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Definition of an energy density and a generalized state of total radiation

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If one is given a time direction **u** then a convenient choice of the three tensors will permit one to define the energy density scalar that is associated with **u**. We define a state of total radiation by imposing some conditions on the curvature tensor of V_4 that generalize the ones that A. Lichnerowicz pointed out ([1]).

1. Let V_4 be the space-time manifold of general relativity endowed with the metric $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ ([2]), let x_0 be a point of V_4, let be the tangent vector space at x_0, and let T_{x_0} be the subspace of $T_{x_0}^{\wedge(2)}$ of $T_{x_0}^{\wedge}$ antisymmetric tensors of order 2. If one is given a basis $\mathbf{e}_{(\alpha)}$ for then we will suppose that T_{x_0} is referred to the basis ([3]):

$$\mathrm{E}_{(\mathrm{I})} = \mathbf{e}_{(\alpha)} \wedge \mathbf{e}_{(\beta)} .$$

One will then have:

$$G_{\mathrm{IJ}} \equiv \mathrm{E}_{(\mathrm{I})}\, \mathrm{E}_{(\mathrm{J})} = \gamma_{\alpha\beta,\,\lambda\mu} = g_{\alpha\beta,\,\lambda\mu} - g_{\alpha\beta,\,\lambda\mu} .$$

It is easy to associate the curvature $R_{\alpha\beta,\,\lambda\mu}$ that is defined at x_0 with the tensors:

$$*R_{\alpha\beta,\,\lambda\mu} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta}\, R^{\gamma\delta}{}_{,\,\lambda\mu} \qquad \text{and} \qquad **R_{\alpha\beta,\,\lambda\mu} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta}\, \eta_{\lambda\mu\nu\sigma}\, R^{\gamma\delta,\,\lambda\mu} ,$$

in which $\eta_{\alpha\beta\gamma\delta}$ is the volume element form. Those three tensors can be considered to be tensors of order 2 in $T_{x_0}^{\wedge(2)}$. In that case, they will denoted by H_{IJ}, $*H_{\mathrm{IJ}}$, and $**H_{\mathrm{IJ}}$, respectively. In what follows, we shall also utilize the three scalars:

([1]) A. LICHNEROWICZ, C. R. Acad. Sci. Paris **246** (1958), pp. 893.

([2]) $\alpha, \beta, \ldots = 0, 1, 2, 3$; $i, j, \ldots = 1, 2, 3$.

([3]) The correspondence between indices α and I conforms to the substitution:

$$\begin{pmatrix} 23 & 31 & 12 & 10 & 20 & 30 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

$$A = \tfrac{1}{2} H_{IJ}\, H^{IJ} = \tfrac{1}{2} {**}H_{IJ}\, {**}H^{IJ} = -\tfrac{1}{2} {*}H_{IJ}\, {*}H^{IJ},$$

$$B = \tfrac{1}{2} H_{IJ}\, {*}H^{IJ} = -\tfrac{1}{2} {*}H_{IJ}\, {**}H^{IJ},$$

$$C = \tfrac{1}{2} H_{IJ}\, {**}H^{IJ}.$$

2. If one is given a vector u^α of square + 1 at a point x_0 then consider the three tensors:

$$Y_{\beta\mu} = R_{\alpha\beta,\,\lambda\mu}\, u^\alpha u^\lambda, \qquad X_{\beta\mu} = {**}R_{\alpha\beta,\,\lambda\mu}\, u^\alpha u^\lambda, \quad Z_{\beta\mu} = -\, {*}R_{\alpha\beta,\,\lambda\mu}\, u^\alpha u^\lambda.$$

With respect to any orthonormal frame such that $\mathbf{e}_{(0)} = \mathbf{u}$, one will have:

$$A = \tfrac{1}{2} (X_{\beta\mu} X^{\beta\mu} + Y_{\beta\mu} Y^{\beta\mu} - Z_{\beta\mu} Z^{\beta\mu}), \quad B = (X_{\beta\mu} - Y_{\beta\mu})\, Z^{\beta\mu}, \quad C = X_{\beta\mu} Y^{\beta\mu} + Z_{\beta\mu} Z^{\beta\mu}.$$

The spatio-temporal square of each of these three tensors is positive or zero; they will be annulled only if the corresponding tensor is zero. If all three are zero then the curvature tensor itself will be zero. Consider the scalar:

$$V = \tfrac{1}{2} (X_{\beta\mu} X^{\beta\mu} + Y_{\beta\mu} Y^{\beta\mu} + 2\, Z_{\beta\mu} Z^{\beta\mu}).$$

This scalar is strictly positive unless $R_{\alpha\beta,\,\lambda\mu}$ is zero. We call it the *energy density that is associated with the time direction* **u** ([4]).

