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On the general equations of elasticity

(Paper by E. BELTRAMI, in Pavia)

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It is known that LAMÉ was the first to transform the equations of elasticity into orthogonal curvilinear coordinates. That transformation, which he presented for the first time in a paper that was published in v. 6 of LIOUVILLE's Journal (first series), was reproduced posthumously in lessons XV and XVI of his *Leçons sur les coordonnées curvilignes.*

C. NEUMANN, and later BORCHARDT, noticeably abbreviated the elegant, but somewhat prolix, calculations of that famous French geometer by procedures that differed in places.

The first of those authors, in his very interesting paper "Zur Theorie der Elastizität" [Berliner Berichte **57** (1859)], has addressed the question again from the beginning by calculating the potential of the molecular forces in isotropic bodies and deduced the known equations for the variations in that potential directly. The simplifications that were obtained in that work resulted mainly from certain relations that were first established in curvilinear coordinates by the author between what one calls the coefficients of variation of that potential before and after the transformation. (Those coefficients are no different from the expressions that are found to multiply the variations of the unknown functions in that part of the variation of the integral that is represented by an integral of order equal to the dimensions of the multiplicity.)

BORCHARDT, in an article that was entitled "Transformation der Elastizitätsgleichungen in allgemeine orthogonale Coordinaten," [page 76 of the aforementioned Journal (1873)], which was reproduced in the Bulletin des Sciences mathématiques et astrodynamiques [v. 6 (1875)], has also based his deductions upon the variation of the integral that represents the potential of the elastic force, but the simplification that he arrived at was in both the suppression of certain parts of the integral that are convertible into surface integrals, and therefore do not contribute to any of the indefinite equations and the direct transformation of the expression that represents the square of the elementary rotation.

At its basis, for all three of the cited authors, the essential innovation in the transformation consisted of grouping three unknown functions and their nine derivatives into just four distinct expressions, which are the ones that represent the cubic dilatation and the three components of rotation. Indeed, LAMÉ started directly from the Cartesian equations between those four expressions, while NEUMANN and BORCHARDT arranged the elementary potential in such a way that only those expressions would provide terms in the transformed equations.

Now that innovation, which permits one to arrive at those equations with the greatest expediency that the nature of the argument allows, nonetheless leaves in the shadows a situation of great interest, which, it would seem, cannot be avoided and which leads to consequences that are totally unexpected.

In order to shed a better light upon that point, we begin by establishing *directly* the general equations of elastic equilibrium in orthogonal coordinates of any type.

Let q_1, q_2, q_3 be the orthogonal curvilinear coordinates of an arbitrary point in a three-dimensional space, and let:

$$ds^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2 + Q_3^2\, dq_3^2 \tag{1}$$

be the expression for the square of any line element in that space.

If one varies the position of any point then one will find that:

$$ds\ \delta ds = Q_1^2\, dq_1\, d\delta q_1 + Q_2^2\, dq_2\, d\delta q_2 + Q_3^2\, dq_3\, d\delta q_3 + Q_1\, \delta Q_1\, dq_1^2 + Q_2\, \delta Q_2\, dq_2^2 + Q_3\, \delta Q_3\, dq_3^2 .$$

However, one has:

$$d\delta q_i = \frac{\partial \delta q_i}{\partial q_1} dq_1 + \frac{\partial \delta q_i}{\partial q_2} dq_2 + \frac{\partial \delta q_i}{\partial q_3} dq_3$$

for $i = 1, 2, 3$; therefore, if one sets:

$$\left.\begin{aligned}
\delta\theta_1 &= \frac{\partial \delta q_1}{\partial q_1} + \frac{\delta Q_1}{Q_1}, \\
\delta\theta_2 &= \frac{\partial \delta q_2}{\partial q_2} + \frac{\delta Q_2}{Q_2}, \\
\delta\theta_3 &= \frac{\partial \delta q_3}{\partial q_3} + \frac{\delta Q_3}{Q_3}, \\
\delta\omega_1 &= \frac{Q_2}{Q_3}\frac{\partial \delta q_2}{\partial q_3} + \frac{Q_3}{Q_2}\frac{\partial \delta q_3}{\partial q_2}, \\
\delta\omega_2 &= \frac{Q_3}{Q_1}\frac{\partial \delta q_3}{\partial q_1} + \frac{Q_1}{Q_3}\frac{\partial \delta q_1}{\partial q_3}, \\
\delta\omega_3 &= \frac{Q_1}{Q_2}\frac{\partial \delta q_1}{\partial q_2} + \frac{Q_2}{Q_1}\frac{\partial \delta q_2}{\partial q_1}
\end{aligned}\right\} \tag{2}$$

then one can write:

$$\frac{\delta ds}{ds} = \lambda_1^2\, \delta\theta_1 + \lambda_2^2\, \delta\theta_2 + \lambda_3^2\, \delta\theta_3 + \lambda_2\, \lambda_3\, \delta\omega_1 + \lambda_3\, \lambda_1\, \delta\omega_2 + \lambda_1\, \lambda_2\, \delta\omega_3 , \tag*{$(2)_a$}$$

in which the three quantities λ_1, λ_2, λ_3, which are defined by:

$$\lambda_i = \frac{Q_i \, dq_i}{ds},$$

are the cosines of the angles that the line element *ds* makes with the three coordinate lines q_1, q_2, q_3 (which denote the lines along which only the coordinate q_1 or q_2 or q_3 varies, for brevity).

We then have a continuous material system that occupies a connected space *S* that is bounded by a surface σ, and we let that system be in equilibrium under the action of:

1) The external forces that are applied to any volume element *dS* and any surface element $d\sigma$.

2) The internal forces that are developed in each element *dS* by the deformation that the external force determines in the system.

That system, *once it has been deformed and equilibrated*, is the one whose points are individuated by the coordinates q_1, q_2, q_3 .

Let:

$$F_1 \, dS, \quad F_2 \, dS, \quad F_3 \, dS$$

be the components along the directions q_1, q_2, q_3 of the external force that acts upon the volume element *dS*, and let:

$$\varphi_1 \, d\sigma, \qquad \varphi_2 \, d\sigma, \qquad \varphi_3 \, d\sigma$$

be the analogous components of the external force that is applied to the surface element $d\sigma$.

In order to express the conditions for equilibrium in the system, imagine that any of its points (q_1, q_2, q_3) is subjected to a new displacement, under which, its coordinates will become $q_1 + \delta q_1$, $q_2 + \delta q_2$, $q_3 + \delta q_3$. The work that is developed under the displacement by the external force that acts upon the volume element *dS* is:

$$(F_1 \, Q_1 \, \delta q_1 + F_2 \, Q_2 \, \delta q_2 + F_3 \, Q_3 \, \delta q_3) \, dS,$$

and the work that is developed by the external force that acts upon the surface element $d\sigma$ is:

$$(\varphi_1 \, Q_1 \, \delta q_1 + \varphi_2 \, Q_2 \, \delta q_2 + \varphi_3 \, Q_3 \, \delta q_3) \, d\sigma .$$

As for the internal forces, which will no do work if that imagined displacement does not alter the lengths of linear elements, it is obvious that the work that they do on the element *dS* can only have an expression of the form:

$$(\Theta_1 \, \delta\theta_1 + \Theta_2 \, \delta\theta_2 + \Theta_3 \, \delta\theta_3 + \Omega_1 \, \delta\omega_1 + \Omega_2 \, \delta\omega_2 + \Omega_3 \, \delta\omega_3) \, dS,$$

since, from $(2)_a$, the variation of the line element depends upon the six $\delta\theta_i$, $\delta\omega_i$, and is annulled by them. The six multipliers Θ_i , Ω_i are functions of q_1, q_2, q_3 whose significance we do not need to investigate, for the moment. From the foregoing, the general equation of equilibrium is the following one:

$$\left.\begin{aligned}&\int (F_1\,Q_1\,\delta q_1 + F_2\,Q_2\,\delta q_2 + F_3\,Q_3\,\delta q_3)\,dS\\ &+\int (\varphi_1\,Q_1\,\delta q_1 + \varphi_2\,Q_2\,\delta q_2 + \varphi_3\,Q_3\,\delta q_3)\,d\sigma\\ &+\int (\Theta_1\,\delta\theta_1 + \Theta_2\,\delta\theta_2 + \Theta_3\,\delta\theta_3 + \Omega_1\,\delta\omega_1 + \Omega_2\,\delta\omega_2 + \Omega_3\,\delta\omega_3)\,dS = 0.\end{aligned}\right\} \tag{3}$$

