Excerpted from J.-L. Lagrange, *Mécanique analytic*, v. I, 1788.

NOTE III

On the equilibrium of an elastic line

(By J. Bertrand)

Translated by D. H. Delphenich

The formulas that Lagrange gave (page 143) suppose that the elastic force at each point is exerted in the osculating plane of the line in equilibrium in such a way that it tends to restore the original radius of curvature. However, such a hypothesis is loath to represent any phenomena, and Binet has remarked that it is essential to add another force to the elastic force that Lagrange considered whose effect is to oppose the variations of the second curvature. While developing the consequences of the last remark, the complexity in the formulas that express that second curvature prevents us from preserving the notation and the path that was followed by Lagrange. We shall confine ourselves to just defining the equations of equilibrium directly by imitating the method that Poisson presented in an article in the *Correspondance sur l'École Polytechnique* (Tome III, page 355).

Consider an elastic line in equilibrium *AMB* whose points are all subjected to given forces. If we suppose that the part *MB* that is found between an arbitrary point *M* and the extremity *B* will become inflexible and fixed and that the other part *MA* will become only inflexible, while preserving its freedom to turn around the point *M*, then equilibrium will not be destroyed, and consequently, the elastic force that is developed at *M* must cancel the *couple* that is equivalent to the forces that act on the portion *MA* of the curve, due to the fixed nature of the point *M*. Now, we shall assume that the elastic force can produce two couples, one of which (namely, the one that Lagrange considered) acts in the osculating plane and tends to restore the curvature to its original value. The other one, which has the tangent to the elastic curve for its axis, tends to cancel the *torsion*, while restoring the second curvature to its original value. Call those two couples θ and *E*. We shall first prove that θ is constant, *no matter what the given forces and original form of the curve.*

Indeed, in order to determine the two couples θ and *E*, one must reduce the forces that act upon the portion *MA* of the curve to a force *F* that passes through the point *M* and a couple *G*. That couple *G* must be equivalent to two couples – viz., θ and *E* – whose axes are the tangent to the proposed curve and a perpendicular to its osculating place. If we begin the same decompositions once more, while replacing the point *M* with an infinitely-close one M', then the force *F* and couple *G* will vary, on the one hand, due to the changes in the point of application of the force, and on the other, due to the influence of the new forces that act on the arc MM'. We first remark that the latter forces can exert no influence upon the value of the couple θ, because their point of application is at a distance from the tangent to the point M' (which is the axis of the couple) that is infinitely-small of order two. It will then suffice to consider the change in position of the fixed point, and that change will obviously have the effect of adding a second couple to the couple G that is produced by the force F and by a force that is equal and opposite and applied at M'. Now, the force F, like the ones that are applied to the arc MM', has its point of application situated at a distance from the tangent at M' that is infinitely-small of order two, in such a way that it will modify the desired coupled whose axis is the tangent by only a quantity of that order. From those remarks, one can calculate the value θ' of the torsion couple that corresponds to the point M' as if neither the magnitude nor the direction of the couple G changed. One must only decompose it into two other ones now, one of which is perpendicular to the tangent at M'. In order to calculate the component of the couple that represents the desired torsion moment, one must replace the couple G with the two couples $-\theta$ and $-E$ that are equivalent to it. Each of these couples must be multiplied by the cosine of the angle that its axis forms with the axis of the couple θ', which is nothing but the tangent to the curve under consideration at the point M'. The axes of the couples θ and θ' form an infinitely-small angle whose cosine is equal to unity, if we neglect the second-order infinitesimals, as above. As for the axis of the couple $-E$, the angle that it forms with the tangent at M' is a right angle if one once more neglects the second-order infinitesimals, because the osculating plane at M is parallel to the tangent at M'. The cosine of that angle can then be considered to be zero, and upon neglecting the second-order infinitesimals, one will have:

$$\theta' = \theta,$$

from which one concludes that the torsion moment is rigorously constant along the elastic curve.

From that remark, one can form the equations of equilibrium by writing down that the forces that are applied to an arbitrary portion MA of the curve that is supposed to be rigid are cancelled by the fixed nature of the point M and by two couples $-\theta$ and $-E$ whose axes are the tangent to the curve and the axis of the osculating plane, respectively; θ is constant and E is proportional to the difference between the present curvature at M and the original curvature at the same point.