3. We propose to say that the point x_0 presents a generalized total radiation state if the following hypotheses are satisfied:

$\mathcal{H}_1$. There exists an isotropic vector l^α such that:

$$R_{\alpha\beta,\,\lambda\mu}\, l^\alpha l^\beta = 0, \qquad {*}R_{\alpha\beta,\,\lambda\mu}\, l^\alpha l^\beta = 0, \qquad {**}R_{\alpha\beta,\,\lambda\mu}\, l^\alpha l^\beta = 0.$$

$\mathcal{H}_2$. There exists a vector u^α of square + 1 such that:

$$R_{\alpha\beta,\,\lambda\mu}\, u^\alpha u^\lambda l^\beta = 0, \qquad {**}R_{\alpha\beta,\,\lambda\mu}\, u^\alpha u^\lambda l^\beta = 0.$$

If that were true then one would find an orthonormal frame $\mathbf{e}_{(\alpha)}$ such that $\mathbf{e}_{(0)} = \mathbf{u}$, $\mathbf{e}_{(0)} + \mathbf{e}_{(1)} = \boldsymbol{l}$ with respect to which one will have:

([4]) For the Schwarzschild case, that definition will coincide with a result of SYNGE, Proc. Roy. Irish Acad. **58**, A4.

$$(H_{IJ}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \beta & 0 & \beta & \sigma_2 \\ 0 & \beta & \alpha_2 & 0 & \sigma_3 & -\beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & \sigma_3 & 0 & \gamma_2 & -\beta \\ 0 & \sigma_2 & -\beta & 0 & -\beta & \gamma_3 \end{pmatrix}, \qquad \sigma_2 = \tfrac{1}{2}(\gamma_3 + \alpha_2), \qquad \sigma_3 = \tfrac{1}{2}(\gamma_2 + \alpha_3).$$

The proper values of (H_{IJ}) with respect to (G_{IJ}) are $\rho_1 = \rho_4 = 0$, $\rho_3 = \rho_5 = (1/2)(\alpha_3 - \gamma_2)$, $\rho_2 = \rho_6 = (1/2)(\alpha_2 - \gamma_3)$. The vectors in the 2-plane $(E_{(1)}, E_{(4)})$, $M \equiv E_{(3)} - E_{(5)}$ and $L \equiv E_{(2)} + E_{(6)}$ are proper vectors that correspond to ρ_1, ρ_3, and ρ_2, resp. H_{IJ} admits the reduction:

$$H_{IJ} = \beta (L_I M_J + L_J M_I) + \tfrac{1}{2}\alpha_2 (E_{(2)\,I} L_J + E_{(2)\,J} L_I) + \tfrac{1}{2}\alpha_3 (E_{(3)\,I} L_J + E_{(3)\,J} L_I)$$
$$- \tfrac{1}{2}\gamma_2 (E_{(5)\,I} L_J + E_{(5)\,J} L_I) + \tfrac{1}{2}\gamma_3 (E_{(6)\,I} L_J + E_{(6)\,J} L_I).$$

Similarly, the proper values of $(R_{\alpha\beta})$ with respect to $(g_{\alpha\beta})$ are $s_0 = s_1 = R / 4$, $s_2 = 2\rho_3$, $s_3 = 2\rho_2$ $(s_0 + s_3 = R / 2)$. $\boldsymbol{l}$, $\mathbf{e}_{(2)}$, and $\mathbf{e}_{(3)}$ are proper vectors that correspond to s_0, s_2, and s_3, resp. $S_{\alpha\beta} = R_{\alpha\beta} - (1/2)\, R\, g_{\alpha\beta}$ admits the reduction:

$$S_{\alpha\beta} = -\tfrac{1}{2}(\alpha_2 + \alpha_3 + \gamma_2 + \gamma_3)\, l_\alpha l_\beta - \tfrac{1}{2}(s_2 + s_3)(l_{(0)\alpha} l_{(0)\beta} - l_{(1)\alpha} l_{(1)\beta})$$
$$+ s_3\, l_{(2)\alpha} l_{(2)\beta} + s_2\, l_{(3)\alpha} l_{(3)\beta}\,.$$

In addition, one will have:

$$A = (\rho_2)^2 + (\rho_3)^2, \qquad B = 0, \qquad C = -\,2\,\rho_2\,\rho_3\,.$$

4. One will deduce from the relations:

$$e^{\mu}_{(2)} \nabla_\beta (*R_{\alpha}{}^{\beta}{}_{,\,\lambda\mu}\, l^\alpha l^\lambda) = 0, \qquad e^{\mu}_{(3)} \nabla_\beta (*R_{\alpha}{}^{\beta}{}_{,\,\lambda\mu}\, l^\alpha l^\lambda) = 0$$

that:

$$s_2\, e_{(3)\alpha}\, l^\beta \nabla_\beta l^\alpha = 0, \qquad s_3\, e_{(2)\alpha}\, l^\beta \nabla_\beta l^\alpha = 0.$$

If $s_2 \neq 0$, $s_3 \neq 0$ then it will result that the vector $l^\beta \nabla_\beta l^\alpha$ is orthogonal to $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$. Therefore:

$$l^\beta \nabla_\beta l^\alpha = a\, l^\alpha.$$

If $s_2 = s_3 = 0$ then $R_{\alpha\beta,\,\lambda\mu}$ will satisfy the conditions:

$$R_{\alpha\beta,\,\lambda\mu}\, l^\alpha = 0, \quad *R_{\alpha\beta,\,\lambda\mu}\, l^\alpha = 0,$$

and the preceding result will remain true, from a theorem of A. Lichnerowicz ([1]). In those two cases, we state: *The trajectories of the vector* ***l*** *that is associated with a* *generalized state of total radiation that is defined in a domain are geodesics of the metric*. We point out that the case for which the theorem is not established is characterized by the relations $A \neq 0$, $B = C = 0$.