"Über die Wechselwirkung von zwei Elektronen," Zeit. Phys. 77 (1932), 296-306.

On the interaction of two electrons

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The connections between the interaction formulas of **Breit** $(^1)$ and **Møller** $(^2)$ and those of quantum electrodynamics will be discussed. It will be shown that the **Breit** formula can be derived from **Møller**'s (§ 1) and from quantum electrodynamics (§ 2), and that **Møller**'s formula also follows from the latter (§ 3).

Introduction. – Two Ansätze exist for the interaction of two electrons, namely, those of **Breit** (¹) and **Møller** (²), which seem to start from entirely-different places. It would seem desirable to investigate the relationships between the two Ansätze and their relationships to quantum electrodynamics. **Breit** derived his differential equation directly from quantum electrodynamics, and **Rosenfeld** (³) showed that the formulas from quantum electrodynamics that were required for **Breit**'s derivation were obtained by the **Heisenberg** (⁴) correspondence process that **Møller** had applied. We will obtain **Breit**'s formula *directly* from **Møller**'s, and for its derivation from quantum electrodynamics, we will choose a form that is completely analogous to our derivation of **Møller**'s, such that the various approximation processes will become more comparable.

As is known, one can derive **Coulomb**'s interaction energy between two electrons from quantum electrodynamics precisely when one considers only the coupling of *longitudinal* waves of the electromagnetic fields with the electrons (5). Any deviation from **Coulomb**'s law must then come about as a result of their interaction with *transversal* waves, which we will refer to briefly as the *radiation field* in what follows. Now, in the *first* approximation, the coupling between matter and the radiation field will create only those transitions for which the quantum state of the electrons will change, and a light quantum will be absorbed or emitted. However, we will be interested in the matrix elements of the interaction energy of the electrons that correspond to a change in the quantum state of the electrons alone without altering the state of the radiation field, in which we can assume, in particular, that no radiation is present in the initial state, as well as the final one.

⁽¹⁾ G. Breit, Phys. Rev. 34 (1929), pp. 553; *ibid.* 39 (1932), pp. 616.

^{(&}lt;sup>2</sup>) C. Møller, Zeit. Phys. 70 (1931), pp. 686.

^{(&}lt;sup>3</sup>) **L. Rosenfeld**, *ibidem* **73** (1931), pp. 253.

^{(&}lt;sup>4</sup>) **W. Heisenberg**, Ann. Phys. (Leipzig) **9** (1931), pp. 338.

^{(&}lt;sup>5</sup>) E. Fermi, Rev Mod. Phys. 4 (1932), pp. 87, cited as *loc. cit.* in what follows.

One will obviously first get such transitions of the electrons alone by a double process, in which a quantum is initially emitted and then the same quantum is reabsorbed.

The derivations of the **Breit** and **Møller** formulas from quantum electrodynamics differ essentially in only the following way: In the first case, one has already added the **Coulomb** interaction energy of the electrons to the "unperturbed Hamiltonian function," so the perturbation is merely the interaction of the electrons with the transverse waves of the electromagnetic field. Now, the calculation will be performed in only the non-relativistic approximation, i.e., only up to terms of order v^2 / c^2 . In addition, all of the transitions of the material part of the system that are coupled with a permanent emission of radiation will be neglected. Naturally, one can treat the latter with the ordinary theory of radiation.

By contrast, in **Møller**'s theory, the Coulomb energy is one part of the *perturbation* and enteron along with the interaction with the radiation field on an equal basis, since the two interactions will, in fact, have the same order of magnitude in the extreme relativistic case. In the zeroth-order approximation, the electrons move independently of each other. **Møller**'s theory will be developed in a relativistically-exact manner, and in order to do that, we develops it in powers of *e* and neglects terms of order higher than e^2 . When one then derives the **Breit** interaction from **Møller**'s theory, that will only justify the fact that one is treating the Coulomb interaction in the first approximation, but when one starts from quantum electrodynamics, the Coulomb force will be considered in the unperturbed system such that its effect can be treated exactly. Note that no radiation occurs in the **Møller** (viz., second) approximation, and that such a thing will first come about only when one considers the radiation interaction in the *third* approximation, so when one excludes the possibility that the electrons will have negative energy in the final state.