In order to get the equations of equilibrium from that formula, properly speaking, we need to duly transform the integrals into the form:

$$\int \Theta_i\,\delta\theta_i\,dS\,, \qquad \int \Omega_i\,\delta\omega_i\,dS\,.$$

Starting with the first one, one has, from (2):

$$\int \Theta_i\,\delta\theta_i\,dS = \int \Theta_i \left(\frac{\partial\,\delta q_i}{\partial q_i} + \frac{\delta Q_i}{Q_i}\right) dS\,,$$

and if one sets $Q_1\,Q_2\,Q_3 = \nabla$, for brevity, that:

$$\begin{aligned}\int \Theta_i\,\delta\theta_i\,dS &= \int \nabla\Theta_i\,\frac{\partial\,\delta q_i}{\partial q_i}\,\frac{dS}{\nabla} + \int \frac{\Theta_i\,\delta Q_i}{Q_i}\,dS\\ &= \int \frac{\partial}{\partial q_i}(\nabla\Theta_i\,\delta q_i)\,\frac{dS}{\nabla} - \int \left\{\frac{\partial\nabla\Theta_i}{\partial q_i}\,\frac{\delta q_i}{\nabla} - \frac{\Theta_i\,\delta Q_i}{Q_i}\right\} dS\,.\end{aligned}$$

Now, from the known equation:

$$\int \frac{\partial f}{\partial q_i}\,\frac{dS}{\nabla} = -\int \frac{Q_i\,f\cos(n\,q_i)}{\nabla}\,d\sigma\,,$$

in which n is the internal normal to the surface σ, one will have:

$$\int \frac{\partial}{\partial q_i}(\nabla\Theta_i\,\delta q_i)\,\frac{dS}{\nabla} = -\int Q_i\,\Theta_i\cos(n\,q_i)\,\delta q_i\,d\sigma\,,$$

and therefore:

$$\int \Theta_i\,\delta\theta_i\,dS = -\int \left\{\frac{\partial\nabla\Theta_i}{\partial q_i}\,\frac{\delta q_i}{\nabla} - \frac{\Theta_i\,\delta Q_i}{Q_i}\right\} dS - \int Q_i\,\Theta_i\cos(n\,q_i)\,\delta q_i\,d\sigma\,.$$

Passing on to the second integral, one has, from (2):

$$\begin{aligned}\int \Omega_1\,\delta\omega_1\,dS &= \int Q_1\,\Omega_1\left(Q_2^2\,\frac{\partial\,\delta q_2}{\partial q_2}+Q_3^2\,\frac{\partial\,\delta q_3}{\partial q_2}\right)\frac{dS}{\nabla}\\ &= \int\left\{\frac{\partial}{\partial q_3}(Q_1\,Q_2^2\,\Omega_1\,\delta q_2)+\frac{\partial}{\partial q_2}(Q_1\,Q_3^2\,\Omega_1\,\delta q_3)\right\}\frac{dS}{\nabla}\\ &\quad-\int\left\{\frac{\partial(Q_1\,Q_2^2\,\Omega_1)}{\partial q_3}\,\delta q_2+\frac{\partial(Q_1\,Q_3^2\,\Omega_1)}{\partial q_2}\,\delta q_3\right\}\frac{dS}{\nabla},\end{aligned}$$

or, from the theorem that was recalled:

$$\begin{aligned}\int \Omega_1\,\delta\omega_1\,dS = &-\int\left\{\frac{\partial(Q_1\,Q_2^2\,\Omega_1)}{\partial q_3}\,\delta q_2+\frac{\partial(Q_1\,Q_3^2\,\Omega_1)}{\partial q_2}\,\delta q_3\right\}\frac{dS}{\nabla}\\ &-\int\{Q_2\cos(n\,q_3)\,\delta q_2+Q_3\cos(n\,q_2)\,\delta q_3\}\,\Omega_1\,d\sigma.\end{aligned}$$

One transforms the other two integrals, viz:

$$\int \Omega_2\,\delta\omega_2\,dS\,,\qquad \int \Omega_3\,\delta\omega_3\,dS$$

analogously.

If one substitutes the values, thus-transformed, of the six integrals:

$$\int \Theta_i\,\delta\theta_i\,dS\,,\qquad \int \Omega_i\,\delta\omega_i\,dS$$

in equation (3) then one will get a result of the form:

$$\int (S_1\,\delta q_1+S_2\,\delta q_1+S_3\,\delta q_3)\,dS+\int(\sigma_1\,\delta q_1+\sigma_2\,\delta q_1+\sigma_3\,\delta q_3)\,d\sigma=0,$$

which will split into three equations:

$$S_1 = 0,\qquad S_2 = 0,\qquad S_3 = 0,$$

due to the arbitrariness in the variations δq_i, which are valid at any point of space S, along with three equations:

$$\sigma_1 = 0,\qquad \sigma_2 = 0,\qquad \sigma_3 = 0$$

that are valid at any point of the surface σ.

The substitutions that were performed give the three indefinite equations:

$$\left.\begin{aligned}
Q_1 F_1 &= \frac{1}{\nabla}\left\{\frac{\partial(\nabla\Theta_1)}{\partial q_1} + \frac{\partial(Q_1^2\,Q_3\,\Omega_3)}{\partial q_2} + \frac{\partial(Q_1^2\,Q_2\,\Omega_2)}{\partial q_3}\right\} - \left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_1} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_1} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_1}\right),\\
Q_2 F_2 &= \frac{1}{\nabla}\left\{\frac{\partial(Q_2^2\,Q_3\,\Omega_3)}{\partial q_1} + \frac{\partial(\nabla\Theta_2)}{\partial q_2} + \frac{\partial(Q_2^2\,Q_1\,\Omega_1)}{\partial q_3}\right\} - \left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_2} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_2} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_2}\right),\\
Q_3 F_3 &= \frac{1}{\nabla}\left\{\frac{\partial(Q_3^2\,Q_2\,\Omega_2)}{\partial q_1} + \frac{\partial(Q_3^2\,Q_1\,\Omega_1)}{\partial q_2} + \frac{\partial(\nabla\Theta_3)}{\partial q_3}\right\} - \left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_3} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_3} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_3}\right),
\end{aligned}\right\} \tag{4}$$

and the three boundary equations:

$$\left.\begin{aligned}
\varphi_1 &= \Theta_1 \cos(n\,q_1) + \Omega_3 \cos(n\,q_2) + \Omega_2 \cos(n\,q_3),\\
\varphi_2 &= \Omega_3 \cos(n\,q_1) + \Theta_2 \cos(n\,q_2) + \Omega_1 \cos(n\,q_3),\\
\varphi_3 &= \Omega_2 \cos(n\,q_1) + \Omega_1 \cos(n\,q_2) + \Theta_3 \cos(n\,q_3).
\end{aligned}\right\} \tag*{(4)$_a$}$$

The latter provides the definitions of the six functions Θ_i, Ω_i. Indeed, it is applicable to any portion of the system if φ_i represent the components of the force that must be applied to the surface of that portion in order to maintain equilibrium when the remaining portion is destroyed. Now, for an element $d\sigma_1$ of a surface q_1 = const., one will have, from $(4)_a$:

$$\varphi_1^{(1)} = \Theta_1, \qquad \varphi_2^{(1)} = \Omega_3, \qquad \varphi_3^{(1)} = \Omega_2;$$

for an element $d\sigma_2$ of a surface q_2 = const., one will have:

$$\varphi_1^{(2)} = \Omega_3, \qquad \varphi_2^{(2)} = \Theta_2, \qquad \varphi_3^{(2)} = \Omega_1;$$

for an element $d\sigma_3$ of a surface q_3 = const., one will have:

$$\varphi_1^{(3)} = \Omega_2, \qquad \varphi_2^{(3)} = \Omega_1, \qquad \varphi_3^{(3)} = \Theta_3.$$

Therefore, the quantities Θ_1, Θ_2, Θ_3 represent the unit tensions that are developed *normally* to the coordinate surfaces q_1 = const., q_2 = const., q_3 = const., and the quantities Ω_1, Ω_2, Ω_3 represent the unit tensions that are developed *tangentially* to the surfaces. The equalities:

$$\varphi_2^{(3)} = \varphi_3^{(2)}, \qquad \varphi_3^{(1)} = \varphi_1^{(3)}, \qquad \varphi_1^{(2)} = \varphi_2^{(1)},$$

which result from the preceding values, are the ones that are ordinarily deduces from the consideration of the elementary tetrahedron.

