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# Memoir on the theory of surfaces

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Translated by D. H. Delphenich

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To me, the general theorems of Euler and Monge on the theory of surfaces seem to be the most beautiful propositions that are known to geometry, because they are the most general. Those simple, elegant laws on the curvature of surfaces, which were proved independently of the particular definition of the surface that one is addressing, are, at the same time, eminently suited to the task of making the true spirit of the very fertile method of infinitesimals and the law of continuity comprehensible when it is transported from analysis to geometry. However, in order to exhibit the nature of those beautiful theorems more clearly, I believe that it is useful to present them in a manner that is somewhat different from the one that one usually adopts.

Both theorems express a general property of the normals to the same surface and can be stated without one having to introduce the surface itself or its sections by different planes. However, although the one is different from the other, the first proposition – viz., that of Euler, which tells us that the sum of the curvatures of two normal sections that are perpendicular to each other will be constant – is a true identity. It result solely from the law of continuity, and cannot by any means serve to characterize the normals to a surface. If (as is quite simple) one makes the words *surface* and *plane sections* disappear from its statement then one will obtain a theorem that applies to lines that are situated in space in an arbitrary manner, provided that their directions are expressed by continuous functions.

On the contrary, Monge’s theorem (viz., the existence of two perpendicular directions along which two normals will meet) is essentially particular to the lines that are normal to a surface, and there exists no corresponding proposition for straight lines that are distributed randomly in space.

As I see it, that beautiful theorem of Monge suffices to characterize surfaces completely, so it is the most general property. However, one can generalize it, as well, and make it more suited to the applications. Indeed, I was led to the following theorem, of which, that of Monge is obviously only a special case:

*If one draws a normal AZ through an arbitrary point A on a surface, and one then makes two perpendicular lines pass through the surface, along which one takes infinitely-small lengths that are equal to AB, AC then the normal to the point B will make an angle with the plane ZAB that is equal to the one that the normal at the point C makes with the plane ZAC. I will add that the two normals are either both inside of the dihedral angle BAC or both outside of that angle.*

In this paper, I shall also give a proposition that can be considered to be the complement of that of Euler, and which, when combined with it, will characterize the law of variation of the normals to a surface around a given point in the most complete manner.

Indeed, Euler showed the law of variation for the curvature of normal sections; i.e., upon adopting the symbols that were employed in the statement of the preceding theorem, the manner by which the projection of the normal at the point *B* onto the plane *ZAB* is inclined towards the principal normal *ZA*. However, in order to know the position of that normal at each point, it will not suffice to be able to determine its projection onto a known plane, since one must also know the angle that it forms with that projection. That angle is subject to a very simple law of variation that is expressed by the following theorem:

*If AZ is the normal to a surface at an arbitrary point A, and AP, AQ denote the directions of the two lines of curvature at that point, and if one take an infinitely-small length AB in a direction AB on the surface then the normal at the point B thus-obtained will make an angle with the plane ZAB that is expressed by the following formula:*

$$\frac{AB}{2}\left(\frac{1}{R}-\frac{1}{r}\right)\sin 2\alpha,$$

*in which R and r denote the two radii of curvature that corresponds to the lines AP, AQ and the angle BAQ.*

The first of the two theorems that we stated is an immediate consequence of the second.

These new properties of surfaces express the necessary condition for lines to be normal to a series of surfaces in a very convenient form. As I see it, it will suffice that the property that is expressed by the first of my new theorems is verified for two perpendicular directions that are taken by starting from each point of space. That being the case, it will necessarily be true for all other directions.

That simple definition of the normal to a surface has allowed me to prove geometrically some beautiful results that Sturm was led to by analysis.

The first of these two theorems of Dupin relates to orthogonal surfaces. It consists of saying that three series of orthogonal surfaces will always intersect along their lines of curvature. I shall give a very simple proof of this, which had been conjectured up to now.

The second application of the results that are obtained in this article relates to optics.

Malus has proved that light rays that start from the same point and reflect from an arbitrary surface will remain normal to the same series of surface after their reflection. Dupin generalized Malus's proposition by applying it to not only rays that start from the same point, but to rays that are directed in an arbitrary manner. Provided that they are normal to the same surface, they will preserve that property after having been reflected or refracted in an arbitrary manner, and for whatever surface separates the media. I shall give geometric proofs of all those theorems, and I shall even point out the most general law of refraction that can arise from them.

Sturm, who addressed the same question, gave some general formulas that permit one to calculate the radii of curvature and the position of the principal sections of the normal surface to the refracted rays, provided that one knows analogous elements for either the normal surface to the incident rays or for the separation surface of the media. I arrived geometrically at some formulas that are analogous to those of Sturm, and which will fulfill the same purpose.

## I.

**1.** Let *A* be a point that is taken on a surface, and let *AZ* be the direction of the normal at that point. Take the *z*-axis to be that line *A*, and take the *x* and *y* axes to be perpendicular that are chosen at random in the tangent plane. If the equation of the surface is:

$$z = \varphi(x, y)$$

then the angles that normals make with the three axes will be cosines that are proportional to the quantities:

$$\frac{dz}{dx} = p, \qquad \frac{dz}{dy} = q, \qquad \text{and} \qquad 1,$$

respectively, and one can represent them by:

$$\lambda p, \qquad \lambda q, \qquad \lambda,$$

resp. If one starts at the point *A* and takes two infinitely-small lengths *AB*, *AC* along the *X* and *Y* axes, resp., then the points *B* and *C* thus-obtained can be considered to be on the surface, and it is easy to calculate the cosines of the angles that the normals at those two points will form with the axes.