§ 1. Deriving Breit's interaction energy from Møller's theory. – According to Møller, in order to calculate the interaction of two electrons, one must construct the retarded potentials that are created by the charge distribution of the first electron, and they must be considered to be a perturbation that acts upon the second electron. We would like to calculate the matrix element of the interaction energy that corresponds to a transition of the first electron from the state $(^1) n_1$ to the state n'_1 , and the second electron from n_2 to n'_2 , under which the total energy in the initial and final state should be equal to:

$$E_1 + E_2 = E_1' + E_2'. \tag{1}$$

The transition $n_1 \rightarrow n'_1$ corresponds to the charge distribution:

$$\rho_{n,n'}(\mathbf{r}_1,t) = e_1 u_1'^*(\mathbf{r}_1) u_1(\mathbf{r}_1) e^{2\pi i (E_1' - E_1)t/h},\tag{2}$$

 $^(^{1})$ In this derivation, we shall *not* consider the Coulomb energy in the unperturbed problem since it is included in the **Møller** interaction. We can then speak of the quantum states of the individual electrons.

in which u_1 and u'_1 are the **Dirac** eigenfunctions of the first electron in the initial and final state and e_1 is its charge. When one considers the retardation at the location \mathfrak{r}_2 at time *t*, the charge distribution (2) will generate the scalar potential:

$$\varphi_{n_{1}n_{1}'}(\mathfrak{r}_{2},t) = \int \frac{\rho_{n_{1}n_{1}'}(\mathfrak{r}_{1},t-|\mathfrak{r}_{2}-\mathfrak{r}_{1}|/c))}{|\mathfrak{r}_{2}-\mathfrak{r}_{1}|} d\tau_{1}
= e_{1} e^{2\pi i (E_{1}'-E_{1})t/\hbar} \int \frac{(u_{1}'^{*}(\mathfrak{r}_{1})u_{1}(\mathfrak{r}_{1}))}{|\mathfrak{r}_{2}-\mathfrak{r}_{1}|} e^{2\pi i (E_{1}'-E_{1})|\mathfrak{r}_{2}-\mathfrak{r}_{1}|/\hbar c} d\tau_{1}.$$
(3)

One correspondingly gets the vector potential in the form:

$$\mathfrak{A}_{n_{1}n_{1}'}(\mathfrak{r}_{2},t) = -e_{1} e^{2\pi i (E_{1}'-E_{1})t/h} \int \frac{(u_{1}'^{*}(\mathfrak{r}_{1})\gamma_{1}u_{1}(\mathfrak{r}_{1}))}{|\mathfrak{r}_{2}-\mathfrak{r}_{1}|} e^{2\pi i (E_{1}'-E_{1})|\mathfrak{r}_{2}-\mathfrak{r}_{1}|/hc} d\tau_{1} .$$

$$\tag{4}$$

 γ_{x_1} , γ_{y_1} , γ_{z_1} , δ_1 are the **Dirac** operators of the first electron. γ_1 is the vector with the components γ_{x_1} , γ_{y_1} , γ_{z_1} .

Now, the **Breit** formula for the interaction energy that we wish to derive is exact only up to terms of order $1/c^2$, inclusive. That suggests that we might develop the exponential functions in (3), (4), which represent the retardation of the potential in powers of 1/c:

$$\varphi_{n_{1}n_{1}'}(\mathfrak{r}_{2},t) = e_{1} e^{2\pi i (E_{1}'-E_{1})t/h} \int (u_{1}'^{*}(\mathfrak{r}_{1})u_{1}(\mathfrak{r}_{1})) \left[\frac{1}{|\mathfrak{r}_{2}-\mathfrak{r}_{1}|} + \frac{2\pi i}{hc} (E_{1}'-E_{1}) - \frac{2\pi^{2}i}{h^{2}c^{2}} (E_{1}'-E_{1})^{2} |\mathfrak{r}_{2}-\mathfrak{r}_{1}| \right] d\tau_{1}, \quad (5)$$

and correspondingly for \mathfrak{A} . The second term in the square bracket in (5) will vanish: That is because either the states n_1 and n'_1 are identical, so $E_1 - E'_1 = 0$, or they are different from each other, in which case, the volume integral will vanish due to the orthogonality of the eigenfunctions. Now, the perturbation energy that acts upon the second electron is:

$$V_{n_1}^{n_1'}(\mathfrak{r}_2,t) = e_2 \,\varphi_{n_1 n_1'}(\mathfrak{r}_2,t) + e_2 \,(\gamma_2 \,\mathfrak{U}_{n_1 n_1'}(\mathfrak{r}_2,t)) \,.$$