Equations (4) coincide with the ones that LAMÉ deduced from the transformation of the analogous equations into Cartesian coordinates (*Leçons sur les coordonées curvilignes*, pp. 272). The only difference consists of the fact that LAMÉ introduced the derivatives with respect to the arcs at the location of Q_1, Q_2, Q_3 ; however, it is easy to pass from one form to the other by means of formulas that will be pointed out below.

However, it is more important to observe (and this will prove to be obvious from the process that was used here in order to establish the equations) that the space to which it is referred is not defined by anything else but the expression (1) for the line element, with no conditions on the functions Q_1, Q_2, Q_3 . Therefore, equations (4), $(4)_a$ possess much greater generality than the analogous ones in Cartesian coordinates, and in particular, one should immediately note that they *are independent of* EUCLID's *postulate*. That fact is intimately connected with the one that I alluded to at the beginning. However, before proceeding to other things, it is necessary to complete the presentation of the theory of the equations of elastic tensions.

Set:

$$\left.\begin{aligned}
\theta_1 &= \frac{\partial \kappa_1}{\partial q_1} + \frac{1}{Q_1}\left(\frac{\partial Q_1}{\partial q_1}\kappa_1 + \frac{\partial Q_1}{\partial q_2}\kappa_2 + \frac{\partial Q_1}{\partial q_3}\kappa_3\right), \\
\theta_2 &= \frac{\partial \kappa_2}{\partial q_2} + \frac{1}{Q_2}\left(\frac{\partial Q_2}{\partial q_1}\kappa_1 + \frac{\partial Q_2}{\partial q_2}\kappa_2 + \frac{\partial Q_3}{\partial q_3}\kappa_3\right), \\
\theta_3 &= \frac{\partial \kappa_3}{\partial q_3} + \frac{1}{Q_3}\left(\frac{\partial Q_3}{\partial q_1}\kappa_1 + \frac{\partial Q_3}{\partial q_2}\kappa_2 + \frac{\partial Q_3}{\partial q_3}\kappa_3\right), \\
\omega_1 &= \frac{Q_2}{Q_3}\frac{\partial \kappa_2}{\partial q_3} + \frac{Q_3}{Q_2}\frac{\partial \kappa_3}{\partial q_2}, \\
\omega_2 &= \frac{Q_3}{Q_1}\frac{\partial \kappa_3}{\partial q_1} + \frac{Q_1}{Q_3}\frac{\partial \kappa_1}{\partial q_3}, \\
\omega_3 &= \frac{Q_1}{Q_2}\frac{\partial \kappa_1}{\partial q_2} + \frac{Q_2}{Q_1}\frac{\partial \kappa_2}{\partial q_1}.
\end{aligned}\right\} \tag{5}$$

If one compares these quantities θ_i, ω_i with the $\delta\theta_i$, $\delta\omega_i$ that were defined in equations (2) then one will perceive that the latter are the variations of the former if one assumes that:

$$\delta\kappa_i = \delta q_i ,$$

and that the coordinates q_i are invariable with respect to δ.

Assume, as usual, that the deformation that is produced by the external force is small enough to allow one to treat the total variations that the coordinates of each point are subjected to as differentials, so it will be legitimate to substitute the initial coordinates for the final ones in the functions Q_i, Θ_i, Ω_i , and if one considers the quantities κ_i to be the total increments in the initial coordinates q_i then one can establish the equation:

$$\frac{\Delta\, ds}{ds} = \theta_1 \lambda_1^2 + \theta_2 \lambda_2^2 + \theta_3 \lambda_3^2 + \omega_1 \lambda_2 \lambda_3 + \omega_2 \lambda_3 \lambda_1 + \omega_3 \lambda_1 \lambda_2 , \tag{5$_a$}$$

which is analogous $(2)_a$, in order to determine the total variation $\Delta\, ds$ that the element ds submits to during the deformation.

The six quantities θ_i, ω_i (like the preceding ones $\delta\theta_i$, $\delta\omega_i$) have a simple geometric significance. Indeed, in order to perform the deformation that is produced by the external forces, the three orthogonal line elements:

$$ds_1 = Q_1\, dq_1\,, \quad ds_2 = Q_2\, dq_2\,, \quad ds_3 = Q_3\, dq_3\,,$$

whose resultant is ds, will become three line elements ds_1', ds_2', ds_3' that are no longer orthogonal, but slightly oblique, while ds becomes the resultant ds' of those three new elements. If one then lets β_1, β_2, β_3 denote the complements of the planar angles:

$$(ds_2',\, ds_3'), \qquad (ds_3',\, ds_1'), \qquad (ds_1',\, ds_2')$$

then one will have:

$$ds'^2 = ds_1'^2 + ds_2'^2 + ds_3'^2 + 2\beta_1\, ds_2'\, ds_3' + 2\beta_2\, ds_3'\, ds_1' + 2\beta_3\, ds_1'\, ds_2'\,,$$

from the elementary formula for the resultant. If one sets:

$$ds_1' = (1 + \alpha_1)\, ds_1\,, \qquad ds_2' = (1 + \alpha_2)\, ds_2\,, \qquad ds_3' = (1 + \alpha_3)\, ds_3\,,$$
$$ds = (1 + a)\, ds$$

then one will get from that:

$$\alpha = \alpha_1\, \lambda_1^2 + \alpha_2\, \lambda_2^2 + \alpha_3\, \lambda_3^2 + \beta_1\, \lambda_2\, \lambda_3 + \beta_2\, \lambda_3\, \lambda_1 + \beta_3\, \lambda_1\, \lambda_2\,.$$

However, it is obvious that one has simply:

$$\alpha = \frac{ds' - ds}{ds} = \frac{\Delta ds}{ds}\,,$$

such that when one compares the preceding value for α with formula $(5)_a$, it will result that:

$$\alpha_i = \beta_i\,, \qquad \beta_i = \omega_i\,.$$

Therefore, the three quantities θ_i and the three quantities ω_i represent the (relative) elongations of the edges and the decrements in the angles of an orthogonal parallelepiped element that is bounded by its six coordinate surfaces.

For well-known reasons, one assumes that the virtual work of the internal forces:

$$\Theta_1\, \delta\theta_1 + \Theta_2\, \delta\theta_2 + \Theta_3\, \delta\theta_3 + \Omega_1\, \delta\omega_1 + \Omega_2\, \delta\omega_2 + \Omega_3\, \delta\omega_3$$

(per unit volume) is an exact variation with respect to the quantity κ_i that defined the deformation that was arrived at before. When one substitutes the values of the variations $\delta\theta_i$, $\delta\omega_i$ that they get from formula (5) in the preceding expression, it will become:

$$\sum_{i=1}^{3}\left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_i}+\frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_i}+\frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_i}\right)\delta\kappa_i$$

$$+\ \Theta_1\delta\frac{\partial\kappa_1}{\partial q_1}+\frac{Q_1\,\Omega_3}{Q_2}\delta\frac{\partial\kappa_1}{\partial q_2}+\frac{Q_1\,\Omega_2}{Q_3}\delta\frac{\partial\kappa_1}{\partial q_3}$$

$$+\ \frac{Q_2\,\Omega_3}{Q_1}\delta\frac{\partial\kappa_2}{\partial q_1}+\Theta_2\,\delta\frac{\partial\kappa_2}{\partial q_2}+\frac{Q_2\,\Omega_1}{Q_3}\delta\frac{\partial\kappa_2}{\partial q_3}$$

$$+\ \frac{Q_3\,\Omega_2}{Q_1}\delta\frac{\partial\kappa_3}{\partial q_1}+\frac{Q_3\,\Omega_1}{Q_2}\delta\frac{\partial\kappa_3}{\partial q_2}+\Theta_3\,\delta\frac{\partial\kappa_3}{\partial q_3}.$$

