"Sull'equilibrio delle superficie flessibili ed inestendibili," Mem. Accad. scienze Istituto Lombardo (4) **3** (1881), 217-265.

On the equilibrium of flexible, inextensible surfaces

Report

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The recent paper by LECORNU "Sur l'équilibre des surfaces flexibles et inextensibles," in volume XLVIII of the *Journal de l'École polytechnique* has, quite opportunely, attracted the attention of the mathematicians to an argument that was never studied to the degree that it deserved and that can be considered to have been forgotten for some time.

LECORNU's assertion that the argument was not made by anyone before him is correct, except for the fact that it referred to the method that was due to him, and above all, to the intimate link that he justifiably recognized between the mechanical question that he treated and the geometric theory of the deformations of surfaces. That viewpoint constitutes the principal benefit of his lengthy work, and it suggests a geometric study. However, the purely mechanical question of the equilibrium of surface, in relation to which LECORNU can be credited for having established the exact differential equations for the first time, I believe, has quite a number of precedents, even if its story is not as true as that of some other questions that are much less interesting and less intricate.

Even if one would, in fact, like to pass over to Giovanni BERNOULLI's *sail* problem (i.e., to the search for the cylindrical surface that is formed by a sail that is inflated by the wind), which is a problem that, in substance, enters the theory of funicular curves, one cannot doubt that LAGRANGE, in his *Meccanica analitica* (Part I, Sec. V, Chap. III, § II) and POISSON, in a paper in 1814 on elastic surfaces (*), have sought to erect a general theory that includes, quite obviously, the case of flexible and inextensible surfaces. Indeed, CISA DE GRESY, in his "Considérations sur l'équilibre des surfaces flexibles et inextensibles" [Memorie della R. Accademia di Torino (1) **23** (1817)], has done nothing more than to reconsider and discuss the hypotheses and formulas of those celebrated authors. Moreover, although they have not properly deduced the true equations of the problem, they have always pointed clearly to the path that one should take, which is a path that would later be made much easier by the use of curvilinear

(*) That paper is included in the volume that contains the memoirs of the French institute from the year 1812, part 2, pp. 167. The first paragraph of that paper has the title: "Équation d'équilibre de la surface flexible et non élastique" (pp. 173-192).

coordinates. However, without insisting upon these more dated papers or citing other more recent ones that are more or less related to the argument that they address, I shall confine myself to recalling one of the well-known *Lezioni di meccanica razionale* by MOSOTTI (Florence, 1851) that is dedicated entirely to the equilibrium of flexible and inextensible surfaces and offers a fairly broad treatment that is accompanied by several examples.

However, MOSSOTTI has made an error (*) that does not invalidate the applications of what he did, but only detracts from the generality of his equations of equilibrium and makes them less adaptable to other applications that one might wish to make that do not present the accidental peculiarity of the ones that he treated.

In order to better clarify the origin and nature of that error, one should go back to the step that was cited before in LAGRANGE's *Meccanica analitica*. If one observes the process of calculation that he adopted and interprets the formulas that he found from the standpoint of flexible and inextensible surfaces then one will see immediately that the *inextensibility* that he refers to should not be interpreted (as would seem natural) in the sense of the *invariability of the line element,* but in the sense of the *invariability of the surface element:* In other words, one needs to associate the surface with an incompressible liquid film of constant and invariable infinitesimal thickness. Under that hypothesis, the surface tension will always be exerted normally to the line element and will be the same in all directions around that point. However, when one reconsiders the study of the question by POISSON, before all others, one will see that this equivalence of the tensions around a point would be a too-restrictive hypothesis, and one would prefer to assume that two line elements that emanate from the same point and are or are not mutually-perpendicular can be subject to tensions that are directed normally to each of them but have different values for each of them. Now, in fact, for any point on the surface, there are two *orthogonal* elements that are subject to only normal tensions that are generally unequal. However, assuming that *any* line element that emanates from a point is subject to only normal tensions will necessarily lead one back to LAGRANGE's hypotheses and will be in contradiction with the other hypothesis that those normal tensions possess values that are different for each element. In particular, the hypothesis that *two oblique* line elements are subject to *normal, unequal* tensions is absolutely contradictory. At any rate, equations that are based upon considerations of unequal normal tensions that act upon pairs of normal elements will be applicable to only those cases in which the special nature of the problem permits one to predict *a priori* what the (orthogonal) lines on the surface are that can be composed of a succession of line elements that are subject to only normal tensions.

Now, MOSSOTTI, who initially supposed that the directions of the tensions were completely unknown, later excluded (by means of an illusory consideration) the possibility that they had a tangential component, which however allowed the difference between the normal tensions on the two systems of orthogonal coordinate lines to persist, and took advantage of the resulting freedom to choose those lines arbitrarily by assuming that one of the two coordinate systems is a system of geodetic lines. It will then follow that his equations of equilibrium are not even applicable in *all* cases in which the lines of

(*) That error is, in part, common to some previous works by BORDONI and CODAZZA that had another theory as their objective (namely, the equilibrium of vaults), but was based, in substance, on the same considerations and had the same differential equations as its focus.

normal tensor are known *a priori*, but also demand that the lines of one of the two systems are geodetics. Those conditions are verified in all applications that he made.

Considering that the work of MOSSOTTI is consulted and justly appreciated by those Italians that deal with the doctrine of rational mechanics, and that on the other hand, LECORNU has passed over the strictly mechanical part of the question in order to give preference to the geometric part, I managed to do some useful work while summarizing *ab initio* the problem of the equilibrium of flexible and inextensible surfaces by establishing all of the fundamental equations with the method that seems simplest to me, as well as the most direct, and above all, the most general, in the sense that it excludes any preconception about the distribution of surface tensions. That method is nothing but that of LAGRANGE, combined with the true analytic definition of *inextensibility*.

Therefore, after having clarified in § **1** with some very simple (and some might say, intuitive) considerations in regard to the imperfections that are inherent to the process that MOSSOTTI employed and other similar ones, I will establish in § **2** the general principle of equilibrium, and from that single principle, I will deduce in §§ **3**, **4**, and **5** the indefinite equations and the boundary equations in completely general curvilinear coordinates. In §§ **6** and **7**, I will arrive at the theory of surface tensions from those equations, which plainly conforms to what LECORNU established *a priori* by taking advantage of geometric considerations. The following sections §§ **8**, **9**, and **10** contain an exposition of some equilibrium cases that are noteworthy for their simplicity and generality, and one of them was mentioned by POISSON, while the other ones do not seem to have been looked at, up to now. In § **11**, I shall point out the conditions under which one can arrive at the general equations that were given by the other authors. Finally, in § **12**, I shall collect some observations in regard to the formulas that relate to infinitely-small deformations of a flexible, inextensible surface.

I was inclined to add the deduction of the equations of *motion* of those surfaces, which are equations that one can put into a form that is analogous to the form of equations (III) in § **4**. However, the necessity of considering many other differential equations in addition to them rendered the problem of integration so complex that it seemed almost impossible for me to be able to arrive at any useful results. I therefore believed it best to leave aside that argument, which can be left to more capable hands, and that might, in particular, give rise to interesting and relatively less arduous research in the case of infinitely-small motions around a figure of equilibrium.

§ 1. Preliminary considerations

Suppose that we have a homogeneous planar rectangle that is subjected to tensions that are distributed uniformly on its opposite sides, and appropriately let P be the absolute value of the tension on a unit of length for two of its opposite sides, while Q is the analogous quantity for the other two. It is obvious that under those conditions, the rectangle will be in equilibrium and that the unit tension P will be transmitted to any line element that is parallel to the first two sides, just as the tension Q will be transmitted to any element that is parallel to the other two sides.

Having said that, let a and b be two arbitrary points that are taken from that rectangle, and let R be the unit tension that prevails on any line element of the line ab. Draw the line *ac* through the point *a* that is parallel to the sides with the tension *P* and draw the line *bc* through the point *b* that is parallel to the sides with tension *Q*, so one will get a right triangle *abc* that one supposes to be rigid and which must be in equilibrium under the action of the forces *P.ac*, *Q.cb*, *R.ab* that are distributed uniformly on its three sides *ac*, *cb*, *ab* with the first two in directions that are normal to the respective sides, while the third one has an unknown direction, and all of them point from the inside of the triangle to the outside. Those forces can be regarded as having been applied to the midpoints of the respective sides, and therefore their directions will concur at the midpoint of the hypotenuse *ab*. Hence, if one draws a line $a'c' = P.ac$ through an arbitrary point a' in the plane in the direction *bc* and one then draws a second line $c'b' = Q.bc$ to the endpoint c' of that line in the direction *ac* then it will be clear that the connecting line $b'a'$ will represent the third force *R.ab* in magnitude and direction. The unknown tension *R* will then be determined completely with that.

Now, if that tension is also normal to *ab* then the triangle of forces $a'b'c'$ will have its sides $a'b'$, $b'c'$, $c'a'$ perpendicular to the sides *ab*, *bc*, *ca*, respectively, of the triangle *abc*, so it will be similar to that triangle, and one will have:

$$P.ac : Q.bc : R.ab = ac : bc : ab, \qquad \text{namely,} \qquad P = Q = R.$$

Hence, the arbitrary line *ab* (and in general any line element that is oblique to the sides of the rectangle) cannot be subjected to *normal* tension unless $P = Q$, and when that happens, the normal tension *R* of the line *ab* cannot differ in magnitude from the one that is common to all sides of the rectangle. That will therefore contradict the supposition that two *arbitrarily-chosen* orthogonal line elements can be subject to *unequal* normal tensions. The tension is generally oblique to the line element upon which it is exerted (*).

However, let us pursue the geometric considerations that led us to that conclusion. If we transport the triangle $a'b'c'$ parallel to itself within the equilibrated rectangle that we assume it to belong to, while operating on it just as we do on the original triangle *abc*, then we propose to determine the tension *R* that prevails on its hypotenuse $a'b'$. In order to construct the new force triangle, one draws the line $b''c'' = P.b'c' = PQ.bc$ through an arbitrary point b'' in the plane in the direction $a'c'$ (viz., *bc*), and then draw the line $c''a'' = Q.a'c' = PQ.ac$ through the point c'' in the direction *bc* (viz., *ca*). The line $a''b''$ that joins them will represent the unknown force $R'.a'b'$ (viz., $RR'.ab$) in magnitude and direction. The new right triangle thus-formed is homothetic to the original one *abc*, because its catheti $b''c''$ and $c''a''$ are proportional to, parallel to, and with the same sense as the catheti *bc*, *ca*, respectively, of the original triangle. The hypotenuse $a''b'' = RR'.ab$ will be parallel to *ab* and will have the same ratio of the catheti with that line, so one will have $RR' = PQ$.

One concludes from this that the tension R' on a line $a'b'$ that is parallel to *R* (i.e., parallel to the tension that prevails on an arbitrary line *ab*) has the same direction as *ab*, and that the product of the unit values of the two *conjugate* tensions *R*, R' will be constant, and therefore necessarily equal to that of the *principal* tensions *P*, *Q*.

(*) See the footnote by BERTRAND on pp. 140 of LAGRANGE's *Meccanica analitica*, 1853 edition, volume I.

There is thus an infinitude of pairs of lines like *ab* and *a′b′* such that the tension on one of them is directed along the other one. However, the directions of those conjugate lines are related to each other in such a way that if one is given one of them then the other one will be determined absolutely.

It then follows that if one imagines an *arbitrary* parallelogram inside of the usual equilibrated rectangle then it would not be legitimate to demand that the tension on one of the pairs of opposite sides acts parallel to the other side. If that is the case then one can decompose that tension into two of them, one of which is directed along the other pair of sides, while the other one is directed along that pair whose tension it refers to, but the second component will be non-zero unless the pairs of sides have conjugate directions and to suppose that it is zero *in any case* will imply a contradiction, at least, as long as the parallelogram is not a rectangle and the tensions are not supposed to be equal on *all* sides of the rectangle and equal to those of *any* other analogous rectangle.

On the other hand, the equilibrium that exists for the total rectangle will necessarily imply the equilibrium of any parallelogram that is partially inside of it, and that equilibrium cannot therefore be at least confirmed (as MOSSOTTI believed) by the existence of (equal and contrary) pairs that are due to the tangential components of the tensions along it.

One easily sees that the considerations that we carried out for a planar rectangle of finite dimensions will be valid for the infinitely-small element of any equilibrated surface, and it was precisely by reasoning with that element that LECORNU established the formulas for the tensions. However, it seems more natural and more consistent with the spirit of analytical mechanics to avoid any preconceptions about the way by which those tensions are generated and distributed, and to deduce the theory from the interpretation of those equations of equilibrium that are established directly on the basis of the concept of inextensibility of any line element on the surface.

That is what we shall proceed to do in the successive §§.

