"Formules fondamentales de cinématique dans les espaces de courbure constante," Bull. des Sci. math. et astrophysiques **11** (1876), 233-240. Extract from a paper presented to the Royal Academy of Lincei in Rome. *Opera matematiche di Eugenio Beltrami*, XLVII.

Fundamental formulas of kinematics in spaces of constant curvature

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I shall take the expression for the square of the line element in the known form:

$$\frac{ds^2}{R^2} = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{x^2}, \tag{1}$$

in which $x_1, x_2, \ldots, x_n$ are the *linear* coordinates of an arbitrary point in *n*-space (i.e., such that each line is represented by $n - 1$ equations of degree one), R is the constant pseudo-spherical radius, and x is an extra variable that is defined by the equation:

$$x^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = a^2, \tag{2}$$

in which a is a bounded constant.

I shall now consider a continuous system of points. I let $\delta x_1, \delta x_2, \ldots, \delta x_n$ denote the infinitely-small variations of the coordinates $x_1, x_2, \ldots, x_n$ of one of those points as a result of an arbitrary elementary displacement, and let δx denote the variation that ensues for x, and I would like to seek an expression of a convenient form for the variations δds that the distance ds between two contiguous points of the system experiences.

One infers from equation (1), when written in this manner:

$$\frac{ds^2}{R^2} = \left(\frac{dx}{x}\right)^2 + \sum\left(\frac{dx_r}{x}\right)^2,$$

that:

$$\frac{ds\,\delta ds}{R^2} = \frac{dx}{x}\,\delta\frac{dx}{x} + \sum\frac{dx_r}{x}\,\delta\frac{dx_r}{x};$$

as a result of the identity:

$$\frac{dx_r}{x} = d\,\frac{x_r}{x} + \frac{x_r\,dx}{x^2},$$

the latter can also be written in the form:

$$\frac{ds\,\delta ds}{R^2} = \frac{dx}{x}\delta\frac{dx}{x} + \sum\frac{dx_r}{x}d\delta\frac{x_r}{x} + \sum\frac{dx_r}{x}\delta\left(\frac{x_r}{x}\frac{dx}{x}\right).$$

However, one also has:

$$\sum\frac{dx_r}{x}\delta\left(\frac{x_r}{x}\frac{dx}{x}\right) = \frac{dx}{x}\sum\frac{dx_r}{x}\delta\frac{x_r}{x} + \delta\frac{dx}{x}\sum\frac{x_r\,dx_r}{x^2},$$

namely, from (2):

$$\sum\frac{dx_r}{x}\delta\left(\frac{x_r}{x}\frac{dx}{x}\right) = \frac{dx}{x}\sum\frac{dx_r}{x}\delta\frac{x_r}{x} - \frac{dx}{x}\delta\frac{dx}{x},$$

so

$$\frac{ds\,\delta ds}{R^2} = \sum\frac{dx_r}{x}\left(d\delta\frac{x_r}{x} + \frac{dx}{x}\delta\frac{x_r}{x}\right),$$

hence:

(3) $$\delta ds = \frac{R^2}{x^2}\sum\frac{dx_r}{ds}d\left(x\,\delta\frac{x_r}{x}\right).$$

That is the form that one agrees to give to the expression for δds.

Due to its generality, that formula can serve as the basis for the search for the fundamental equations of kinematics of systems of variable form. However, for the moment, I shall confine myself to the consideration of rigid systems, so I will set $\delta ds = 0$, which will give the following necessary and sufficient condition for a displacement to not be accompanied by a deformation:

(4) $$\sum dx_r\,d\left(x\,\delta\frac{x_r}{x}\right) = 0.$$

It now comes down inferring the most general values of the variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ as functions of the coordinates $x_1, x_2, \ldots, x_n$ from that equation.

First set:

$$X_r = x\,\delta\frac{x_r}{x} \qquad (r = 1, 2, \ldots, n).$$

One sees that, by virtue of equation (4), the n unknown functions $X_1, X_2, \ldots, X_n$ must satisfy the identity:

$$\sum_r\sum_s\frac{\partial X_r}{\partial x_s}dx_r\,dx_s \qquad \begin{cases}(r = 1, 2, \ldots, n)\\(s = 1, 2, \ldots, n),\end{cases}$$

which demands that one must have:

