded as a line-field, corresponding to a right-handed generator R_2 of M_4^2.

Theorem 14: *Two left-bushes* [*right-bushes*] *that have a common soma intersect in a pencil of somas.*

We can directly assume that $\mathfrak{X}_0 \neq 0$ for the common soma. One must then show that when two motions [transfers] S and T simultaneously take x to x', there will be ∞^1 other points that transform both of them in the same way. Any such point is then the fixed point of a *motion* ST^{-1} that can only be a rotation, since it already possesses the fixed point x. However, the existence of ∞^1 fixed points will be confirmed with that.

One can also prove it directly. If one regards the α, β in (66) as given and replaces the x, x' according to (62) then the coordinates of the soma of the *left-bush* that is associated with the motion (α, β) will satisfy the four equations:

$$(66a)\qquad \begin{aligned}
\beta_0\,\mathfrak{X}_0 + \alpha_1\,\mathfrak{X}_{23} + \alpha_2\,\mathfrak{X}_{31} + \alpha_3\,\mathfrak{X}_{12} &= 0,\\
\beta_1\,\mathfrak{X}_0 - \alpha_0\,\mathfrak{X}_{23} + \alpha_2\,\mathfrak{X}_{03} - \alpha_3\,\mathfrak{X}_{02} &= 0,\\
\beta_2\,\mathfrak{X}_0 - \alpha_0\,\mathfrak{X}_{31} + \alpha_3\,\mathfrak{X}_{01} - \alpha_1\,\mathfrak{X}_{03} &= 0,\\
\beta_3\,\mathfrak{X}_0 - \alpha_0\,\mathfrak{X}_{12} + \alpha_1\,\mathfrak{X}_{02} - \alpha_2\,\mathfrak{X}_{01} &= 0.
\end{aligned}$$

Our theorem will now be proved (to the extent that it relates to left-bushes) by considering two such systems of equations. For right-bushes, one correspondingly employs (67):

The coordinates of the somas of right-bushes that is associated with the transfer (γ, δ) satisfy the equations:

$$(67a)\qquad \begin{aligned}
\delta_0\,\mathfrak{X}_0 + \gamma_1\,\mathfrak{X}_{01} + \gamma_2\,\mathfrak{X}_{02} + \gamma_3\,\mathfrak{X}_{03} &= 0,\\
\delta_1\,\mathfrak{X}_0 - \gamma_0\,\mathfrak{X}_{01} + \gamma_2\,\mathfrak{X}_{12} - \gamma_3\,\mathfrak{X}_{31} &= 0,\\
\delta_2\,\mathfrak{X}_0 - \gamma_0\,\mathfrak{X}_{02} + \gamma_3\,\mathfrak{X}_{23} - \gamma_1\,\mathfrak{X}_{12} &= 0,\\
\delta_3\,\mathfrak{X}_0 - \gamma_0\,\mathfrak{X}_{03} + \gamma_1\,\mathfrak{X}_{31} - \gamma_2\,\mathfrak{X}_{23} &= 0.
\end{aligned}$$

If one abbreviates (§ 1) then the four equations (66a) will express the condition for *the straight line* $\mathfrak{X}$ *to run through the point* $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$. With that, on the one hand, the stated parallelism of left-bushes and bundles of lines will be established, while on the other hand, one can also glimpse a proof of the convenience of the notation for the parameters of motion. Namely, instead of the parameters (α, β) that are employed here, which were introduced by Study, other symbols were employed by other authors, in succession:

Study:	α_0	α_1	α_2	α_3	β_0	β_1	β_2	β_3
Combebiac:	α_0	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	β_0	$-\beta_1$	$-\beta_2$	$-\beta_3$
Klein:	D	$-A$	$-B$	$-C$	D'	$-A'$	$-B'$	$-C'$
Bricard:	λ	μ	ν	ρ	l	m	n	p (!)
Schoenflies ([13]):	$-D$	A	B	C	$+\frac{1}{2}D_1$	$+\frac{1}{2}A_1$	$+\frac{1}{2}B_1$	$+\frac{1}{2}C_1$

The Bricard notation is completely useless in practice. For all of the other ones, there is a discrepancy between (66a) and the corresponding system of formulas in line geometry. Nevertheless, we have summarized them here in order to build a bridge to the papers of the aforementioned authors.

If one abbreviates the system (67a) then the abbreviated formulas will express *the united position of the straight line* $\mathfrak{X}$ *with the plane* $\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3$.