§ 2. General principle of equilibrium

Refer the surface to an arbitrary system of curvilinear coordinates *u* and *v*, and if one considers the Cartesian coordinates *x*, *y*, *z* of its points to be functions of those two independent variables, then one takes:

$$(1)\qquad \begin{cases} E = \left(\dfrac{\partial x}{\partial u}\right)^2 + \left(\dfrac{\partial y}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial u}\right)^2, \\[2ex] F = \dfrac{\partial x}{\partial u}\dfrac{\partial x}{\partial v} + \dfrac{\partial y}{\partial u}\dfrac{\partial y}{\partial v} + \dfrac{\partial z}{\partial u}\dfrac{\partial z}{\partial v}, \\[2ex] G = \left(\dfrac{\partial x}{\partial v}\right)^2 + \left(\dfrac{\partial y}{\partial v}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2, \end{cases}$$

as usual, such that if one lets ds denote the line element that emanates from the point (u, v) and corresponds to the increments du, dv then one will have:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{1$_a$}$$

If one assumes that the lines u, v (*) are real then the two quantities E, G will be greater than zero, and if it becomes necessary to consider square roots, it shall always be intended that $\sqrt{E}$ and $\sqrt{G}$ denote the absolute value or the positive root. Hence, if one takes:

$$H = \sqrt{EG - F^2}\,,$$

for brevity, then H will be meant to denote the absolute value of the indicated radical. The expression $EG - F^2$ inside the radical is also always greater than zero, as long as the lines u and v always intersect at an angle that is different from 0 and 180°, as one supposes. In particular, assume that these conditions are verified at any point of the piece σ of the surface whose equilibrium must be considered. The area of an element $d\sigma$ of that surface that is included between the lines v = const., u = const., $v + dv$ = const., $u + du$ = const. will be given by:

$$d\sigma = H\,du\,dv.$$

Let α, β, γ denote the cosines of the angles that the normal w to the surface s makes with the three axes of x, y, z, resp. That normal is intended to be directed in such a way that the rotation of the line u towards the line v will take place around it in the same sense as the rotation of the positive x-axis to the positive y-axis, when the angle traverses is $< \pi$, and will occur around the positive z-axis when the angle traversed is a right angle. With that convention, one will have, as is known:

$$H\alpha = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}, \qquad H\beta = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}, \qquad H\gamma = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Having said that, let:

$$X\,d\sigma, \quad Y\,d\sigma, \quad Z\,d\sigma$$

be the components along the three axes of the external force that acts upon the surface element $d\sigma$, and let:

$$X_s\,ds, \quad Y_s\,ds, \quad Z_s\,ds$$

be the analogous components of the external force that acts upon the line element ds of the contour s of σ.

If the surface σ, which is assumed to already be equilibrated, submits to an infinitely-small virtual deformation, by virtue of which, any one if its points (x, y, z) will pass to the

(*) By saying "the line u," we intend to refer to a line along which only u will vary (and therefore v will remain constant), and accordingly regard that line as being traversed in the sense of increasing u. An analogous situation will be true for "the line v."

position ($x + \delta x$, $y + \delta y$, $z + \delta z$), then that force will do a virtual work that is represented by:

$$\int (X\,\delta x + Y\,\delta y + Z\,\delta z)\,d\sigma + \int (X_s\,\delta x + Y_s\,\delta y + Z_s\,\delta z)\,ds\,.$$

The variations δx, δy, δz are continuous, finite, monodromic functions of the variables u, v. In order for the surface to be inextensible, those variations must satisfy the three conditions:

(2) $$\delta E = 0, \qquad \delta F = 0, \qquad \delta G = 0,$$

in which:

$(2)_a$ $$\begin{cases} \tfrac{1}{2}\delta E = \sum \dfrac{\partial x}{\partial u}\dfrac{\partial \delta x}{\partial u}, \\ \tfrac{1}{2}\delta F = \sum \left(\dfrac{\partial x}{\partial u}\dfrac{\partial \delta x}{\partial v} + \dfrac{\partial x}{\partial u}\dfrac{\partial \delta x}{\partial v}\right), \\ \tfrac{1}{2}\delta G = \sum \dfrac{\partial x}{\partial v}\dfrac{\partial \delta x}{\partial v}, \end{cases}$$

By virtue of LAGRANGE's principle, the general equation of equilibrium will then be the following one:

(I) $$\int (X\,\delta x + Y\,\delta y + Z\,\delta z)\,d\sigma + \int (X_s\,\delta x + Y_s\,\delta y + Z_s\,\delta z)\,ds$$
$$\tfrac{1}{2}\int (\lambda\,\delta E + 2\mu\,\delta F + \nu\,\delta G)\frac{d\sigma}{H} = 0,$$

in which λ, μ, ν are three multipliers that are functions of u and v. (The divisor $2H$ is introduced in the last integral for the sake of later calculations.)

Finally, observe that if one assumes, with LAGRANGE, only the invariability of the surface element – i.e. if one sets the single condition:

(3) $$\delta H = 0,$$

in place of the three conditions (2) – then the last integral in equation (I) would contain just one multiplier κ; and it would have the form:

$$\int \kappa\,\delta H\,\frac{d\sigma}{H}, \qquad \text{namely,} \qquad \tfrac{1}{2}\int \frac{\kappa(G\,\delta E - 2F\,\delta F + E\,\delta G)}{H}\frac{d\sigma}{H}\,.$$

Therefore, assuming only the invariability of the surface element is equivalent to setting:

$$\lambda = \frac{\kappa G}{H}, \qquad \mu = -\frac{\kappa F}{H}, \qquad \nu = \frac{\kappa E}{H}$$

in the general formula (I), and therefore in all of the equations that one deduces from it, as well, or more simply:

$(3)_a$ $$\lambda : \mu : \nu = G : -F : E.$$

Conversely: If the multipliers λ, μ, ν prove to be proportional to G, $-F$, E, resp., in a given case of equilibrium then one can conclude with no further discussion that the equilibrium will also persist when the surfaces loses its linear inextensibility, while keeping its surface inextensibility.

§ 3. Deduction of the equations of equilibrium

In order to deduce the equations of equilibrium, properly speaking, from formula (I), one needs to duly transform the last of the three integrals that are contained in the left-hand side of that formula.

To that end, for the sake of brevity, consider only the part of that integral that contains the variation δx and that can be written as:

$$\int\left\{\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)\frac{\partial\delta x}{\partial u}+\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\frac{\partial\delta x}{\partial v}\right\}\frac{d\sigma}{H},$$

by virtue of formulas $(2)_a$. That expression can be transformed into the following one:

$$\int\left\{\frac{\partial}{\partial u}\left[\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)\delta x\right]+\frac{\partial}{\partial v}\left[\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\delta x\right]\right\}\frac{d\sigma}{H}$$
$$-\int\left\{\frac{\partial}{\partial u}\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\right\}\delta x\frac{d\sigma}{H}.$$

Now, for any function $\phi(u, v)$, as long as it is continuous, finite, and monodromic, one will have (*):

$$\int\frac{\partial\phi}{\partial u}\frac{\partial\sigma}{H}=-\int\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)\frac{\phi}{H}ds,$$

$$\int\frac{\partial\phi}{\partial v}\frac{\partial\sigma}{H}=-\int\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)\frac{\phi}{H}ds,$$

in which n is the direction of the line element of σ that is normal to the contour s and directed towards the interior of the region σ. Therefore, the preceding expression can be converted into this one:

$$-\int\left\{\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)\right\}\frac{\delta x}{H}ds$$

(*) See art. V in my paper "Delle variabili complesse sopra una surpeficie qualunque," Annali di Matematica, new series, vol. I.

$$-\int\left\{\frac{\partial}{\partial u}\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\right\}\frac{\delta x}{H}d\sigma .$$

Similar things will be true for the two analogous integrals that contain the variations δy and δz.

If one substitutes the expressions thus-transformed in formula (I) and annuls the coefficients of δx, δy, δz separately in the surface integral, as well as in the contour integral, then one will get the following equations:

$$\text{(II)}\qquad\left\{\begin{aligned}
HX &= \frac{\partial}{\partial u}\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right),\\
HY &= \frac{\partial}{\partial u}\left(\lambda\frac{\partial y}{\partial u}+\mu\frac{\partial y}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mu\frac{\partial y}{\partial u}+\nu\frac{\partial y}{\partial v}\right),\\
HZ &= \frac{\partial}{\partial u}\left(\lambda\frac{\partial z}{\partial u}+\mu\frac{\partial z}{\partial v}\right)+\frac{\partial}{\partial v}\left(\mu\frac{\partial z}{\partial u}+\nu\frac{\partial z}{\partial v}\right);
\end{aligned}\right.$$

$$\text{(II)}_s\qquad\left\{\begin{aligned}
HX_s &= \left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right),\\
HY_s &= \left(\lambda\frac{\partial y}{\partial u}+\mu\frac{\partial y}{\partial v}\right)\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial y}{\partial u}+\nu\frac{\partial y}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right),\\
HZ_s &= \left(\lambda\frac{\partial z}{\partial u}+\mu\frac{\partial z}{\partial v}\right)\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial z}{\partial u}+\nu\frac{\partial z}{\partial v}\right)\left(G\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right).
\end{aligned}\right.$$

These are the desired equations of equilibrium. The first three (II) are valid for any point of the surface *s*, and are therefore the so-called *indefinite equations* of equilibrium. The last three $(\text{II})_s$ are valid for any point of the contour, or more precisely, for any point of that part of the contour that is not fixed invariably (since one obviously has $\delta x = \delta y = \delta z = 0$, for any fixed point): They are then the so-called *boundary equations* (*).

When the figure of equilibrium has already been assigned *a priori*, the preceding equations will serve to determine the unknown functions λ, μ, ν, if one assumes that equilibrium is possible. However, when the figure of equilibrium is not assigned *a priori*, one needs to associate equations (II), $(\text{II})_s$, in which the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ will also become unknown, with the three equations (I), which express the idea that those functions are the coordinates of the points of a mappable surface (by flexion, without extension) on which the line element $(\text{I})_a$ is given.

The three conditions (2) are obviously equivalent to the single one:

$$(2)_b\qquad dx\, d\delta x + dy\, d\delta y + dz\, d\delta z = 0,$$

(*) For the fixed part of the contour, equations $(\text{II})_s$ will yield the reactions that are exerted by the supports.

which is satisfied identically when the variations δx, δy, δz have values that correspond to the most general infinitesimal displacement of a rigid body. It will then result (I) that the external forces must always be such that they are equilibrated in the surface *s*, which is supposed to be rigid, which is obvious, moreover, and at the basis for the method that MOSSOTTI followed.

§ 4. Transformation of the equations of equilibrium

Equations (II), $(\mathrm{II})_s$ that were just obtained contain the components of the external forces along the three x, y, z axes and therefore refer to the system of those axes. It is good to consider other equivalent equations along with them that contain the components of that force along three directions that are more intimately connected with the nature of the surface. For each point of the surface, those directions are those of the line u, the line v, and the normal w. Let:

$$U\,d\sigma,\quad V\,d\sigma,\quad W\,d\sigma$$

be the components along those three directions of the external force that acts upon the element $d\sigma$ of the surface and let:

$$U_s\,d\sigma,\ V_s\,d\sigma,\ W_s\,d\sigma$$

be the analogous components of the external force that acts upon the element ds of the contour (*).

The new equations that we speak of can be obtained in two ways: Namely, one can deduce them from the ones that were established already or establish them directly on the basis of the principle that was contained in formula (I).

We commence with the first of these two ways and observe that the deduction can be made immediately with respect to the boundary equations $(\mathrm{II})_s$, since it is enough to set:

$$(\mathrm{III})_s,\qquad \left\{\begin{aligned} U_s &= \frac{\sqrt{E}}{H}\left\{\lambda\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\mu\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)\right\},\\ V_s &= \frac{\sqrt{G}}{H}\left\{\mu\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\nu\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)\right\},\\ W_s &= 0. \end{aligned}\right.$$

Indeed, equations $(\mathrm{II})_s$ have the form:

$$X_s = \frac{U_s}{\sqrt{E}}\frac{\partial x}{\partial u}+\frac{V_s}{\sqrt{G}}\frac{\partial x}{\partial v} = U_s\cos(ux) + V_s\cos(vx),$$

(*) It is almost pointless to caution that we are speaking of *oblique* components.

$$Y_s = \frac{U_s}{\sqrt{E}}\frac{\partial y}{\partial u} + \frac{V_s}{\sqrt{G}}\frac{\partial y}{\partial v} = U_s \cos(uy) + V_s \cos(vy),$$

$$Z_s = \frac{U_s}{\sqrt{E}}\frac{\partial z}{\partial u} + \frac{V_s}{\sqrt{G}}\frac{\partial z}{\partial v} = U_s \cos(uz) + V_s \cos(vz)\,;$$

i.e., they express the idea that the resultant of the force X_s, Y_s, Z_s is identical to that of the force U_s, V_s, W_s.