It results from the form of this expression that there exists a function Π such that the expression is its exact variation, and that function can depend upon only the q_i , the κ_i , and the κ_{ij} , in which we have set:

$$\kappa_{ij} = \frac{\partial\kappa_i}{\partial q_j},$$

for brevity, and one that must properly have:

$$\left.\begin{aligned}
&\frac{\partial\Pi}{\partial\kappa_i}=\sum_{j=1}^{3}\frac{\Theta_i}{Q_j}\frac{\partial Q_j}{\partial q_i}, && \frac{\partial\Pi}{\partial\kappa_i}=\Theta_i && (i=1,2,3),\\
&\frac{\partial\Pi}{\partial\kappa_{12}}=\frac{Q_1\,\Omega_3}{Q_2}, && \frac{\partial\Pi}{\partial\kappa_{13}}=\frac{Q_1\,\Omega_2}{Q_3}, &&\\
&\frac{\partial\Pi}{\partial\kappa_{23}}=\frac{Q_2\,\Omega_1}{Q_3}, && \frac{\partial\Pi}{\partial\kappa_{21}}=\frac{Q_2\,\Omega_3}{Q_1}, &&\\
&\frac{\partial\Pi}{\partial\kappa_{31}}=\frac{Q_3\,\Omega_2}{Q_1}, && \frac{\partial\Pi}{\partial\kappa_{32}}=\frac{Q_3\,\Omega_1}{Q_2}. &&
\end{aligned}\right\}\qquad(6)$$

That implies the six relations:

$$\left.\begin{aligned}
\frac{Q_3}{Q_2}\frac{\partial\Pi}{\partial\kappa_{23}}&=\frac{Q_2}{Q_3}\frac{\partial\Pi}{\partial\kappa_{32}}(=\Omega_1),\\
\frac{Q_1}{Q_3}\frac{\partial\Pi}{\partial\kappa_{31}}&=\frac{Q_3}{Q_1}\frac{\partial\Pi}{\partial\kappa_{12}}(=\Omega_2),\\
\frac{Q_2}{Q_1}\frac{\partial\Pi}{\partial\kappa_{12}}&=\frac{Q_1}{Q_2}\frac{\partial\Pi}{\partial\kappa_{21}}(=\Omega_3),\\
\frac{\partial\Pi}{\partial\kappa_i}&=\sum_{j=1}^{3}\frac{1}{Q_j}\frac{\partial Q_j}{\partial q_i}\frac{\partial\Pi}{\partial\kappa_{ii}},\quad(i=1,2,3),
\end{aligned}\right\}\qquad(6)_a$$

which express the idea that the functions κ_1, κ_2, κ_3, and their first derivatives enter into Π only in the six combinations:

$$\theta_1,\ \theta_2,\ \theta_3,\ \omega_1,\ \omega_2,\ \omega_3,$$

and one will therefore have:

$$\delta\Pi = \frac{\partial\Pi}{\partial\theta_1}\delta\theta_1 + \frac{\partial\Pi}{\partial\theta_2}\delta\theta_2 + \frac{\partial\Pi}{\partial\theta_3}\delta\theta_3 + \frac{\partial\Pi}{\partial\omega_1}\delta\omega_1 + \frac{\partial\Pi}{\partial\omega_2}\delta\omega_2 + \frac{\partial\Pi}{\partial\omega_3}\delta\omega_3$$

or

$$\Theta_i = \frac{\partial\Pi}{\partial\theta_i}, \qquad \Omega_i = \frac{\partial\Pi}{\partial\omega_i} \qquad (i = 1, 2, 3). \tag{7}$$

That conclusion can be based upon the simple observation that the six quantities θ_i, ω_i that are defined by equations (5) are not coupled by any linear relation that is independent of the κ_i, κ_{ij}. However, the preceding deduction exhibits some relations that permit one to immediately give a new form to equations (4) and $(4)_a$. Indeed, by virtue of formulas (6), $(6)_a$, the aforementioned equations will become:

$$\begin{aligned} Q_i\,F_i &= \frac{1}{\nabla}\sum_{j=1}^{3}\frac{\partial\left(\nabla\dfrac{\partial\Pi}{\partial\kappa_{ij}}\right)}{\partial q_i} - \frac{\partial\Pi}{\partial\kappa_i}, \\ Q_i\,\varphi_i &= \sum_{j=1}^{3} Q_j\,\frac{\partial\Pi}{\partial\kappa_{ij}}\cos(n\,q_j), \end{aligned} \qquad (i = 1, 2, 3) \tag{8}$$

and it is in precisely that form that the general equations of elasticity were given by C. NEUMANN in the cited paper.

Properly speaking, the functions that NEUMANN introduced (as well as LAMÉ) are not the κ_i, but the $Q_i\,\kappa_i$; i.e., they are the components of the displacements: However, it is easy to see that if one sets:

$$k_i = Q_i\,\kappa_i,$$

and therefore:

$$k_{ij} = Q_i\,\kappa_{ij} + \frac{\partial Q_i}{\partial q_j}\kappa_i,$$

then when one considers Π to be a function of k_i and k_{ij}, one will have:

$$\begin{aligned} \frac{\partial\Pi}{\partial\kappa_i} &= \frac{\partial\Pi}{\partial k_i}Q_i + \sum_{j=1}^{3}\frac{\partial\Pi}{\partial k_{ij}}\frac{\partial Q_i}{\partial q_j}, \\ \frac{\partial\Pi}{\partial\kappa_{ij}} &= \frac{\partial\Pi}{\partial k_{ij}}Q_i, \end{aligned}$$

and by means of these relations, equations (8) will immediately reduce to the following ones:

$$\left.\begin{aligned} F_i &= \frac{1}{\nabla}\sum_{j=1}^{3}\frac{\partial\left(\nabla\dfrac{\partial\Pi}{\partial k_{ij}}\right)}{\partial q_j} - \frac{\partial\Pi}{\partial k_i}, \\ \varphi_i &= \sum_{j=1}^{3} Q_j\,\frac{\partial\Pi}{\partial k_{ij}}\cos(n\,q_j), \end{aligned}\right\} \tag{8$_a$}$$

which are those of NEUMANN.

We shall now address the problem of establishing the equations of elasticity for isotropic media, namely, the ones that will give Π the form:

$$\Pi = -\tfrac{1}{2}(A\,\vartheta^2 + B\,\varpi), \tag{9}$$

in which:

$$\vartheta = \theta_1 + \theta_2 + \theta_3 ,$$
$$\varpi = \omega_1^2 + \omega_2^2 + \omega_3^2 - 4\,(\theta_2\,\theta_3 + \theta_3\,\theta_1 + \theta_1\,\theta_2).$$

The constants A and B, which depend upon the nature of the medium, are the ones that GREEN used ("On the laws of reflexion and refraction of light, etc.," 1837). In the usual theory, the ratios of those two constants to the density of the medium represent the squares of the velocities of propagation of longitudinal and transversal waves, resp.

One should immediately note that the quantity ϑ – i.e., the cubic dilatation – has a very simple expression. Indeed, one easily deduces from the first three equations (5) that:

$$\vartheta = \frac{1}{\nabla}\left\{\frac{\partial(\nabla\kappa_1)}{\partial q_1} + \frac{\partial(\nabla\kappa_2)}{\partial q_2} + \frac{\partial(\nabla\kappa_3)}{\partial q_3}\right\}. \tag{10}$$

By virtue of (7), one deduces from equation (9) that:

$$\left.\begin{aligned} \Theta_1 &= -A\,\vartheta + 2B(\theta_1 + \theta_3), & \Omega_1 &= -B\omega_1, \\ \Theta_2 &= -A\,\vartheta + 2B(\theta_3 + \theta_1), & \Omega_2 &= -B\omega_2, \\ \Theta_3 &= -A\,\vartheta + 2B(\theta_1 + \theta_2), & \Omega_3 &= -B\omega_3, \end{aligned}\right\} \tag{11}$$

and those are the values that must be substituted in the right-hand side of equations (4), $(4)_a$.

Such substitutions offer no difficulty in regard to equations $(4)_a$.