Theorem 15: *A left-bush and a right-bush either intersect in a single soma or they have an entire bundle of somas in common.*

In fact, the system of eight equations (66a), (76a) will always be satisfied by the soma:

$$(68)\quad \begin{aligned} \mathfrak{X}_0 &= \alpha_0\gamma_0 + \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3, & \mathfrak{X}_{01} &= \alpha_0\delta_1 - \alpha_1\delta_0 - \beta_2\gamma_3 + \beta_3\gamma_2, \\ \mathfrak{X}_{123} &= -(\beta_0\delta_0 + \beta_1\delta_1 + \beta_2\delta_2 + \beta_3\delta_3), & \mathfrak{X}_{23} &= -(\beta_0\gamma_1 - \beta_1\gamma_0 - \alpha_2\delta_3 + \alpha_3\delta_2), \end{aligned}$$

and all that remains to be examined is the case in which that soma is undetermined.

We denote the left-hand sides of the equations in (66*a*) by L_0, L_1, L_2, L_3 and the ones in (67*a*) by R_0, R_1, R_2, R_3 . Furthermore, let:

$$\begin{aligned} -\gamma_1 L_0 + \gamma_0 L_1 - \alpha_3 R_2 + \alpha_2 L_3 &= P_1, \ldots \\ -\alpha_1 R_0 + \alpha_0 R_1 - \gamma_3 L_2 + \gamma_2 L_3 &= Q_1, \ldots \end{aligned}$$

For $\gamma_0 \neq 0$, we then have a system that is equivalent to the original one in the form of $P_1 = P_2 = P_3 = R_1 = R_2 = R_3 = L_0 = 0$. R_0 is then linearly independent of R_1, R_2, R_3 . However, $P_1 = 0$, $P_2 = 0$, $P_3 = 0$ are then satisfied identically, such that four more independent, homogeneous equations will be needed for the determination of the seven unknowns.

For $\gamma_0 = 0$, $\alpha_0 \neq 0$, we take the equivalent system to be the following one:

$$Q_1 = Q_2 = Q_3 = L_1 = L_2 = L_3 = R_0 = 0.$$

([13]) Combébiac, "Calcul des Triquaternions"; Klein and Sommerfeld, *Theorie des Kreisels*; Bricard, Nouv. Ann. de Math. **10** (1910); Schoenflies, Rend. Circ. Mat. Palermo **29** (1910).

L_0 will then be linearly independent of L_1, L_2, L_3 ; $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$ will be fulfilled identically. Finally, if $\gamma_0 = 0$, $\alpha_0 = 0$ then one will have:

$$\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 = 0, \qquad \gamma_1 R_1 + \gamma_2 R_2 + \gamma_3 R_3 = 0,$$
$$\beta_0 R_0 + \delta_1 L_1 + \delta_2 L_2 + \delta_3 L_3 = 0, \qquad \delta_0 L_0 + \beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 = 0,$$

such that once more only four independent quantities remain.

In the latter case, where the left-bush and the right-bush intersect in a bundle of somas, the motion (α, β) and the transfer (γ, δ) transform a plane u into a plane u' in the same way:

$$(69)\qquad \begin{aligned} u_0 &= \alpha_0\,\delta_0 + \alpha_1\,\delta_1 + \ldots - \beta_0\,\gamma_0 - \beta_1\,\gamma_1 - \ldots, \\ u_1 &= \alpha_0\,\gamma_1 - \alpha_1\,\gamma_0 + \alpha_2\,\gamma_3 - \alpha_3\,\gamma_2 , \ldots, \\ u_0' &= \alpha_0\,\delta_0 + \alpha_1\,\delta_1 + \ldots + \beta_0\,\gamma_0 + \beta_1\,\gamma_1 + \ldots, \\ u_1' &= \alpha_0\,\gamma_1 - \alpha_1\,\gamma_0 - \alpha_2\,\gamma_3 - \alpha_3\,\gamma_2 , \ldots \end{aligned}$$

If the transfer is given in addition to u then the motion will be determined by:

$$(70a)\qquad \begin{aligned} (\gamma_0^2+\gamma_1^2+\gamma_2^2+\gamma_3^2)\,\alpha_0 &= + u_1\gamma_1 + u_2\gamma_2 + u_3\gamma_3 , \\ (\gamma_0^2+\gamma_1^2+\gamma_2^2+\gamma_3^2)\,\alpha_1 &= - u_1\gamma_0 + \quad + u_3\gamma_2 - u_2\gamma_3 , \ldots \\ (\gamma_0^2+\gamma_1^2+\gamma_2^2+\gamma_3^2)\,\beta_0 &= - u_1\gamma_0 - u_1\delta_1 - u_2\delta_2 - u_3\delta_3 , \\ (\gamma_0^2+\gamma_1^2+\gamma_2^2+\gamma_3^2)\,\beta_1 &= - u_1\gamma_0 + u_1\delta_0 - u_3\delta_2 + u_3\delta_3 , \ldots \end{aligned}$$

