# On the interaction of two electrons 

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(Received on 9 June 1932)
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#### Abstract

The connections between the interaction formulas of Breit ( ${ }^{1}$ ) and Møller ( ${ }^{2}$ ) and those of quantum electrodynamics will be discussed. It will be shown that the Breit formula can be derived from Møller's (§ 1) and from quantum electrodynamics (§ 2), and that Møller's formula also follows from the latter (§ 3).


Introduction. - Two Ansätze exist for the interaction of two electrons, namely, those of Breit $\left({ }^{1}\right)$ and Møller $\left({ }^{2}\right)$, which seem to start from entirely-different places. It would seem desirable to investigate the relationships between the two Ansätze and their relationships to quantum electrodynamics. Breit derived his differential equation directly from quantum electrodynamics, and Rosenfeld $\left({ }^{3}\right)$ showed that the formulas from quantum electrodynamics that were required for Breit's derivation were obtained by the Heisenberg $\left({ }^{4}\right)$ correspondence process that Møller had applied. We will obtain Breit's formula directly from Møller's, and for its derivation from quantum electrodynamics, we will choose a form that is completely analogous to our derivation of Møller's, such that the various approximation processes will become more comparable.

As is known, one can derive Coulomb's interaction energy between two electrons from quantum electrodynamics precisely when one considers only the coupling of longitudinal waves of the electromagnetic fields with the electrons $\left({ }^{5}\right)$. Any deviation from Coulomb's law must then come about as a result of their interaction with transversal waves, which we will refer to briefly as the radiation field in what follows. Now, in the first approximation, the coupling between matter and the radiation field will create only those transitions for which the quantum state of the electrons will change, and a light quantum will be absorbed or emitted. However, we will be interested in the matrix elements of the interaction energy of the electrons that correspond to a change in the quantum state of the electrons alone without altering the state of the radiation field, in which we can assume, in particular, that no radiation is present in the initial state, as well as the final one.

[^0]One will obviously first get such transitions of the electrons alone by a double process, in which a quantum is initially emitted and then the same quantum is reabsorbed.

The derivations of the Breit and Møller formulas from quantum electrodynamics differ essentially in only the following way: In the first case, one has already added the Coulomb interaction energy of the electrons to the "unperturbed Hamiltonian function," so the perturbation is merely the interaction of the electrons with the transverse waves of the electromagnetic field. Now, the calculation will be performed in only the non-relativistic approximation, i.e., only up to terms of order $v^{2} / c^{2}$. In addition, all of the transitions of the material part of the system that are coupled with a permanent emission of radiation will be neglected. Naturally, one can treat the latter with the ordinary theory of radiation.

By contrast, in Møller's theory, the Coulomb energy is one part of the perturbation and enteron along with the interaction with the radiation field on an equal basis, since the two interactions will, in fact, have the same order of magnitude in the extreme relativistic case. In the zeroth-order approximation, the electrons move independently of each other. Møller's theory will be developed in a relativistically-exact manner, and in order to do that, we develops it in powers of $e$ and neglects terms of order higher than $e^{2}$. When one then derives the Breit interaction from Møller's theory, that will only justify the fact that one is treating the Coulomb interaction in the first approximation, but when one starts from quantum electrodynamics, the Coulomb force will be considered in the unperturbed system such that its effect can be treated exactly. Note that no radiation occurs in the Møller (viz., second) approximation, and that such a thing will first come about only when one considers the radiation interaction in the third approximation, so when one excludes the possibility that the electrons will have negative energy in the final state.
§ 1. Deriving Breit's interaction energy from Møller's theory. - According to Møller, in order to calculate the interaction of two electrons, one must construct the retarded potentials that are created by the charge distribution of the first electron, and they must be considered to be a perturbation that acts upon the second electron. We would like to calculate the matrix element of the interaction energy that corresponds to a transition of the first electron from the state $\left({ }^{1}\right) n_{1}$ to the state $n_{1}^{\prime}$, and the second electron from $n_{2}$ to $n_{2}^{\prime}$, under which the total energy in the initial and final state should be equal to:

$$
\begin{equation*}
E_{1}+E_{2}=E_{1}^{\prime}+E_{2}^{\prime} . \tag{1}
\end{equation*}
$$

The transition $n_{1} \rightarrow n_{1}^{\prime}$ corresponds to the charge distribution:

$$
\begin{equation*}
\rho_{n_{1} n_{1}^{\prime}}\left(\mathfrak{r}_{1}, t\right)=e_{1} u_{1}^{\prime *}\left(\mathfrak{r}_{1}\right) u_{1}\left(\mathfrak{r}_{1}\right) e^{2 \pi i\left(E_{1}^{\prime}-E_{1}\right) t / h} \tag{2}
\end{equation*}
$$

