“Note sur la surface des ondes,” Comptes rendus **47** (1858), 817-819.

Note on the wave surface

By **J. BERTRAND**

Translated by D. H. Delphenich

In the last issue of *Quarterly Journal of Mathematics*, Cayley studied the developable surface that circumscribes both the Fresnel wave surface and a concentric sphere. From a theorem that is due to a German geometer and is cited without proof, that developable surfaces will touch the wave surface along a line of curvature.

That result seemed sufficiently remarkable to me that I was inspired to look for a proof of it. However, it did not take long for me to see that it was, unfortunately, incorrect. The authority that is carried by the name of the illustrious geometer that I cited made me think that it would not be pointless to show here how it is impossible.

THEOREM I. – *If one drops perpendiculars from a point O to the tangent planes to a surface then the locus of their feet will be a new surface. Let P be the surface of that surface that corresponds to the point M of the first one, so the normal at P will pass through the midpoint of OM.*

THEOREM II. – *If a line of curvature of a surface is such that the tangent planes to the different points of that line are equidistant from a point O then that line will be situated in a sphere that is described with the point O as its center.*

Indeed, drop perpendiculars from the point *O* to the tangent planes that are drawn on the surface through points of the line of curvature considered. The locus of their feet will be a spherical curve such that a normal at an arbitrary point *P* will obviously pass through the point *P*. Furthermore, by virtue of the preceding theorem, another normal will pass through the midpoint of the radius *OM* that ends at the corresponding point on the given surface. The tangent to the curve that is the locus of the point *P* will then be perpendicular to the plane *MOP*, and in turn, to the line *MP*. Now, when a developable surface is circumscribed by a sphere, the perpendiculars that are dropped from the center of the that sphere onto the tangent planes to the surfaces have their feet on the generators, as a result, the curve that is the locus of points *P* is situated on the developable surface and cuts the generators at a right angle. However, since the curve that is the locus of point *M* is, by hypothesis, a line of curvature that also cuts the generators of the same surface at a right angle, those two curves will then be equidistant, *MP* will be constant, and since *OP* is also constant, by hypothesis, the same thing will be true for *OM*, which proves the stated proposition.

Having said that, recall that the wave surface can be generated in two different ways:

1. It is the locus of extremities of perpendiculars that are raised from the diametral sections of an ellipsoid *E* and are equal to the axes of those sections.

2. It is the envelope of planes that are parallel to the diametral sections of a second ellipsoid E' and drawn to distances that are inversely proportional to the axes of those sections.

In order to obtain all the tangent planes to the wave surface that are situated at a distance h from the center, one must look for diametral sections of the ellipsoid E' that have an axis of length $1 / h$. In order to do that, one cuts that ellipsoid with a concentric sphere of radius $1 / h$, and one draws the tangent planes to the cone that has its summit at the center of the sphere and its intersection with the sphere as its base. The tangent planes to the wave surface that are parallel to the tangent planes to that cone, respectively, will be at a distance h from the center, and if they touch the wave surface at points of the same line of curvature then it will result from the preceding theorem that their points of contact will also be at the same distance from the center. However, the distance from the center of the wave surface to the point of contact to one of its tangent planes is inversely proportional to the perpendicular that is dropped from the center of the ellipsoid E' to the tangent plane to the corresponding point of that ellipsoid. Now, the curve of intersection of an ellipsoid with a concentric sphere is not such that the tangent planes to its various points are at the same distance from the center, and as a result the tangent planes considered will not determine a line of curvature of the wave surface by their points of contact.