In regard to equations (4), one must first of all separate the part that is multiplied by A from the one that is multiplied by B. As far as the first part is concerned, one will see immediately that the right-hand sides of equations (4) reduce to:

$$-A\left\{\frac{1}{\nabla}\frac{\partial(\nabla\vartheta)}{\partial q_i} - \frac{\vartheta}{\nabla}\frac{\partial\nabla}{\partial q_i}\right\},$$

namely, to:

$$-A\frac{\partial\vartheta}{\partial q_i} \qquad (i = 1, 2, 3). \tag{α}$$

As for the part that contains the factor B, it has the following expression in the right-hand side of the first equation in (4):

$$-\frac{B}{\nabla}\left\{-2\frac{\partial[\nabla(\vartheta_2+\vartheta_3)]}{\partial q_1}+\frac{\partial(Q_1^2\,Q_3\,\omega_3)}{\partial q_2}+\frac{\partial(Q_1^2\,Q_2\,\omega_2)}{\partial q_3}\right\}$$
$$-\;2B\left\{\frac{\vartheta_2+\vartheta_3}{Q_1}\frac{\partial Q_1}{\partial q_1}+\frac{\vartheta_3+\vartheta_1}{Q_2}\frac{\partial Q_2}{\partial q_1}+\frac{\vartheta_1+\vartheta_2}{Q_3}\frac{\partial Q_3}{\partial q_1}\right\},$$

namely, after some obvious reductions:

$$-\frac{B}{\nabla}\left\{Q_1\,\vartheta_1\frac{\partial(Q_2\,Q_3)}{\partial q_1}-Q_3\,Q_1\frac{\partial(Q_2\,\theta_2)}{\partial q_1}-Q_1\,Q_2\frac{\partial(Q_2\,\theta_3)}{\partial q_1}+\frac{1}{2}\frac{\partial(Q_1^2\,Q_3\,\omega_3)}{\partial q_2}+\frac{1}{2}\frac{\partial(Q_1^2\,Q_2\,\omega_2)}{\partial q_3}\right\}. \tag{β}$$

The direct substitution of the values (5) in that expression (β) will lead to an especially prolix calculation, as LAMÉ knew already (in the two cited places), and in order to avoid precisely that prolixity, he preferred to start from some opportunely-arranged Cartesian equations. However, that stopgap will not be admissible, from the previous observation in regard to the greater generality of equations (4) in comparison to the Cartesian ones. We therefore need to perform the indicated calculation, which can be abbreviated somewhat, above all, on the basis of a reasonable induction. Therefore, if one knows that in ordinary space the final equations of isotropy contain only the components of the elementary rotation in the terms that are multiplied by B then it would be natural to think that those components must also figure in the equations that relate to a more general space, since the concept of elementary rotation – with the definition of W. THOMSON – will persist in any space.

In my "Cinematica dei fluidi" (§ 11), I already gave the general values of the components of rotation in arbitrary curvilinear coordinates. With the present orthogonal coordinates q_1, q_2, q_3, those formulas will become:

$$\left.\begin{aligned}
\vartheta_1 &= \frac{1}{Q_2\,Q_3}\left\{\frac{\partial(Q_3^2\,\kappa_3)}{\partial q_2}-\frac{\partial(Q_2^2\,\kappa_2)}{\partial q_3}\right\},\\
\vartheta_2 &= \frac{1}{Q_3\,Q_1}\left\{\frac{\partial(Q_1^2\,\kappa_1)}{\partial q_3}-\frac{\partial(Q_3^2\,\kappa_3)}{\partial q_1}\right\},\\
\vartheta_3 &= \frac{1}{Q_1\,Q_2}\left\{\frac{\partial(Q_2^2\,\kappa_2)}{\partial q_1}-\frac{\partial(Q_1^2\,\kappa_1)}{\partial q_2}\right\},
\end{aligned}\right\} \tag{12}$$

in which ϑ_1, ϑ_2, ϑ_3 denote *twice the components of the elementary rotation* that accompanies the deformation of the system or elastic medium. These are just the expressions that figure in the transformed equations of LAMÉ, NEUMANN, and BORCHARDT. The presence of the products $Q_i^2\,\kappa_i$ in those formulas suggests that one should set:

$$Q_i^2\,\kappa_i = K_i\,,$$

and to write equations (5) in the following form:

$$Q_1\,\theta_1 = \frac{1}{Q_1}\frac{\partial K_1}{\partial q_1} - \frac{1}{Q_1^2}\frac{\partial Q_1}{\partial q_1}K_1 + \frac{1}{Q_2^2}\frac{\partial Q_1}{\partial q_2}K_2 + \frac{1}{Q_3^2}\frac{\partial Q_1}{\partial q_3}K_3\,,$$

$$Q_2\,\theta_2 = \frac{1}{Q_2}\frac{\partial K_2}{\partial q_2} + \frac{1}{Q_1^2}\frac{\partial Q_2}{\partial q_1}K_1 - \frac{1}{Q_2^2}\frac{\partial Q_2}{\partial q_2}K_2 + \frac{1}{Q_3^2}\frac{\partial Q_2}{\partial q_3}K_3\,,$$

$$Q_3\,\theta_3 = \frac{1}{Q_3}\frac{\partial K_3}{\partial q_2} - \frac{1}{Q_1^2}\frac{\partial Q_3}{\partial q_1}K_1 + \frac{1}{Q_2^2}\frac{\partial Q_3}{\partial q_2}K_2 - \frac{1}{Q_3^2}\frac{\partial Q_3}{\partial q_3}K_3\,,$$

$$Q_2\,Q_3\,\omega_1 = \frac{\partial K_2}{\partial q_3} + \frac{\partial K_3}{\partial q_2} - 2\left(\frac{1}{Q_2}\frac{\partial Q_2}{\partial q_3}K_2 + \frac{1}{Q_3}\frac{\partial Q_3}{\partial q_2}K_3\right),$$

$$Q_3\,Q_1\,\omega_2 = \frac{\partial K_3}{\partial q_1} + \frac{\partial K_1}{\partial q_3} - 2\left(\frac{1}{Q_3}\frac{\partial Q_3}{\partial q_1}K_3 + \frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}K_1\right),$$

$$Q_1\,Q_2\,\omega_3 = \frac{\partial K_1}{\partial q_2} + \frac{\partial K_2}{\partial q_1} - 2\left(\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_2}K_1 + \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_1}K_2\right).$$

The substitution of those values in the expression (β) can be accomplished quite easily if one keeps separate the terms that contain the partial derivatives of first and second order of the functions K_i from the ones that contain those functions. The former can be grouped into the expression:

$$-\frac{B\,Q_1}{Q_2\,Q_3}\left\{\frac{\partial(Q_2\,\vartheta_2)}{\partial q_3} - \frac{\partial(Q_3\,\vartheta_3)}{\partial q_2}\right\} \tag{γ}$$

with no great difficulty. The latter constitute a homogeneous linear function of the quantities κ_1, κ_2, κ_3 . The coefficients of those functions are somewhat complicated. However, with a little bit of attention, one can easily reduce them to a form whose symmetry makes immediately obvious the law that governs the composition of all three analogous linear functions that enter into equations (4). Namely, set:

$$
\left.\begin{aligned}
H &= \frac{\partial}{\partial q_1}\left\{\frac{1}{Q_1}\frac{\partial(Q_2\,Q_3)}{\partial q_1}\right\}+\frac{\partial}{\partial q_2}\left\{\frac{1}{Q_2}\frac{\partial(Q_3\,Q_1)}{\partial q_2}\right\}+\frac{\partial}{\partial q_3}\left\{\frac{1}{Q_3}\frac{\partial(Q_1\,Q_2)}{\partial q_3}\right\}\\
&\quad-\left\{\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_2}{\partial q_3}\right\},\\
H_{11} &= Q_1\left\{\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\right)\right\}+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1},\\
H_{22} &= Q_2\left\{\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\right)+\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right)\right\}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2},\\
H_{33} &= Q_3\left\{\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)\right\}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_2}{\partial q_3},\\
H_{23} &= H_{32} = \frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_3}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_2}-\frac{\partial^2 Q_1}{\partial q_2\,\partial q_3},\\
H_{31} &= H_{13} = \frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3}{\partial q_1}+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial q_3}-\frac{\partial^2 Q_2}{\partial q_3\,\partial q_1},\\
H_{12} &= H_{21} = \frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial q_2}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2}{\partial q_1}-\frac{\partial^2 Q_3}{\partial q_1\,\partial q_2},
\end{aligned}\right\}\tag{13}
$$

and if one takes into account the identity:

$$H_{11} + H_{22} + H_{33} = H \tag{13$_a$}$$

then one will find that the linear function of the κ_i that relates to the first of equations (4) can be put into the form:

$$-\frac{2B}{Q_2\,Q_3}\{(H_{11} - H)\,Q_1\,\kappa_1 + H_{12}\,Q_2\,\kappa_2 + H_{13}\,Q_3\,\kappa_3\},$$

namely:

$$-\frac{2B}{Q_2\,Q_3}\frac{\partial\Phi}{\partial(Q_1\,\kappa_1)}, \tag{δ}$$

in which one sets:

$$\Phi = \sum_{ij} H_{ij}\,Q_i\,Q_j\,\kappa_i\,\kappa_j - H\sum_i Q_i^2\,\kappa_i^2\,. \tag{14}$$