[^1]in which $u_{1}$ and $u_{1}^{\prime}$ are the Dirac eigenfunctions of the first electron in the initial and final state and $e_{1}$ is its charge. When one considers the retardation at the location $\mathfrak{r}_{2}$ at time $t$, the charge distribution (2) will generate the scalar potential:
\[

$$
\begin{align*}
& \varphi_{n_{1} n_{1}^{\prime}}\left(\mathfrak{r}_{2}, t\right)=\int \frac{\left.\rho_{n_{1} n_{1}^{\prime}}\left(\mathfrak{r}_{1}, t-\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right| / c\right)\right)}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|} d \tau_{1} \\
& =e_{1} e^{2 \pi i\left(E_{1}^{\prime}-E_{1}\right) t / h} \int \frac{\left(u_{1}^{\prime *}\left(\mathfrak{r}_{1}\right) u_{1}\left(\mathfrak{r}_{1}\right)\right)}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|} e^{2 \pi i\left(E_{1}^{\prime}-E_{1}\right) \mid \mathfrak{r}_{2}-\mathfrak{r}_{1} / h c} d \tau_{1} . \tag{3}
\end{align*}
$$
\]

One correspondingly gets the vector potential in the form:

$$
\begin{equation*}
\mathfrak{A}_{n_{1} n_{1}}\left(\mathfrak{r}_{2}, t\right)=-e_{1} e^{2 \pi i\left(E_{1}^{\prime}-E_{1}\right) t / h} \int \frac{\left(u_{1}^{\prime *}\left(\mathfrak{r}_{1}\right) \gamma_{1} u_{1}\left(\mathfrak{r}_{1}\right)\right)}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|} e^{2 \pi i\left(E_{1}^{\prime}-E_{1}\right)\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right| / h c} d \tau_{1} . \tag{4}
\end{equation*}
$$

$\gamma_{x_{1}}, \gamma_{y_{1}}, \gamma_{z_{1}}, \delta_{1}$ are the Dirac operators of the first electron. $\gamma_{1}$ is the vector with the components $\gamma_{x_{1}}, \gamma_{y_{1}}, \gamma_{z_{1}}$.

Now, the Breit formula for the interaction energy that we wish to derive is exact only up to terms of order $1 / c^{2}$, inclusive. That suggests that we might develop the exponential functions in (3), (4), which represent the retardation of the potential in powers of $1 / c$ :

$$
\begin{align*}
& \varphi_{n_{1} n_{1}^{\prime}}\left(\mathfrak{r}_{2}, t\right) \\
& =e_{1} e^{2 \pi i\left(E_{1}^{\prime}-E_{1}\right) t / h} \int\left(u_{1}^{\prime *}\left(\mathfrak{r}_{1}\right) u_{1}\left(\mathfrak{r}_{1}\right)\right)\left[\frac{1}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|}+\frac{2 \pi i}{h c}\left(E_{1}^{\prime}-E_{1}\right)-\frac{2 \pi^{2} i}{h^{2} c^{2}}\left(E_{1}^{\prime}-E_{1}\right)^{2}\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|\right] d \tau_{1}, \tag{5}
\end{align*}
$$

and correspondingly for $\mathfrak{A}$. The second term in the square bracket in (5) will vanish: That is because either the states $n_{1}$ and $n_{1}^{\prime}$ are identical, so $E_{1}-E_{1}^{\prime}=0$, or they are different from each other, in which case, the volume integral will vanish due to the orthogonality of the eigenfunctions. Now, the perturbation energy that acts upon the second electron is:

$$
V_{n_{1}}^{n_{1}^{\prime}}\left(\mathfrak{r}_{2}, t\right)=e_{2} \varphi_{n_{1} n_{1}^{\prime}}\left(\mathfrak{r}_{2}, t\right)+e_{2}\left(\gamma_{2} \mathfrak{U}_{n_{1} n_{1}^{\prime}}\left(\mathfrak{r}_{2}, t\right)\right) .
$$

We form its matrix element, which corresponds to the transition $n_{2} \rightarrow n_{2}^{\prime}$ of the second electron:

$$
\begin{align*}
V_{n_{1} n_{2}}^{n_{2}^{\prime} n_{2}^{\prime}}= & e_{1} e_{2} e^{2 \pi i\left(E_{1}^{\prime}+E_{2}^{\prime}-E_{1}-E_{2}\right) t / h} \int u_{1}^{\prime *}\left(\mathfrak{r}_{1}\right) u_{2}^{\prime *}\left(\mathfrak{r}_{2}\right) \\
& \times\left[\frac{1}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|}-\frac{2 \pi^{2}}{h^{2} c^{2}}\left(E_{1}^{\prime}-E_{1}\right)^{2}\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|-\frac{\left(\gamma_{1} \gamma_{2}\right)}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|}\right] u_{2}\left(\mathfrak{r}_{2}\right) u_{1}\left(\mathfrak{r}_{1}\right) d \tau_{1} d \tau_{2} . \tag{6}
\end{align*}
$$

