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On gravitational radiation

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We define a tensor $T_{\alpha\beta,\lambda\mu}$, a scalar $V(\mathbf{u})$, and a vector $\mathbf{P}(\mathbf{u})$, and we show that they play roles that are analogous to the ones that are played by the Maxwell tensor, the energy density, and the Poynting vector, resp., in the study of electromagnetic radiation. We propose a definition of the state of gravitational radiation.

1. Let V_4 be the space-time manifold of general relativity, let $ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta$ (1) be the (normal hyperbolic) metric, and let $R_{\alpha\beta,\lambda\mu}$ be the curvature tensor. Consider the tensor:

$$M_{\alpha\beta,\lambda\mu} = R^{\alpha}{}_{\beta,}{}^{\lambda}{}_{\mu}\,R_{\alpha\gamma,\lambda\nu} + R^{\alpha}{}_{\beta,}{}^{\lambda}{}_{\nu}\,R_{\alpha\gamma,\lambda\mu} \tag{1}$$

($M_{\beta\gamma,\mu\nu} = M_{\gamma\beta,\mu\nu} = M_{\beta\gamma,\nu\mu} = M_{\mu\nu,\beta\gamma}$). We suppose that the equations $R_{\alpha\beta} = \lambda\,g_{\alpha\beta}$ (λ = const., $R_{\alpha\beta} = g^{\mu\nu}R_{\alpha\mu,\beta\nu}$) are satisfied. Therefore (2):

$$\nabla_\beta R^{\alpha\beta,\lambda\mu} = 0, \tag{2a}$$

$$R^{\alpha\beta,\lambda}{}_{\mu}\,R_{\alpha\beta,\lambda\nu} = 2\,A\,g_{\mu\nu} \qquad (A = \tfrac{1}{8}R^{\alpha\beta,\lambda\mu}R_{\alpha\beta,\lambda\mu}). \tag{2b}$$

From (1) and (2*a*), one will have:

$$\nabla_\beta M^{\beta}{}_{\gamma,\mu\nu} = R^{\alpha\beta,\lambda}{}_{\mu}\,\nabla_\beta R_{\alpha\gamma,\lambda\nu} + R^{\alpha\beta}{}_{,}{}^{\lambda}{}_{\nu}\,\nabla_\beta R_{\alpha\gamma,\lambda\mu}\,.$$

Take into account the antisymmetry in α and β, the Bianchi identities ($\nabla_\beta R_{\alpha\gamma,\lambda\nu} + \nabla_\alpha R_{\gamma\beta,\lambda\nu} + \nabla_\gamma R_{\beta\alpha,\lambda\nu} = 0$), and (2*b*), and let:

$$\nabla_\beta M^{\beta}{}_{\gamma,\mu\nu} = g_{\mu\nu}\,\partial_\gamma A \qquad \left(\partial_\gamma \equiv \partial/\partial x^\gamma\right). \tag{3}$$

2. Now consider the tensor:

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(1) $\alpha, \beta, \ldots = 0, 1, 2, 3$; $i, j, \ldots = 1, 2, 3$.

(2) LANCZOS, Ann. Math. **39** (1938), pp. 842.

$$T_{\beta\gamma,\,\mu\nu} = g_{\beta\gamma}\, g_{\mu\nu}\, A - M_{\beta\gamma,\mu\nu}\,.$$

From (3), we find that:

(4) $$\nabla_\beta\, T^{\beta}{}_{\gamma,\,\mu\nu} = 0.$$

Let $(\mathbf{e}_{(\alpha)})$ be an orthonormal frame ($\mathbf{e}_0^2 = 1$). If (H_{IJ}) ([3]) is the matrix of components of $R_{\alpha\gamma,\lambda\mu}$ for that frame then the sub-matrices (X_{ij}), (Y_{ij}), (Z_{ij}), such that:

$$(H_{\mathrm{IJ}}) = \begin{pmatrix} X & Z \\ Z' & X \end{pmatrix} \qquad\qquad (Z'\text{transposed of } Z)$$

will be the matrices of spatial components for the three tensors that are associated with $\mathbf{e}_{(0)}$ ([4]). The equations $R_{\alpha\beta} = \lambda\, g_{\alpha\beta}$ imply $X = -\,Y$, $Z = Z'$. The scalar A will become:

$$A = X_{ij}\, X^{ij} - Z_{ij}\, Z^{ij}.$$

Therefore:

(5) $$T_{00,00} = X_{ij}\, X^{ij} - Z_{ij}\, Z^{ij} = -\,2V,$$

in which V is a scalar that is associated with $\mathbf{e}_{(0)}$ and enjoys the property of being positive or zero, and that will be true only if $R_{\alpha\gamma,\lambda\mu} = 0$ ([4]).