The relationship between left-bushes and motions, and right-bushes and transfers that is mediated by the formulas (66a) and (67a) will be single-valued and invertible *with no gaps* when one drops the requirements (9) and (19) (i.e., "degenerate" motions and transfers) or when one goes from motions and transfers to somas:

Theorem 16: *The totality of left-bushes can be related to the totality of (proper and improper) right-handed somas in a single-valued, invertible way without gaps, while the total of right-bushes can be related to the left-handed somas.*

Admittedly, in order to do that, one must necessarily extend formulas (66a), (67a) to the following ones:

$$(66b)\qquad \begin{aligned} \alpha_0\,\mathfrak{X}_{123} + \beta_1\,\mathfrak{X}_{01} + \beta_2\,\mathfrak{X}_{02} + \beta_3\,\mathfrak{X}_{03} &= 0, \\ \alpha_1\,\mathfrak{X}_{123} - \beta_0\,\mathfrak{X}_{01} + \beta_2\,\mathfrak{X}_{12} - \beta_3\,\mathfrak{X}_{31} &= 0, \ldots \end{aligned}$$

(67*b*)
$$\gamma_0\,\mathfrak{X}_{123} + \delta_1\,\mathfrak{X}_{23} + \delta_2\,\mathfrak{X}_{31} + \delta_3\,\mathfrak{X}_{12} = 0,$$
$$\gamma_1\,\mathfrak{X}_{123} - \delta_0\,\mathfrak{X}_{23} + \delta_2\,\mathfrak{X}_{02} - \delta_3\,\mathfrak{X}_{02} = 0, \ldots$$

One then obtains, for example, the left-bush of all improper somas when:

$$\alpha_0 = 0,\ \alpha_1 = 0,\ \ldots \qquad \beta_0 = 1,\ \beta_1 = 0,\ \ldots$$

In this, one has the possibility of also representing a left-bush, and likewise a right-bush by eight parameters that are coupled by a quadratic relation (§ 3).

All of the remaining questions about linear manifolds of somas can be conveniently addressed by the map to point-pairs. For the applications to differential geometry, we consider the figure of two pencils of somas.

In the most general case, two pencils of somas cannot be linked by a bush of somas; they shall then be called "bush-foreign" (*gebüschfremd*). They will then be "bundle-foreign" (*bündelfremd*), as well (Theorem 13). It can happen that:

a) Every soma of one pencil is incident with a single soma of the other pencil. That will always happen for the two pencils:

$$\sigma_1\,\mathfrak{A} + \sigma_2\,\mathfrak{B} \quad \text{and} \quad \tau_1\mathfrak{A}' + \tau_2\mathfrak{B}'$$

when the matrix:

$$\begin{Vmatrix} (\mathfrak{A}\mathfrak{A}') & (\mathfrak{A}\mathfrak{B}') \\ (\mathfrak{B}\mathfrak{A}') & (\mathfrak{B}\mathfrak{B}') \end{Vmatrix}$$

has rank two. The totality of all somas that have an incident soma in each of the two pencils defines a nonlinear, irreducible manifold that possesses a connection to a singularity-free M_2^2.

b) However, if the rank is one then there will be a single soma in each pencil that is incident with all somas of the other pencil. All somas that have an incident soma in each pencil then divide into two bundles of somas that intersect in a pencil of somas, or:

c) The two original pencils of somas have a common soma. We give an example for each of the three cases:

a)
$$\begin{matrix} \sigma_1 : \sigma_2 : 0 : 0 : & 0 : 0 : 0 : 0 : \\ 0 : 0 : 0 : 0 : & \tau_1 : \tau_2 : 0 : 0 : \end{matrix}$$

b)
$$\begin{matrix} \sigma_1 : \sigma_2 : 0 : 0 : & 0 : 0 : 0 : 0 : \\ 0 : 0 : \tau_1 : 0 : & 0 : \tau_2 : 0 : 0 : \end{matrix}$$

c)
$$\begin{matrix} \sigma_1 : \sigma_2 : 0 : 0 : & 0 : 0 : 0 : 0 : \\ \tau_1 : 0 : 0 : 0 : & 0 : \tau_2 : 0 : 0 : \end{matrix}$$

It is, moreover, possible that two pencils of somas can be coupled by *a single* bush of somas. They will then be bundle-foreign (Theorem 13) and have no common soma. There are two cases (left-bush, right-bush).

It can then happen that two pencils of somas can be coupled by precisely *two* bushes of somas, namely, a left-bush and a right-bush. They will then have a common soma and can be coupled by a bundle of somas.