The first term in the bracket here is the ordinary Coulomb potential, while the second one comes from the retarding of the scalar potential, and the third one reflects the influence of the (unretarded) vector potential: Since the $\gamma$ operators themselves have order of magnitude $v / c$, we need do not need to consider the retardation of the vector potential in our approximation. The first and third terms are symmetric in the two electrons, while the second one is not, which is based upon the asymmetry of Møller's method. However, since the total energy should be the same in the initial and final state (1), we can symmetrize the term by writing - $\left(E_{1}-E_{1}^{\prime}\right)\left(E_{2}-E_{2}^{\prime}\right)$ in place of $\left(E_{2}-E_{2}^{\prime}\right)^{2}$. With that, (6) will become the matrix element of:

$$
\begin{equation*}
V=e_{1} e_{2} \frac{1-\left(\gamma_{1} \gamma_{2}\right)}{\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|}+\frac{2 \pi^{2} e_{1} e_{2}}{h^{2} c^{2}}\left[H_{2}\left(H_{1}\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|-\mathfrak{r}_{2}-\mathfrak{r}_{1} \mid H_{1}\right)-\left(H_{1}\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|-\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right| H_{1}\right) H_{2}\right], \tag{7}
\end{equation*}
$$

in which $H_{1}$ and $H_{2}$ are the unperturbed Hamiltonian functions of the two electrons in the absence of interaction, so:

$$
\begin{equation*}
H_{1}=-e_{1} \varphi_{0}\left(\mathfrak{r}_{1}\right)-\frac{h c}{2 \pi i}\left(\gamma_{1}, \operatorname{grad}_{1}\right)-e_{1}\left(\gamma_{1}, \mathfrak{A}_{0}\left(\mathfrak{r}_{1}\right)\right)-m_{1} c^{2} \delta_{1} . \tag{8}
\end{equation*}
$$

$\varphi_{0}$ and $\mathfrak{A}_{0}$ are the scalar and vector potentials of any external static field in which the two electrons move. All components of $H_{1}$ commute with $\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|$, except for the gradient, so:

$$
H_{1}\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right|-\left|\mathfrak{r}_{2}-\mathfrak{r}_{1}\right| H_{1}=-\frac{h c}{2 \pi i}\left(\gamma_{1}, \frac{\mathfrak{r}_{1}-\mathfrak{r}_{2}}{\left|\mathfrak{r}_{1}-\mathfrak{r}_{2}\right|}\right)=F,
$$

and

$$
\begin{equation*}
H_{2} F-F H_{2}=+\frac{h^{2} c^{2}}{4 \pi^{2}}\left(\frac{\left(\gamma_{1} \gamma_{2}\right)}{\left|\mathfrak{r}_{1}-\mathfrak{r}_{2}\right|}-\frac{\left(\gamma_{1}, \mathfrak{r}_{1}-\mathfrak{r}_{2}\right)\left(\gamma_{2}, \mathfrak{r}_{1}-\mathfrak{r}_{2}\right)}{\left|\mathfrak{r}_{1}-\mathfrak{r}_{2}\right|^{3}}\right) . \tag{9}
\end{equation*}
$$

If one substitutes that in (7) then one will get precisely the Breit formula for the interaction energy:

$$
\begin{equation*}
V=\frac{e_{1} e_{2}}{\left|\mathfrak{r}_{1}-\mathfrak{r}_{2}\right|}\left(1-\frac{1}{2}\left(\gamma_{1} \gamma_{2}\right)-\frac{1}{2} \frac{\left(\gamma_{1}, \mathfrak{r}_{1}-\mathfrak{r}_{2}\right)\left(\gamma_{2}, \mathfrak{r}_{1}-\mathfrak{r}_{2}\right)}{\left|\mathfrak{r}_{1}-\mathfrak{r}_{2}\right|^{2}}\right) . \tag{10}
\end{equation*}
$$

§ 2. Deriving the Breit law of interaction from quantum electrodynamics. - We take our starting point for this derivation to be the Hamiltonian function of the system composed of the electrons and the electromagnetic field in the form [loc. cit. (166)] that one obtains when one eliminates the coordinates from the scalar potential and the longitudinal components of the vector potential with the help of the continuity equation. The Hamiltonian function will already include the Coulomb interaction of the particles in that form. When one observes [loc. cit. (167)], the Hamiltonian function can be written in the form:
$R=\sum_{s}\left(\frac{1}{2} p_{s}^{2}+2 \pi^{2} v_{s}^{2} q_{s}^{2}\right)+\sum_{i<j} \frac{e_{i} e_{j}}{r_{i j}}-\sum_{i} c\left(\gamma_{i} p_{i}\right)-\sum_{i} m_{i} c^{2} \delta_{i}+c \sqrt{\frac{8 \pi}{\Omega}} \sum_{i, s} e_{i}\left(\gamma_{i} A_{s}\right) q_{s} \sin \Gamma_{s i}$.
The notations in that formula are the same as the ones in loc. cit.. However, the radiation components $q_{s}$ are enumerated with a single index $s$, and not with two $s_{1}$ and $s_{2}$. The Coulomb interaction of the electrons is written out explicitly, while the infinitely-large constant electrostatic self-energy is ignored.