If one recalls the partial expressions (α), (γ), (δ) and forms the analogous expressions for the second and third of equations (4) then one will get the following indefinite equations for isotropic elastic media:

$$\left.\begin{aligned}
&\frac{A}{Q_1}\frac{\partial\vartheta}{\partial q_1}+\frac{B}{Q_2\,Q_3}\left\{\frac{\partial(Q_2\,\vartheta_2)}{\partial q_3}-\frac{\partial(Q_3\,\vartheta_3)}{\partial q_2}\right\}+\frac{B}{Q_1\,Q_2\,Q_3}\frac{\partial\Phi}{\partial(Q_1\,\kappa_1)}+F_1=0,\\
&\frac{A}{Q_2}\frac{\partial\vartheta}{\partial q_2}+\frac{B}{Q_3\,Q_1}\left\{\frac{\partial(Q_3\,\vartheta_3)}{\partial q_1}-\frac{\partial(Q_1\,\vartheta_1)}{\partial q_3}\right\}+\frac{B}{Q_1\,Q_2\,Q_3}\frac{\partial\Phi}{\partial(Q_2\,\kappa_2)}+F_2=0,\\
&\frac{A}{Q_3}\frac{\partial\vartheta}{\partial q_3}+\frac{B}{Q_1\,Q_2}\left\{\frac{\partial(Q_1\,\vartheta_1)}{\partial q_2}-\frac{\partial(Q_2\,\vartheta_2)}{\partial q_1}\right\}+\frac{B}{Q_1\,Q_2\,Q_3}\frac{\partial\Phi}{\partial(Q_3\,\kappa_3)}+F_3=0.
\end{aligned}\right\}\tag{15}$$

As for the boundary equations $(4)_a$, they will not give rise to any reductions that are worthy of note or differ from the ordinary ones, and that is why it does not seem necessary to transcribe them explicitly.

One deduces from the form of equations (15) that in order to form those equations by the method of variation of the potential, it is enough to take that potential in the form:

$$-\int\left\{\tfrac{1}{2}A\vartheta^2+\tfrac{1}{2}B(\vartheta_1^2+\vartheta_2^2+\vartheta_3^2)+\frac{B\,\Phi}{Q_1\,Q_2\,Q_3}\right\}dS, \tag*{$(15)_a$}$$

from which, one can conclude directly that the expression:

$$\frac{\Phi}{Q_1\,Q_2\,Q_3}$$

possesses the same invariant character as the expressions:

$$\vartheta \qquad \text{and} \qquad \vartheta_1^2+\vartheta_2^2+\vartheta_3^2 .$$

Comparing the preceding equations (15) with the ones that LAMÉ gave, which are generally assumed, one will recognize that the former ones will not agree with latter, unless the function Φ is *zero* independently of any hypothesis on the functions κ_i , which demands that one must have:

$$H_{11}=0, \qquad H_{22}=0, \qquad H_{33}=0, \qquad H_{23}=0, \qquad H_{31}=0, \qquad H_{12}=0,$$

because of the identity $(13)_a$. Now, those six equations are precisely the ones that the very same LAMÉ, in v. 5 of LIOUVILLE's Journal and later in the fifth of his *Leçons sur les coordonées curvilignes*, showed to be necessary for the expression (1) to be a transform of:

$$ds^2 = dx^2 + dy^2 + dz^2,$$

or, in other words, for the space in which the elastic medium considered exists to be Euclidian space. Therefore, the usual equations of isotropy are subordinate to the truth of EUCLID's postulate, while the general equations (4) are independent of it, as we have observed before.

That fact, which is the one that was alluded to at the beginning of the present article, explains the necessity of the various gimmicks that were adopted by the cited authors in order to deduce the equations of isotropy in the general equation when the form of the line element, from the indeterminacy in its coefficients, does not include the Euclidian hypothesis *a priori*. Hence, BORCHARDT, for example, could profit from the form that the integral $(15)_a$ takes when the coordinates are Cartesian ones to reduce the quantity under the integral to:

$$\tfrac{1}{2}A\vartheta^2 + \tfrac{1}{2}B(\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2).$$

If one abandons the Euclidian hypothesis then equations (15) will become the *equations of isotropy in a space of constant curvature.* It is called *constant* curvature because if the curvature of the space varied then it would not be legitimate to consider the coefficients A and B in the expression (9) to be constant quantities *a priori*. To that end, one can observe that if the quantity A varies with q_i then the part of Π that corresponds to the term $\tfrac{1}{2}A\vartheta^2$ in the right-hand side of equations (15) will be even simpler – i.e., it will be represented by:

$$\frac{1}{Q_i}\frac{\partial(A\vartheta)}{\partial q_i} \qquad (i = 1, 2, 3),$$

as is easy to verify. Things are different for the part that relates to the other term $\tfrac{1}{2}B\varpi$.

Now, the function Φ assumes a simple form in spaces of constant curvature.

Indeed, the line element of a space of constant curvature ($= \alpha$) can always be put into the form that RIEMANN indicated:

$$ds = \frac{\sqrt{dq_1^2 + dq_2^2 + dq_3^2}}{1 + \frac{\alpha}{4}(q_1^2 + q_2^2 + q_3^2)},$$

which promises to be quite useful, due to its symmetry. Set:

$$Q = \frac{1}{1 + \frac{\alpha}{4}(q_1^2 + q_2^2 + q_3^2)} \qquad (= Q_1 = Q_2 = Q_3),$$

so one finds (13) that:

$$H = -Q^3\,\alpha,$$

hence:

$$H_{11} = H_{22} = H_{33} = -Q^3\,\alpha,$$

and finally:

$$H_{11} = H_{22} = H_{33} = 0.$$

It will then result that when the coordinates q_i are those of RIEMANN – i.e., the ones that I called *stereographic* in my "Teoria fondamentale degli spazii di curvature costante," (v. 2 of these Annali) – one will have:

$$\frac{\Phi}{Q_1\,Q_2\,Q_3} = 2\alpha\,Q^2\,(\kappa_1^2 + \kappa_2^2 + \kappa_3^2)\,.$$

Now, the quantity $Q^2\,(\kappa_1^2 + \kappa_2^2 + \kappa_3^2)$ is the square of the displacement of the point (q_1, q_2, q_3), that is to say, the quantity that is represented by $Q_1^2\,\kappa_1^2 + Q_2^2\,\kappa_2^2 + Q_3^2\,\kappa_3^2$ with the general orthogonal coordinates that the expression (1) is referred to. Therefore, in any space of constant curvature α that is referred to orthogonal coordinates, one will have:

$$\frac{\Phi}{Q_1\,Q_2\,Q_3} = 2\alpha\,(Q_1^2\,\kappa_1^2 + Q_2^2\,\kappa_2^2 + Q_3^2\,\kappa_3^2)\,, \tag{16}$$

and consequently:

$$\left.\begin{array}{c} H = -3\alpha\,Q_1\,Q_2\,Q_3, \\ H_{11} = H_{22} = H_{33} = -\alpha\,Q_1\,Q_2\,Q_3, \\ H_{23} = H_{31} = H_{12} = 0. \end{array}\right\} \tag{16$_a$}$$

These last six formulas can be transformed, like an analogue of LAMÉ's, into just as many geometric relations between the curvatures of the orthogonal surfaces.

Indeed, if one lets $1 / r_{ij}$ denote the geodetic curvatures of the lines of intersection of the two surfaces q_i = const., q_j = const., when those lines are considered to exist on the *first* surface (so the geodetic curvatures of its lines, when they are considered to exist in the *second* surface, will, however, be denoted by $1 / r_{ji}$), then one will get the following relations:

$$\begin{array}{ll} \dfrac{\partial Q_1}{\partial q_2} = \dfrac{Q_1\,Q_2}{r_{22}}, & \dfrac{\partial Q_1}{\partial q_3} = \dfrac{Q_1\,Q_2}{r_{23}}, \\ \dfrac{\partial Q_2}{\partial q_3} = \dfrac{Q_2\,Q_3}{r_{13}}, & \dfrac{\partial Q_2}{\partial q_1} = \dfrac{Q_2\,Q_1}{r_{31}}, \\ \dfrac{\partial Q_3}{\partial q_1} = \dfrac{Q_3\,Q_2}{r_{21}}, & \dfrac{\partial Q_3}{\partial q_2} = \dfrac{Q_3\,Q_2}{r_{12}}, \end{array}$$

from known formulas.