Finally, the two pencils of somas can coincide; there are then ∞^1 coupling left-bushes and just as many right-bushes.

The matrix that was just considered will have rank zero for all non-bush-foreign pairs of pencils of somas.

We are now in a position to be able to rectify a deferred proof. Since there are ∞^6 left-bushes, each of them will be fixed by a group of soma transformations that depends upon 28 – 6 = 22 parameters. Their transformations all belong to G_{28} (i.e., and not to H_{28}). If the left-bush is that of all improper somas, in particular, then one will be dealing with the group G_{22} .

11. The map (61), (62) of somas to point-pairs was endowed with singularities; the somas with the property that $\mathfrak{X}_0 = 0$, which we briefly call *critical*, could not be mapped. These exceptions can be ignored when one appeals to a new conceptual image and speaks of *critical point-pairs.* Admittedly, such a thing is accessible by only part of one's intuition. As we anticipate, it consists of *an ordered pair of oriented directions that is endowed with a number,* or rather two such "figures" that are coupled in some way.

A pencil of somas can consist of nothing but critical somas. In the other cases, there will be a single critical soma, and we consider all ∞^4 pencils of somas that possess the same critical soma. From § 9, each of them will be mapped to a pair of isometrically-related lines that will be casually distinguished as the initial line and the final line. However, for the aforementioned ∞^4 pencils of somas, the initial lines will all be parallel to each other, and likewise for the final lines. However, the two bundles of parallels that are determined in that way given rise to ∞^5 pencils of somas, so the critical soma is still not defined; one must be given the type of isometric relationship.

If $\mathfrak{P}$ is a *proper* initial line, and $\mathfrak{P}'$ is a *proper* final line (Plücker coordinates) then let:

$$\mathfrak{P}_{11} = \mathfrak{P}_{02}\,\mathfrak{P}_{12} - \mathfrak{P}_{03}\,\mathfrak{P}_{31}, \ldots \qquad \mathfrak{P}^2 = \mathfrak{P}_{01}^2 + \mathfrak{P}_{02}^2 + \mathfrak{P}_{03}^2 .$$

The ∞^1 point-pairs:

$$(71)\qquad \begin{aligned} P:&\quad x = \mathfrak{P}_{11} + (t-c)\,\mathfrak{P}\,\mathfrak{P}_{01} : \mathfrak{P}^2, \ldots \\ P':&\quad x' = \mathfrak{P}'_{11} + (t+c)\,\mathfrak{P}'\,\mathfrak{P}'_{01} : \mathfrak{P}'^2, \ldots \end{aligned}$$

then give an isometric relationship between the *spears* $\mathfrak{P} : \mathfrak{P}_{0i} : \mathfrak{P}_{kl}$ and $\mathfrak{P}' : \mathfrak{P}'_{0i} : \mathfrak{P}'_{kl}$ for varying t. If one then lets c vary then another isometric relationship will result. From (61), the soma that is associated with the point-pair $\mathfrak{P} \to \mathfrak{P}'$ will be:

$$
(72)\quad
\begin{aligned}
\mathfrak{X}_0 &= 2\,\mathfrak{P}^2\,\mathfrak{P}'^2,\\
\mathfrak{X}_{01} &= \mathfrak{P}'^2\,\mathfrak{P}_{11} + \mathfrak{P}^2\,\mathfrak{P}'_{11} + (t-c)\,\mathfrak{P}'^2\,\mathfrak{P}\,\mathfrak{P}_{01} + (t+c)\,\mathfrak{P}^2\,\mathfrak{P}'\,\mathfrak{P}'_{01}, \ldots\\
\mathfrak{X}_{123} &= \tfrac{1}{2}\mathfrak{P}^2\,\{\mathfrak{P}'^2_{23} + \mathfrak{P}'^2_{31} + \mathfrak{P}'^2_{12}\} - \tfrac{1}{2}\mathfrak{P}'^2\{\mathfrak{P}^2_{23} + \mathfrak{P}^2_{31} + \mathfrak{P}^2_{12}\} + 2\,c\,t\,\mathfrak{P}^2\,\mathfrak{P}'^2,\\
\mathfrak{X}_{23} &= \mathfrak{P}'^2\,\mathfrak{P}_{11} - \mathfrak{P}^2\,\mathfrak{P}'_{11} + (t-c)\,\mathfrak{P}'^2\,\mathfrak{P}\,\mathfrak{P}_{01} - (t+c)\,\mathfrak{P}^2\,\mathfrak{P}'\,\mathfrak{P}'_{01}, \ldots
\end{aligned}
$$