In order to derive the Breit formula, we must consider the Coulomb force in the unperturbed system. We then consider the unperturbed Hamiltonian function to be the sum of the energy of radiation:

$$
\begin{equation*}
H_{s}=\sum_{s}\left(\frac{1}{2} p_{s}^{2}+2 \pi^{2} v_{s}^{2} q_{s}^{2}\right) \tag{12}
\end{equation*}
$$

and the energy of the material particles:

$$
\begin{equation*}
H_{M}=\sum_{i}-c\left(\gamma_{i} p_{i}\right)-\sum_{i} m_{i} c^{2} \delta_{i}+\sum_{i<j} \frac{e_{i} e_{j}}{r_{i j}} . \tag{13}
\end{equation*}
$$

We regard the coupling energy of the particles and the field to be:

$$
\begin{equation*}
H=c \sqrt{\frac{8 \pi}{\Omega}} \sum_{i, s} e_{i}\left(\gamma_{i} A_{s}\right) q_{s} \sin \Gamma_{s i} . \tag{14}
\end{equation*}
$$

The states of the unperturbed system are characterized by a quantum number $\mathbf{n}_{i}$ that determines the state of the material particle of the system with the Hamiltonian function $H_{M}$, and the quantum numbers $n_{1}, n_{2}, \ldots, n_{s}, \ldots$ of the radiating oscillators. Let the corresponding probability amplitudes be:

$$
\begin{equation*}
a_{n n_{1} n_{2} \cdots n_{s} \cdots} \tag{15}
\end{equation*}
$$

The $a$ vary in time as a result of the effect of the coupling energy (14), which corresponds to the known equations:

$$
\begin{equation*}
a_{n^{\prime} n_{1}^{\prime} n_{2}^{\prime} \ldots n_{s}^{\prime}}=-\frac{2 \pi i}{h} \sum_{n n_{1} n_{2} \ldots} H_{n^{2} n_{1}^{n} n_{2}^{2} \ldots n^{2}}^{n n_{1}, \ldots} a_{n n_{1} n_{2} \ldots} e^{2 \pi i\left(E_{n n_{1} n_{1} \ldots}-E_{n n_{1} \ldots}\right) t / h} . \tag{16}
\end{equation*}
$$

As is known, the only non-zero matrix elements of $H$ are the following ones:

$$
H_{n^{\prime} n_{1} n_{2} \cdots n_{s}+1 \cdots}^{n n_{1} n_{2} \cdots n_{s} \ldots}=c \sqrt{\frac{8 \pi}{\Omega}} Q_{n^{\prime} n}(s) \sqrt{\frac{h}{8 \pi^{2} v_{s}}} \cdot\left\{\begin{array}{l}
\sqrt{n_{s}+1}  \tag{17}\\
\sqrt{n_{s}}
\end{array}\right.
$$

in which $Q_{n^{\prime} n}(s)$ represents the matrix element $n^{\prime} n$ of the quantity:

$$
\begin{equation*}
Q(s)=\sum_{i} e_{i}\left(\gamma_{i} A_{s}\right) \sin \Gamma_{s i} . \tag{18}
\end{equation*}
$$

We would now like to assume that the material system is found in the state $n$ to begin with, and no radiating oscillator has been excited. We will then have:

$$
\left|a_{n 00} \ldots 0 \ldots\right|=1 .
$$

We would like to see how a probability amplitude for the state $n^{\prime} 00 \ldots 0 \ldots$ arises from the effect of the perturbation. Now, the perturbation matrix (17) has no element that would couple the two states $n 000 \ldots$ and $n^{\prime} 000 \ldots$ directly. Such a transition can occur only indirectly by the detour through an intermediate term. From (17), the states $m 0 \ldots 1_{s} \ldots$ will come into question as such things since they combine with the initial and final states. Upon applying (16) to the transition $n 0$ $00 \ldots \rightarrow m 00 \ldots 1_{s} \ldots$ :

$$
\dot{a}_{m 0 \ldots 1_{s} \ldots}=-\frac{2 \pi i}{h} H_{m 00 \ldots 1_{s} \ldots}^{n 00.0} a_{n 00 \ldots 0 \ldots} e^{2 \pi i\left(v_{m n}+v_{s}\right) t}
$$

in which:

$$
\begin{equation*}
v_{m n}=\frac{E_{m}-E_{n}}{h} \tag{19}
\end{equation*}
$$

Since $a_{n 00 \ldots 0} \ldots$ is practically constant, we can integrate and find:

$$
\begin{equation*}
a_{m 0 \ldots 11_{s} \ldots}=-\frac{H_{m 00 \ldots 1_{s} \ldots}^{n 00 \ldots 0.1}}{h\left(v_{m n}+v_{s}\right)} a_{n 00 \ldots \ldots \ldots} e^{2 \pi i\left(v_{m n}+v_{s}\right) t} \tag{20}
\end{equation*}
$$