The angles that relate to the normal at the point *B* will have the cosines:

$$\frac{d\cdot\lambda p}{dx}\cdot AB, \qquad \frac{d\cdot\lambda q}{dx}\cdot AB, \qquad 1,$$

and the ones that define the normal to the point *C* have the cosines:

$$\frac{d\cdot\lambda p}{dy}\cdot AC, \qquad \frac{d\cdot\lambda q}{dy}\cdot AC, \qquad 1.$$

Upon remarking that *p* and *q* are zero for the points considered then those cosines will become:

$$\lambda\frac{dp}{dx}\cdot AB, \qquad \lambda\frac{dq}{dx}\cdot AB, \qquad 1$$

for the point *B* and:

$$\lambda \frac{dp}{dy} \cdot AC, \qquad \lambda \frac{dq}{dy} \cdot AC, \qquad 1$$

for the point *C*. However, as one knows, one has:

$$\frac{dp}{dy} = \frac{dq}{dx}.$$

Hence, if *AB* is equal to *AC*, which conforms to our hypothesis, then the angle that the normal at *B* makes with the *y*-axis will be the same as the one that the normal at *C* makes with the *x*-axis, or, what amounts to the same thing:

*The normal at B is inclined in the ZAB plane by the same quantity as the normal C is inclined in the plane ZAC.*

**2.** Since the cosines $\lambda \frac{dp}{dy} \cdot AC$, $\lambda \frac{dq}{dx} \cdot AB$ are equal and have the same sign, it will result that the two normals are either both inside the dihedral angle *BAC* or both outside of that angle; i.e., one of the two of them will be skew to the normal section that passes through its point of departure, while the second one will be perpendicular, and conversely. Hence, by virtue of the law of continuity, one can conclude that there must necessarily exist an intermediate direction to *AB* and *AC*, such that the corresponding normal is in the normal plane that is drawn through that direction. If the direction *AD* enjoys that property then our theorem will show that the same thing must be true for the perpendicular direction *AD′*, and in turn, that there must exist two mutually-perpendicular lines at each point of the surface such that the normals that are infinitely-close to the point considered and drawn in the direction of those two lines will meet the original normal.

**3.** The cosines of the angles that the normal to the point *B* make with the *X*-axis and the normal at the point *C* makes with the *Y*-axis can be considered to be equal (up to second-order infinitesimals) to the angles that the projections of those normals onto the planes *ZAB*, *ZAC* define with the *Z*-axis. That is, when those cosines are divided by the infinitely-small lengths *AB*, *AC* can be considered as representing curvatures of the normal sections *ZAB*, *ZAC*.

Upon calling the curvatures $1 / R$, $1 / r$, one will then have:

$$\frac{1}{R} = \frac{d \cdot \lambda p}{dx}, \qquad \frac{1}{r} = \frac{d \cdot \lambda p}{dy}.$$

**4.** Now suppose that, without changing the *Z*-axis, one takes the *X* and *Y* axes to be new lines *AX′*, *AY′*, which are mutually-perpendicular and located in the tangent plane. If

$\alpha$ denotes the angle between the axes $XX'$ then one will have the following transformation formulas:

$$x = y' \sin \alpha + x' \cos \alpha, \qquad x' = x \cos \alpha - y \sin \alpha,$$
$$y = y' \cos \alpha - x' \sin \alpha, \qquad y' = y \cos \alpha + x \sin \alpha,$$

and in turn, upon letting $p'$, $q'$, $\lambda'$ denote the quantities that we called $p$, $q$, $\lambda$ in the old system of axes:

$$p' = \frac{dz}{dx'} = \frac{dz}{dx}\cdot\frac{dx}{dx'} + \frac{dz}{dy}\cdot\frac{dy}{dx'} = p \cos \alpha - q \sin \alpha,$$

$$q' = \frac{dz}{dy'} = \frac{dz}{dx}\cdot\frac{dx}{dy'} + \frac{dz}{dy}\cdot\frac{dy}{dy'} = p \sin \alpha + q \cos \alpha,$$

$$\lambda'^2 = (p'^2 + q'^2 + 1)^{-1} = (p^2 + q^2 + 1)^{-1} = \lambda^2.$$

It results from this that upon letting $R'$, $r'$ denote the radii of curvature of the normal sections that were drawn through the $x'$ and $y'$ axes, one will have:

$$\frac{1}{R'} = \frac{d\cdot\lambda' p'}{dx'} = \cos \alpha \, \frac{d\cdot\lambda p}{dx'} - \sin \alpha \, \frac{d\cdot\lambda p}{dx'}$$

$$= \cos \alpha \left(\frac{d\cdot\lambda p}{dx}\frac{dx}{dx'} + \frac{d\cdot\lambda p}{dy}\frac{dy}{dx'}\right) - \sin \alpha \left(\frac{d\cdot\lambda q}{dx}\frac{dx}{dx'} + \frac{d\cdot\lambda q}{dy}\frac{dy}{dx'}\right)$$

$$= \cos \alpha \left(\frac{1}{R}\cos\alpha + \frac{d\cdot\lambda p}{dy}\sin\alpha\right) - \sin \alpha \left(\frac{d\cdot\lambda q}{dx}\cos\alpha - \frac{1}{r}\sin\alpha\right).$$

One will similarly find that:

$$\frac{1}{r'} = \frac{d\cdot\lambda' p'}{dx'} = \sin \alpha \, \frac{d\cdot\lambda p}{dy'} + \cos \alpha \, \frac{d\cdot\lambda q}{dy'}$$

$$= \sin \alpha \left(\frac{1}{r}\sin\alpha + \frac{d\cdot\lambda p}{dy}\cos\alpha\right) + \cos \alpha \left(\frac{d\cdot\lambda q}{dx}\sin\alpha + \frac{1}{R}\cos\alpha\right).$$

Upon adding them, one will get:

$$\frac{1}{R'} + \frac{1}{r'} = \frac{1}{R} + \frac{1}{r}.$$

If one supposes that the original $x$ and $y$ axes are in the direction of the lines of curvature, whose existence was proved above, then one will have:

$$\frac{d\cdot\lambda p}{dy} = 0, \qquad \frac{d\cdot\lambda q}{dx} = 0,$$

and in turn:

$$\frac{1}{R'} = \frac{1}{R}\cos^2\alpha + \frac{1}{r}\sin^2\alpha,$$

$$\frac{1}{r'} = \frac{1}{R}\sin^2\alpha + \frac{1}{r}\cos^2\alpha.$$

These are the known formulas that give the curvatures of the normal sections.