(5) $$\frac{\partial X_r}{\partial x_s} + \frac{\partial X_s}{\partial x_r} = 0$$

for all values of the indices r and s, whether they are equal or unequal. One infers from this equation that:

$$\frac{\partial}{\partial x_s}\left(\frac{\partial X_r}{\partial x_t}\right)+\frac{\partial}{\partial x_r}\left(\frac{\partial X_s}{\partial x_t}\right) = 0$$

for any third index t, namely, due to the same equation (5), when applied to the indices r, t and s, t in succession:

$$\frac{\partial}{\partial x_s}\left(\frac{\partial X_t}{\partial x_r}\right)+\frac{\partial}{\partial x_r}\left(\frac{\partial X_t}{\partial x_s}\right) = 0,$$

or finally:

$$\frac{\partial^2 X_t}{\partial x_r\,\partial x_s} = 0.$$

Since r, s, t are three arbitrary indices here (equal or unequal) from the series 1, 2, …, n, one will see from the last formula that the n functions X_1, X_2, …, X_n have all of their second derivatives equal to zero. They are then necessarily of the linear form:

$$X_r = c_r + c_{1r}\,x_1 + c_{2r}\,x_2 + \ldots + c_{nr}\,x_n\,,$$

in which the quantities c_r , as well as the c_{rs} , are constants with respect to the coordinates (and functions of time, in general); however, since the functions X must further satisfy the original conditions (5), the quantities c_{rs} are not absolutely arbitrary. One must have:

$$c_{rs} + c_{sr} = 0 \tag{6}$$

for all values (equal or unequal) of the indices r and s.

If those conditions are assumed to have been satisfied then one will have:

$$x\,\delta\frac{x_r}{x} = c_r + \sum_i c_{ir}\,x_i\,,$$

from which, one will infer that:

$$\delta x_r = c_r + \sum_i c_{ir}\,x_i + \frac{x_r}{x}\,\delta x\,, \qquad r = 1, 2, \ldots, n.$$

Multiplying this by x_r and summing over r, while keeping equations (2) and (6) in mind, one will find that:

$$\delta x = -\frac{x}{a^2}\sum c_r\,x_r\,,$$

which is a value that will finally give:

(7) $$\delta x_r = c_r + \sum_i c_{ir}\, x_i - \frac{x_r}{a^2} \sum_i c_i\, x_i$$

for $r = 1, 2, \ldots, n$ when it is substituted in the preceding formula. One must complete these n expressions with the one for δx:

(8) $$\delta x = -\frac{x}{a^2} \sum c_i\, x_i\ .$$

The n equations (7) are the fundamental differential formulas (analogous to those of Euler) of the kinematics of solid bodies in an n-space of constant curvature. The n (n + 1) / 2 arbitrary quantities c_r and c_{rs}, which one must consider, generally speaking, to be arbitrary functions of time t, multiplied by δt (viz., the infinitely-small duration of the elementary displacement), are the analogues of the six components of the translation and rotation in the usual theory.

One can infer a very important consequence from the complementary equation (8), which is necessary adjunct to formulas (7). Indeed, it results that for all points of the limit $(n - 1)$-space $x = 0$ (which is supposed to be linked with the solid system), one will have $\delta x = 0$. That is, that those points do not leave that $(n - 1)$-space, or (what amounts to the same thing) that the space is displaced within itself, while remaining invariable with respect to the n-space that is being considered. That property, which is only a corollary to the invariability that one supposes for the line element here, will, on the contrary, become the definition of the *special* homographic transformation that one calls a *motion of the invariable system* when the geometric of spaces of constant curvature is imagined, as Cayley and Klein did, to be a general projective theory; the projective conception of distance is the key to that identity, which is admirable, as well as fundamental.

Let $u_1, u_2, \ldots, u_n$ denote the coordinates of a point (or pole), while the linear equation in $x_1, x_2, \ldots, x_n$:

(9) $$u_1\, x_1 + u_2\, x_2 + \ldots + u_n\, x_n = a^2$$

represents what one can call the $(n - 1)$-plane that is polar to that point with respect to the limit space $x = 0$. If the point (u) is real then I would like to call it *interior* to $x = 0$, while the plane (9) is ideal – i.e., *exterior* to $x = 0$. On the contrary, the point (u) is ideal and the plane (9) is real; i.e., it possesses a simply-connected region that is indefinite in every sense that is interior to $x = 0$. Moreover, since equation (9) can represent an arbitrary $(n-1)$-plane, one can also define the coefficients $u_1, u_2, \ldots, u_n$ on the left-hand side of that equation to be the (tangential) coordinates of an $(n - 1)$-plane. Now, if one considers the limit locus $x = 0$ and the arbitrary plane (9) to be invariably coupled with each other then the pole (u) of the plane will also itself become invariably coupled to the locus $x = 0$, and since that locus can only slide along itself when it makes up part of an invariable system that moves in n-space, it will be obvious that the pole (u) itself must displace with the system, and as a result that the variations $\delta u_1, \delta u_2, \ldots, \delta u_n$ of the tangential coordinates of an $(n - 1)$-plane that makes up part of an invariable system that moves in n-space will be functions of $u_1, u_2, \ldots, u_n$ with the same form that the $\delta x_1, \delta x_2, \ldots, \delta x_n$ have with respect to the $x_1, x_2, \ldots$