We must now apply equation (16) to the transition $m 0 \ldots 1_{s \ldots} \rightarrow n^{\prime} 00 \ldots 0 \ldots$ We find that:

$$
\begin{align*}
& \dot{a}_{n^{\prime} 0 \ldots 0 \ldots}=-\frac{2 \pi i}{h} \sum_{m, s} H_{n^{\prime} 00 \ldots 0 \ldots}^{m 00 . .1_{s} \ldots} a_{m 00 \ldots 1_{s} \ldots} e^{2 \pi i\left(v_{h^{\prime} m}-V_{s}\right) t} \\
& =-\frac{2 \pi i}{h} \sum_{m, s}\left(-\frac{1}{h} \frac{H_{m \ldots \ldots 1}^{n \ldots 0 . \ldots} H_{n^{\prime} \ldots \ldots}^{m \ldots 1_{s} \ldots}}{v_{m n}+v_{s}}\right) \cdot a_{n \ldots \ldots . \ldots} e^{2 \pi i v_{n n} t} . \tag{21}
\end{align*}
$$

We can regard the quantity:

$$
\begin{equation*}
K_{n^{\prime} n}=-\frac{1}{h} \sum_{m, s} \frac{H_{m, \ldots}^{n \ldots \ldots \ldots} H_{n^{\prime} \ldots 0 . \ldots}^{m} v_{1} \ldots \ldots}{v_{m n}+v_{s}} \tag{22}
\end{equation*}
$$

as a matrix element that mediates the transition between the states $n 0 \ldots 0 \ldots \rightarrow n^{\prime} 00 \ldots 0 \ldots$ directly. The corresponding quantities $K$ represent the correction that we must make to the Coulomb interaction in order to include the retardation in our calculation of the potential.

With the help of (17), we will find from (22) that:

$$
\begin{equation*}
K_{n^{\prime} n}=-\frac{c^{2}}{\pi \Omega} \sum_{m, s} \frac{Q_{n^{\prime} m}(s) Q_{m n}(s)}{v_{s}\left(v_{s}+v_{m n}\right)} . \tag{23}
\end{equation*}
$$

We must now express the quantities $K$ as functions of the coordinates and momenta of the particle. That is easy to do when we neglect $v_{m n}$ in comparison to $v_{s}$ in the denominator of (23) as a first approximation. (Observe that the ratio $v_{m n} / v_{s}$ has order of magnitude $v / c$.) We will then get:

$$
\left.\begin{array}{rl}
K_{n^{\prime} n} & =-\frac{c^{2}}{\pi \Omega} \sum_{s} \frac{1}{v_{s}^{2}} Q_{n^{\prime} m}(s) Q_{m n}(s) \\
& =-\frac{c^{2}}{\pi \Omega} \sum_{s} \frac{1}{v_{s}^{2}}\left[Q^{2}(s)\right]_{n^{\prime} m} \tag{24}
\end{array}\right\}
$$

We then have:

$$
\begin{equation*}
K=-\frac{c^{2}}{\pi \Omega} \sum_{s} \frac{1}{v_{s}^{2}} Q^{2}(s) . \tag{25}
\end{equation*}
$$

All that remains is to perform the summation over $s$ in order for us to prove that the quantity $K$ that we have found is identical to the Breit interaction function.

We substitute the expression (18) for $Q(s)$ in (25) and find that:

$$
\begin{equation*}
K=-\sum_{i, j} e_{i} e_{j} \frac{c^{2}}{\pi \Omega} \sum_{s} \frac{1}{v_{s}^{2}}\left(\gamma_{i} A_{s}\right)\left(\gamma_{j} A_{s}\right) \sin \Gamma_{s i} \sin \Gamma_{s j} . \tag{26}
\end{equation*}
$$

The sum over $s$ will be transformed into an integral in a known way by replacing $\sum_{s}$ with $\int_{0}^{\infty} \frac{8 \pi}{c^{3}} \Omega v_{s}^{2} d v_{s}$ and taking the mean of the expression:

$$
\left(\gamma_{i} A_{s}\right)\left(\gamma_{j} A_{s}\right) \sin \Gamma_{s i} \sin \Gamma_{s j}
$$

over all phases and all directions of propagation and polarization. We then find that:

$$
\begin{equation*}
K=-\sum_{i, j} e_{i} e_{j} \cdot \frac{8}{c} \int_{0}^{\infty} \overline{\left(\gamma_{i} A_{s}\right)\left(\gamma_{j} A_{s}\right) \sin \Gamma_{s i} \sin \Gamma_{s j}} d \nu_{s} . \tag{27}
\end{equation*}
$$

Taking the mean can now be performed with no further analysis. After a calculation that will involve no insurmountable difficulties, when we observe (147) in loc. cit., we will find that:

$$
\overline{\left(\gamma_{i} A_{s}\right)\left(\gamma_{j} A_{s}\right) \sin \Gamma_{s i} \sin \Gamma_{s j}}=
$$

$$
\begin{equation*}
=\frac{\left(\gamma_{i} \gamma_{j}\right)}{4}\left(\frac{\sin \vartheta}{\vartheta}+\frac{\cos \vartheta}{\vartheta^{2}}-\frac{\sin \vartheta}{\vartheta^{3}}\right)-\frac{\left(\gamma_{i} r_{i j}\right)\left(\gamma_{j} r_{i j}\right)}{4 r_{i j}^{2}}\left(\frac{\sin \vartheta}{4}+3 \frac{\cos \vartheta}{\vartheta^{2}}-3 \frac{\sin \vartheta}{\vartheta^{3}}\right) \tag{28}
\end{equation*}
$$

in which:

$$
\begin{equation*}
\vartheta=\frac{2 \pi r_{i j}}{c} v_{s} . \tag{29}
\end{equation*}
$$