**5.** Now, study the law of variation of the angle that is defined by the normal at a point and the let us section through $AZ$ that is drawn through that point. As we have seen, that angle has the same value for two points that are equidistant from $A$ and taken in two rectangular directions, and it will be represented by:

$$AB\cdot\frac{d\cdot\lambda p}{dy} \qquad \text{or} \qquad \frac{d\cdot\lambda q}{dx}\cdot AC, \qquad \text{resp.}$$

If we look for the values for the points that are located at an equal distance on the $x'$ and $y'$ axes then we must calculate:

$$AB\cdot\frac{d\cdot\lambda p'}{dy'}.$$

Now, since $p' = 0$, one will have:

$$\frac{d\cdot\lambda p'}{dy'} = p'\frac{d\lambda}{dy'} + \lambda\frac{dp'}{dy'} = \lambda\frac{dp'}{dy'} = \lambda\cos\alpha\,\frac{dp}{dy'} - \lambda\sin\alpha\,\frac{dq}{dy'}$$

$$= \lambda\cos\alpha\left(\frac{dp}{dx}\sin\alpha + \frac{dp}{dy}\cos\alpha\right) - \lambda\sin\alpha\left(\frac{dq}{dy}\cos\alpha + \frac{dq}{dx}\sin\alpha\right).$$

If one then takes $AX$ and $AY$ to be the directions of the lines of curvature in such a way that one has:

$$\frac{dp}{dy} = 0, \qquad \frac{dq}{dx} = 0$$

then one will have:

$$\frac{d\cdot\lambda p'}{dy'} = \cos\alpha\sin\alpha\left(\lambda\frac{dp}{dx} - \lambda\frac{dq}{dy}\right) = \tfrac{1}{2}\sin 2\alpha\left(\frac{1}{R} - \frac{1}{r}\right),$$

in such a way that the desired angle will have the very simple expression:

$$\tfrac{1}{2}\,AB\cdot\sin 2\alpha\cdot\left(\frac{1}{R}-\frac{1}{r}\right),$$

which says that the normal to the surface, which is situated in the plane of the section for principal sections, is inclined in that section for points that are infinitely close to *A* according to a law that is independent of the nature of the surface considered. The maximum will exist for the direction that inclined at 45 degrees in those principal sections; it will be proportional to the difference between the curvatures of the surface.

## II.

**1.** The preceding formulas exhibit a very simple law by which the positions of the normals to a given surface will vary around a point. However, lines in space that are chosen at random will not generally be normal to the same surface, and their law of variation will not be subject to all of the conditions of the preceding paragraph. Among those conditions, we shall look for the ones that we can consider to be sufficient to characterize the normals to a surface.

Let *X*, *Y*, *Z* be the functions of three variables *x*, *y*, *z* that represent the cosines of the angles that a line that starts at a point whose coordinates are *x*, *y*, *z* defines with the three rectangular axes to which that point is referred, in such a way that a line will correspond to each point in space. The condition for there to exist a series of surfaces that are normal to all of those lines is the same as the condition of integrability for the total differential equation:

(1) $$X\,dx + Y\,dy + Z\,dz = 0.$$

Now, as one knows, that integrability condition is expressed by the equation:

(2) $$0 = X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right).$$

That identity must persist no matter what change of variables is performed in equation (1). For example, if one changes the axes in such a manner as to make the *z*-axis parallel to the direction of the line that corresponds to the point in space whose coordinates are $\alpha$, $\beta$, $\gamma$ then equation (1) will take the form:

(3) $$X_1\,dx_1 + Y_1\,dy_1 + Z_1\,dz_1 = 0,$$

in which $x_1$, $y_1$, $z_1$ are the coordinates of the points in space when referred to the new axes, and $X_1$, $Y_1$, $Z_1$ are the cosines of the angles between the corresponding lines and those axes. One must then have:

(2) $$0 = X_1\left(\frac{dY_1}{dz_1}-\frac{dZ_1}{dy_1}\right)+Y_1\left(\frac{dZ_1}{dx_1}-\frac{dX_1}{dz_1}\right)+Z_1\left(\frac{dX_1}{dy_1}-\frac{dY_1}{dx_1}\right)$$

identically. For the particular point whose coordinates are $\alpha$, $\beta$, $\gamma$ in the old system, the quantities $X_1$, $Y_1$ will obviously become equal to zero, and $Z_1$ will become unity. As a result, that equation will become:

$$\frac{dX_1}{dy_1} = \frac{dY_1}{dx_1}, \tag{5}$$

and for each point in space one can define a condition that is analogous to equation (5) for conveniently-chosen axes. We shall see that if that condition is satisfied for all points in space then equation (2) will be satisfied identically, and as a result equation (1) will be integrable.

Indeed, consider the difference:

$$\frac{dX_1}{dy_1} - \frac{dY_1}{dx_1},$$

which, by hypothesis, is annulled for the points whose coordinates are $\alpha$, $\beta$, $\gamma$ in the old system of axes. We seek to express that difference in the old system of axes.