In order to reduce the transformation of the indefinite equations (II) to that principle, develop the differentiations that are indicated in them in such a way that one will see the first equation in the form:

$$(a) \qquad HX = \left(\frac{\partial\lambda}{\partial u} + \frac{\partial\mu}{\partial v}\right)\frac{\partial x}{\partial u} + \left(\frac{\partial\mu}{\partial u} + \frac{\partial\nu}{\partial v}\right)\frac{\partial x}{\partial v} + \lambda\frac{\partial^2 x}{\partial u^2} + 2\mu\frac{\partial^2 x}{\partial u\,\partial v} + \nu\frac{\partial^2 x}{\partial v^2}.$$

Then recall that the second derivatives of the coordinates x, y, z with respect to the u, v can be expressed in the following way:

$$(4) \qquad \begin{cases} \dfrac{\partial^2 x}{\partial u^2} = E_1\dfrac{\partial x}{\partial u} + E_2\dfrac{\partial x}{\partial v} + A\alpha, \\[2ex] \dfrac{\partial^2 y}{\partial u^2} = E_1\dfrac{\partial y}{\partial u} + E_2\dfrac{\partial y}{\partial v} + A\beta, \\[2ex] \dfrac{\partial^2 z}{\partial u^2} = E_1\dfrac{\partial z}{\partial u} + E_2\dfrac{\partial z}{\partial v} + A\gamma, \end{cases} \qquad \begin{cases} \dfrac{\partial^2 x}{\partial u\,\partial v} = F_1\dfrac{\partial x}{\partial u} + F_2\dfrac{\partial x}{\partial v} + B\alpha, \\[2ex] \dfrac{\partial^2 y}{\partial u\,\partial v} = F_1\dfrac{\partial y}{\partial u} + F_2\dfrac{\partial y}{\partial v} + B\beta, \\[2ex] \dfrac{\partial^2 z}{\partial u\,\partial v} = F_1\dfrac{\partial z}{\partial u} + F_2\dfrac{\partial z}{\partial v} + B\gamma, \end{cases} \qquad \begin{cases} \dfrac{\partial^2 x}{\partial v^2} = G_1\dfrac{\partial x}{\partial u} + G_2\dfrac{\partial x}{\partial v} + C\alpha, \\[2ex] \dfrac{\partial^2 y}{\partial v^2} = G_1\dfrac{\partial y}{\partial u} + G_2\dfrac{\partial y}{\partial v} + C\beta, \\[2ex] \dfrac{\partial^2 z}{\partial v^2} = G_1\dfrac{\partial z}{\partial u} + G_2\dfrac{\partial z}{\partial v} + C\gamma. \end{cases}$$

In order to convince oneself of the legitimacy of these formulas *a priori*, it is enough to observe that, for example, the three derivatives:

$$\frac{\partial^2 x}{\partial u^2}, \quad \frac{\partial^2 y}{\partial u^2}, \quad \frac{\partial^2 z}{\partial u^2}$$

can be considered to be the components along the three x, y, z axes of a certain force that is applied to the point (x, y, z) – namely, (u, v) – and that this force can also be decomposed along the three directions u, v, w. If one lets:

$$E_1\sqrt{E}, \qquad E_2\sqrt{G}, \qquad A$$

denote those three new components, then one will have precisely the three relations (4). Similar things are true for the other two triples. In regard to the determination of the coefficients E_1, E_2, F_1, F_2, G_1, G_2, A, B, C, in the first place, when one differentiates each of equations (1) with respect to u and v and substitutes the values (4), one will get the following relations:

$$\begin{cases} \dfrac{\partial E}{\partial u} = 2(EE_1 + FE_2), \\ \dfrac{\partial E}{\partial v} = 2(EF_1 + FF_2), \end{cases}$$

$$(4)_a \qquad \begin{cases} \dfrac{\partial F}{\partial u} = EF_1 + F(E_1 + F_2) + GE_2, \\ \dfrac{\partial F}{\partial v} = EG_1 + F(F_1 + G_2) + GF_2, \end{cases}$$

$$\begin{cases} \dfrac{\partial G}{\partial u} = 2(FF_1 + GF_2), \\ \dfrac{\partial G}{\partial v} = 2(FG_1 + GG_2), \end{cases}$$

to which, one should add the two following ones:

$$(4)_b \qquad \frac{\partial H}{\partial u} = H\,(E_1 + F_2), \qquad \frac{\partial H}{\partial v} = H\,(F_1 + G_2),$$

which are consequences of them. The group of equations $(4)_a$ determines the quantities E_1, E_2, F_1, F_2, G_1, G_2, which do not depend upon the functions E, F, G and their first derivatives, as one sees, and therefore they are independent of any deformation of the surface (by flexion, without extension). In the second place, one obviously has from equations (4):

$$(4)_c \qquad \begin{cases} A = \alpha \dfrac{\partial^2 x}{\partial u^2} + \beta \dfrac{\partial^2 y}{\partial u^2} + \gamma \dfrac{\partial^2 z}{\partial u^2}, \\ B = \alpha \dfrac{\partial^2 x}{\partial u\,\partial v} + \beta \dfrac{\partial^2 y}{\partial u\,\partial v} + \gamma \dfrac{\partial^2 z}{\partial u\,\partial v}, \\ C = \alpha \dfrac{\partial^2 x}{\partial v^2} + \beta \dfrac{\partial^2 y}{\partial v^2} + \gamma \dfrac{\partial^2 z}{\partial v^2}, \end{cases}$$

and the three quantities thus-defined, which are well-known in the theory of surfaces, have a very important geometric significance that can be summarized completely by the formula:

$$(4)_d \qquad \frac{ds^2}{R} + A\,du^2 + 2B\,du\,dv + C\,dv^2 = 0,$$

in which the differentials du, dv, ds are coupled by the equations $(1)_a$, and R is the radius of curvature of the normal section that goes through the line element ds.

Introduce the expressions (4), whose coefficients prove to be perfectly determined, into equations (a) and find that:

$$\begin{aligned} HX = {} & \left\{ \frac{\partial \lambda}{\partial u} + \frac{\partial \mu}{\partial v} + E_1 \lambda + 2F_1 \mu + G_1 \nu \right\} \frac{\partial x}{\partial u} \\ & + \left\{ \frac{\partial \mu}{\partial u} + \frac{\partial \nu}{\partial v} + E_2 \lambda + 2F_2 \mu + G_2 \nu \right\} \frac{\partial x}{\partial v} \\ & + (A\lambda + 2B\mu + C\nu)\,\alpha, \end{aligned}$$

and operating likewise on the other two analogous equations, one will get two other formulas by first replacing X, x, α with Y, y, β and then Z, z, γ.

One can deduce the following equations from the form of the equations thus-obtained and by virtue of the considerations that have already allowed us to pass from equations $(\mathrm{II})_s$ to $(\mathrm{III})_s$, with no further discussion:

$$(\mathrm{III}) \qquad \begin{cases} HU = \sqrt{E} \left(\dfrac{\partial \lambda}{\partial u} + \dfrac{\partial \mu}{\partial v} + E_1 \lambda + 2F_1 \mu + G_1 \nu \right), \\ HV = \sqrt{G} \left(\dfrac{\partial \mu}{\partial u} + \dfrac{\partial \nu}{\partial v} + E_2 \lambda + 2F_2 \mu + G_2 \nu \right), \\ HW = A\lambda + 2B\mu + C\nu. \end{cases}$$

These, along with $(\mathrm{III})_s$, are the equations that we alluded to at the beginning of this §.

The new components U, V, W are obviously coupled with the original ones by the relations:

(5)
$$\begin{cases} X = \dfrac{U}{\sqrt{E}}\dfrac{\partial x}{\partial u} + \dfrac{V}{\sqrt{G}}\dfrac{\partial x}{\partial v} + W\alpha, \\ Y = \dfrac{U}{\sqrt{E}}\dfrac{\partial y}{\partial u} + \dfrac{V}{\sqrt{G}}\dfrac{\partial y}{\partial v} + W\beta, \\ Z = \dfrac{U}{\sqrt{E}}\dfrac{\partial z}{\partial u} + \dfrac{V}{\sqrt{G}}\dfrac{\partial z}{\partial v} + W\gamma, \end{cases}$$

from which, one infers, conversely:

$(5)_a$
$$\begin{cases} U = \dfrac{\sqrt{E}}{H^2}\left\{G\sum X\dfrac{\partial x}{\partial u} - F\sum X\dfrac{\partial x}{\partial v}\right\}, \\ V = \dfrac{\sqrt{G}}{H^2}\left\{E\sum X\dfrac{\partial x}{\partial v} - F\sum X\dfrac{\partial x}{\partial u}\right\}, \\ W = \alpha X + \beta Y + \gamma Z. \end{cases}$$

§ 5. Another way of deriving the transformed equations

In order to establish the equations of equilibrium in the form (III), $(\mathrm{III})_s$, one should regard the coordinates x, y, z of an arbitrary point in space as functions of the three variables u, v, w ; i.e., the normal distance w between that point and the surface σ and the curvilinear coordinates u, v of the foot of that normal. Since the points in space that we need to consider are infinitely-close to the surface s, it is impossible for there to be any ambiguity in regard to the values of the variables u, v, w that correspond to the given values of the coordinates x, y, z.

If one considers the quantities x, y, z from that standpoint then one will have:

$$\delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v + \frac{\partial x}{\partial w}\delta w,$$

and if one sets $w = 0$ then:

(6)
$$\delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v + \alpha\,\delta w,$$

and one gets similar results for δy and δz. In the last equations (6), the variations δx, δy, δz, as well as the derivatives $\dfrac{\partial x}{\partial u}$, $\dfrac{\partial x}{\partial v}$, etc, and the cosines α, β, γ are now those of the same quantities that were previously denoted with the same symbols. As for the new variations δu, δv, δw, they must be considered to be finite, continuous, monodromic functions of the variables u, v.

It results from the expressions (6) that if δs is the line element whose projections onto the three x, y, z axes are δx, δy, δz then one will have:

$$\delta s^2 = E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2 + \delta w^2,$$

and also that if δs_1 is another similar element that has the origin (u, v) in common with the first one, but corresponds to the other variations δu_1, δv_1, δw_1, then one will have:

$$\delta s\,\delta s_1 \cos(\delta s, \delta s_1) = E\,du\,du_1 + F\,(\delta u\,\delta v + \delta v\,\delta u) + G\,\delta v\,\delta v_1 + \delta w\,\delta w_1 .$$

Now, if the second element δs_1 is in the direction of a force R whose components along the u, v, w directions are U, V, W then one will obviously have:

$$U : V : W : R = \delta u_1 \sqrt{E} : \delta v_1 \sqrt{G} : \delta w_1 : \delta s_1 ,$$

and the preceding formula will give:

$$R\,\delta s \cos(R, \delta s) = \frac{U}{\sqrt{E}}(E\,\delta u + F\,\delta v) + \frac{V}{\sqrt{E}}(F\,\delta u + G\,\delta v) + W\,\delta w. \tag{6)$_a$}$$

Therefore, the left-hand side of the last equation represents the work that is done by the force R when it displaces by δs from its point of application, and the right-hand side represents the expression for that work as a function of the components of both the force and displacement along the directions u, v¸ w. That result can be obtained, less directly, from equations (5).

If one substitutes the expressions:

$$\frac{\partial \delta x}{\partial u} = \frac{\partial^2 x}{\partial u^2}\delta u + \frac{\partial^2 x}{\partial u\,\partial v}\delta v + \frac{\partial \alpha}{\partial u}\delta w + \frac{\partial x}{\partial u}\frac{\partial \delta u}{\partial u} + \frac{\partial x}{\partial v}\frac{\partial \delta v}{\partial u} + \alpha\frac{\partial \delta w}{\partial u},$$

$$\frac{\partial \delta x}{\partial v} = \frac{\partial^2 x}{\partial u\,\partial v}\delta u + \frac{\partial^2 x}{\partial v^2}\delta v + \frac{\partial \alpha}{\partial v}\delta w + \frac{\partial x}{\partial u}\frac{\partial \delta u}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial \delta v}{\partial v} + \alpha\frac{\partial \delta w}{\partial v},$$

which one gets from equation (6), in the right-hand sides of equations $(2)_a$, then one will find that:

$$\tag{7} \left\{ \begin{aligned} \tfrac{1}{2}\delta E &= E\frac{\partial \delta u}{\partial u} + F\frac{\partial \delta v}{\partial u} + \tfrac{1}{2}\delta E - A\,\delta w, \\ \delta F &= E\frac{\partial \delta u}{\partial v} + F\left(\frac{\partial \delta u}{\partial u} + \frac{\partial \delta v}{\partial v}\right) + G\frac{\partial \delta v}{\partial u} + \delta F - 2B\,\delta w, \\ \tfrac{1}{2}\delta G &= F\frac{\partial \delta u}{\partial v} + G\frac{\partial \delta v}{\partial v} + \tfrac{1}{2}\delta G - C\,\delta w, \end{aligned} \right.$$

in which the characteristic δ represents the operation:

$$\delta = \delta u \frac{\partial}{\partial u} + \delta v \frac{\partial}{\partial v}.$$

It is appropriate to add the following equation to these:

$$(7)_a \qquad \delta H = \frac{\partial (H\,\delta u)}{\partial u} + \frac{\partial (H\,\delta v)}{\partial v} - \frac{AG - 2BF + CE}{H}\,\delta w,$$

which is a consequence of them.

If one sets:

$$(7)_b \qquad E\,\delta u + F\,\delta v = \delta u', \qquad F\,\delta u + G\,\delta v = \delta v'$$

and makes use of the relations $(4)_a$ then it will be easy to give the following form to the preceding equations (7):

$$(7)_c \qquad \begin{cases} \tfrac{1}{2}\delta E = \dfrac{\partial \delta u'}{\partial u} - (E_1\,\delta u' + E_2\,\delta v' + A\,\delta w), \\ \delta F = \dfrac{\partial \delta u'}{\partial v} + \dfrac{\partial \delta v'}{\partial u} - 2(F_1\,\delta u' + F_2\,\delta v' + B\,\delta w), \\ \tfrac{1}{2}\delta G = \dfrac{\partial \delta v'}{\partial v} - (G_1\,\delta u' + G_2\,\delta v' + C\,\delta w). \end{cases}$$

By using the formulas that were established before, it will become clear that the fundamental equation (I) is equivalent to this one:

$$(\mathrm{I}') \qquad \begin{aligned} &\int \left\{ \frac{U\,\delta u'}{\sqrt{E}} + \frac{V\,\delta v'}{\sqrt{G}} + W\,\delta w \right\} d\sigma + \int \left\{ \frac{U_s\,\delta u'}{\sqrt{E}} + \frac{V_2\,\delta v'}{\sqrt{G}} + W_s\,\delta w \right\} ds \\ &+ \tfrac{1}{2}\int (\lambda\,\delta E + 2\mu\,\delta F + \nu\,\delta G)\frac{d\sigma}{H} = 0, \end{aligned}$$

in which δE, δF, δG have the values $(7)_c$. Since the quantities $\delta u'$, $\delta v'$ are arbitrary, like the δu, δv, all that remains to be done is to develop the last of the three integrals, while considering the $\delta u'$, $\delta v'$ to be the arbitrary variations.