We form its matrix element, which corresponds to the transition $n_2 \rightarrow n'_2$ of the second electron:

$$V_{n_{1}n_{2}}^{n'_{1}n'_{2}} = e_{1}e_{2}e^{2\pi i (E'_{1}+E'_{2}-E_{1}-E_{2})t/h}\int u_{1}^{\prime*}(\mathfrak{r}_{1})u_{2}^{\prime*}(\mathfrak{r}_{2})$$

$$\times \left[\frac{1}{|\mathfrak{r}_{2}-\mathfrak{r}_{1}|} - \frac{2\pi^{2}}{h^{2}c^{2}}(E'_{1}-E_{1})^{2}|\mathfrak{r}_{2}-\mathfrak{r}_{1}| - \frac{(\gamma_{1}\gamma_{2})}{|\mathfrak{r}_{2}-\mathfrak{r}_{1}|}\right]u_{2}(\mathfrak{r}_{2})u_{1}(\mathfrak{r}_{1})d\tau_{1}d\tau_{2}.$$
(6)

The first term in the bracket here is the ordinary Coulomb potential, while the second one comes from the retarding of the scalar potential, and the third one reflects the influence of the (unretarded) vector potential: Since the γ operators themselves have order of magnitude v/c, we need do not need to consider the retardation of the vector potential in our approximation. The first and third terms are symmetric in the two electrons, while the second one is not, which is based upon the asymmetry of **Møller**'s method. However, since the total energy should be the same in the initial and final state (1), we can symmetrize the term by writing $-(E_1 - E'_1)(E_2 - E'_2)$ in place of $(E_2 - E'_2)^2$. With that, (6) will become the matrix element of:

$$V = e_1 e_2 \frac{1 - (\gamma_1 \gamma_2)}{|\mathfrak{r}_2 - \mathfrak{r}_1|} + \frac{2\pi^2 e_1 e_2}{h^2 c^2} \left[H_2(H_1 | \mathfrak{r}_2 - \mathfrak{r}_1 | - \mathfrak{r}_2 - \mathfrak{r}_1 | H_1) - (H_1 | \mathfrak{r}_2 - \mathfrak{r}_1 | - | \mathfrak{r}_2 - \mathfrak{r}_1 | H_1) H_2 \right], \quad (7)$$

in which H_1 and H_2 are the unperturbed Hamiltonian functions of the two electrons in the absence of interaction, so:

$$H_{1} = -e_{1} \varphi_{0}(\mathfrak{r}_{1}) - \frac{hc}{2\pi i}(\gamma_{1}, \operatorname{grad}_{1}) - e_{1}(\gamma_{1}, \mathfrak{A}_{0}(\mathfrak{r}_{1})) - m_{1}c^{2}\delta_{1}.$$
(8)

 φ_0 and \mathfrak{A}_0 are the scalar and vector potentials of any external static field in which the two electrons move. All components of H_1 commute with $|\mathfrak{r}_2 - \mathfrak{r}_1|$, except for the gradient, so:

$$H_1 \mid \mathfrak{r}_2 - \mathfrak{r}_1 \mid - \mid \mathfrak{r}_2 - \mathfrak{r}_1 \mid H_1 = -\frac{hc}{2\pi i} \left(\gamma_1, \frac{\mathfrak{r}_1 - \mathfrak{r}_2}{|\mathfrak{r}_1 - \mathfrak{r}_2|} \right) = F$$

and

$$H_2 F - F H_2 = + \frac{h^2 c^2}{4\pi^2} \left(\frac{(\gamma_1 \gamma_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{(\gamma_1, \mathbf{r}_1 - \mathbf{r}_2)(\gamma_2, \mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right).$$
(9)