If $a, b, c, a', b', c', a'', b'', c''$ denote the cosines of the angles that the old axes define with the new ones then one will have:

$$x_1 = ax + by + cz, \qquad y_1 = a'x + b'y + c'z, \qquad z_1 = a''x + b''y + c''z,$$
$$X_1 = aX + bY + cZ, \qquad Y_1 = a'X + b'Y + c'Z, \qquad Z_1 = a''X + b''Y + c''Z,$$

and in turn:

$$\left.\begin{aligned} \frac{dY_1}{dx_1} = {} & aa'\frac{dX}{dx} + ba'\frac{dX}{dy} + ca'\frac{dX}{dz} \\ & + ab'\frac{dY}{dx} + bb'\frac{dY}{dy} + cb'\frac{dY}{dz} \\ & + ac'\frac{dZ}{dx} + bc'\frac{dZ}{dy} + cc'\frac{dX}{dz}, \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{dX_1}{dy_1} = {} & aa'\frac{dX}{dx} + b'a\frac{dX}{dy} + c'a\frac{dX}{dz} \\ & + ba'\frac{dY}{dx} + bb'\frac{dY}{dy} + bc'\frac{dY}{dz} \\ & + ca'\frac{dZ}{dx} + cb'\frac{dZ}{dy} + cc'\frac{dX}{dz}, \end{aligned}\right\}$$

$$\frac{dY_1}{dx_1} - \frac{dX_1}{dy_1} = (ba' - ab')\left(\frac{dX}{dy} - \frac{dY}{dx}\right) + (ca' - c'a)\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + (cb' - bc')\left(\frac{dY}{dz} - \frac{dZ}{dy}\right).$$

If one remarks that $a'b - b'a$, $ca' - c'a$, $cb' - bc'$ are equal to $c''$, $b''$, $a''$, resp., then that equation will become:

$$\frac{dY_1}{dx_1} - \frac{dX_1}{dy_1} = c''\left(\frac{dX}{dy} - \frac{dY}{dx}\right) + b''\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + a''\left(\frac{dY}{dz} - \frac{dZ}{dy}\right).$$

The left-hand side is annulled for the point whose coordinates are $\alpha$, $\beta$, $\gamma$, so the same thing must be true for the right-hand side, which is identical to it. However, $a''$, $b''$, $c''$ are proportional to the values $Z$, $Y$, $X$, resp., take at that point. Hence, one will have:

$$0 = Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)+Y\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right). \tag{6}$$

Thus, if equation (5) is satisfied for all points in space then the same thing will be true for equation (6), and equation (1) will be integrable.

**2.** The theorem that was involved with equation (5) is susceptible to a very simple geometric interpretation. Since that equation is true at each point as long as one takes the $Z$-axis to be parallel to the line that corresponds to the point considered, the cosines $X$ and $Y$ will be annulled for that point, in such a way that if, upon taking it to be the starting point, one draws two infinitely-small lengths that are parallel to the $x$ and $y$ axes, resp., then if one lets $\sigma$ denote the common value of the two lengths then the angle that is defined by the line that corresponds to the extremity of the former one and the $Y$-axis then one will have:

$$\sigma\frac{dY}{dx}$$

for its cosine, and the angle that is defined by the line that corresponds to the extremity of the second length and the $X$-axis will have:

$$\sigma\frac{dX}{dy}$$

for its cosine. From equation (5), those two angles must be equal, and conversely if that is true for all points in space then equation (1) must be integrable. We then have the following theorem:

*In order for lines whose directions are given as functions of the coordinates of their starting points to be normal to a series of surfaces, it is necessary and sufficient that upon taking a point A in space and the line AZ that corresponds to that point, and then drawing two infinitely-small lengths from the starting point A that are equal to AB, AC and perpendicular to AZ, the line that corresponds to the point B will make an angle with the plane ZAB that is equal to the one that the line that starts from the point C will make with the plane ZAC.*

Therefore, it will suffice, for example, that there shall exist two perpendicular directions at each point for which those angles are annulled; i.e., such that the normals to the points $C$ and $D$ are in the planes $ZAB$, $ZAC$.

It is, moreover, very easy to verify that if the condition is fulfilled for two directions $AB$, $AC$ then the same thing will be true for two other arbitrary directions that are perpendicular to each other. However, in any case, in order to prove that the lines are normal to the same surface, it will suffice to see that there exist two directions at each point in space that are perpendicular to the line that corresponds to it, and for which the condition is fulfilled.

## III.

As an application, I will show how one can use the preceding results to prove the beautiful theorem of Dupin on orthogonal surfaces.

Consider three series of orthogonal surfaces, and let *AX*, *AY*, *AZ* be the tangents at a point *A* to the curves of intersection of the surfaces that pass through it. Let *M*, *N*, *P* be three points that are taken from the three directions *AX*, *AY*, *AZ*, respectively, at equal, infinitely-small distances from the point *A*. Upon considering the normals to the surfaces that cut along *AX* at the point *M* and calling the angles that they form with the axes $\alpha$, $\beta$, $\gamma$, $\alpha'$, $\beta'$, $\gamma'$, one will have:

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' = 0.$$

However, $\alpha$, $\alpha'$ differ from a right angle by only infinitely little, and $\beta$, $\gamma$ are infinitely-small, so upon neglecting second-order infinitesimals, that equation will become:

$$\cos \beta' + \cos \gamma = 0. \tag{1}$$

Similarly, upon letting $\alpha_1$, $\beta_1$, $\gamma_1$, $\alpha_1'$, $\beta_1'$, $\gamma_1'$ denote the angle that the axes form with the normals to the surfaces that intersect along *AN*, one will have:

$$\cos \gamma_1 + \cos \alpha_1' = 0. \tag{2}$$

Finally, upon letting $\alpha_2$, $\beta_2$, $\gamma_2$, $\alpha_2'$, $\beta_2'$, $\gamma_2'$ denote the angles that axes define with the normals that are drawn the point *p* to the two surfaces that pass through that point, one will have:

$$\cos \alpha_2 + \cos \beta_2' = 0. \tag{3}$$

However, from a theorem that we stated above:

$$\cos \beta' = \cos \alpha_1', \qquad \cos \gamma = \cos \alpha_2, \qquad \cos \gamma_2 = \cos \beta_2', \tag{4}$$

so it will result that upon appending equations (1) and (2), we will have:

$$\cos \beta' + \cos \alpha_2 + \cos \beta_2' = 0, \tag{5}$$

which will give:

$$\cos \beta' = 0,$$

when it is combined with equation (3). One will similarly have:

$$\cos \gamma = 0, \qquad \cos \gamma_1 = 0,$$

which will prove that the normals at the point *M*, *N*, *P* to each of the surfaces that cross at *A* will be situated in planes that pass through those points and the normal that corresponds to *A*. It will result from this that the points *M*, *N*, *P* are on the lines of curvature of the three surfaces, which is Dupin's theorem.