Now, one gets from equations $(7)_c$ that:

$$\tfrac{1}{2}(\lambda\,\delta E + 2\mu\,\delta F + \nu\,\delta G)$$

$$= \lambda \frac{\partial \delta u'}{\partial u} + \mu \left(\frac{\partial \delta u'}{\partial v} + \frac{\partial \delta v'}{\partial u} \right) + \nu \frac{\partial \delta v'}{\partial v}$$

$$-(E_1\,\lambda + 2F_1\,\mu + G_1\,\nu)\,\delta u' - (E_2\,\lambda + 2F_2\,\mu + G_2\,\nu)\,\delta v' - (A\,\lambda + 2B\,\mu + C\,\nu)\,\delta w.$$

However, one has:

$$\lambda\frac{\partial\delta u'}{\partial u}+\mu\left(\frac{\partial\delta u'}{\partial v}+\frac{\partial\delta v'}{\partial u}\right)+\nu\frac{\partial\delta v'}{\partial v}$$

$$=\frac{\partial(\lambda\,\delta u'+\mu\,\delta v')}{\partial u}+\frac{\partial(\mu\,\delta u'+\nu\,\delta v')}{\partial v}-\left(\frac{\partial\lambda}{\partial u}+\frac{\partial\mu}{\partial v}\right)\delta u'-\left(\frac{\partial\mu}{\partial u}+\frac{\partial\nu}{\partial v}\right)\delta v',$$

so with the transformation of the integrals that was adopted already in § **3**, it will result that:

$$-\int(\lambda\,\delta E+2\mu\,\delta F+\nu\,\delta G)\frac{d\sigma}{H}$$

$$=$$

$$\int\left\{\left(\frac{\partial\lambda}{\partial u}+\frac{\partial\mu}{\partial v}+E_1\lambda+2F_1\mu+G_1\nu\right)\delta u'+\left(\frac{\partial\mu}{\partial u}+\frac{\partial\nu}{\partial v}+E_2\lambda+2F_2\mu+G_2\nu\right)\delta v'+(A\lambda+2B\mu+C\nu)\right\}\frac{d\sigma}{H}$$

$$+\int\left\{(\lambda\,\delta u'+\mu\,\delta v')\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+(\mu\,\delta u'+\nu\,\delta v')\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)\right\}\frac{ds}{H}.$$

If one substitutes that expression in formula (I) and equates the coefficients of $\delta u'$, $\delta v'$, δw to zero then one will get equations (III), $(\text{III})_s$ of § **4**.

§ 6. Determination of the surface tensions

Trace out a closed line *s* arbitrarily on the surface *s*, which is supposed to be equilibrated, call the line element *ds* and the normal element *dn*, which is directed towards the interior of the region that is bounded by *s*. If *s* means the variable arc length of that line when measured from an arbitrary origin then one needs to fix the sense of positive increase along that arc in such a way that when the element *ds* is traversed in that sense, it will be arranged with respect to *dn* and *w* in the same way that the *x*-axis is arranged with respect to the *y*-axis and *z*-axis, respectively. For us, that will be the sense of *positive* circulation along the line *s*. With those conventions, one will have the following relations (*):

$$(8)\qquad\left\{\begin{array}{ll} E\dfrac{\partial u}{\partial s}+F\dfrac{\partial v}{\partial s}=H\dfrac{\partial v}{\partial n}, & F\dfrac{\partial u}{\partial s}+G\dfrac{\partial v}{\partial s}=-H\dfrac{\partial u}{\partial n},\\ E\dfrac{\partial u}{\partial n}+F\dfrac{\partial v}{\partial n}=-H\dfrac{\partial v}{\partial s}, & F\dfrac{\partial u}{\partial n}+G\dfrac{\partial v}{\partial n}=\;H\dfrac{\partial u}{\partial s},\end{array}\right.$$

(*) "Della variabili complesse, etc.," art. V.

two of which are consequences of the other two. In order to distinguish them, let s' denote the line of the contour that was previously denotes by s.

Having said that, imagine that for a given system of forces (X, Y, Z; $X_{s'}$, $Y_{s'}$, $Z_{s'}$) that is applied to the surface σ and the contour s' such that it is capable of producing equilibrium on that surface, one must determine the three functions λ, μ, ν in such a way that the indefinite equations and the boundary equations will be satisfied identically. If one substitutes the three functions thus-found in equations $(\mathrm{II})_s$, which are referred to the new closed line s, then one will recover the values that are determined for the quantities that are denoted by X_s, Y_s, Z_s therein, and it is clear that when the new system of forces (X, Y, Z; X_s, Y_s, Z_s) is applied to the portion of σ that is inside of s and to its contour s, it must maintain equilibrium in that portion when it is considered in isolation, since the indefinite equations and the boundary equations for that portion of the surface and for that system of forces will be satisfied identically. On the other hand, when that portion of the surface is considered to be part of σ, it will already be equilibrium under the action of the force (X, Y, Z) that is applied to that portion and some other unknown forces that arise when one connects that portion with the residual portion of σ. Hence, the system of the latter force is equivalent to the system of the force (X_s, Y_s, Z_s) that is determined from equations $(\mathrm{II})_s$, and since one can suppose that the line s becomes rigid in all of its extension, except for the element ds, without perturbing the equilibrium, one must then conclude that the two equal and opposite forces:

$$(X_s\,ds,\ Y_s\,ds,\ Z_s\,ds) \qquad \text{and} \qquad (-X_s\,ds,\ -Y_s\,ds,\ -Z_s\,ds)$$

that are applied to the element ds represent the mutual action that exists in the equilibrium state between the two surface regions that are contiguous to that element. That mutual action is what one calls the *tension* of the surface along the element ds.

Although the tension thus-defined is not properly a force, but the result of the coexistence of two equal and opposite forces, one can usually use one or the other force interchangeably. That will cause no inconvenience when one unambiguously establishes what one must take for the two forces. We shall agree to always take the second one; namely, the one that, under the preceding hypotheses, will be exerted by the portion of the surface that is inside of s on the residual portion, and therefore if $T_s\,ds$ denotes the absolute value of the tension along the element ds and $T_{sr}\,ds$ denotes the (normal or oblique, according to the case) component of that tension in an arbitrary direction r then we will have, from equations $(\mathrm{II})_s$, that:

$$\left\{\begin{aligned}
&HT_{sx}+\left(\lambda\frac{\partial x}{\partial u}+\mu\frac{\partial x}{\partial v}\right)\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial x}{\partial u}+\nu\frac{\partial x}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)=0,\\
&HT_{sy}+\left(\lambda\frac{\partial y}{\partial u}+\mu\frac{\partial y}{\partial v}\right)\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial y}{\partial u}+\nu\frac{\partial y}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)=0,\\
&HT_{sz}+\left(\lambda\frac{\partial z}{\partial u}+\mu\frac{\partial z}{\partial v}\right)\left(E\frac{\partial u}{\partial n}+F\frac{\partial v}{\partial n}\right)+\left(\mu\frac{\partial z}{\partial u}+\nu\frac{\partial z}{\partial v}\right)\left(F\frac{\partial u}{\partial n}+G\frac{\partial v}{\partial n}\right)=0,
\end{aligned}\right.$$

in which T_{sx}, T_{sy}, T_{sz} are the *normal* components of T_s, and we will also have from equations $(\mathrm{III})_s$ that:

$$\begin{cases} HT_{su} + \sqrt{E}\left\{\lambda\left(E\dfrac{\partial u}{\partial n} + F\dfrac{\partial v}{\partial n}\right) + \mu\left(F\dfrac{\partial u}{\partial n} + G\dfrac{\partial v}{\partial n}\right)\right\} = 0, \\ HT_{sv} + \sqrt{G}\left\{\mu\left(E\dfrac{\partial u}{\partial n} + F\dfrac{\partial v}{\partial n}\right) + \nu\left(F\dfrac{\partial u}{\partial n} + G\dfrac{\partial v}{\partial n}\right)\right\} = 0, \\ T_{sw} = 0, \end{cases}$$

in which T_{su}, T_{sv} are the *oblique* components of T_s and the *normal* components, resp.

The last of those equations shows that the surface tension is always directed tangentially to the surface, which can be regarded as obvious *a priori*. We shall no longer refer to the component T_{sw} then.

From the conventions that were made, the surface region where the tension *emanates from* will always be the one towards which the normal n is directed.

By virtue of the relations (8), one can give the following very simple form to the values of the components of the tension along the lines u and v:

$$(\mathrm{IV}) \qquad \begin{cases} T_{su} = \sqrt{E}\left(\lambda\dfrac{\partial v}{\partial s} - \mu\dfrac{\partial u}{\partial s}\right), \\ T_{sv} = \sqrt{G}\left(\mu\dfrac{\partial v}{\partial s} - \nu\dfrac{\partial u}{\partial s}\right). \end{cases}$$

Since the direction n of the normal does not appear in these formulas, it might seem, on first glance, that the region of the surface where the tension acts upon the element ds will remain indeterminate, while it is rather obvious that when one passes from one of the regions that are contiguous to the element to the other one, the components of the tension must change sign, while preserving their absolute values. One should therefore not forget that the relations (8) that led to equations (IV) presuppose that there is a well-defined relation between the directions ds and dn, so the sense of increasing arc length s, and therefore the sign of the derivatives of u and v with respect to that arc length, will determine the direction of dn implicitly. By virtue of the conventions that were made in regard to them, equations (IV) define the components of the tension in the element ds, such that the tension proceeds from *that* region to the one with the arc length s, which is traversed in the sense of its increase, which is the contour or part of the contour, when traversed *positively*, and that should remove any ambiguity (and also when the line s is not closed).

Consider, for example, the angular region (of width $< \pi$) that is found between the two lines u, v that start from a point (u, v) of the surface in the directions of increasing u and v. From the conventions that were made in § **2**, it is clear that the first of those lines, when considered to be part of the contour of that region, is traversed positively when u increases, while the second one is traversed positively when v decreases. If we would then like to calculate the tensions that proceed from that region into two elements that are contiguous to the vertex of the angle from formulas (IV) then we will need to set:

$$\frac{\partial u}{\partial s} = \frac{1}{\sqrt{E}}, \qquad \frac{\partial v}{\partial s} = 0$$

when we treat the line *u*, while we will need to set:

$$\frac{\partial u}{\partial s} = 0, \qquad \frac{\partial v}{\partial s} = -\frac{1}{\sqrt{G}}$$

when we treat the line *v*. If T_u , T_v denote the two tensions thus-defined, for the sake of convenience, then we will have:

$$(9) \qquad \begin{cases} T_{uu} = -\mu, & T_{uv} = -\nu\sqrt{\dfrac{G}{E}}, \\ T_{vu} = -\lambda\sqrt{\dfrac{E}{G}}, & T_{vv} = -\mu, \end{cases}$$

and therefore:

$$(9)_a \qquad \lambda = -T_{vu}\sqrt{\frac{G}{E}}, \qquad \mu = -T_{uu} = -T_{vv}, \qquad \nu = -T_{uv}\sqrt{\frac{E}{G}}.$$

These last formulas summarize the mechanical significance of the multipliers λ, μ, ν, along with the necessary relation:

$$(9)_b \qquad T_{uu} = T_{vv} .$$

In order to interpret that relation, observe that if ds_u , ds_v denote the absolute values of the two line elements to which the tensions T_u, T_v refer, and if θ denotes the angle between them then one can write:

$$T_{uu}\, ds_u \cdot \sin\theta\, ds_v = T_{vv}\, ds_v \cdot \sin\theta\, ds_u .$$

In that form, it will become obvious that there is a spontaneous equivalence of the pairs that arise from the tangential components of the tensions on opposite sides of the parallelogram whose sides are ds_u , ds_v , which are pairs that act in opposite senses, and that is precisely by virtue of the equality $(9)_b$.

§ 7. Study of the surface tensions

If one lets *dt* denote the line element that issues from the origin of *ds* in the direction of the tension $T_s\, ds$ then one will obviously have:

$$T_s\, ds : T_{su}\, ds : T_{sv}\, ds = dt : du\sqrt{E} : dv\sqrt{G}\ ,$$

in which du, dv are the increments in u, v that correspond to the new element dt. It will then follow that:

$$T_{su} = T_s\sqrt{E}\,\frac{\partial u}{\partial t}\,, \qquad T_{sv} = T_s\sqrt{G}\,\frac{\partial v}{\partial t}\,,$$

and therefore if one substitutes that in formulas (IV) then one will get:

$$\text{(IV}'\text{)} \qquad \begin{cases} T_s\dfrac{\partial u}{\partial t} = \lambda\dfrac{\partial u}{\partial s} - \mu\dfrac{\partial u}{\partial s}, \\[2ex] T_s\dfrac{\partial v}{\partial t} = \mu\dfrac{\partial v}{\partial s} - \nu\dfrac{\partial u}{\partial s}. \end{cases}$$

If one recalls that any pair of derivatives (such as $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial v}{\partial t}$, for example) will satisfy the relation:

$$\text{(10)} \qquad E\left(\frac{\partial u}{\partial t}\right)^2 + 2F\,\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + G\left(\frac{\partial v}{\partial t}\right)^2 = 1,$$

by virtue of equations $(1)_a$, then one will see that if the direction s is given then the preceding formulas (IV′) will define the absolute value $T_s\, ds$ and the direction t of the tension on the element ds, which is understood in the sense that was agreed upon in the preceding §. However, since equations (10), which must intervene in the determination of the derivatives $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial v}{\partial t}$, will remain unaltered when one changes t into $-\,t$ (i.e., when one inverts the direction of the element dt), that will yield an opportunity to remove the restriction that T_s must always represent the absolute value of the unitary tension, and to allow T_s to take one or the other sign indifferently. Since changing t into $-\,t$ and T_s into $-T_s$ will leave formulas (IV′) unaltered, that would be equivalent to agreeing that a tension T_s in the direction t is equivalent to a tension $-\,T_s$ in the direction $-\,t$, and that is the usual convention in mechanics.