If one substitutes that in (7) then one will get precisely the **Breit** formula for the interaction energy:

$$V = \frac{e_1 e_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \left(1 - \frac{1}{2} (\gamma_1 \gamma_2) - \frac{1}{2} \frac{(\gamma_1, \mathbf{r}_1 - \mathbf{r}_2)(\gamma_2, \mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \right).$$
(10)

§ 2. Deriving the Breit law of interaction from quantum electrodynamics. – We take our starting point for this derivation to be the Hamiltonian function of the system composed of the electrons and the electromagnetic field in the form [*loc. cit.* (166)] that one obtains when one eliminates the coordinates from the scalar potential and the longitudinal components of the vector potential with the help of the continuity equation. The Hamiltonian function will already include the Coulomb interaction of the particles in that form. When one observes [*loc. cit.* (167)], the Hamiltonian function can be written in the form:

$$R = \sum_{s} \left(\frac{1}{2} p_{s}^{2} + 2\pi^{2} v_{s}^{2} q_{s}^{2} \right) + \sum_{i < j} \frac{e_{i} e_{j}}{r_{ij}} - \sum_{i} c \left(\gamma_{i} p_{i} \right) - \sum_{i} m_{i} c^{2} \delta_{i} + c \sqrt{\frac{8\pi}{\Omega}} \sum_{i,s} e_{i} \left(\gamma_{i} A_{s} \right) q_{s} \sin \Gamma_{si}.$$
(11)

The notations in that formula are the same as the ones in *loc. cit.*. However, the radiation components q_s are enumerated with a single index s, and not with two s_1 and s_2 . The Coulomb interaction of the electrons is written out explicitly, while the infinitely-large constant electrostatic self-energy is ignored.

In order to derive the **Breit** formula, we must consider the Coulomb force in the unperturbed system. We then consider the unperturbed Hamiltonian function to be the sum of the energy of radiation:

$$H_{s} = \sum_{s} \left(\frac{1}{2} p_{s}^{2} + 2\pi^{2} v_{s}^{2} q_{s}^{2} \right)$$
(12)

and the energy of the material particles:

$$H_{M} = \sum_{i} -c(\gamma_{i}p_{i}) - \sum_{i} m_{i}c^{2}\delta_{i} + \sum_{i < j} \frac{e_{i}e_{j}}{r_{ij}}.$$
 (13)

We regard the coupling energy of the particles and the field to be:

$$H = c \sqrt{\frac{8\pi}{\Omega}} \sum_{i,s} e_i (\gamma_i A_s) q_s \sin \Gamma_{si}.$$
(14)

The states of the unperturbed system are characterized by a quantum number \mathbf{n}_i that determines the state of the material particle of the system with the Hamiltonian function H_M , and the quantum numbers $n_1, n_2, ..., n_s, ...$ of the radiating oscillators. Let the corresponding probability amplitudes be:

$$a_{nn_1n_2\cdots n_s\cdots} \tag{15}$$

The a vary in time as a result of the effect of the coupling energy (14), which corresponds to the known equations:

$$a_{n'n'_{1}n'_{2}\cdots n'_{s}} = -\frac{2\pi i}{h} \sum_{nn_{1}n_{2}\cdots} H^{nn_{1}n_{2}\cdots}_{n'n'_{1}n'_{2}\cdots} a_{nn_{1}n_{2}\cdots} e^{2\pi i (E_{n'n'_{1}\cdots}-E_{nn_{1}\cdots})t/h}.$$
 (16)

As is known, the only non-zero matrix elements of *H* are the following ones:

$$H_{n'n_1n_2\cdots n_s\pm 1\cdots}^{nn_1n_2\cdots n_s \pm 1\cdots} = c \sqrt{\frac{8\pi}{\Omega}} Q_{n'n}(s) \sqrt{\frac{h}{8\pi^2 v_s}} \cdot \begin{cases} \sqrt{n_s+1}, \\ \sqrt{n_s}, \end{cases}$$
(17)

in which $Q_{n'n}(s)$ represents the matrix element n'n of the quantity:

$$Q(s) = \sum_{i} e_i(\gamma_i A_s) \sin \Gamma_{si} .$$
(18)

We would now like to assume that the material system is found in the state *n* to begin with, and no radiating oscillator has been excited. We will then have:

$$|a_{n\,0\,0\,\ldots\,0\,\ldots}|=1$$
.