We should remark that we made use of the fact that the three surfaces must pass through *A* in our proof. We have actually proved the following theorem then, which has Dupin's theorem as an immediate consequence:

*If three surfaces intersect in such a manner that they are normal at all points where they meet then the curves of intersection will be tangent to the lines of curvature that are drawn through the common point of the three surfaces on each of those three surfaces.*

## IV.

Malus proved that light rays that start from a point and reflect from an arbitrary surface will remain normal to that surface after reflection. Dupin generalized that theorem by showing that light rays that are directed normal to the same surface can be reflected or refracted upon traversing an arbitrary surface without losing the property of being normal to the same surface.

More recently, Sturm has reprised the proof of that theorem and used analysis to arrive at an expression for the radii of curvature and the position of the lines of curvature of the surface that is normal to the refracted rays as functions of the corresponding elements of the surface that is normal to the incident rays.

All of those results can be deduced geometrically from the theorems that were proved in this paper.

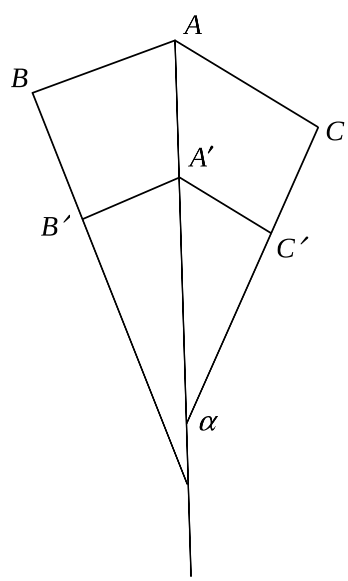


Figure 1.

We first remark that if the light rays are normal to the same surface then they will be, at the same time, normal to an infinitude of surfaces that one can obtain by drawing a constant length on each ray by starting with the original normal surface. Indeed, let (Fig. 1) *AZ* be the normal at a point of a surface, and let *AB*, *AC* be the directions of the lines of curvature at the point *A*. Draw normals to the point *B* and *C*, which will be in the planes *ZAB*, *ZAC*, respectively, and if one takes an arbitrary point *A′* on *AZ* then I say that the sheaf of light rays around the point *A′* will satisfy the necessary and sufficient condition for them to all be normal to the same surface that passes through *A′*. Indeed, if one takes two elements *A′B′*, *A′C′* that are perpendicular to *A′Z* and situated in the planes *ZAB*, *ZAC*, resp., then the rays that correspond to the points *B′*, *C′* will be *BB′*, *CC′*, precisely. They will then both meet the normal *A′Z*, and consequently, from our theorem, all of the light rays will be normal to the same surface around the point *A′*.

It is obvious that the normals *AA′*, *BB′*, which are found between the two surfaces, will differ by only second-order infinitesimals, and that as a result, the portions that are intercepted along two normals at a finite distance along the same line of curvature will be rigorously equal to each other, and since one can always pass from one arbitrary point to another arbitrary point on the same surface by advancing along two lines of curvature successively, it will result that the two surfaces will intercept equal portions of all the normals between them.

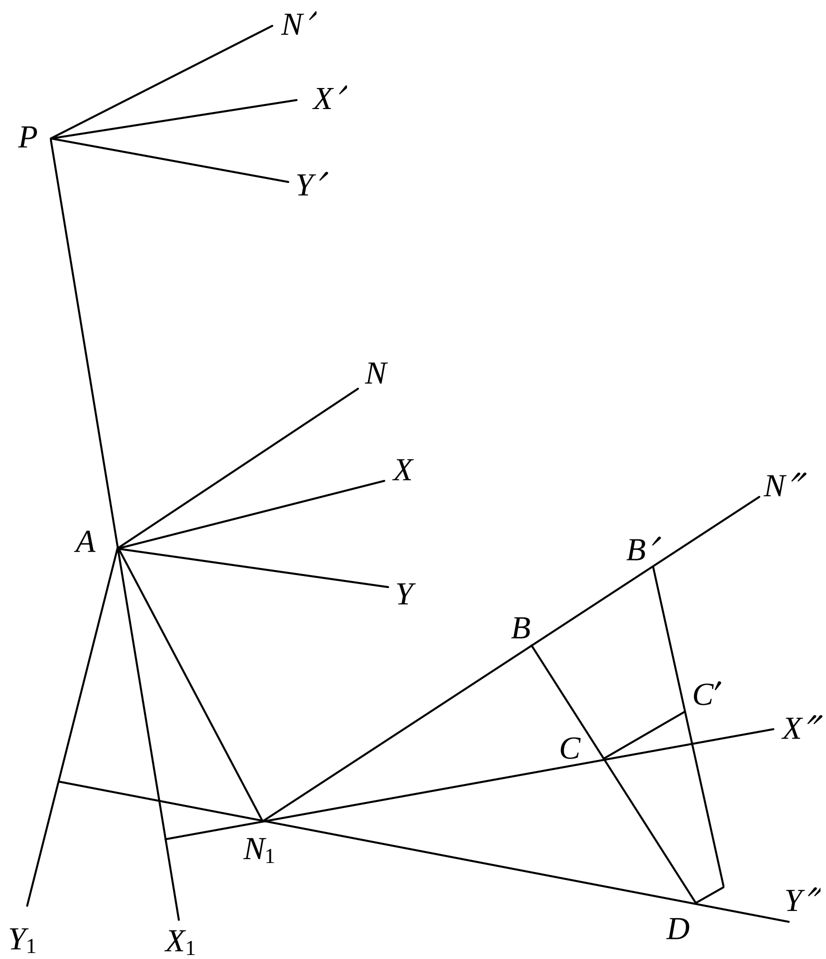


Figure 2.