In particular, we shall consider the case in which the curve is originally straight and the only force that is applied to it acts on its extremity A, while the extremity B is fixed. Upon supposing that one fixes a point M whose coordinates are x, y, z, the moments of the given forces with respect to that point will have components of the form:

$$\begin{aligned} &cy - bz + a_1, \\ &az - cx + b_1, \\ &bx - ay + c_1, \end{aligned}$$

in which a, b, c, a_1, b_1, c_1 are constants that depend upon the direction of the force and the position of its point of application. Upon equating those moments to the elastic couples that are decomposed perpendicularly to the same three axes, we will have the equations:

$$p\frac{dy\,d^2z - dz\,d^2y}{ds^2} = \theta\frac{dx}{ds} + cy - bz + a_1,$$

$$p\frac{dy\,d^2z - dz\,d^2y}{ds^2} = \theta\frac{dy}{ds} + az - cx + b_1,$$

$$p\frac{dx\,d^2y - dy\,d^2x}{ds^2} = \theta\frac{dz}{ds} + bx - ay + c_1,$$

which differs from those of Lagrange (page 148) only by the notation and the introduction of the terms in θ.

After having obtained these equations, Lagrange added that: *Their integration is probably impossible, in general.* We shall show that this is, on the contrary, *always* possible, and in order to do that, we shall follow the path that was indicated by Binet (*) and simplified shortly after that by Wantzel.

If one takes the z-axis to have the same direction as the given force then, as one will easily see, the preceding formulas will take on the form:

$$(1)\qquad \begin{cases} p\dfrac{dy\,d^2z - dz\,d^2y}{ds^2} = \theta\dfrac{dx}{ds} + gy, \\ p\dfrac{dz\,d^2x - dx\,d^2z}{ds^2} = \theta\dfrac{dy}{ds} - gx, \\ p\dfrac{dx\,d^2y - dy\,d^2x}{ds^2} = \theta\dfrac{dz}{ds}, \end{cases}$$

in which g is a constant.

The last equation shows that if one neglects θ, as Lagrange did, then the curve will be necessarily planar. Upon multiplying those equations by dx, dy, dz and adding them, one will get (**):

$$(2)\qquad 0 = \theta\,ds + g\,(y\,dx - x\,dy)\,.$$

Upon adding the first two, when multiplied by x and y, respectively, one will then find that:

$$(3)\qquad \frac{p}{ds^2}d^2z\,(x\,dy - y\,dx) - \frac{p\,dz\,(x\,d^2y - y\,d^2x)}{ds^2} = \theta\left(\frac{x\,dx + y\,dy}{ds}\right),$$

or, by virtue of the preceding, if one takes s to be the independent variables then:

(*) See the *Comptes rendus de l'Académie des Sciences* for 1844, pages 1115 and 1197.

(**) One can remark that if one can suppose that $x = 0$, $y = 0$ in formula (2) then one can conclude that $\theta = 0$. In order for there to be torsion, it will then be necessary that the force should not be applied directly to the point of the curve on which it exerts its action. (*J. Bertrand*)

(4) $$\frac{p}{g}\frac{d^2 z}{ds^2} = \frac{x\,dx + y\,dy}{ds},$$

and upon integrating this:

(5) $$\frac{2p}{g}\frac{dz}{ds} = x^2 + y^2 - \frac{c}{g}.$$

If one replaces x and y with polar coordinates by setting:

$$x^2 + y^2 = r^2, \qquad \frac{y}{x} = \tan\,\omega,$$

then the preceding equations will become:

$$r^2\,d\omega = \frac{\theta}{g}ds, \qquad \frac{dz}{ds} = \frac{gr^2 - c}{2p}.$$

Hence, upon setting $dz / dt = \cos\,\varphi$ and appealing to the known formula:

$$ds^2 = dr^2 + r^2\,d\omega^2 + dz^2,$$

one will deduce that:

$$ds = \frac{p \sin\varphi\, d\varphi}{\sqrt{g \sin^2\varphi(2p\cos\varphi + c) - \theta^2}},$$

$$d\omega = \frac{\theta \sin\varphi\, d\varphi}{(2p\cos\varphi + c)\sqrt{g \sin^2\varphi(2p\cos\varphi + c) - \theta^2}}.$$

One will then have:

$$dz = \int ds \cos\varphi,$$

$$x = r\cos\,\omega,$$
$$y = r\sin\,\omega,$$

$$r^2 = \frac{\theta}{g}\frac{ds}{d\omega},$$

in such a way that x, y, z can be expressed as functions of the angle φ by quadratures.

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