We would like to see how a probability amplitude for the state $n' 0 0 \dots 0 \dots$ arises from the effect of the perturbation. Now, the perturbation matrix (17) has no element that would couple the two states $n \ 0 \ 0 \ 0 \dots$ and $n' \ 0 \ 0 \ 0 \dots$ directly. Such a transition can occur only indirectly by the detour through an intermediate term. From (17), the states $m \ 0 \ 0 \dots \ 1_s \dots$ will come into question as such things since they combine with the initial and final states. Upon applying (16) to the transition $n \ 0 \ 0 \ \dots \ \rightarrow m \ 0 \ 0 \dots \ 1_s \dots$:

$$\dot{a}_{m\,0...1_{s}...} = -\frac{2\pi i}{h} H^{n00...0...}_{m00...1_{s}...} a_{n00...0...} e^{2\pi i (v_{mn} + v_{s})t} ,$$

$$v_{mn} = \frac{E_{m} - E_{n}}{h} .$$
(19)

Since $a_{n \ 0 \ 0... 0...}$ is practically constant, we can integrate and find:

$$a_{m\,0...l_{s}...} = -\frac{H_{m00...l_{s}...}^{n\,00...0...}}{h(v_{mn} + v_{s})} a_{n00...0...} e^{2\pi i (v_{mn} + v_{s})t}.$$
(20)

We must now apply equation (16) to the transition $m \ 0...1_s... \rightarrow n' \ 0 \ 0...0...$ We find that:

$$\dot{a}_{n'0...0...} = -\frac{2\pi i}{h} \sum_{m,s} H_{n'00...1_{s}...}^{m\,00...1_{s}...} a_{m00...1_{s}...} e^{2\pi i (v_{n'm} - v_{s}) t}$$

$$= -\frac{2\pi i}{h} \sum_{m,s} \left(-\frac{1}{h} \frac{H_{m...1_{s}...}^{n...0...} H_{n'...0...}^{m...1_{s}...}}{v_{mn} + v_{s}} \right) \cdot a_{n...0...} e^{2\pi i v_{n'n} t} .$$
(21)

We can regard the quantity:

in which:

$$K_{n'n} = -\frac{1}{h} \sum_{m,s} \frac{H_{m..l_s...}^{n...0...} H_{n'...l_s...}^{m...l_s...}}{v_{mn} + v_s}$$
(22)

as a matrix element that mediates the transition between the states $n \ 0...0... \rightarrow n' \ 0 \ 0...0...$ directly. The corresponding quantities K represent the correction that we must make to the Coulomb interaction in order to include the retardation in our calculation of the potential.

With the help of (17), we will find from (22) that:

$$K_{n'n} = -\frac{c^2}{\pi\Omega} \sum_{m,s} \frac{Q_{n'm}(s)Q_{mn}(s)}{v_s (v_s + v_{mn})}.$$
(23)

We must now express the quantities K as functions of the coordinates and momenta of the particle. That is easy to do when we neglect v_{mn} in comparison to v_s in the denominator of (23) as a first approximation. (Observe that the ratio v_{mn} / v_s has order of magnitude v / c.) We will then get:

$$K_{n'n} = -\frac{c^2}{\pi \Omega} \sum_{s} \frac{1}{v_s^2} Q_{n'm}(s) Q_{mn}(s)$$

= $-\frac{c^2}{\pi \Omega} \sum_{s} \frac{1}{v_s^2} [Q^2(s)]_{n'm}.$ (24)

We then have:

$$K = -\frac{c^2}{\pi\Omega} \sum_{s} \frac{1}{v_s^2} Q^2(s) .$$
 (25)

All that remains is to perform the summation over *s* in order for us to prove that the quantity *K* that we have found is identical to the **Breit** interaction function.