One can eliminate all of the derivatives of the three functions Q_i from the last six of equations $(16)_a$ by means of these relations, and if one also sets:

$$Q_i\,dq_i = ds_i$$

then one can actually eliminate those Q_i . Operating in that way, one will find the three equations:

$$H_{11} = H_{22} = H_{33} = -\,\alpha\,Q_1\,Q_2\,Q_3\,,$$

which are equivalent to the following ones:

$$\left.\begin{aligned}
&\frac{\partial \frac{1}{r_{12}}}{\partial s_2}+\frac{\partial \frac{1}{r_{13}}}{\partial s_3}+\frac{1}{r_{12}^2}+\frac{1}{r_{13}^2}+\frac{1}{r_{21}\,r_{31}}+\alpha=0,\\
&\frac{\partial \frac{1}{r_{23}}}{\partial s_3}+\frac{\partial \frac{1}{r_{21}}}{\partial s_1}+\frac{1}{r_{23}^2}+\frac{1}{r_{21}^2}+\frac{1}{r_{32}\,r_{12}}+\alpha=0,\\
&\frac{\partial \frac{1}{r_{31}}}{\partial s_1}+\frac{\partial \frac{1}{r_{32}}}{\partial s_2}+\frac{1}{r_{31}^2}+\frac{1}{r_{33}^2}+\frac{1}{r_{12}\,r_{23}}+\alpha=0.
\end{aligned}\right\} \tag{16$_b$}$$

As for the other three equations:

$$H_{23} = H_{31} = H_{12} = 0,$$

which are identical to the three LAMÉ equations, they translate into the corresponding relations (*Coordonées curvilignes*, pp. 80) between the radii r_{ij} , except that they must naturally be considered to be radii of geodetic curvature and not radii of principal curvature. In addition, one should note that LAMÉ took the opposite sign for his curvature.

If one lets α_1, α_2, α_3 denote the measures of the curvature (according to GAUSS) of the three surfaces q_1 = const., q_2 = const., q_3 = const. at the point (q_1 , q_2 , q_3) and compares the preceding equations (16)$_b$ with the known equations of BONNET then one will get:

$$\left.\begin{aligned}
\alpha_1&=\frac{1}{r_{21}\,r_{31}}+\alpha,\\
\alpha_2&=\frac{1}{r_{32}\,r_{12}}+\alpha,\\
\alpha_3&=\frac{1}{r_{13}\,r_{23}}+\alpha.
\end{aligned}\right\} \tag{16$_c$}$$

When $\alpha = 0$ (i.e., when the space is Euclidian), the radii of geodetic curvature (r_{11}, r_{31}), (r_{32}, r_{12}), (r_{13}, r_{23}) will coincide with the radii of principal curvature of the three orthogonal surfaces q_1 = const., q_2 = const., q_3 = const., and the preceding values of α_1 , α_2 , α_3 will coincide with the ones that are given by GAUSS's theorem.

By virtue of the form (16) that was found for the function Φ, the indefinite equations of isotropy in a space of constant curvature α can be definitively put into the following form:

$$\left.\begin{aligned}
&\frac{A}{Q_1}\frac{\partial\vartheta}{\partial q_1}+\frac{B}{Q_1\,Q_3}\left\{\frac{\partial(Q_2\,\vartheta_2)}{\partial q_3}-\frac{\partial(Q_3\,\vartheta_3)}{\partial q_2}\right\}+4\alpha BQ_1\,\kappa_1+F_1=0,\\
&\frac{A}{Q_2}\frac{\partial\vartheta}{\partial q_2}+\frac{B}{Q_3\,Q_1}\left\{\frac{\partial(Q_3\,\vartheta_3)}{\partial q_1}-\frac{\partial(Q_1\,\vartheta_1)}{\partial q_3}\right\}+4\alpha BQ_2\,\kappa_2+F_2=0,\\
&\frac{A}{Q_3}\frac{\partial\vartheta}{\partial q_3}+\frac{B}{Q_1\,Q_2}\left\{\frac{\partial(Q_1\,\vartheta_1)}{\partial q_2}-\frac{\partial(Q_2\,\vartheta_2)}{\partial q_1}\right\}+4\alpha BQ_3\,\kappa_3+F_3=0.
\end{aligned}\right\}\tag{17}$$

One can predict *a priori* that the curvature of space must not be devoid of influence on the equations of elasticity. However, it is undoubtedly quite noteworthy that its influence would manifest itself in a very simple way.

Despite that simplicity, the theory of elastic media in spaces of constant curvature presents relevant differences in comparison to the ordinary one, and it seems to me that the consequences to which they would lead would merit a rigorous study.

For now, I shall confine myself to giving a summary of some results that relate to the case in which the elastic deformation has a rotational sense.

The three quantities ϑ_i that are defined by equations (12) are zero in that case, so one can set:

$$\kappa_i = \frac{1}{Q_i^2}\frac{\partial U}{\partial q_i}, \tag{18}$$

and therefore (10):

$$\vartheta = \Delta_2\,U, \tag*{(18)$_a$}$$

in which:

$$\Delta_2\,U = \frac{1}{Q_1\,Q_2\,Q_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{Q_2\,Q_3}{Q_1}\frac{\partial U}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{Q_3\,Q_1}{Q_2}\frac{\partial U}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{Q_1\,Q_2}{Q_3}\frac{\partial U}{\partial q_3}\right)\right\}. \tag*{(18)$_b$}$$

In that way, equations (17) will become:

$$\frac{\partial}{\partial q_i}\{A\,\Delta_2\,U + 4\alpha BU\} + Q_i\,F_i = 0 \qquad (i = 1, 2, 3),$$

and will show that the force F must have a potential V; i.e., that one must have:

$$F_i = \frac{1}{Q_i}\frac{\partial V}{\partial q_i}. \tag*{(18)$_c$}$$

With that, the three aforementioned equations will be equivalent to the single one:

$$A\,\Delta_2\,U + 4\alpha BU + V = 0, \tag{19}$$

in which it is intended that U subsumes the quantity that was introduced by the integration, which is independent of q_1, q_2, q_3.

If one supposes that $\vartheta = 0$ (i.e., that $\Delta_2 U = 0$) then one will have:

$$V = -4\alpha BU, \qquad \Delta_2 V = 0, \tag{19$_a$}$$

and one will then get a deformation that is devoid of either rotation or dilatation, in which the force and the displacement have the same (or opposite) direction and magnitudes that are constantly proportional to each other. That result, which is not found in Euclidian space, presents a singular analogy with certain modern concepts about the action of dielectric media (Maxwell, *Treatise on Electricity and Magnetism*, v. 1, pp. 63). If one assumes the equivalence of the directions for the force and displacement then one will need to suppose that the curvature of space is negative.

In order to better fix the idea, one should consider a particular form of the line element in a space of constant curvature α; namely, one should set:

$$ds^2 = d\xi^2 + \frac{1}{\alpha}\sin^2(\xi\sqrt{\alpha})\,(d\eta^2 + \sin^2\eta\, d\zeta^2), \tag{20}$$

in which ξ is the radius vector that leads from a fixed center to an arbitrary point of space, and η, ζ are two angles that determine the direction of that radius. The quantities ξ, η, ζ are the *spherical* coordinates of the space of constant curvature. With those coordinates, one has:

$$\Delta_2 U = \frac{\alpha}{\sin^2(\xi\sqrt{\alpha})}\left\{\frac{1}{\alpha}\frac{\partial}{\partial\xi}\left(\sin^2(\xi\sqrt{\alpha})\frac{\partial U}{\partial\xi}\right) + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial U}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2 U}{\partial\xi^2}\right\}, \tag{20$_a$}$$

and if the equation $\Delta_2 U = 0$ is satisfied then one can take:

$$U = \mu \cot\,(\xi\sqrt{\alpha}), \tag{21}$$

in which μ is a constant. That solution corresponds to the usual elementary Newtonian potential.