We then find from (27) that when we introduce $\vartheta$ in place of $v_{s}$ as the integration variable:

$$
\begin{align*}
K= & -\sum_{i, j} \frac{e_{i} e_{j}}{\pi r_{i j}}\left\{\left(\gamma_{i} \gamma_{j}\right) \int_{0}^{\infty}\left(\frac{\sin \vartheta}{\vartheta}+\frac{\cos \vartheta}{\vartheta^{2}}-\frac{\sin \vartheta}{\vartheta^{3}}\right) d \vartheta\right. \\
& \left.-\frac{\left(\gamma_{i} r_{i j}\right)\left(\gamma_{j} r_{i j}\right)}{r_{i j}^{2}} \cdot \int_{0}^{\infty}\left(\frac{\sin \vartheta}{4}+3 \frac{\cos \vartheta}{\vartheta^{2}}-3 \frac{\sin \vartheta}{\vartheta^{3}}\right) d \vartheta\right\} \tag{30}
\end{align*}
$$

The two integrals have the values $+\pi / 4$ and $-\pi / 4$. We then find that:

$$
\begin{equation*}
K=-\sum_{i, j} \frac{e_{i} e_{j}}{4 r_{i j}}\left\{\left(\gamma_{i} \gamma_{j}\right)+\frac{\left(\gamma_{i} r_{i j}\right)\left(\gamma_{j} r_{i j}\right)}{r_{i j}^{2}}\right\} . \tag{31}
\end{equation*}
$$

That formula included an infinitely-large constant self-energy for the electrons ( ${ }^{1}$ ). If we overlook that constant then we will find that the expression for the interaction of two electrons is:

$$
\begin{equation*}
-\frac{e_{i} e_{j}}{2 r_{i j}}\left\{\left(\gamma_{i} \gamma_{j}\right)+\frac{\left(\gamma_{i} r_{i j}\right)\left(\gamma_{j} r_{i j}\right)}{r_{i j}^{2}}\right\}, \tag{32}
\end{equation*}
$$

which is identical to the Breit interaction.
§ 3. Deriving the Møller formula from quantum electrodynamics. - We shall once more take our starting point to be the Hamiltonian function (11) and restrict ourselves to the case of two particles 1 and 2. However, this time we shall regard the Coulomb energy $e_{1} e_{2} / r_{12}$ as the perturbation, such that the unperturbed Hamiltonian function will be:

$$
\begin{equation*}
H_{0}=\sum_{s}\left(\frac{1}{2} p_{s}^{2}+2 \pi^{2} v_{s}^{2} q_{s}^{2}\right)+\sum_{i=1}^{2}\left[-c\left(\gamma_{i} p_{i}\right)-m c^{2} \delta_{i}\right] . \tag{33}
\end{equation*}
$$

[^2]That transition can be produced, on the one hand, in the direct manner by the Coulomb interaction or on the other hand by the mechanism that we discussed in the previous section, namely, through the detour of an intermediate state:

$$
\begin{equation*}
m_{1} m_{2} 0 \ldots 1_{s} \ldots \tag{35}
\end{equation*}
$$

The first process corresponds to the matrix elements:

$$
\begin{equation*}
\left(\frac{e_{1} e_{2}}{r_{12}}\right)_{n_{1} n_{2}}^{n_{1}^{\prime} n_{2}^{\prime}} \tag{36}
\end{equation*}
$$

and from (23), the second process corresponds to:

$$
\begin{equation*}
K_{n_{1} n_{2}}^{n_{1}^{\prime} n_{2}^{\prime}}=-\frac{c^{2}}{\pi \Omega} \sum_{m_{1}, m_{2}, s} \frac{Q_{m_{1}, s}^{n_{1}^{\prime} n_{2}^{\prime}}(s) Q_{s}\left(v_{s}+v_{n_{1} n_{2} m_{2}}^{m_{1} n_{1}}+v_{m_{2} n_{2}}\right)}{v_{1}} . \tag{37}
\end{equation*}
$$

It is reasonable to restrict oneself to the first approximation in the first case and go up to the second in the second case since (36) and (37) are both proportional to $e_{1} e_{2}$.