From that, in order to prove that the rays that are normal to the same surface will preserve that property after having been refracted in an arbitrary manner, it will suffice to prove that the condition that we gave as necessary and sufficient will be fulfilled for some point on each ray. We choose the point that is situated on the surface that separates the two media: Let (Fig. 2):

| | |
|---|---|
| *AN* | be the normal to the separation surface, |
| *AX* | be the incident ray, |
| *AY* | be the refracted ray, |
| $x$ | be the angle *XAN*, |
| $y$ | be the angle *YAN*, |
| $l$ | be the index of refraction $\frac{\sin x}{\sin y}$. |

Draw an infinitely-small length *AP* along a line that is normal to the three lines *AN*, *AX*, *AY*, which will be in the same plane, from the known law of refraction. Let *PN′*, *PX′*, *PY′* be three lines that correspond to *AN*, *AX*, *AY*, which are drawn through the point *P*, which can be considered to belong to the separation surface. If we project those lines *PN′*, *PX′*, *PY′* onto a plane that is perpendicular to *AP* then the angle that they make between them will change only by second-order infinitesimals, and consequently, the ratio of the two sines will remain equal to $l$.

Let $\alpha$ be the inclination of the projection of *PX′* onto a parallel to *AX*, or, in other words, the inclination of *PX′* above the plane *PAX*.

Let $b$ be the inclination of the projection of *PY′* onto a plane that is parallel to *AY*; i.e., the inclination of *Y′* above the plane *PAY*.

Finally, let $\gamma$ be the angle that projection of *PN′* makes with a parallel to *AN*; i.e., the inclination of *PN′* above the plane *PAN*; one must have:

$$\frac{\sin(x+\alpha-\gamma)}{\sin(y+\beta-\gamma)} = \frac{\sin x}{\sin y} = l\ ;$$

hence, one infers, upon remarking that $\alpha-\gamma$ and $\beta-\gamma$ are infinitely small, that:

$$\frac{(\alpha-\gamma)\cos x}{(\beta-\gamma)\cos y} = l, \qquad \beta = \frac{(\alpha-\gamma)\cos x + l\gamma\cos y}{l\cos y}.$$

As one sees, these formulas allow one to calculate $\beta$ in terms of $\alpha$ and $\gamma$, which are assumed to be known. From our theorem, in order for the rays *AY* to be normal to the same surface, it is necessary and sufficient that upon taking a length $AY_1 = AP$ in the plane that is perpendicular to *AY* and normal to *AP*, the angle that the ray *YY″*, which passes through the point $Y_1$, makes with the plane $YAY_1$ will be equal to the angle $\beta$ that was calculated above. We then look for the value of that new angle, which we denote by $\beta'$.

Draw three lines $AN_1$, $AX_1$, $AY_1$ through the point $A$ in the plane of the lines $AN$, $AX$, $AY$ that are perpendicular to them, respectively. Let $AN_1$ have a length that that is equal to $AP$. The point $N_1$ can be considered to be situated on the separation surface of the two media. Let $N_1N''$ be the normal at that point, and let $N_1X''$, $N_1Y''$ be the incident and refracted rays that correspond to them. Those rays will make infinitely-small angles with the plane of the lines $AX_1$, $AY_1$, $AN_1$, and can consequently be regarded as meeting the lines $AX_1$, $AY_1$ at $m$ and $n$, resp., up to second-order infinitesimals.

Cut the three lines $N_1N''$, $N_1X''$, $N_1Y''$ with a plane so that $BCD$ represents the trace of the lines $AX$, $AY$ on that plane, which are perpendicular to the projection of $N_1N''$. Suppose, to fix ideas, that the length $N_1B$ is equal to unity, so that plane will be cut by the lines $N_1N''$, $N_1X''$, $N_1Y''$ at points whose rabattement around $BD$ as a hinge [†] will be represented by $B'$, $C'$, $D'$. Let $\gamma_1$ be the angle that is defined by the line $N_1N''$ and the plane $XAN$, while $\beta_1$, $\alpha_1$ are the angles that $Y_1Y''$, $X_1X''$ make with that plane, resp. One will obviously have:

$$BB' = \gamma_1, \qquad CC' = \frac{\alpha_1}{\cos x}, \qquad DD' = \frac{\beta_1}{\cos y}.$$

However, the points $B'$, $C'$, $D'$ are in a straight line, since the lines on which they are found are in the same plane; consequently, one will have:

$$\frac{BB' - DD'}{BB' - CC'} = \frac{BD}{BC} = \frac{\tan y}{\tan x} = \frac{\cos x}{l \cos y},$$

and as a result:

$$\frac{\gamma_1 - \dfrac{\beta_1}{\cos y}}{\gamma_1 - \dfrac{\alpha_1}{\cos x}} = \frac{\cos x}{l \cos y}.$$

However, from our theorem, and since, by hypothesis, the lines $N$, $X$ are normal to the same surface, one will have $\gamma_1 = \gamma$. Moreover, the angle $\alpha_1$ that is defined by the normal at $m$ to the plane $XAm$ is equal to:

$$\alpha \times \frac{Am}{AP} = \alpha \cos x, \quad \text{so} \quad \frac{\alpha_1}{\cos x} = \alpha,$$

and as a result:

$$\frac{\beta_1}{\cos y} = \frac{(\alpha - \gamma)\cos x + l\gamma \cos y}{l \cos y}, \quad \text{so} \quad \frac{\beta_1}{\cos y} = \beta.$$

[†] Translator: Rabattement around a line (viz., the "hinge") is a technique from descriptive geometry that involves rotating a certain plane around a line until it coincides with another plane, thus taking points of one plane to points of the other one.