That conclusion can be verified directly by inferring from equation (9) that:

$$\sum u_r\ \delta x_r + \sum x_r\ \delta u_r = 0,$$

namely:

$$\sum c_r\, u_r + \sum_r \sum_i c_{ir}\, u_r\, x_i - \sum c_r\, u_r + \sum x_r\ \delta u_r = 0,$$

or rather:

$$\sum_r (\delta u_r - \sum_i c_{ir}\, u_i - c_r)\, x_r + \sum_i c_i\, u_i = 0.$$

The relation that this formula establishes between the $x_1, x_2, \ldots, x_n$ obviously cannot differ from the one (9) that one started from; one will then have:

$$\frac{\delta u_r - c_r - \sum_i c_{ir}\, u_i}{u_r} + \frac{\sum_i c_i\, u_i}{a^2} = 0,$$

so

(7)
$$\delta u_r = c_r + \sum_i c_{ir}\, u_i - \frac{u_r}{a^2} \sum_i c_i\, u_i$$

for $r = 1, 2, \ldots, n$. These n formulas are perfectly similar to formulas (7).

If there is some point $(x_1, x_2, \ldots, x_n)$ that remains immobile during the elementary motion of the invariable system then the variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ of its coordinates must all be = 0 at the instant considered, and therefore one will also have $x\ \delta x = 0$ for that same point ; i.e., $\delta x = 0$, if one supposes that the point is not found on the limit $x = 0$. Now, due to (8), those condition $x > 0$, $\delta x = 0$ will give:

$$\sum c_i\, x_i = 0,$$

and as a result, the conditions $\delta x_r = 0$ give, in turn:

(10)
$$c_{ir} + \sum c_{ir}\, x_i = 0, \qquad r = 1, 2, \ldots, n,$$

which are equations that imply the preceding one.

When there exists a system of values of the $x_1, x_2, \ldots, x_n$ that satisfies these n linear equations there will be a point (which will be real or ideal according to whether $x_1^2 + x_2^2 + \ldots + x_n^2$ is $<$ or $>$ a^2, resp.) that possesses the character of an *instantaneous center of rotation*, and whose polar $(n - 1)$-plane with respect to $x = 0$ is an *instantaneous glide* $(n - 1)$*-plane* (which will be ideal or real according to whether the pole is real or ideal, resp.).

Now, due to (6), the determinant:

$$\sum \pm\ c_{11}\, c_{22} \ldots c_{nn}$$

of equations (10) will be equal to zero or a positive quantity that is generally non-zero according to whether the number n is odd or even, resp. Therefore:

When n is even there will always exist either a real instantaneous center of rotation or a real instantaneous glide $(n - 1)$-plane in an n-space of constant curvature for each (completely general) elementary motion of the rigid system.

When n is odd there will never exist, in general, either a center of rotation or a glide $(n - 1)$-plane in an n-space of constant curvature. However, if the motion is such that it has an instantaneous center [or instantaneous $(n - 1)$-plane] then it will have an infinitude of them that will form a line or pencil.

For the moment, I shall stop with these conclusions of an absolutely-general nature whose development and discussion has led me quite far, moreover. I shall add simply the remark that the fundamental theorems of ordinary kinematics already provide us with some special examples of the preceding general properties. Indeed, it teaches us that there always exists an instantaneous center of motion in the plane, while there will not generally exist an analogous point in *three*-dimensional space, or if it does exists then there will be an infinitude of them that lie along a straight line. On the contrary, there will always exist an instantaneous line in that space that one calls the *central axis* of the motion. Now, that fact is in perfect accord with the preceding theorems, because Euclidian space, when one considers the line to be the primitive element (analytical point), is an n-space of constant curvature for which n is even and equal to 4. It must always have an invariable instantaneous element then, and that element, which is in a line in this case, will be precisely the central axis. In the same case of $n = 4$, one will have:

$$\sum (\pm\, c_{11}\, c_{22} \ldots c_{nn}) = (c_{14}\, c_{23} + c_{24}\, c_{31} + c_{34}\, c_{12})^2,$$

as one knows, and under the special hypothesis that $c_{14}\, c_{23} + c_{24}\, c_{31} + c_{34}\, c_{12} = 0$, the number of invariable elements can become infinite. That condition answers to that of (ordinary) simple rotation, as one can convince oneself.

Upon adopting, with Schering, the term of *Gaussian* and *Riemannian* space for the spaces of constant curvature whose measure of curvature is negative or positive, respectively, one will see that the preceding results refer to the Gaussian spaces. There is a completely similar theory for the Riemannian spaces, and it would be easy for the reader to formulate it from the foregoing. There is no essential difference in regard to n-spaces for which n is odd, but when n is even, the center of rotation and the glide $(n - 1)$-plane will always exist *simultaneously* in the real state, no matter what the elementary motion is. The simplest example, which is drawn from ordinary kinematics, is provided by the displacement of a spherical figure on its own sphere. There will always be a center of rotation then, and at the same time, a great circle of sliding (whose center is the pole).