3. Suppose that one can find a coordinate system in the neighborhood U of a point x of V_4 such that:

$$ds^2 = (dx^0)^2 + g_{ij}\, dx^i\, dx^j, \qquad 0 < |\,\partial_t\, g_{ij}\,| < \varepsilon_{ij} \qquad (x^0 = ct,\ \varepsilon_{ij} \text{ given}).$$

Let W_3 be a spatial section that is assumed to be endowed with the metric $\widehat{ds}^2 = \hat{g}_{ij}\, dx^i\, dx^j$ ($\hat{g}_{ij} = g_{ij}$). The relation $\nabla_\alpha\, T^{\alpha}{}_{0,\,00} = 0$ will then give:

(6*a*) $$\partial_t\, V = \frac{c}{2}\hat{\nabla}_t P^i + [g^{ik}\partial_t g_{ik} T^0{}_{0,00} - g^{jk}\partial_t g_{ik}(T^i{}_{j,00} + 2T^i{}_{0,j0})]\,,$$

in which $P^i \equiv T^i{}_{0,\,00}$. If one supposes that the ε_{ij} are sufficiently small that the bracket is is negligible in comparison to the first term then one will have:

(6*b*) $$\partial_t\, V = \frac{c}{2}\hat{\nabla}_t P^i$$

([3]) The correspondence between indices α and I conforms to the substitution:

$$\begin{pmatrix} 23 & 31 & 12 & 10 & 20 & 30 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

([4]) BEL, C. R. Acad. Sci. Paris **246** (1958), pp. 3015.

in that approximation, and integrating it over a volume C in W_3 :

(7) $$\partial_t \int_C V\sqrt{|\hat{g}|}\, dx^1 \wedge dx^2 \wedge dx^3 = \frac{c}{2}\int_{\partial C} P_{ij}\, dx^i \wedge dx^j \qquad (P_{ij} = \tfrac{1}{2}\hat{\eta}_{ijk} P^k),$$

in which $\hat{\eta}_{ijk}$ is the volume element form in W_3 . The results (4), (5), (6), and (7) show us that the *tensor* $T_{\beta\gamma,\,\mu\nu}$, *the energy density* V, *and the vector* **P** *play the role that is played by the Maxwell tensor, the energy density, and the Poynting vector, resp.*, in the study of electromagnetic radiation.

4. The vector **P** that we just defined and the scalar V are associated with a time direction. In general, let **u** be that direction. It is easy to verify that:

$$P_\rho(\mathbf{u}) = (g^\mu_\rho - u_\rho u^\mu)\, T_{\beta\gamma,\,\mu\nu} u^\beta u^\gamma u^\nu, \qquad V(\mathbf{u}) = -\tfrac{1}{2} T_{\beta\gamma,\,\mu\nu} u^\beta u^\gamma u^\mu u^\nu.$$

If **P** (**u**) = 0.

$$T_{\beta\gamma,\,\mu\nu} u^\beta u^\gamma u^\nu = -2\, V u_\rho .$$

We shall also show that this implies that the matrices X and Z that are associated with **u** are simultaneously diagonalizable.

5. Always by analogy with electromagnetism, we propose to say that *there will be a state of gravitational radiation in a neighborhood U of x if* **P** (**u**) *is non-zero for any* **u** ($\mathbf{u}^2$ = 1) *at x*. Suppose that $\lambda = 0$. It results from paragraph **4** that the cases 2 and 3 that were pointed out by Petrov ([5]) will characterize such a radiation state. The case in which there exists an isotropic vector l^α such that ([6]):

$$l^\alpha R_{\alpha\beta,\,\lambda\mu} = 0, \qquad l_\alpha R_{\beta\gamma,\,\lambda\mu} + l_\beta R_{\gamma\alpha,\,\lambda\mu} + l_\gamma R_{\alpha\beta,\,\lambda\mu} = 0, \qquad (R_{\alpha\beta} = 0)$$

is particularly interesting (viz., a special case of Petrov's case 2). In that case, one will have:

$$T_{\alpha\beta,\,\lambda\mu} = 4\sigma^2\, l_\alpha l_\beta l_\gamma l_\mu ,$$

which justifies the statement that one is dealing with a "pure" case.

([5]) PETROV, Sci. Not. Kazan. St. Univ. **114** (1954), pp. 55; *see* also PIRANI, Phy. Rev. **10**, no. 3 (1957), pp. 1089-1099.

([6]) LICHNEROWICZ, C. R. Acad. Sci. Paris **246** (1958), pp. 893.