We would now like to calculate the matrix element (37). We first remark that $Q$ ( $s$ ) [cf., (18)] consists of two summands $Q(1 s)$ and $Q(2 s)$, each of which depends upon only the coordinates of one electron. Since the unperturbed eigenfunctions are products of eigenfunctions of the individual electrons, the matrix elements of $Q(s)$ can correspond to only those transitions that happen to a single electron. In that way, the sum over $m_{1} m_{2}$ will reduce to two terms $m_{1} m_{2}=n_{1} n_{2}^{\prime}$ and $m_{1} m_{2}=n_{1}^{\prime} n_{2}$. (37) will then become:

$$
\begin{equation*}
K_{n_{1} n_{2}}^{n_{n}^{\prime} n_{2}^{\prime}}=-\frac{c^{2}}{\pi \Omega} \sum_{m_{1}, m_{2}, s} \frac{Q_{n_{1}^{\prime} n_{1}}(1 s) Q_{n_{2}^{\prime} n_{2}}(2 s)}{v_{s}}\left(\frac{1}{v_{s}+v_{n_{1}^{\prime} n_{1}}}+\frac{1}{v_{s}+v_{n_{2}^{\prime} n_{2}}}\right) . \tag{38}
\end{equation*}
$$

$Q(1 s)$ has the form [cf., (18) and loc. cit. (147)]:

$$
\begin{equation*}
Q(1 s)=\frac{e_{1}}{2 i}\left(\gamma_{1} A_{s}\right)\left[e^{2 \pi i v_{s}\left(\alpha_{s} \mathfrak{r}_{1}\right) / c+i \beta_{s}}-e^{-2 \pi i v_{s}\left(\alpha_{s} \mathfrak{\varsigma}_{1}\right) / c-i \beta_{s}}\right] \tag{39}
\end{equation*}
$$

We write the eigenfunctions of the states $n_{1}$ and $n_{1}^{\prime}$ in the form:

$$
\begin{equation*}
u_{1}=\frac{a_{1}}{\sqrt{\Omega}} e^{2 \pi i\left(p_{1} \mathrm{r}_{1}\right) / h}, \quad u_{1}^{\prime}=\frac{a_{1}^{\prime}}{\sqrt{\Omega}} e^{2 \pi i\left(\mathrm{p}_{1}^{\prime} \mathrm{r}_{1}\right) / h} \tag{40}
\end{equation*}
$$

in which the $a$ represent four-component constants that are normalized to unity.

The matrix elements of $Q(1 s)$ will then be non-zero only when:

$$
\begin{equation*}
\frac{h v_{s}}{c} \alpha_{s}= \pm\left(\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right) . \tag{41}
\end{equation*}
$$

Their values in this case are:

$$
\begin{equation*}
\mp \frac{e_{1}}{2 i}\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, A_{s}\right) e^{\mp i \beta_{s}} . \tag{42}
\end{equation*}
$$

We get a corresponding expression for $Q(2 s)$. Now, since one has to take a mean over the phases $\beta_{s}$, the only terms that will yield non-zero contributions are the ones for which the signs in $Q(1 s)$ and $Q(2 s)$ are opposite. One then has:

$$
\begin{gather*}
\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}=-\left(\mathfrak{p}_{2}-\mathfrak{p}_{2}^{\prime}\right)= \pm \frac{h v_{s}}{c} \boldsymbol{\alpha}_{s}, \quad v_{s}=\frac{c}{h}\left|\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right|,  \tag{43}\\
K_{n_{1} n_{2}}^{n_{1}^{\prime} n_{2}^{\prime}}=-\frac{e_{1} e_{2} h^{2}}{4 \pi \Omega} \sum_{s} \frac{\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, A_{s}\right)\left(\tilde{a}_{2}^{\prime} \gamma_{2} a_{2}, A_{s}\right)}{\left|\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right|}\left(\frac{1}{\left|\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right|+\frac{\left(E_{1}^{\prime}-E_{1}\right)}{c}}+\frac{1}{\left|\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right|+\frac{\left(E_{2}^{\prime}-E_{2}\right)}{c}}\right) . \tag{44}
\end{gather*}
$$

Due to (43), the sum over $s$ consists of only four terms that correspond to the doubled sign in (43) and two mutually-perpendicular directions of polarization. When we sum over $s$, we can then take the mean over all directions of the unit vector $A_{s}$ that are perpendicular to $\mathfrak{p}-\mathfrak{p}^{\prime}$ and multiply by 4. If we do that and further observe that due to the law of energy:

$$
E_{1}-E_{1}^{\prime}=E_{2}^{\prime}-E_{2},
$$

then we will find that:

$$
\begin{equation*}
K_{n_{1} n_{2}}^{n_{1}^{\prime} n_{2}^{\prime}}=-\frac{e_{1} e_{2} h^{2}}{\pi \Omega}\left[\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, \tilde{a}_{2}^{\prime} \gamma_{2} a_{2}\right)-\frac{\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, \mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)\left(\tilde{a}_{2}^{\prime} \gamma_{2} a_{2}, \mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)}{\left(\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)^{2}}\right] \cdot \frac{1}{\left(\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)^{2}+\left(\frac{E_{1}-E_{1}^{\prime}}{c}\right)^{2}} . \tag{45}
\end{equation*}
$$

We must still add the matrix element (36) to (45). According to Møller, it has the value:

$$
\begin{equation*}
\left(\frac{e_{1} e_{2}}{r_{12}}\right)_{n_{1} n_{2}}^{n_{1}^{\prime} n_{2}^{\prime}}=\frac{e_{1} e_{2} h^{2}}{\pi \Omega} \frac{\left(\tilde{a}_{1}^{\prime} a_{1}\right)\left(\tilde{a}_{2}^{\prime} a_{2}\right)}{\left(\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)^{2}} . \tag{46}
\end{equation*}
$$