If one eliminates T_s from the two equations (IV′) then one will find the fundamental relation:

$$\text{(11)} \qquad \nu\frac{\partial u}{\partial s}\frac{\partial u}{\partial t} - \mu\left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right) + \lambda\frac{\partial v}{\partial s}\frac{\partial v}{\partial t} = 0,$$

which establishes the necessary dependency between the direction of an arbitrary line element and that of the tension to which it is subject. As one sees, that dependency is reciprocal, since if one traces out the system of lines u arbitrarily then it will always be possible (*) to associate another system of lines v such that the tension at any point of the

(*) Except in a case that will be discussed below.

surface on the element of the line *u* will be directed along the line *v*, and conversely the tension on the element of the line *v* will be directed along the line *u*. The characteristic property of such associated systems is that the function μ remains identically zero for them. In general, annulling μ at a point of the surface will indicate that the lines *u* and *v* are mutually conjugate at that point, in the sense that was defined by equations (11).

One deduces from formulas (IV′) that:

$$T_s\left(\lambda\frac{\partial v}{\partial t}-\mu\frac{\partial u}{\partial t}\right)+(\lambda\nu-\mu^2)\frac{\partial u}{\partial s}=0,$$

$$T_s\left(\mu\frac{\partial v}{\partial t}-\nu\frac{\partial u}{\partial t}\right)+(\lambda\nu-\mu^2)\frac{\partial v}{\partial s}=0.$$

However, due to the reciprocity of the directions *s*, *t*, (IV′) will also give:

$$T_t\frac{\partial u}{\partial s}=\lambda\frac{\partial v}{\partial t}-\mu\frac{\partial u}{\partial t},$$
$$T_t\frac{\partial v}{\partial s}=\mu\frac{\partial v}{\partial t}-\nu\frac{\partial u}{\partial t},$$

in which T_t is the unit tension on the element *dt*, which will be positive or negative according to whether its direction agrees with that of *ds* or is opposite to it, resp. It then results that the tensions T_s, T_t on two conjugate line elements will be coupled by the relation:

(11)$_a$ $$T_s\,T_t+\lambda\nu-\mu^2=0,$$

and we have already encountered a special case of that in § **1**.

The infinitude of pairs of conjugate direction *s* and *t* at one of the points of the surface form (11) a quadratic involution whose unit elements are given by the equation:

(11)$_b$ $$\nu\left(\frac{\partial u}{\partial s}\right)^2-2\mu\frac{\partial u}{\partial s}\frac{\partial v}{\partial s}+\lambda\left(\frac{\partial v}{\partial s}\right)^2=0.$$

These elements are real, coincident, or imaginary according to whether one has:

$$\lambda\nu-\mu^2<0,\quad =0,\quad >0,$$

resp. In the first case, each of those elements is subject to only tangential tensions (*) – i.e., to a tension that acts in the sense of that element – and the value of that tension is given (11)$_a$ by:

(*) When that case is verified at any point of σ, there will exist an infinitude of lines that are subject to only tangential tensions, and therefore ones that are *conjugate to themselves*. It will then be impossible to associate the system of those lines with a second system that is distinct from it and its conjugate. For that, one should refer to the exception that was pointed out in the previous note.

$$T_s^2 + \lambda\nu - \mu^2 = 0.$$

In the second case – i.e., when the two unit elements coincide in one – the corresponding tension will be *zero*. Conversely, those unit elements, when they exist, will be the only ones that are free of tension, since for $T_s = 0$, the two equations (IV′) will not be mutually consistent unless they are subject to the condition that $\lambda\nu - \mu^2 = 0$, and when that is satisfied, they will define the same direction as equations $(11)_b$.

A pair of orthogonal elements will always exist for any quadratic involution. Hence, for any point of the surface, there will be two mutually-perpendicular line elements, and each of them will be subject to normal tension.

In order to determine the directions of those *principal elements* and the tensions to which they are subject, observe that the two directions *s* and *t* must be mutually perpendicular for them, so one will have:

$$E\frac{\partial u}{\partial s} + F\frac{\partial v}{\partial s} = H\frac{\partial u}{\partial t}, \qquad F\frac{\partial u}{\partial s} + G\frac{\partial v}{\partial s} = -H\frac{\partial u}{\partial t},$$

on the basis of equations (8), and formulas (IV′) will give:

$$(12) \qquad \begin{cases} T_s\left(E\dfrac{\partial u}{\partial s} + F\dfrac{\partial v}{\partial s}\right) = H\left(\mu\dfrac{\partial v}{\partial s} - \nu\dfrac{\partial u}{\partial s}\right), \\[2ex] T_s\left(F\dfrac{\partial u}{\partial s} + G\dfrac{\partial v}{\partial s}\right) = H\left(\mu\dfrac{\partial u}{\partial s} - \lambda\dfrac{\partial v}{\partial s}\right). \end{cases}$$

If one recalls what was said in regard to the relations (8) then it will be clear that the *principal tension* T_s will prove to be positive when it is directed towards the interior of the region where it comes from. If one eliminates T_s from the two equations (12) then one will have:

$$\left(E\frac{\partial u}{\partial s} + F\frac{\partial v}{\partial s}\right)\left(\lambda\frac{\partial v}{\partial s} - \mu\frac{\partial u}{\partial s}\right) + \left(F\frac{\partial u}{\partial s} + G\frac{\partial v}{\partial s}\right)\left(\mu\frac{\partial v}{\partial s} - \nu\frac{\partial u}{\partial s}\right) = 0.$$

However, if one eliminates the derivatives of *u*, *v* then one will have:

$$(E\,T_s + H\nu)\,(GT_s + H\lambda) - (F\,T_s - H\mu)^2 = 0.$$

One can give the first of these two equations the form:

$$(12)_a \qquad \begin{vmatrix} \left(\frac{\partial u}{\partial s}\right)^2 & \frac{\partial u}{\partial s}\frac{\partial v}{\partial s} & \left(\frac{\partial v}{\partial s}\right)^2 \\ \lambda & \mu & \nu \\ G & -F & E \end{vmatrix} = 0,$$

and give the form:

$$(12)_b \qquad T_s^2 + \frac{E\lambda + 2F\mu + G\nu}{H} T_s + \lambda\nu - \mu^2 = 0$$

to the second of them.

The solutions to these two equations are always real, since the expression:

$$(E\lambda - G\nu)^2 + 4\,(E\mu + F\nu)(F\lambda + G\mu),$$

as well as the other one:

$$(E\lambda + 2F\mu + G\nu)^2 - 4\,H^2\,(\lambda\nu - \mu^2),$$

are equivalent to the single one:

$$(12)_c \qquad \frac{(EF\lambda + 2EG\mu + FG\nu)^2 + H^2(E\lambda - G\nu)^2}{EG},$$

which cannot become negative.

In addition, equations $(12)_a$ effectively define two orthogonal directions, since if one calls the arc lengths that correspond to them s', s'' then one will have:

$$\frac{\partial u}{\partial s'}\frac{\partial u}{\partial s''} : \left(\frac{\partial u}{\partial s'}\frac{\partial v}{\partial s''} + \frac{\partial u}{\partial s''}\frac{\partial v}{\partial s'}\right) : \frac{\partial v}{\partial s'}\frac{\partial v}{\partial s''} = \begin{vmatrix} F & G \\ -\mu & \lambda \end{vmatrix} : \begin{vmatrix} G & E \\ \lambda & \nu \end{vmatrix} : \begin{vmatrix} E & F \\ \nu & -\mu \end{vmatrix},$$

from the aforementioned equations, and it will follow from this that:

$$E\frac{\partial u}{\partial s'}\frac{\partial u}{\partial s''} + F\left(\frac{\partial u}{\partial s'}\frac{\partial v}{\partial s''} + \frac{\partial u}{\partial s''}\frac{\partial v}{\partial s'}\right) + G\frac{\partial v}{\partial s'}\frac{\partial v}{\partial s''} = 0,$$

which is a relation that expresses precisely the orthogonality of the arc lengths s', s'' at the point (u, v).

The two principal tensions $T_{s'}$, $T_{s''}$ that are defined by equation $(12)_b$ prove to be equal to each other when one has $(12)_c$:

$$EF\,\lambda + 2EG\,\mu + FG\,\nu = 0, \qquad E\lambda - G\nu = 0\,;$$

i.e.:

$$(12)_d \qquad \lambda : \mu : \nu = G : -F : E\,.$$

In that case, equation $(12)_a$ will become an identity, and *any* line element that emanates from the point at which the relations $(12)_d$ are verified will then be subject to *normal* and *constant* tensions. Indeed, equation $(12)_b$ gives:

$$T_{s'} = T_{s''} = -\frac{H\lambda}{G} = \frac{H\mu}{F} = -\frac{H\nu}{E},$$

and equations (IV′) will become (8):

$$T_s \frac{\partial u}{\partial t} = T_{s'} \frac{\partial u}{\partial n}, \qquad T_s \frac{\partial v}{\partial t} = T_{s'} \frac{\partial v}{\partial n},$$

so:

$$t = n, \quad T_s = T_{s'} .$$

When the proportionality $(12)_d$ is verified at any point of σ, one will get back to the hypotheses that have been mentioned many times of inextensibility according to LAGRANGE, which are inferred from that proportionality precisely [see formulas $(3)_a$ at the end of § **1**]. That is to say, under those hypotheses, the tension will always be normal to the element and constant for one point of the surface. Conversely, when the tension obeys that law, equilibrium will demand only the inextensibility of the surface element.

Let:

$$T_{sx} = \frac{T_{su}}{\sqrt{E}} \frac{\partial x}{\partial u} + \frac{T_{sv}}{\sqrt{G}} \frac{\partial x}{\partial v},$$

$$T_{sy} = \frac{T_{su}}{\sqrt{E}} \frac{\partial y}{\partial u} + \frac{T_{sv}}{\sqrt{G}} \frac{\partial y}{\partial v},$$

$$T_{sz} = \frac{T_{su}}{\sqrt{E}} \frac{\partial z}{\partial u} + \frac{T_{sv}}{\sqrt{G}} \frac{\partial z}{\partial v}.$$

If one projects the tension onto the directions s and n then one will have:

$$T_{ss} = \frac{T_{su}}{\sqrt{E}} \left(E \frac{\partial u}{\partial s} + F \frac{\partial v}{\partial s} \right) + \frac{T_{sv}}{\sqrt{G}} \left(F \frac{\partial u}{\partial s} + G \frac{\partial v}{\partial s} \right),$$

$$T_{sn} = \frac{T_{su}}{\sqrt{E}} \left(E \frac{\partial u}{\partial n} + F \frac{\partial v}{\partial n} \right) + \frac{T_{sv}}{\sqrt{G}} \left(F \frac{\partial u}{\partial n} + G \frac{\partial v}{\partial n} \right)$$

$$= H \left(\frac{T_{su}}{\sqrt{E}} \frac{\partial u}{\partial s} - \frac{T_{sv}}{\sqrt{G}} \frac{\partial v}{\partial s} \right).$$

When one substitutes the values (IV) in this, one will deduce that:

$$(13)\qquad \left\{\begin{aligned} T_{ss} &= -H\left\{\nu\left(\frac{\partial u}{\partial s}\right)^2 - 2\mu\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + \lambda\left(\frac{\partial v}{\partial s}\right)^2\right\}, \\ T_{sn} &= -\begin{vmatrix} \left(\frac{\partial u}{\partial s}\right)^2 & \frac{\partial u}{\partial s}\frac{\partial v}{\partial s} & \left(\frac{\partial v}{\partial s}\right)^2 \\ \lambda & \mu & \nu \\ G & -F & E \end{vmatrix}. \end{aligned}\right.$$

In these expressions, one has a confirmation, along a different path, of the fact that the equations (11), $(12)_a$ define the directions of the line elements that are subject to only tangential tension or only to normal tension, respectively.

Suppose that line elements at a point (u, v) on the surface that are directed along u and along v are principal elements. That will imply the two conditions $\mu = 0$, $F = 0$ for that point. In that case, the equations that precede (IV) will give:

$$T_{su}\sqrt{G} = -E\lambda\frac{\partial u}{\partial n}, \qquad T_{sv}\sqrt{E} = -G\lambda\frac{\partial v}{\partial n},$$

or $(9)_a$:

$$T_{su} = \frac{\partial u}{\partial n}T_{vu}\sqrt{E}, \qquad T_{sv} = \frac{\partial v}{\partial n}T_{uv}\sqrt{G}.$$

The T_{uv} , T_{vu} will then be the principal tensions. One deduces from them that:

$$\frac{T_{su}^2}{T_{vu}^2} + \frac{T_{sv}^2}{T_{uv}^2} = 1,$$

and if one then draws the ellipse in the tangent plane at (u, v) that has its center at that point and its semi-axes T_{vu} , T_{uv} directed along u and v, respectively, then the magnitude of any semi-diameter of that ellipse will represent the tension that is directed along that semi-diameter. That ellipse will not teach one the direction of the element to which that tension belongs in a simple way. That direction is inferred from equation (11), which will become:

$$E\,T_{uv}\frac{\partial u}{\partial s}\frac{\partial u}{\partial t} + G\,T_{vu}\frac{\partial v}{\partial s}\frac{\partial v}{\partial t} = 0,$$

in the present case, or also:

$$T_{su}\,T_{uv}\cos(su) + T_{sv}\,T_{vu}\cos(sv) = 0.$$

It results from the first of equations (13) that in order for the normal component T_{su} to never be negative, one must have:

$$\lambda \le 0, \qquad \nu \le 0, \qquad \lambda\nu - \mu^2 \ge 0.$$

Those conditions are necessary, *in general*, since the internal tensions counteract the inextensibility of the surface. However, if its contour is fixed, totally or partially, then the tensions can also become negative without perturbing the equilibrium. That must be examined in each particular case, moreover.