That is precisely what must be true in order for the lines $Y$ to be normal to the same surface, since $\alpha$, $\alpha_1$ are the angles that two normals that are drawn in two perpendicular directions, but at distances from the point $A$ whose ratio is cos $y$, make with the corresponding planes. Consequently, if one takes equal lengths along $AY$ and $AP$ then the refracted rays that correspond to the points thus-obtained will make equal angles with the planes $XAX_1$, $XAP$; *those rays will then be normal to the same surface.*

Recall the equation:

$$\beta = \gamma + \frac{(\alpha - \gamma)\cos x}{l \cos y}.$$

Let $\theta$ be the angle between the line $AP$ and one of the lines of curvature of the surface that is normal to the incident ray $AX$. Let $\omega$ be the angle between that same line $AP$ and the line of the curvature of the surface that is normal to $AY$. Finally, let $u$ be the angle that it defines with the line of curvature of the separation surface that is normal to $AN$.

If $R$, $r$, $R'$, $r'$, $R''$, $r''$ are the radii of curvature of those three surfaces then we will have:

$$\alpha = \tfrac{1}{2}\sin 2\theta \cdot AP\left(\frac{1}{R} - \frac{1}{r}\right),$$

$$\beta = \tfrac{1}{2}\sin 2\omega \cdot AP\left(\frac{1}{R'} - \frac{1}{r'}\right),$$

$$\gamma = \tfrac{1}{2}\sin 2u \cdot AP\left(\frac{1}{R''} - \frac{1}{r''}\right),$$

and in turn:

$$(a) \qquad \left(\frac{1}{R'} - \frac{1}{r'}\right)\sin 2\omega = \left(\frac{1}{R''} - \frac{1}{r''}\right)\sin 2u + \frac{\cos x}{l \cos y}\left[\left(\frac{1}{R} - \frac{1}{r}\right)\sin 2\theta - \left(\frac{1}{R''} - \frac{1}{r''}\right)\sin 2u\right].$$

This is a first relation between the elements that relate to the curvatures of the surface that is normal to the refracted rays. One can easily find two other ones, and as a result, determine the three quantities $R'$, $r'$, and $\omega$. Once more, consider three normals (Fig. 3) $AN$, $AX$, $AY$ that are drawn from the point $A$ to three surfaces. As before, draw the normal $AP$, and at the point $P$, which can be considered to be found on each of the three surfaces, draw the normals $PN$, $PX$, $PY$, which must be found in the same plane and must satisfy the condition:

$$\frac{\sin X'P'N'}{\sin Y'PN'} = l.$$

It is easy to see that the curvatures of the normal sections that are made to the three surfaces along $AP$ are equal to the angles that are formed by the lines $PN'$, $PX'$, $PY'$ and a plane that is perpendicular to $AP$, divided by the length $AP$. Let $\alpha_n$ , $\alpha_x$ , $\alpha_y$ be those angles, and let $1 / \rho_n$, $1 / \rho_x$, $1 / \rho_y$ be the curvatures of the sections, so one will have:

$$\frac{1}{\rho_n} = \frac{\alpha_n}{AP}, \qquad \frac{1}{\rho_x} = \frac{\alpha_x}{AP}, \qquad \frac{1}{\rho_y} = \frac{\alpha_y}{AP}.$$

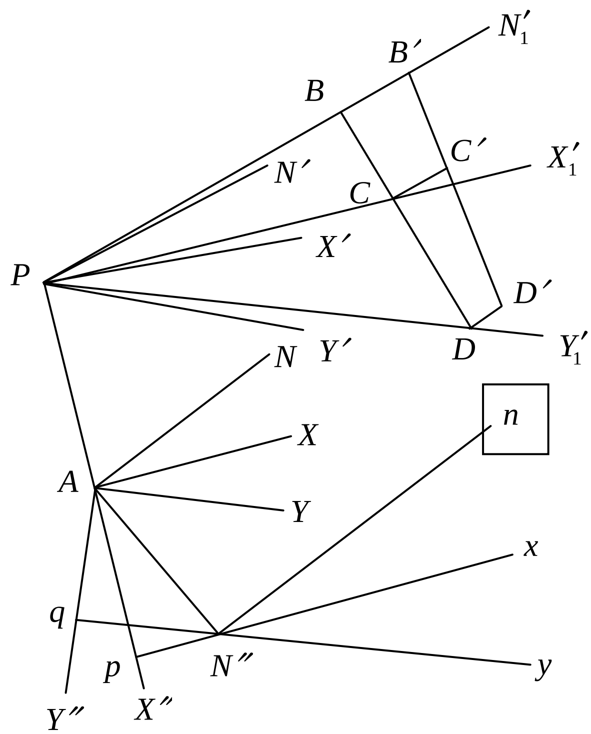


Figure 3.

We seek a relation between $\alpha_n$, $\alpha_x$, $\alpha_y$. In order to do that, we consider three projections of the lines $AN'$, $AX'$, $AY'$ onto the plane that is perpendicular to $AP$. Let $PX_1'$, $PY_1'$, $PN_1'$ be those projections. Imagine a plane $BCD$ that is drawn perpendicular to the plane of those lines and along a line $BD$ that is normal to $PN_1'$. With $BD$ as a hinge, rabatt the points where the lines $PN'$, $PX'$, $PY'$ pierce that plane. Let $B'$, $C'$, $D'$ be those rabattements, which will be in a straight line. We will obviously have:

$$\frac{BB' - DD'}{BB' - CC'} = \frac{DB}{BC} = \frac{\tan y}{\tan x} = \frac{\cos x}{l \cos y}.$$

However:

$$BB' = \alpha_n, \qquad CC' = \frac{\alpha_x}{\cos x}, \quad DD' = \frac{\alpha_y}{\cos y},$$

so

$$\frac{\alpha_n - \dfrac{\alpha_y}{\cos y}}{\alpha_n - \dfrac{\alpha_x}{\cos x}} = \frac{\cos x}{l \cos y},$$

so one can infer that:

$$\alpha_x = \alpha_n \cos x - l\,\alpha_n \cos y + l\,\alpha_y\,,$$

and as a result:

$$(b) \qquad \frac{1}{\rho_x} = \frac{1}{\rho_n}\cos x - \frac{l\cos y}{\rho_n} + \frac{l}{\rho_y}\,.$$