The sum is:

$$
\left.\begin{array}{rl}
H_{n_{1} n_{2}}^{n_{n}^{\prime} n_{2}^{\prime}} & =K_{n_{1} n_{2}}^{n_{2}^{\prime} n_{2}^{\prime}}+\left(\frac{e_{1} e_{2}}{r_{12}}\right)_{n_{1} n_{2}}^{n_{1}^{\prime} n_{2}^{\prime}} \\
& =\frac{e_{1} e_{2} h^{2}}{\pi \Omega} \cdot \frac{\left(\tilde{a}_{1}^{\prime} a_{1}\right)\left(\tilde{a}_{2}^{\prime} a_{2}\right)-\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}\right)\left(\tilde{a}_{2}^{\prime} \gamma_{2} a_{2}\right)}{\left(\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)^{2}-\left(\frac{E_{1}-E_{1}^{\prime}}{c}\right)^{2}}  \tag{47}\\
& +\frac{e_{1} e_{2} h^{2}}{\pi \Omega\left|\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right|} \cdot \frac{-\left(\frac{E_{1}-E_{1}^{\prime}}{c}\right)^{2}\left(\tilde{a}_{1}^{\prime} a_{1}\right)\left(\tilde{a}_{2}^{\prime} a_{2}\right)+\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, \mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)\left(\tilde{a}_{2}^{\prime} \gamma_{2} a_{2}, \mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)}{\left(\mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)^{2}-\left(\frac{E_{1}-E_{1}^{\prime}}{c}\right)^{2}} .
\end{array}\right\}
$$

The first summand on the right-hand side is precisely the Møller interaction energy. We must then prove that the second summand will vanish. We next write that summand in a symmetric form by using the energy and impulse theorem:

$$
\text { Factor } \cdot\left[+\frac{E_{1}-E_{1}^{\prime}}{c}\left(\tilde{a}_{1}^{\prime} a_{1}\right) \frac{E_{2}-E_{2}^{\prime}}{c}\left(\tilde{a}_{2}^{\prime} a_{2}\right)-\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, \mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right)\left(\tilde{a}_{2}^{\prime} \gamma_{2} a_{2}, \mathfrak{p}_{2}-\mathfrak{p}_{2}^{\prime}\right)\right]
$$

In order to prove that this vanishes, it suffices to show that one has:

$$
\begin{equation*}
\left(E_{1}-E_{1}^{\prime}\right) \tilde{a}_{1}^{\prime} a_{1}=-c\left(\tilde{a}_{1}^{\prime} \gamma_{1} a_{1}, \mathfrak{p}_{1}-\mathfrak{p}_{1}^{\prime}\right), \tag{49}
\end{equation*}
$$

and a corresponding equation for the second particle. However, (49) follows immediately from the Dirac equation:

$$
\begin{equation*}
E_{1} a_{1}=-c\left(\mathfrak{p}_{1} \gamma_{1} a_{1}\right)-m_{1} c^{2} \delta_{1} a_{1}, \tag{50}
\end{equation*}
$$

and the corresponding equation for $\tilde{a}_{1}^{\prime}$.
Furthermore, one easily convinces oneself that the Møller formula can be derived in precisely the same way when one of the particles is bound.

It is satisfying that no radiation will be emitted in our approximation. One can imagine that it is possible for processes to exist in which the two electrons change their states, and two quanta are emitted. However, in such processes, one must satisfy the law of impulse twice (viz., the impulse of one quantum must be equal to the change in impulse of one electron, and the same thing must be true for the other quantum and the other electron) and the law of energy once. However, that is impossible as long as the no electron has a negative energy in the final state.

One of us (H. Bethe) would like to thank the Rockefeller Foundation for having granted us a stipend that made it possible for him to travel to Rome.

Rome, Institute for theoretical physics.


[^0]:    $\left.{ }^{1}{ }^{1}\right)$ G. Breit, Phys. Rev. 34 (1929), pp. 553; ibid. 39 (1932), pp. 616.
    ( ${ }^{2}$ ) C. Møller, Zeit. Phys. 70 (1931), pp. 686.
    $\left(^{3}\right)$ L. Rosenfeld, ibidem 73 (1931), pp. 253.
    $\left(^{4}\right)$ W. Heisenberg, Ann. Phys. (Leipzig) 9 (1931), pp. 338.
    $\left({ }^{5}\right)$ E. Fermi, Rev Mod. Phys. 4 (1932), pp. 87, cited as loc. cit. in what follows.

[^1]:    $\left({ }^{1}\right)$ In this derivation, we shall not consider the Coulomb energy in the unperturbed problem since it is included in the Møller interaction. We can then speak of the quantum states of the individual electrons.

[^2]:    $\left({ }^{1}\right)$ It will cancel the electrostatic self-energy precisely. However, one must not think that the difficulties associated with infinite self-energies would be eliminated in that way. One will once more find an infinitely-large self-energy for a free electron when one calculates the diagonal elements of (23) and in so doing, one does not neglect $v_{m n}$, and one also considers negative-energy states to be intermediate states $m$.