Let c be fixed and let t be variable. (72) then represents a pencil of somas whose critical soma is ascertained by passing to the limit of $t = \infty$:

$$
(61a)\quad
\begin{aligned}
\mathfrak{X}_0 &= 0, & \mathfrak{X}_{01} &= \mathfrak{P}'\,\mathfrak{P}_{01} + \mathfrak{P}\,\mathfrak{P}'_{01}, \ldots\\
\mathfrak{X}_{123} &= 2\,c\,\mathfrak{P}\mathfrak{P}', & \mathfrak{X}_{23} &= \mathfrak{P}'\,\mathfrak{P}_{01} - \mathfrak{P}\,\mathfrak{P}'_{01}, \ldots
\end{aligned}
$$

c probably enters into that, but no longer the line coordinates $\mathfrak{P}_{23}, \ldots, \mathfrak{P}'_{23}, \ldots$ If one then varies the line $\mathfrak{P}$ or $\mathfrak{P}'$ inside of their bundle of parallel then the critical soma will remain the same. If one changes the orientation of *both* lines, *as well as changing* the sign of the weight c, then one will still arrive at the same critical soma. Conversely, if its coordinates are given then it will, in fact, be *double-valued:*

$$
(62a)\quad
\begin{aligned}
&\mathfrak{P}_{01} : \mathfrak{P} = \rho\,(\mathfrak{X}_{01} + \mathfrak{X}_{23}), \ldots,\ \mathfrak{P}'_{01} : \mathfrak{P}' = \rho\,(\mathfrak{X}_{01} - \mathfrak{X}_{23}), \ldots,\ c = \rho\,\mathfrak{X}_{123}\,;\\
&1 : \rho^2 = \mathfrak{X}^2_{01} + \mathfrak{X}^2_{02} + \mathfrak{X}^2_{03} + \mathfrak{X}^2_{23} + \mathfrak{X}^2_{31} + \mathfrak{X}^2_{12}\,.
\end{aligned}
$$

One can now define a critical point-pair as the "figure" of two ordered spear direction that are weighted with a number and do not change when one changes all of the orientations and signs of the weights. Every soma can now be assigned a unique point-pair invertibly and *without exceptions* ([14]).

The weight c mediates an association between the (oriented) normal planes of the oriented directions $\mathfrak{P}$ and $\mathfrak{P}'$. The normal plane to $\mathfrak{P}$ at the zero distance $t - c$ will be associated with the normal plane to $\mathfrak{P}'$ at the zero distance $t + c$ [cf., (71)]. Any point P can be linked with all ∞^2 points P' of a plane (II), and likewise every point P' with all ∞^2 points P of a plane (I), when I and II are such associated normal planes of $\mathfrak{P}$ and $\mathfrak{P}'$. One will then obtain ∞^5 point-pairs ($P \to P'$) that one calls "isometric to the critical pair ($\mathfrak{P}$, $\mathfrak{P}'$, c)", so the incidence condition for the associated somas will imply the condition for the *isometric position of the pairs* $x \to x'$ *and* ($\mathfrak{P}$, $\mathfrak{P}'$, c):

([14]) When one introduces an improper critical pair that corresponds to the soma 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0. However, it would probably not pay to go into that any further.

(73) $$\mathfrak{P}\{x'_1\,\mathfrak{P}'_{01}+x'_2\,\mathfrak{P}'_{02}+x'_3\,\mathfrak{P}'_{03}\}-\mathfrak{P}'\{x_1\,\mathfrak{P}_{01}+x_2\,\mathfrak{P}_{02}+x_3\,\mathfrak{P}_{03}\}=2c\,\mathfrak{P}\,\mathfrak{P}'.$$

If one cancels the factor $\mathfrak{P}\mathfrak{P}'$ on the right then the two terms on the left will give the projections of the vector at the endpoint (starting point, resp.) along the oriented final direction (initial direction, resp.) when the two vectors are attached to the origin.

The isometric position of the two critical pairs ($\mathfrak{P}$, $\mathfrak{P}'$, c_1) and ($\mathfrak{Q}$, $\mathfrak{Q}'$, c_2) will now be defined by:

(74) $$\mathfrak{P}'\mathfrak{Q}'(\mathfrak{P}_{01}\mathfrak{Q}_{01}+\mathfrak{P}_{02}\mathfrak{Q}_{02}+\mathfrak{P}_{03}\mathfrak{Q}_{03})=\mathfrak{P}\mathfrak{Q}\{\mathfrak{P}'_{01}\mathfrak{Q}'_{01}+\mathfrak{P}'_{02}\mathfrak{Q}'_{02}+\mathfrak{P}'_{03}\mathfrak{Q}'_{03}\}.$$

The angle (which is given by just its cosine) *between the two initial directions is equal to the angle between the two final directions.*

The critical pairs that are associated with *improper* somas are characterized by the fact that initial direction is opposite to the final direction.