§ 8. Noteworthy first case of equilibrium

One deduces from formulas $(2)_a$ that:

$$\tfrac{1}{2}(G\,\delta E - 2F\,\delta F + E\,\delta G) = \sum\left(G\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}\right)\frac{\partial \delta x}{\partial u} + \sum\left(E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u}\right)\frac{\partial \delta x}{\partial v},$$

or

$$\frac{G\,\delta E - 2F\,\delta F + E\,\delta G}{2H}$$

$$= \sum\left\{\frac{\partial}{\partial u}\left(\frac{G\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}}{H}\,\delta x\right) + \frac{\partial}{\partial v}\left(\frac{E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u}}{H}\,\delta x\right)\right\}$$

$$-\sum \delta x\left\{\frac{\partial}{\partial u}\left(\frac{G\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u}}{H}\right)\right\}.$$

Now the expression:

$$\frac{1}{H}\left\{\frac{\partial}{\partial u}\left(\frac{G\frac{\partial \phi}{\partial u} - F\frac{\partial \phi}{\partial v}}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E\frac{\partial \phi}{\partial v} - F\frac{\partial \phi}{\partial u}}{H}\right)\right\}$$

is what I have been referring to for quite some time (*) by the name of "second differential parameter" of the function $\phi(u, v)$ and denoted by the symbol $\Delta_2\,\phi$. I have shown (**), in addition, that for $\phi = x, y, z$, one will have the formulas:

$$\Delta_2 x = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\alpha, \qquad \Delta_2 y = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\beta, \qquad \Delta_2 z = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\gamma,$$

(*) "Ricerche di analisi applicata alla geometria," Giornale di BATTAGLINI **2** (1864).
(**) "Sulla teoria generale delle superficie d'area minima," Memorie dell' Accademia di Bologna (2) **7** (1868).

in which R_1, R_2 are the two principal radii of curvature of the surface at the point (u, v), which are considered to be positive or negative according to whether their directions [from the respective center of curvature to the point (u, v)] agree with that of the normal w or not. Therefore, if one sets:

$$h = \frac{1}{R_1} + \frac{1}{R_2},$$

for brevity – i.e., if one lets h / 2 denote the *mean curvature* of the surface – then one will have:

$$\frac{G\,\delta E - 2F\,\delta F + E\,\delta G}{2H}$$

$$= h\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) + \frac{1}{H}\sum \left\{\frac{\partial}{\partial u}\left(\frac{G\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}}{H}\delta x\right) + \frac{\partial}{\partial v}\left(\frac{E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u}}{H}\delta x\right)\right\}.$$

When that equality is multiplied by $d\sigma$ and integrated over an arbitrary piece σ of the surface considered, it will give (with the usual transformations):

$$\tfrac{1}{2}\int \frac{G\,\delta E - 2F\,\delta F + E\,\delta G}{H^2}\,d\sigma = \int (\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)\,h\,d\sigma$$

$$-\int\left\{\left(E\frac{\partial u}{\partial n} + F\frac{\partial v}{\partial n}\right)\sum\left(G\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}\right)\delta x + \left(F\frac{\partial u}{\partial n} + G\frac{\partial v}{\partial n}\right)\sum\left(E\frac{\partial x}{\partial u} - F\frac{\partial x}{\partial v}\right)\delta x\right\}\frac{ds}{H^2},$$

or more simply, by virtue of formulas (6), $(7)_b$:

$$-\int h\,\delta w\,d\sigma + \int\left(\frac{\partial u}{\partial n}\delta u' + \frac{\partial v}{\partial n}\delta v'\right)ds + \tfrac{1}{2}\int \frac{G\,\delta E - 2F\,\delta F + E\,\delta G}{H^2}\,d\sigma = 0.$$

Now that equation will have the general type (I′) if one sets:

$$(14)\qquad \begin{cases} U = 0, & V = 0, & W = \rho h, \\ U_s = -\rho\sqrt{E}\,\dfrac{\partial u}{\partial n}, & V_s = -\rho\sqrt{G}\,\dfrac{\partial v}{\partial n}, & W_s = 0, \\ \lambda = -\dfrac{\rho G}{H}, & \mu = \dfrac{\rho F}{H}, & \nu = -\dfrac{\rho E}{H}, \end{cases}$$

in which ρ is a constant factor. However, that equation will be satisfied identically for any system of values for the variations δu, δv, δw : Hence, the system (14) of forces (U, V, W) and (U_s, V_s) that maintains equilibrium in the piece of the surface over which the integration is extended will generate tensions whose values will result from the general formulas when one gives the values (14) to λ, μ, ν.

The forces that are applied at the various points of the surface are normal to it and proportional to the local mean curvature in the case of equilibrium.

The forces that are applied along the contour have the *constant* intensity ρ and are directed in the opposite sense to n (since the factor ρ is supposed to be positive) – i.e., along the external normal to that surface.

In addition, equation (11) becomes the condition for the orthogonality of the directions s, t : Hence, any line element is subject to only normal tension, and that tension will be same at any point and equal to the one that prevails along the contour. The first part of that property depends upon the fact that the present values of the quantities λ, μ, ν satisfy the conditions $(3)_a$, so the inextensibility can be interpreted in the Lagrangian sense in that equilibrium case.

We then have the following theorem:

Any piece of a flexible, inextensible surface is maintained in equilibrium by a constant tension that is normal to the contour and a normal force over the entire surface that is proportional to the local mean curvature. The constant tension in the contour is transmitted equably to any point of the surface.

Among the special cases that are worthy of note, we point out the *surfaces of constant mean curvature*, for which the normal force to the surface is everywhere constant, like the tension on the contour, and that of the surface of minimum area, for which one has the theorem:

An arbitrary piece of a surface of minimum area that is subjected to constant tensions that are normal along the contour will always be equilibrium and will present the same tension at any point and in any direction (*).

If the values (14) of the quantities U, V, W, λ, μ, ν are substituted in equations (III) then the first two of them [while being mindful of the relations $(4)_a$] will be satisfied identically, and the third one will reproduce the known expression:

$$\frac{h}{2} = -\frac{AG - 2BF + CE}{2H^2}$$

for the mean curvature.

§ 9. Noteworthy second case of equilibrium

Let us preface our discussion with a lemma.

From the expressions:

(*) That equilibrium case was known already to POISSON in the paper that was cited in the beginning of this one. POISSON had also considered the more general case (which will be discussed in § **10**), but in a very incomplete way. Moreover, the fundamental equations that he started with are not exact, and will not give rise to correct applications unless one compensates for the errors in some way.

$$A = -\sum \frac{\partial x}{\partial u}\frac{\partial \alpha}{\partial u}, \qquad B = -\sum \frac{\partial x}{\partial u}\frac{\partial \alpha}{\partial v} = -\sum \frac{\partial x}{\partial v}\frac{\partial \alpha}{\partial u}, \qquad C = -\sum \frac{\partial x}{\partial v}\frac{\partial \alpha}{\partial v},$$

which are equivalent to $(4)_c$, one will easily deduce the following equalities:

$$\frac{C\dfrac{\partial x}{\partial u} - B\dfrac{\partial x}{\partial v}}{H} = \beta\frac{\partial \gamma}{\partial v} - \gamma\frac{\partial \beta}{\partial v},$$

$$\frac{A\dfrac{\partial x}{\partial v} - B\dfrac{\partial x}{\partial u}}{H} = \gamma\frac{\partial \beta}{\partial u} - \beta\frac{\partial \gamma}{\partial v},$$

and one will infer from this that:

$$\frac{\partial}{\partial u}\left(\frac{C\dfrac{\partial x}{\partial u} - B\dfrac{\partial x}{\partial v}}{H}\right) + \frac{\partial}{\partial v}\left(\frac{A\dfrac{\partial x}{\partial v} - B\dfrac{\partial x}{\partial u}}{H}\right) = 2\left(\frac{\partial \beta}{\partial u}\frac{\partial \gamma}{\partial v} - \frac{\partial \beta}{\partial v}\frac{\partial \gamma}{\partial u}\right).$$

One will get two analogous formulas by permuting x with y and z, and α with β and γ. Now, one deduces three relations from the two identities:

$$\alpha\frac{\partial \alpha}{\partial u} + \beta\frac{\partial \beta}{\partial u} + \gamma\frac{\partial \gamma}{\partial u} = 0, \qquad \alpha\frac{\partial \alpha}{\partial v} + \beta\frac{\partial \beta}{\partial v} + \gamma\frac{\partial \gamma}{\partial v} = 0,$$

the first of which is:

$$\frac{\partial \beta}{\partial u}\frac{\partial \gamma}{\partial v} - \frac{\partial \beta}{\partial v}\frac{\partial \gamma}{\partial u} = D\alpha,$$

in which D is a factor that is common to all three of them, and in order to make $\alpha^2 + \beta^2 + \gamma^2 = 1$, it will be represented by:

$$D = \begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\partial \alpha}{\partial u} & \dfrac{\partial \beta}{\partial u} & \dfrac{\partial \gamma}{\partial u} \\ \dfrac{\partial \alpha}{\partial v} & \dfrac{\partial \beta}{\partial v} & \dfrac{\partial \gamma}{\partial v} \end{vmatrix}.$$

However, one will obviously have:

$$H = \begin{vmatrix} \alpha & \beta & \gamma \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix},$$

so one will have:

$$HD = AC - B^2,$$

and therefore:

$$D = Hk,$$

in which k is the *measure of the curvature*, according to GAUSS.

It then results that if one introduces the symbol:

$$\nabla\phi = \frac{1}{H}\left\{\frac{\partial}{\partial u}\left(\frac{C\frac{\partial\phi}{\partial u} - B\frac{\partial\phi}{\partial v}}{H}\right) + \frac{\partial}{\partial v}\left(\frac{A\frac{\partial\phi}{\partial v} - B\frac{\partial\phi}{\partial u}}{H}\right)\right\},$$

for brevity, which is analogous, in a way, to the $\Delta_2\phi$ in the preceding §, one will have the new formulas:

$$\nabla x = 2k\alpha, \qquad \nabla y = 2k\beta, \qquad \nabla z = 2k\gamma,$$

which can be compared to what was said in that §.

Having said that, go back to formulas $(2)_c$ and deduce the following one:

$$\tfrac{1}{2}(C\,\delta E - 2B\,\delta F + A\,\delta G) = \sum\left\{\left(C\frac{\partial x}{\partial u} - B\frac{\partial x}{\partial v}\right)\frac{\partial\delta x}{\partial u} + \left(A\frac{\partial x}{\partial v} - B\frac{\partial x}{\partial u}\right)\frac{\partial\delta x}{\partial v}\right\},$$

or, by virtue of the formulas that were proved just now:

$$\frac{C\,\delta E - 2B\,\delta F + A\,\delta G}{2H^2}$$

$$= -\,2k\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) + \frac{1}{H}\sum\left\{\frac{\partial}{\partial u}\left(\frac{C\frac{\partial x}{\partial u} - B\frac{\partial x}{\partial v}}{H}\,\delta x\right) + \frac{\partial}{\partial v}\left(\frac{A\frac{\partial x}{\partial v} - B\frac{\partial x}{\partial u}}{H}\,\delta x\right)\right\}.$$

When that equality is multiplied by $d\sigma$ and integrated over an arbitrary piece σ of the surface considered, with the usual transformations and making use of formulas (6), (7), (8), that will give:

$$\int 2k\,\delta w\,d\sigma + \int \left\{ \left(A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s} \right)\delta v' - \left(B\frac{\partial u}{\partial s} + C\frac{\partial v}{\partial s} \right)\delta u' \right\} \frac{ds}{H}$$

$$+\tfrac{1}{2}\int \frac{C\,\delta E - 2B\,\delta F + A\,\delta G}{H^2}\,d\sigma = 0.$$

Now that equation will get back to the general type (I′) when one sets:

$$(15)\quad \begin{cases} U = 0, & V = 0, & W = \rho k, \\ U_s = -\dfrac{\rho\sqrt{E}}{2H}\left(B\dfrac{\partial u}{\partial s} + C\dfrac{\partial v}{\partial s} \right), & V_s = \dfrac{\rho\sqrt{G}}{2H}\left(A\dfrac{\partial u}{\partial s} + B\dfrac{\partial v}{\partial s} \right) & W_s = 0, \\ \lambda = \dfrac{\rho C}{2H}, & \mu = \dfrac{\rho B}{2H}, & \nu = \dfrac{\rho A}{2H}, \end{cases}$$

in which ρ is a constant. On the other hand, the aforementioned equation will be satisfied identically for any system of values for the variations δu, δv, δz : Therefore, the system (15) of forces (U, V, W) and (U_s, V_s) that maintains the equilibrium in a piece of the surface over which the integration is extended will generate tensions whose values result from the general formulas when one gives the values (15) to λ, μ, ν.