If one continues to let $\kappa_1, \kappa_2, \kappa_3$ denote the increments in the three variables ξ, η, ζ that are due to the elastic deformation then one will have:

$$\kappa_1 = \frac{dU}{d\xi} = -\frac{\mu\sqrt{\alpha}}{\sin^2(\xi\sqrt{\alpha})}, \qquad \kappa_2 = 0, \qquad \kappa_3 = 0$$

under the hypothesis, and therefore (5):

$$\theta_1 = \frac{\partial^2 U}{\partial\xi^2}, \qquad \theta_2 = \theta_3 = -\frac{1}{2}\frac{d^2 U}{d\xi^2},$$

$$\omega_1 = \omega_2 = \omega_3 = 0.$$

The internal tensions of the medium are therefore determined (11) by the components:

$$\left.\begin{array}{c} \Theta_1 = -2B\dfrac{d^2U}{d\xi^2}, \quad \Theta_2 = \Theta_3 = B\dfrac{d^2U}{d\xi^2}, \\ \Omega_1 = \Omega_2 = \Omega_3 = 0. \end{array}\right\} \tag{21$_a$}$$

That is to say, they are represented by a force that acts like of pressure or a traction in the direction of the line of force, and will give a force that acts in the opposite sense – i.e., as a traction or pressure, respectively, in the directions that are perpendicular to that line.

That result is also in harmony with the known concepts of FARADAY. In truth, MAXWELL mathematically developed that concept (*loc. cit.*, v. 1, pp. 128) by supposing that the absolute values of the pressure in the sense of the line of force and the traction in the normal sense were equal. However, more recently, HELMHOLTZ, in a new theory of dielectrics (Monatsberichte der Berliner Akademie, February, 1881) was already led to assume the possibility of a ratio that was different from unity by other considerations.

Another simple solution of the equation $\Delta_2 U = 0$, when considered in the form $(20)_a$, is given by:

$$U = \mu\,\zeta, \tag{22}$$

in which μ is a constant. That solution corresponds to, and is also equal to, the usual electromagnetic potential of a rectilinear current that traverses the polar axis $\eta = 0$. However, for the calculation of the internal tensions that are verified in that case, it is better to take another form of the line element, and in fact, the following one:

$$ds^2 = du^2 + \cos^2(u\sqrt{\alpha})\,dz^2 + \frac{1}{\alpha}\sin^2(u\sqrt{\alpha})\,d\zeta^2,$$

in which u is the distance from an arbitrary point of space to a fixed axis, z is the distance from the foot of that perpendicular to a fixed point on that axis, and ζ is the angle that the plane through the fixed axis and the arbitrary point makes with a fixed plane. Those quantities u, z, ζ are the *cylindrical* coordinates of the space of constant curvature.

By means of those coordinates, one will find (while supposing that the current flows along the fixed axis $u = 0$):

$$\kappa_1 = 0, \qquad \kappa_2 = 0, \qquad \kappa_3 = \frac{\mu\,\alpha}{\sin^2(u\sqrt{\alpha})},$$

and therefore one gets from equations (5) that:

$$\theta_1 = \theta_2 = \theta_3 = 0,$$

$$\omega_1 = 0, \qquad \omega_2 = -\frac{2\mu\alpha\cos(u\sqrt{\alpha})}{\sin^2(u\sqrt{\alpha})}, \qquad \omega_3 = 0.$$

The internal tensions of the medium are then determined (11) by the components:

$$\left.\begin{array}{c} \Theta_1 = \Theta_2 = \Theta_3 = 0, \\ \Omega_1 = 0, \quad \Omega_2 = \dfrac{2B\mu\alpha\cos(u\sqrt{\alpha})}{\sin^2(u\sqrt{\alpha})}, \quad \Omega_3 = 0. \end{array}\right\} \tag{22$_a$}$$

That is to say, they are represented uniquely by an internal force of torsion around the line u = const., ζ = const., namely, around the line that is in the same plane with the line that the current flows through and that has its points equidistant from it.

If one would like to consider the vibratory motion of the elastic medium in the absence of external accelerating forces, while maintaining the special hypothesis (18), then one would need to assume that the function U depends upon time t, along with the coordinates q_i , and set:

$$F_i = -\rho\, Q_i\, \frac{\partial^2 \alpha_i}{\partial t^2},$$

namely, from (18):

$$F_i = \frac{1}{Q_i}\frac{\partial}{\partial q_i}\left(-\rho\frac{\partial^2 U}{\partial t^2}\right),$$

in which ρ is the density. When the last relation is compared with $(18)_c$, that will give:

$$V = -\rho\frac{\partial^2 U}{\partial t^2},$$

and therefore the general equation of vibratory motion that one gets from (19) will be:

$$\rho\frac{\partial^2 U}{\partial t^2} = A\,\Delta_2 U + 4\alpha BU. \tag{23}$$

If one sets:

$$U = \Psi\cos\left(\frac{2\pi t}{\tau} + \mu\right), \tag{24}$$

in order to consider a simple stationary vibration, in which Ψ is a function of just the coordinates, and τ, μ are two constants, the first of which represents the period of a complete vibration, and the second of which represents the phase. If one substitutes that value of U in equation (23) then one will get:

$$A\,\Delta_2\,\Psi + 4\left(\frac{\pi^2\rho}{\tau^2}+\alpha B\right)\Psi = 0. \tag{24$_a$}$$

When the curvature α is zero (viz., *Euclidian* space) or positive (viz., RIEMANN or *spherical* space), there will be no admissible value of τ that annuls the coefficient of Ψ. However, when the curvature α is negative (viz., GAUSSian or *pseudo-spherical* space), i.e., when one has:

$$\alpha = -\frac{1}{R^2},$$

in which R is the radius of constant curvature, if one takes:

$$\tau = \pi R\sqrt{\frac{\rho}{B}} \tag{24$_b$}$$

then the coefficient of Ψ will be zero, and one will get a singular class of vibrations that are defined by:

$$U = \Psi\cos\left(\frac{2t}{R}\sqrt{\frac{B}{\rho}}+\mu\right), \tag{24$_d$}$$

and for which the function Ψ of the three coordinates q_i will satisfy the equation:

$$\Delta_2\Psi = 0. \tag{24$_e$}$$

Those vibrations, which are devoid of rotations and dilatations at one particular time, and which will not be found, as such, in ordinary space (except for the so-called incompressible fluids), will take place everywhere in the direction of the force that is due to the potential and will have amplitudes that are proportional to that force. Such vibratory motions give birth to internal tensions in the vibrating medium that can be calculated with formulas (5) and (11), as in the case of equilibrium, and will contain all of the periodic factors. If one takes Ψ to have the values (21), (22), for example, which satisfy equation $(24)_d$ then one will once more find the tensions $(21)_a$, $(22)_a$, multiplied by that factor.

If one supposes that the U in equation (23) depends upon only ξ and t [in which ξ has the same meaning as in equation (20)] then one will get the differential equation of spherical waves in the form:

$$\rho\frac{\partial^2 U}{\partial t^2} = \frac{A}{\sin^2(\xi\sqrt{\alpha})}\frac{\partial}{\partial\xi}\left\{\sin^2(\xi\sqrt{\alpha})\frac{\partial U}{\partial\xi}\right\} + 4\alpha B\,U. \tag{25}$$

One can satisfy that equation by taking:

$$U = \frac{E\cos(g\xi + ht + k)}{\sin(\xi\sqrt{\alpha})}, \tag{25$_a$}$$

in which g, h, k, E are four constants, the first two of which are coupled by the relation:

$$h^2 = \frac{A}{\rho}g^2 - \frac{A+4B}{\rho}\alpha. \tag{25$_b$}$$

One obtains progressive spherical waves in that way, whose velocity of propagation:

$$a = \pm\frac{h}{g}$$

and wave length:

$$\lambda = \pm\frac{2\pi}{g}$$

are coupled by the relation:

$$a^2 = \frac{A}{\rho} - \frac{A+4B}{\rho}\frac{\alpha\lambda^2}{4\pi^2}. \tag{25$_c$}$$

If one supposes that $g^2 = \alpha$ then one will be back to the case that was considered just now.

That result, which was discussed with a haste for which I must apologize to the reader, seems to me to be the one that justifies all of the attention that one should give to the new equations (17).

Pavia, 5 June 1881.