If that is not the case then the weight c will be closely related to the step size $2H_0$ of the unscrewing that takes the protosoma to the associated (now proper) soma. If one then orients the unscrewing axis by the requirement that:

$$\sqrt{2\mathfrak{P}^2\mathfrak{P}'^2+2\mathfrak{P}\mathfrak{P}'\{\mathfrak{P}_{01}\mathfrak{P}'_{01}+\mathfrak{P}_{02}\mathfrak{P}'_{02}+\mathfrak{P}_{03}\mathfrak{P}'_{03}\}}=2\,\mathfrak{P}\mathfrak{P}'\cos\tfrac{1}{2}(\mathfrak{P},\,\mathfrak{P}')$$

then one will have:

$$c=H_0\cos\tfrac{1}{2}(\mathfrak{P},\,\mathfrak{P}').$$

It shall now be shown how the pencils of somas that consist of nothing but critical somas are mapped. Let the images of the basic somas in the pencils be the pairs ($\mathfrak{P}$, $\mathfrak{P}'$, c_1) and ($\mathfrak{Q}$, $\mathfrak{Q}'$, c_2). Since they must be isometric, the rectilinear carriers of $\mathfrak{P}'$ and $\mathfrak{Q}'$ will be separate, as long as that is true for $\mathfrak{P}$ and $\mathfrak{Q}$, and conversely. $\mathfrak{P}$ and $\mathfrak{Q}$ are parallel to the planes of a pencil of parallel planes, and similarly, $\mathfrak{P}'$ and $\mathfrak{Q}'$. One takes an arbitrary $\mathfrak{R}$ that is parallel to the first of these two pencils of planes, and likewise a direction $\mathfrak{R}'$ that is parallel to the second one. $\mathfrak{R}$ and $\mathfrak{R}'$, when suitably weighted, will then determine a variable soma in a pencil of somas.

One must distinguish the same two cases when the pencils of somas consist of nothing but *improper* somas. $\mathfrak{P}'$ must then be opposite to $\mathfrak{P}$, and $\mathfrak{Q}'$ must be opposite to $\mathfrak{Q}$.

One makes the relationship clearest when one appeals to the spherical map of the oriented direction to the unit sphere.

12. The map of somas to point-pairs also proves to be so fruitful and intuitive for the study of nonlinear soma figures that we shall next derive it in another way and would then like to bring about its constructive implementation.

One sets:

(75)
$$\mathfrak{X}_0 + \kappa^2\, \mathfrak{X}_{123} = \xi_0, \qquad \mathfrak{X}_{01} + \mathfrak{X}_{23} = \xi_1, \ldots,$$
$$\mathfrak{X}_0 - \kappa^2\, \mathfrak{X}_{123} = \xi_0', \qquad \mathfrak{X}_{01} - \mathfrak{X}_{23} = \xi_1', \ldots,$$

where κ^2 is an arbitrary positive constant that will first be set to zero. One will then have:

(76)
$$\mathfrak{X}_0 = \tfrac{1}{2}(\xi_0 + \xi_0'), \qquad \mathfrak{X}_{01} = \tfrac{1}{2}(\xi_1 + \xi_1'), \ldots$$
$$\kappa^2\, \mathfrak{X}_{123} = \tfrac{1}{2}(\xi_0 - \xi_0'), \qquad \mathfrak{X}_{23} = \tfrac{1}{2}(\xi_1 - \xi_1'), \ldots$$

(77)
$$2\,(\mathfrak{X}\mathfrak{X}) = \frac{1}{\kappa^2}\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{1}{\kappa^2}\xi_0'^2 - \xi_1'^2 - \xi_2'^2 - \xi_3'^2 = 0.$$

If we accordingly set:

(78)
$$\frac{1}{\kappa^2}\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{1}{\kappa^2}\xi_\omega^2 = \frac{1}{\kappa^2}\xi_0'^2 + \xi_1'^2 + \xi_2'^2 + \xi_3'^2$$

then the formulas (75) and (76) will mediate a *gapless*, invertible, single-valued association of somas with pairs $(\xi_\omega : \xi_0 : \xi_1 : \xi_1 : \xi_2 : \xi_3) \to (\xi_\omega : \xi_0' : \xi_1' : \xi_2' : \xi_3')$ of points of R_4 that be selected from the singularity-free M_3^2 whose equation is contained in (78), or even to *double-pairs*, so one can also replace the given pair with the pair $(-\xi_\omega : \xi_0 : \xi_1 : \xi_1 : \xi_2 : \xi_3) \to (-\xi_\omega : \xi_0' : \xi_1' : \xi_2' : \xi_3')$. We construct a projective metric on the M_3^2:

(79)
$$\cos(\xi, \eta) = \frac{1}{\kappa^2}\,\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 : \frac{1}{\kappa^2}\,\xi_\omega\eta_\omega\,.$$

Since:

$$2\,(\mathfrak{X}\mathfrak{Y}) = \left\{\frac{1}{\kappa^2}\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3\right\} - \left\{\frac{1}{\kappa^2}\xi_0'\eta_0' + \xi_1'\eta_1' + \xi_2'\eta_2' + \xi_3'\eta_3'\right\},$$

incident somas $\mathfrak{X}$ and $\mathfrak{Y}$ will then be mapped to double-pairs that are isometric, in the sense of non-Euclidian geometry that is defined by (78) and (79):

$$\cos\,\kappa(\xi, \eta) = \cos\,\kappa(\xi', \eta').$$

We would now like to pass to the limit of $\kappa^2 = 0$, and then calculate:

$$\frac{1}{\kappa^2}\sin^2\,\kappa(\xi, \eta) = \frac{(\xi\,|\,\xi)\eta_0^2 + (\eta\,|\,\eta)\xi_0^2 - 2(\xi\,|\,\eta)\,\xi_0\eta_0 + \kappa^2\{(\xi\,|\,\xi)(\eta\,|\,\eta) - (\xi\,|\,\eta)^2\}}{\xi_0^2\eta_0^2 + \kappa^2\{(\xi\,|\,\xi)\eta_0^2 + (\eta\,|\,\eta)\xi_0^2\} + \kappa^4(\xi\,|\,\xi)(\eta\,|\,\eta)},$$

in which we have get:

$$(\xi \mid \eta) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$

to abbreviate.

In the limit, one will then have:

$$(\xi, \eta)^2 = \left(\frac{\xi_1}{\xi_0} - \frac{\eta_1}{\eta_0}\right)^2 + \left(\frac{\xi_2}{\xi_0} - \frac{\eta_2}{\eta_0}\right)^2 + \left(\frac{\xi_3}{\xi_0} - \frac{\eta_3}{\eta_0}\right)^2,$$

such that the symbol (ξ, η) will be the *Euclidian* distance between the points $(\xi_0 : \xi_1 : \xi_2 : \xi_3)$ and $(\eta_0 : \eta_1 : \eta_2 : \eta_3)$. In the limit, the M_3^2 will become:

$$\xi_\omega^2 - \xi_0^2 = \xi_\omega^2 - \xi_0'^2 = 0.$$

If we set $\xi_\omega - \xi_0 = \xi_\omega - \xi_0' = 0$, so $\xi_0 = \xi_0'$, then (76) will go to:

$$\mathfrak{X}_0 = \xi_0, \qquad \mathfrak{X}_{01} = \tfrac{1}{2}(\xi_1 + \xi_1'), \ldots, \ \mathfrak{X}_{23} = \tfrac{1}{2}(\xi_1 - \xi_1'), \ldots,$$

while $\mathfrak{X}_{123} = \lim \dfrac{1}{\kappa^2}(\xi_0 - \xi_0')$ will be implied by (2). With that, formulas (61) are once more obtained. The assumption that $\xi_\omega + \xi_0 = \xi_\omega + \xi_0' = 0$ will yield the same thing, while the two otherwise conceivable possibilities will yield nothing useful. In the limit, one will no longer have double-pairs in M_3^2, or what amounts to the same thing, in *conformal* three-dimensional space, but simple pairs in *projective* space. For that, the singularities will appear that we had to address in § 11.

The constructive implementation of the map can result in a more or less satisfactory way when we start with point-pairs $P \to P'$ $(x \to x')$ and each time seek the defining data of the motion that takes the protosoma to the desired associated soma. There is an entire series of cases to distinguish; that relations that are thus introduced will remain valid in each of the following cases:

Let M be the midpoint of PP', and let O be the midpoint of the protosoma.

1. P and P' coincide at O. Identity motion. Protosoma.

2. P and P' are distinct, but M coincides with O. The length and direction of OP' gives the step size and direction of translation.

3. P' coincides with P, but M is distinct from O. Rotation around the axis OM. For an arbitrary orientation, one will have $\tan \Theta_0 = MO = - OM$.