However, one has:

$$\frac{1}{\rho_x} = \frac{1}{R}\sin^2\theta + \frac{1}{r}\cos^2\theta,$$

$$\frac{1}{\rho_y} = \frac{1}{R'}\sin^2\omega + \frac{1}{r'}\cos^2\omega,$$

$$\frac{1}{\rho_n} = \frac{1}{R''}\sin^2 u + \frac{1}{r''}\cos^2 u,$$

which will transform equation ($b$) into the following one:

$$(c) \qquad \frac{\sin^2\theta}{R} + \frac{\cos^2\theta}{r} =$$
$$= \cos x\left(\frac{\sin^2 u}{R''} + \frac{\cos^2 u}{r''}\right) - \cos y\left(\frac{\sin^2 u}{R''} + \frac{\cos^2 u}{r''}\right) + l\left(\frac{\sin^2\omega}{R'} + \frac{\cos^2\omega}{r'}\right).$$

In order to find a third relation between those quantities $R'$, $r'$, and $\omega$, we draw three lines $AN''$, $AX''$, $AY''$ through the point $A$ in the plane of the three normals $AN$, $AX$, $AY$ that are perpendicular to those normals. Draw the normal $N''n$ to the separation surface through the point $N''$ that is situated on the perpendicular to $AN$ at an infinitely-small distance $AN''$, and the incident and refracted rays $N''x$, $N''y$, resp. Those rays can be considered to meet the lines $AX''$, $AY''$ at $p$ and $q$, and it is easy to see that if $\alpha_n'$, $\alpha_x'$, $\alpha_y'$ denote the angles that the projection of the three lines $N''n$, $N''x$, $N''y$ define with the corresponding lines that pass through the point $A$ then one will have, upon denoting the radii of curvature of the normal sections that were made in the three surfaces by the plane $XANY$ by $\rho_n'$, $\rho_x'$, $\rho_y'$:

$$\frac{1}{\rho_n'} = \frac{\alpha_n'}{AN''}, \quad \frac{1}{\rho_x'} = \frac{\alpha_x'}{AN''\cos x}, \qquad \frac{1}{\rho_y'} = \frac{\alpha_y'}{AN''\cos y}.$$

However, since one of the three lines $N''n$, $N''x$, $N''y$ is normal to the separation surface of the media, and the other two are directed along the incident and refracted rays, one must have:

$$\frac{\sin(x + \alpha_x' - \alpha_n')}{\sin(y + \alpha_y' - \alpha_n')} = l,$$

from which, one will infer, upon remarking that $\frac{\sin x}{\sin y} = l$:

$$\frac{(\alpha'_x - \alpha'_n)\cos x}{(\alpha'_y - \alpha'_n)\cos y} = l,$$

$$\alpha' x = \frac{l\,(\alpha'_y - \alpha'_x)\cos y + \alpha'_n \cos x}{\cos y},$$

so

$$(d) \qquad \frac{\cos x}{\rho'_x} = \frac{l\left(\frac{\cos y}{\rho'_y} - \frac{1}{\rho'_x}\right) + \frac{1}{\rho'_n}\cos x}{\cos x}.$$

However, from a known formula:

$$\frac{1}{\rho'_x} = \frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{r}, \qquad \frac{1}{\rho'_y} = \frac{\cos^2\omega}{R'} + \frac{\sin^2\omega}{r'}, \qquad \frac{1}{\rho'_n} = \frac{\cos^2 u}{R''} + \frac{\sin^2 u}{r''},$$

in such a way that the formula (*d*) becomes:

$$(e) \quad \cos^2 x \left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{r}\right) = (l - \cos x)\left(\frac{\cos^2 u}{R''} + \frac{\sin^2 u}{r''}\right) + l\cos y\left(\frac{\cos^2\omega}{R'} + \frac{\sin^2\omega}{r'}\right).$$

The three formulas (*a*), (*c*), (*e*) allow one to calculate the quantities $R'$, $r'$, $\omega$ as functions of $R$, $r$, $R''$, $r''$, $\theta$, and $u$.

## V.

One can easily find all of the laws of refraction that can exist from Dupin's theorem. Suppose that the refracted ray always remains in the plane that passes through incident ray and the normal to the separation surface, and $y = \varphi(x)$ represents the relation that exists between the two angles of incidence and refraction. Upon recalling Fig. 2 and making absolutely the same argument, while replacing the relation $y = \varphi(x)$ with just the equation:

$$\frac{\sin x}{\sin y} = l,$$

one will find that:

$$y + \beta - \gamma = \varphi(x + \alpha - \gamma),$$
$$\beta - \gamma = (\alpha - \gamma)\,\varphi'(x),$$
$$\beta = \gamma + (\alpha - \gamma)\,\varphi'(x).$$

One will likewise find, by means of the constructions that were employed above and upon adopting the same notations that:

$$\frac{\beta_1}{\cos y} = \gamma + (\alpha - \gamma)\,\frac{\tan y}{\tan x}.$$

In order for the lines $Y$ to be normal to the same surface, it is necessary and sufficient that one must have:

$$\beta = \frac{\beta_1}{\cos y};$$

i.e.:

$$\gamma\left(1 - \frac{\tan y}{\tan x}\right) + \alpha\,\frac{\tan y}{\tan x} = \gamma[1 - \varphi'(x)] + \alpha\,\varphi'(x).$$

That equation must be true for any quantities $\alpha$, $\gamma$, which are obviously mutually-independent, so it will be necessary that one must have:

$$\varphi'(x) = \frac{\tan y}{\tan x};$$

the integral is:

$$\varphi(x) = \text{arc sin } C \sin x$$

or

$$\frac{\sin \varphi(x)}{\sin x} = C.$$

The law of nature is, consequently, the only one that Dupin's theorem will permit to be exact in full generality.