The tensions thus-calculated will be:

$$T_{su} = \frac{\rho\sqrt{E}}{2H}\left(B\frac{\partial u}{\partial s} + C\frac{\partial v}{\partial s} \right), \qquad T_{sv} = -\frac{\rho\sqrt{G}}{2H}\left(A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s} \right)$$

for an arbitrary line element ds, and it will result naturally that around the contour, they will be equal and opposite to the external force (U_s, V_s). Equations (11) will become:

$$A\frac{\partial u}{\partial s}\frac{\partial u}{\partial t} + B\left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial s} \right) + C\frac{\partial v}{\partial s}\frac{\partial v}{\partial t} = 0$$

in this case and will coincide with the known relations between the (Dupinian) *conjugate tangents* of the surface, so the tension in any element will be directed along the conjugate tangent to it. It will then follow that the lines whose elements are subject to only normal tensions will be the *lines of curvature*, and that the lines whose elements are subject to only tangential tensions will be the *asymptotic lines.* Formulas (13) will become:

$$T_{sn} = -\frac{\rho}{2}\left\{ A\left(\frac{\partial u}{\partial s} \right)^2 + 2B\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + C\left(\frac{\partial v}{\partial s} \right)^2 \right\},$$

$$T_{ss} = -\frac{\rho}{2H}\begin{vmatrix} \left(\frac{\partial v}{\partial s}\right)^2 & -\frac{\partial v}{\partial s}\frac{\partial u}{\partial s} & \left(\frac{\partial u}{\partial s}\right)^2 \\ A & B & C \\ E & F & G \end{vmatrix},$$

namely:

$$(15)_a \qquad T_{sn} = \frac{\rho}{2R_s}, \qquad T_{ss} = \frac{\rho}{2S_s},$$

in which $1 / R_s$ is the normal curvature of the arc length s, and $1 / S_s$ is the geodetic torsion of that arc length. When the direction of the tension T_s is known, it will be enough to know the first of these components in order to determine its magnitude.

We then have the following theorem:

An arbitrary piece of a flexible, inextensible surface is kept in equilibrium by a force that is normal everywhere on the surface and proportional to the measure of local curvature, and it will give a tension along the contour that is directed along the conjugate tangent to that contour and have a normal component that is proportional to the normal curvature of the contour. The lines of normal tension are the lines of curvature of the surface, while those of tangential tension are the asymptotic lines of that surface.

Among the particular cases that are worthy of note, we recall the case of surfaces of constant curvature, for which the normal force is everywhere constant, and that of developable surfaces, for which that force is everywhere zero, while the tensions along the contour will be directed along the generators.

If the values (15) of the quantities U, V, W, λ, μ, ν are substituted in equations (III) then they will become:

$$(15)_b \qquad \left\{ \begin{aligned} &\frac{\partial \frac{C}{H}}{\partial u} - \frac{\partial \frac{B}{H}}{\partial v} + \frac{AG_1 - 2BF_1 + CE_1}{H} = 0, \\ &\frac{\partial \frac{A}{H}}{\partial v} - \frac{\partial \frac{B}{H}}{\partial u} + \frac{AG_2 - 2BF_2 + CE_2}{H} = 0, \\ &\frac{AC - B^2}{H^2} = k. \end{aligned} \right.$$

The last of these formulas reproduces the known expression for the measure of curvature. The first two constitute the known differential relations between the quantities A, B, C, which are relations that can present themselves spontaneously when one seeks the values of the four expressions:

$$\sum \alpha \frac{\partial^3 x}{\partial u^3}, \quad \sum \alpha \frac{\partial^3 x}{\partial u^2\, \partial v}, \quad \sum \alpha \frac{\partial^3 x}{\partial u\, \partial v^2}, \quad \sum \alpha \frac{\partial^3 x}{\partial v^3}.$$

Indeed, if one differentiates the values (4) with respect to u and v and substitutes the derivatives in those expressions then one will find that:

$$\sum \alpha \frac{\partial^3 x}{\partial u^3} = \frac{\partial A}{\partial u} + AE_1 + BE_2 ,$$

$$\sum \alpha \frac{\partial^3 x}{\partial u^2\, \partial v} = \frac{\partial B}{\partial u} + AF_1 + BF_2 = \frac{\partial A}{\partial v} + BE_1 + CE_2 ,$$

$$\sum \alpha \frac{\partial^3 x}{\partial u\, \partial v^2} = \frac{\partial C}{\partial u} + AG_1 + BG_2 = \frac{\partial B}{\partial v} + BF_1 + CF_2 ,$$

$$\sum \alpha \frac{\partial^3 x}{\partial v^3} = \frac{\partial C}{\partial v} + BG_1 + CG_2 ,$$

which give the two relations:

$$\frac{\partial C}{\partial u} - \frac{\partial B}{\partial v} + AG_1 + B\,(G_2 - F_1) - CF_2 = 0,$$

$$\frac{\partial A}{\partial v} - \frac{\partial B}{\partial u} - AF_1 + B\,(E_1 - F_2) + CE_2 = 0,$$

which coincide with the first two equations in $(15)_b$, by virtue of formulas $(4)_b$.

§ 10. An outline of some other cases of equilibrium

If one combines the two cases of equilibrium that were discussed in the preceding two §§ then one will immediately obtain a third one in which the force that is applied normally to the surface is given by:

$$W = \rho_1\, h + \rho_2\, k,$$

in which ρ_1 and ρ_2 are two constants, and in which the force that is applied along the contour will likewise prove to be the sum of the homologous components that relate to the first and second case, when one changes ρ into ρ_1 for those of the first one and changes ρ into ρ_2 for those of the second.

However, with respect to the possibility of deducing new cases of equilibrium from the cases that are known already, it is good to make the following general observation:

Suppose that one has already determined the functions λ, μ, ν for given external forces (U, V, W) and (U_s , V_s , W_s), and then set:

$$\lambda' = \rho\lambda, \qquad \mu' = \rho\mu, \qquad \nu' = \rho\nu,$$

in which ρ is a function of u and v. If one sets λ', μ', ν' in place of λ, μ, ν and U', V', W' in place of U, V, W in equations (III) then one will find that:

$$(16) \qquad \begin{cases} U' = \dfrac{\sqrt{E}}{H}\left(\lambda\dfrac{\partial\rho}{\partial u} + \mu\dfrac{\partial\rho}{\partial v}\right) + \rho U, \\ V' = \dfrac{\sqrt{G}}{H}\left(\lambda\dfrac{\partial\rho}{\partial u} + \nu\dfrac{\partial\rho}{\partial v}\right) + \rho V, \\ W' = \rho W, \end{cases}$$

and if one puts U'_s, V'_s, W'_s in place of U_s, V_s, W_s in $(\text{III})_s$ then one will find that:

$$(16)_s \qquad U'_s = \rho\, U_s, \qquad V'_s = \rho\, V_s, \qquad W'_s = 0.$$

It will result from this that the problem of equilibrium with respect to the new forces (U', V', W'), (U'_s, V'_s, W'_s) can be solved by the functions λ', μ', ν'.

Suppose, for example, that the quantities λ, μ, ν are the ones that correspond to the first equilibrium case (§ **8**), when the constant ρ is equal to unity; in that case, one will have:

$$\lambda' = -\frac{\rho G}{H}, \quad \mu' = -\frac{\rho F}{H}, \quad \nu' = -\frac{\rho E}{H}.$$

Those values are the most general ones possible (for the multipliers λ, μ, ν) when the inextensibility is intended in the Lagrangian sense. Assume orthogonal coordinates, for simplicity, set $F = 0$, and therefore $\mu = 0$, as well. It will result from (16) that:

$$U' = -\frac{\partial\rho}{\partial s_u}, \qquad V' = -\frac{\partial\rho}{\partial s_v}, \qquad W' = \rho\, h,$$

and one will then get a new case of equilibrium that is naturally valid under the hypotheses of purely-superficial inextensibility, and therefore also under that of linear inextensibility. In that equilibrium case, other than the normal force $\rho\, h$, a tangential force with potential ρ will intervene, while the force that acts along the contour will also be normal to it, but vary from point to point like that potential.

If the surface σ is part of one of the external level surfaces that relate to the Newtonian potential Π then the LAPLACE equation will translate into the known relation:

$$\frac{\partial^2\Pi}{\partial w^2} + h\frac{\partial\Pi}{\partial w} = 0$$

for points on that surface. If one therefore sets:

$$\rho = \frac{\partial \Pi}{\partial w}$$

then one will have:

$$U' = -\frac{\partial}{\partial s_u}\left(\frac{\partial \Pi}{\partial w}\right), \qquad V' = -\frac{\partial}{\partial s_v}\left(\frac{\partial \Pi}{\partial w}\right), \qquad W' = -\frac{\partial}{\partial w}\left(\frac{\partial \Pi}{\partial w}\right),$$

and one will then conclude the following theorem:

Any portion σ *of an external level surface that relates to a Newtonian potential* Π, *when considered as a flexible, inextensible surface of density* = 1 *can be maintained in equilibrium by a force that is due to the potential* $\partial\Pi / \partial w$ *and a force that is normal to the contour and equal in value to that potential. The tension in any internal line element will always be normal to it, and its magnitude will be represented by that potential* $\partial\Pi / \partial w$.

If the surface considered is imagined to be a fluid film (see above, in the foreword) then the pressure in that fluid will be $-\partial\Pi / \partial w$.

§ 11. On various special forms for the equations of equilibrium

When the system of curvilinear coordinates u and v is supposed to be *oblique*, the only special case that is worthy of note is the one in which the lines u and the lines v are mutually conjugate with respect to the tension – i.e., in which the quantity μ is zero at any point of the surface (§ **7**). Under that hypothesis, when one recalls $(9)_d$, equations (II) will become:

$$HX = -\frac{\partial}{\partial u}\left(T_v \frac{\partial x}{\partial u}\sqrt{\frac{G}{E}}\right) - \frac{\partial}{\partial v}\left(T_u \frac{\partial x}{\partial v}\sqrt{\frac{E}{G}}\right),$$

$$HY = -\frac{\partial}{\partial u}\left(T_v \frac{\partial y}{\partial u}\sqrt{\frac{G}{E}}\right) - \frac{\partial}{\partial v}\left(T_u \frac{\partial y}{\partial v}\sqrt{\frac{E}{G}}\right),$$

$$HZ = -\frac{\partial}{\partial u}\left(T_v \frac{\partial z}{\partial u}\sqrt{\frac{G}{E}}\right) - \frac{\partial}{\partial v}\left(T_u \frac{\partial z}{\partial v}\sqrt{\frac{E}{G}}\right),$$

in which, for the sake of convenience, T_u is written in place of T_{uv} and T_v is written in place of T_{vu} in order to make $T_{uu} = T_{vv} = 0$.

These equations agree with the ones that BRIOSCHI has given incidentally in note 1 of his paper "Intorno ad alcuni punti della teoria delle superficie" (*). Under the same hypotheses, after some convenient reductions, equations (III) will take the second form that the author in question gave to the preceding equations. As we have already cautioned to begin with, they can be usefully invoked only in the case in which a system of lines that are mutually-conjugate with respect to the tension is known *a priori*.

The most obvious (and always legitimate) simplification is then one that one gets by supposing that the lines u and v are orthogonal. Under those hypotheses, equations (II) will suffer no alteration, but (III) can be easily and entirely developed with the intervention of only the functions E, G, since equations $(4)_d$ will give:

$$E_1 = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}, \qquad F_1 = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}, \qquad G_1 = -\frac{\sqrt{G}}{E}\frac{\partial\sqrt{G}}{\partial u},$$

$$E_2 = -\frac{\sqrt{E}}{G}\frac{\partial\sqrt{E}}{\partial v}, \qquad F_2 = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}, \qquad G_2 = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial v}$$

for $F = 0$. By virtue of these equalities, equations (III) will initially become:

$$U = \frac{1}{\sqrt{G}}\left\{\frac{1}{\sqrt{E}}\frac{\partial(\lambda\sqrt{E})}{\partial u} + \frac{\partial\mu}{\partial v} + \frac{2}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\mu - \frac{\sqrt{G}}{E}\frac{\partial\sqrt{G}}{\partial u}\nu\right\},$$

$$V = \frac{1}{\sqrt{E}}\left\{\frac{1}{\sqrt{G}}\frac{\partial(\nu\sqrt{G})}{\partial v} + \frac{\partial\mu}{\partial u} + \frac{2}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\mu - \frac{\sqrt{E}}{G}\frac{\partial\sqrt{E}}{\partial v}\lambda\right\},$$

$$W = \frac{1}{\sqrt{EG}}(A\,\lambda + 2B\mu + C\nu).$$

If one substitutes the values $(9)_a$ in place of λ, μ, ν, writes T_u in place of T_{uv}, T_v in place of T_{vu}, T in place of $T_{uu} = T_{vv}$, and sets:

$$\sqrt{E}\,du = ds_u, \qquad \sqrt{G}\,dv = ds_v$$

then one will get:

$$U = -\frac{\partial T}{\partial s_v} - \frac{\partial T_v}{\partial s_u} - \frac{2}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v}T + \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u}(T_u - T_v),$$

(*) Annali di TORTOLINI, 1852.

$$V = -\frac{\partial T}{\partial s_u} - \frac{\partial T_u}{\partial s_v} - \frac{2}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u}T + \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v}(T_v - T_u)\,,$$

$$W = -\left(\frac{AT_v}{E} + \frac{2BT}{\sqrt{EG}} + \frac{CT_u}{G}\right).$$

Finally, if one recalls that if:

$$\frac{1}{R_u}, \quad \frac{1}{r_u}, \quad \frac{1}{R_v}, \quad \frac{1}{r_v}$$

denote the normal and tangential (namely, geodetic) curvatures of the lines u and v, respectively, then one will have:

$$A = -\frac{E}{R_u}, \qquad C = -\frac{G}{R_v}, \qquad \frac{\partial\sqrt{E}}{\partial v} = \frac{\sqrt{EG}}{r_u}, \qquad \frac{\partial\sqrt{G}}{\partial u} = \frac{\sqrt{EG}}{r_v},$$

and that the quantity:

$$\frac{1}{S} = \frac{B}{\sqrt{EG}}$$

is the geodetic torsion of the line u (which is equal and opposite in sign to that of the line v) then one can give the following form to the preceding equations:

$$\begin{cases} U = -\dfrac{\partial T}{\partial s_v} - \dfrac{\partial T_v}{\partial s_u} - \dfrac{2T}{r_u} + \dfrac{T_u - T_v}{r_v}, \\[2ex] V = -\dfrac{\partial T}{\partial s_u} - \dfrac{\partial T_u}{\partial s_v} - \dfrac{2T}{r_v} + \dfrac{T_v - T_u}{r_u}, \\[2ex] W = \dfrac{T_v}{R_u} - \dfrac{2T}{S} + \dfrac{T_u}{R_v}. \end{cases}$$

These are (if one abstracts from the difference in sign, which is due to differing conventions) the equations of equilibrium that were given for the first time by LECORNU, which are equations in which, as one sees, no restrictive hypothesis is made upon the choice of the orthogonal lines u and v ; i.e., no relation between the course of those lines and the distribution of the tensions is assumed *a priori*, so they are perfectly general.