4. Let P' be distinct from P, and M, from O, but PP' runs through O. Screwing motion around OM. P^* means the mirror image of P relative to O, and M^* means the midpoint between P and P^*. With an arbitrary orientation of the screw axis, one will then have:

$$\tan \Theta_0 = MO, \quad H_0 = M^*O.$$

5. Now, let PP' not run through O. For $OP' = OP$, one is dealing with a rotation, and otherwise a screwing motion whose axis is parallel to OM, but no longer runs through O. The altitude of O to P^*P has the base point Q. For any syntactic orientation of OM and P^*P, one will have:

$$\tan \Theta_0 = MO, \quad H_0 = M^*Q.$$

Now, OQ is oriented arbitrarily, and defines the second axis of a right-handed soma whose first axis is defined by OM. The third axis will then be determined uniquely. One measures out the segment $OQ : OM$ along it from O. A point of the screw axis is given by that, as well as the axis itself (cf., Fig. 1).

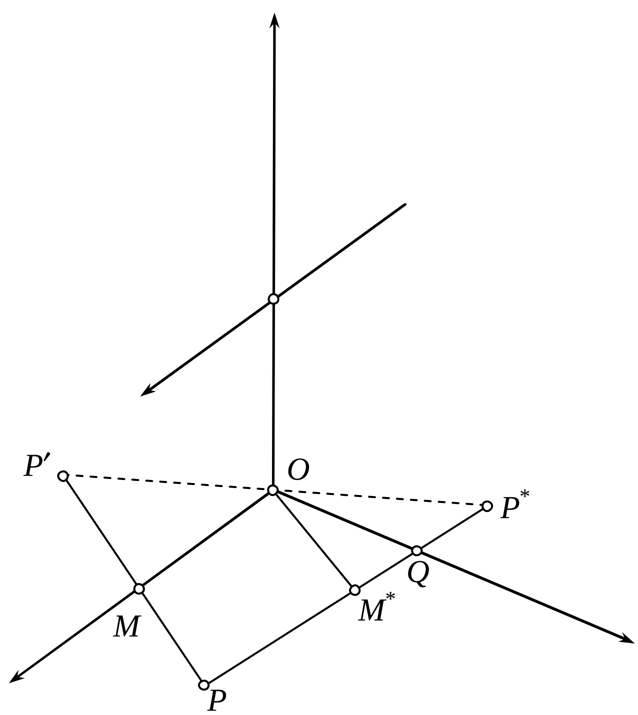

Figure 1.

6. Critical pair $(\mathfrak{P}, \mathfrak{P}', c)$, where $\mathfrak{P}$ and $\mathfrak{P}'$ coincide. The spear through O that is syntactic to it is the unscrewing (reversal) axis, which makes $H_0 = c$.

7. Critical pair $(\mathfrak{P}, \mathfrak{P}', c)$, where $\mathfrak{P}$ and $\mathfrak{P}'$ are opposite. An improper soma is associated. One will obtain the associated spear direction (§ 5) when one rotates the protosoma of the rotation around any spear $\mathfrak{P}'$ through $2\Theta_0$, where $\cot \Theta_0 = c$.

8. The directions $\mathfrak{P}$ and $\mathfrak{P}'$ of the critical pair belong to different bundles of parallels. $\cos(\mathfrak{P}, \mathfrak{P}')$ is then given uniquely. We decree that $\frac{1}{2}(\mathfrak{P}, \mathfrak{P}')$ should mean two arbitrary steps. However, $\cos\frac{1}{2}(\mathfrak{P}, \mathfrak{P}')$ and $\sin\frac{1}{2}(\mathfrak{P}, \mathfrak{P}')$ will then be determined uniquely.

We construct a right-sided trihedron through O. Let the direction cosines be:

a) $\mathfrak{P}' \mathfrak{P}_{01} + \mathfrak{P}\, \mathfrak{P}'_{01} : 2\, \mathfrak{P}\, \mathfrak{P}' \cos\frac{1}{2}(\mathfrak{P}, \mathfrak{P}'), \ldots$ (“true” bisector) *b*) $\mathfrak{P}' \mathfrak{P}_{01} - \mathfrak{P} \mathfrak{P}'_{01} : 2 \mathfrak{P} \mathfrak{P}' \sin\frac{1}{2}(\mathfrak{P}, \mathfrak{P}'), \ldots$ (“false” bisector)

c) $\mathfrak{P}_{03} \mathfrak{P}'_{02} - \mathfrak{P}_{02} \mathfrak{P}'_{03} : 2 \mathfrak{P} \mathfrak{P}' \sin(\mathfrak{P}, \mathfrak{P}'), \ldots$ (common normal direction).

One measures out the segment tan $\frac{1}{2}(\mathfrak{P}, \mathfrak{P}')$ around O along the third, uniquely-determined axis, and thus has a point of the unscrewing (reversal) axis. It is oriented syntactically to the true bisector direction, which then implies that:

$$H_0 = c : \cos\frac{1}{2}(\mathfrak{P}, \mathfrak{P}') .$$

With that, we have once more arrived at the last formula of § 11.

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