If one assumes that the orthogonal lines u and v are those of normal tension then one must set $T = 0$, and the preceding equations will become:

$$\begin{cases} U = -\dfrac{\partial T_v}{\partial s_u} + \dfrac{T_u - T_v}{r_v}, \\ V = -\dfrac{\partial T_u}{\partial s_v} + \dfrac{T_v - T_u}{r_u}, \\ W = \dfrac{T_v}{R_u} + \dfrac{T_u}{R_v}, \end{cases}$$

under that hypothesis. These equations coincide with the ones in the article that was cited before that BRIOSCHI had deduced from the equations that were referred to above in the case of orthogonality. They are usefully applicable in all cases (which are certainly not infrequent) in which the nature of the question will indicate the disposition of the lines of normal tension *a priori*.

Finally, if one supposes that one of the two systems of lines of normal tensions – for example, that of the lines v – is composed geodetic lines (which obviously can happen only in special cases) then one must set $r_v = \infty$, and one will obtain the MOSSOTTI equations from the preceding:

$$\begin{cases} U = -\dfrac{\partial T_v}{\partial s_u}, \\ V = -\dfrac{\partial T_u}{\partial s_v} + \dfrac{T_v - T_u}{r_u}, \\ W = \dfrac{T_v}{R_u} + \dfrac{T_u}{R_v}. \end{cases}$$

The equations that POISSON gave (on pp. 179 of the cited paper) cannot be deduced from the general equations in any way, since they are based upon the inadmissible hypothesis of *unequal normal* tensions acting upon elements that are *oblique* to each other, in general. It is only in the case of equality of the tensions that those equations will become the translation of the LAGRANGE hypothesis.

CISA DE GRESY obtained the same thing for equations with two tensions in the paper that was cited above, but under hypotheses that were not very plausible, and quite artificial, in any case; his equations led back to those of POISSON. CISA DE GRESY did not know how to derive the maximum benefit from starting from the considerations (which were justified, for the most part) that one reads in the foreword to his work. In particular, he observed therein that "in order to get a general solution of the problem of surfaces in equilibrium, one must be able to *express the inextensibility of the surface in the calculations in a general manner*." That general manner of expressing the inextensibility consists simply of imposing the conditions (2), which is an observation that might now seem quite obvious, but which is, in reality, a long way from being the first one that GAUSS's doctrine made known.

§ 12. On the infinitesimal deformation of a flexible, inextensible surface

The conditions (2) of inextensibility translate into the following equations:

$$\text{(17)}\qquad \begin{cases} E\dfrac{\partial \delta u}{\partial u}+F\dfrac{\partial \delta v}{\partial u}+\frac{1}{2}\delta E = A\,\delta w, \\ E\dfrac{\partial \delta u}{\partial v}+F\left(\dfrac{\partial \delta u}{\partial u}+\dfrac{\partial \delta v}{\partial v}\right)+G\dfrac{\partial \delta v}{\partial u}+\delta F = 2B\,\delta w, \\ F\dfrac{\partial \delta u}{\partial v}+G\dfrac{\partial \delta v}{\partial v}+\frac{1}{2}\delta G = C\,\delta w, \end{cases}$$

by virtue of formulas (7).

Other equations that are entirely equivalent to these can be obtained from formulas $(7)_c$ by setting $\delta E = \delta F = \delta G = 0$.

First of all, we would like to show how the three equations (17) can be summarized in just one supremely simple formula.

To that end, observe that by the definitions of the variations δu, δv (§ **5**), the quantities $u + \delta u$, $v + \delta v$ will be the coordinates of the point at which the original surface s is met by the normal w that passes through the point in space to which the point (u, v) will be transported when the surface suffers an infinitely-small deformation. However, those same variations δu, δv can also be considered from another standpoint, i.e., as the increments that the variables u and v receive when the point (u, v) changes position *on the original surface* by passing to the position that is occupied by the foot of the aforementioned normal. When considered from this second standpoint, denote the variations by δu and δv and notice immediately that in order to perform that displacement on the surface, the quantities E, F, G will have to take the increments that were already denoted by δE, δF, δG in equations (17) when referred to the point that is displaced. Having said that, if one lets du, dv denote other *arbitrary* increments of the variables u, v and one sums the aforementioned equations, after having multiplied them by du^2, $du\,dv$, dv^2, then one will get:

$$(E\,du + F\,dv)\,d\delta u + (F\,du + G\,dv)\,d\delta v + \tfrac{1}{2}(\delta E\,du^2 + 2\delta F\,du\,dv + \delta G\,dv^2)$$

$$= (A\,du^2 + 2B\,du\,dv + C\,dv^2)\,\delta w\,.$$

However, from the significance of the symbols δu, δv , one has:

$$d\delta u = \delta du, \qquad d\delta v = \delta dv,$$

so the last equation will be equivalent to the following one:

$$\tfrac{1}{2}\,\delta(E\,du^2 + 2F\,du\,dv + G\,dv^2) = (A\,du^2 + 2B\,du\,dv + C\,dv^2)\,\delta w\,;$$

hence, from formulas $(1)_a$, $(4)_d$, one will finally have:

$$\frac{\delta ds}{dx} + \frac{\delta w}{R} = 0. \tag{17$_a$}$$

That very simple formula (*), which is easy to interpret geometrically, subsumes all three equations (17). Indeed, is it obvious that when one performs the transformation again in the opposite sense and observes that the direction of the element *ds* (viz., the value of the ratio *du* : *dv*) is arbitrary, it will again resolve into three equations, namely, (2).

JELLETT already considered some relations that were analogous to (17), and one can basically confer his interesting paper "On the properties of inextensible surfaces," Transactions of the Royal Irish Academy **22** (1853), 343-377. That author's equations (*B*) correspond precisely to equations (17), just as his equations (*C*) correspond to the ones that result from formulas $(7)_c$, except that JELLETT assumed Cartesian coordinates *x*, *y*, in place of our variables *u* and *v*, so the verification of that correspondence cannot be achieved by a simple substitution, but will require some caveats. Indeed, supposing that $u = x$, $v = y$ will not exactly imply that the variations δu, δv can be identified with δx, δy with no further discussion, and in fact, equations (6) will give:

$$\delta x = \delta u + \alpha\,\delta w, \qquad \delta y = \delta v + \beta\,\delta w, \qquad \delta z = p\;\delta u + q\;\delta v + \gamma\,\delta w,$$

under those hypotheses, in which $p = \partial z / \partial x$, $q = \partial z / \partial x$. One will then observe that if:

$$E = 1 + p^2, \qquad F = pq, \qquad G = 1 + q^2$$

then the first equation (17) will become:

$$\frac{\partial \delta u}{\partial x} + p\,\frac{\partial (p\,\delta u + q\,\delta v)}{\partial x} = \frac{\partial p}{\partial x}\,\gamma\,\delta w,$$

namely:

$$\frac{\partial (\delta x - \alpha\,\delta w)}{\partial x} + p\,\frac{\partial (\delta x - \gamma\,\delta w)}{\partial x} = \frac{\partial p}{\partial x}\,\gamma\,\delta w,$$

or also:

$$\frac{\partial \delta x}{\partial x} + p\,\frac{\partial \delta z}{\partial x} = \frac{\partial [(\alpha + p\gamma)\,\delta w]}{\partial x}.$$

However, one has $\alpha + p\gamma = 0$, so:

$$\frac{\partial \delta x}{\partial x} + p\,\frac{\partial \delta z}{\partial x} = 0,$$

and one similarly gets:

(*) In order to make the significance of this formula as precise as possible, let *ds*′ be the line element that corresponds to *ds* on the deformed surface – i.e., the element that *ds* is converted into (such that *ds*′ = *ds*). Project *ds*′ normally to the original surface (by means of the line *w*) and let *ds*″ be the projected element. The symbol δds will then represent the difference *ds*″ – *ds*.

$$\frac{\partial \delta x}{\partial y}+\frac{\partial \delta y}{\partial x}+p\frac{\partial \delta z}{\partial y}+q\frac{\partial \delta z}{\partial x}=0,$$

$$\frac{\partial \delta y}{\partial y}+q\frac{\partial \delta z}{\partial y}=0,$$

from the other two equations (17). These are JELLETT's three equations (A), whose direct proof is naturally simpler.

LECORNU also gave three equations that were analogous to the preceding one at the end of his Chap. I, but in a different form. In order to obtain his formulas, one needs to make use of the relations:

$$\cos\,\theta=\frac{F}{\sqrt{EG}}, \qquad \sin\,\theta=\frac{H}{\sqrt{EG}},$$

which defines the angle between the lines u and v, along with:

$$\frac{d\sqrt{E}}{\partial v}=\frac{H}{r_u}+\frac{\partial\frac{F}{\sqrt{E}}}{\partial u}, \qquad \frac{d\sqrt{G}}{\partial u}=\frac{H}{r_u}+\frac{\partial\frac{F}{\sqrt{G}}}{\partial v},$$

which define the tangential curvatures of those lines. If one sets:

$$\sqrt{E}\;\delta u=\delta s_u, \qquad \sqrt{G}\;\delta v=\delta s_v,$$

in addition, then equations (17) will be easily transformed into the following ones:

$$\frac{\partial \delta s_u}{\partial s_u}+\frac{\partial \delta s_v}{\partial s_u}\cos\theta+\left(\frac{1}{r_u}-\frac{\partial\theta}{\partial s_u}\right)\delta s_v\sin\theta=\frac{A\,\delta w}{E},$$

$$\frac{\partial \delta s_u}{\partial s_v}+\frac{\partial \delta s_v}{\partial s_u}+\left(\frac{\partial \delta s_u}{\partial s_u}+\frac{\partial \delta s_v}{\partial s_v}\right)\cos\theta-\left(\frac{\delta s_u}{r_u}+\frac{\partial \delta s_v}{r_v}\right)\sin\theta=\frac{2B\,\delta w}{\sqrt{EG}},$$

$$\frac{\partial \delta s_v}{\partial s_v}+\frac{\partial \delta s_u}{\partial s_v}\cos\theta+\left(\frac{1}{r_v}-\frac{\partial\theta}{\partial s_v}\right)\delta s_u\sin\theta=\frac{C\,\delta w}{G}.$$

When the lines u and v are mutually-orthogonal, those equations will become:

$$\frac{\partial \delta s_u}{\partial s_u}+\frac{\delta s_v}{r_u}=\frac{A\,\delta w}{E},$$

$$\frac{\partial \delta s_u}{\partial s_v} + \frac{\partial \delta s_v}{\partial s_u} - \frac{\delta s_u}{r_u} - \frac{\partial \delta s_v}{r_v} = \frac{2B\,\delta w}{\sqrt{EG}},$$

$$\frac{\partial \delta s_v}{\partial s_v} + \frac{\delta s_u}{r_v} = \frac{C\,\delta w}{G},$$

and these coincide with the ones in LECORNU except for the difference in symbols.

When one must operate on equations (17), it is good to keep in mind some second relations that are consequences of them and which will serve to ease the calculations. We cite only two of them, due to their special importance. The first one is the following one:

$$\frac{\partial (H\,\delta u)}{\partial u} + \frac{\partial (H\,\delta v)}{\partial v} + Hh\,\delta w = 0,$$

which one deduces immediately from equation $(7)_a$. The second one, which requires a bit more artifice, is this one:

$$\partial k - hk\,\delta w + \nabla(\delta w) = 0.$$

The symbols h, k, ∇ are the ones that were adopted already in §§ **8**, **9**. The last relation will serve, for example, to verify the invariability of the measure of curvature k, since if one looks for the variation of that quantity as a result of an *arbitrary* infinitesimal deformation (*) then one will find precisely:

$$\delta k = \partial k - hk\,\delta w + \nabla(\delta w).$$

That verification can be achieved more expediently by JELLETT's simple formulas. However, the too-special choice of the independent variables will make those formulas less adapted to other applications, such as, for example, the ones that we have in mind in the present article.

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(*) I.e., it is not constrained by the conditions (2).