We substitute the expression (18) for Q(s) in (25) and find that:

$$K = -\sum_{i,j} e_i e_j \frac{c^2}{\pi \Omega} \sum_s \frac{1}{v_s^2} (\gamma_i A_s) (\gamma_j A_s) \sin \Gamma_{si} \sin \Gamma_{sj}.$$
(26)

The sum over *s* will be transformed into an integral in a known way by replacing \sum_{s} with $\int_{0}^{\infty} \frac{8\pi}{c^3} \Omega v_s^2 dv_s$ and taking the mean of the expression:

 $(\gamma_i A_s)(\gamma_j A_s) \sin \Gamma_{si} \sin \Gamma_{sj}$

over all phases and all directions of propagation and polarization. We then find that:

$$K = -\sum_{i,j} e_i e_j \cdot \frac{8}{c} \int_0^\infty \overline{(\gamma_i A_s)(\gamma_j A_s) \sin \Gamma_{si} \sin \Gamma_{sj}} \, d\nu_s \,. \tag{27}$$

Taking the mean can now be performed with no further analysis. After a calculation that will involve no insurmountable difficulties, when we observe (147) in *loc. cit.*, we will find that:

 $\overline{(\gamma_i A_s)(\gamma_j A_s) \sin \Gamma_{si} \sin \Gamma_{sj}} =$

$$=\frac{(\gamma_i\,\gamma_j)}{4}\left(\frac{\sin\vartheta}{\vartheta}+\frac{\cos\vartheta}{\vartheta^2}-\frac{\sin\vartheta}{\vartheta^3}\right)-\frac{(\gamma_i\,r_{ij})(\gamma_j\,r_{ij})}{4\,r_{ij}^2}\left(\frac{\sin\vartheta}{4}+3\frac{\cos\vartheta}{\vartheta^2}-3\frac{\sin\vartheta}{\vartheta^3}\right),\tag{28}$$

in which:

$$\mathcal{G} = \frac{2\pi r_{ij}}{c} v_s. \tag{29}$$

We then find from (27) that when we introduce \mathcal{G} in place of v_s as the integration variable:

$$K = -\sum_{i,j} \frac{e_i e_j}{\pi r_{ij}} \left\{ (\gamma_i \gamma_j) \int_0^\infty \left(\frac{\sin \vartheta}{\vartheta} + \frac{\cos \vartheta}{\vartheta^2} - \frac{\sin \vartheta}{\vartheta^3} \right) d\vartheta - \frac{(\gamma_i r_{ij})(\gamma_j r_{ij})}{r_{ij}^2} \cdot \int_0^\infty \left(\frac{\sin \vartheta}{4} + 3 \frac{\cos \vartheta}{\vartheta^2} - 3 \frac{\sin \vartheta}{\vartheta^3} \right) d\vartheta \right\} .$$
(30)

The two integrals have the values + $\pi/4$ and – $\pi/4$. We then find that:

$$K = -\sum_{i,j} \frac{e_i e_j}{4r_{ij}} \left\{ (\gamma_i \gamma_j) + \frac{(\gamma_i r_{ij})(\gamma_j r_{ij})}{r_{ij}^2} \right\}.$$
 (31)

That formula included an infinitely-large constant self-energy for the electrons (¹). If we overlook that constant then we will find that the expression for the interaction of two electrons is:

$$-\frac{e_i e_j}{2 r_{ij}} \left\{ (\gamma_i \gamma_j) + \frac{(\gamma_i r_{ij})(\gamma_j r_{ij})}{r_{ij}^2} \right\},$$
(32)

which is identical to the **Breit** interaction.

§ 3. Deriving the Møller formula from quantum electrodynamics. – We shall once more take our starting point to be the Hamiltonian function (11) and restrict ourselves to the case of two particles 1 and 2. However, this time we shall regard the Coulomb energy $e_1 e_2 / r_{12}$ as the perturbation, such that the unperturbed Hamiltonian function will be:

$$H_0 = \sum_{s} \left(\frac{1}{2} p_s^2 + 2\pi^2 v_s^2 q_s^2 \right) + \sum_{i=1}^{2} \left[-c \left(\gamma_i p_i \right) - mc^2 \delta_i \right] .$$
(33)

⁽¹⁾ It will cancel the electrostatic self-energy precisely. However, one must not think that the difficulties associated with infinite self-energies would be eliminated in that way. One will once more find an infinitely-large self-energy for a free electron when one calculates the diagonal elements of (23) and in so doing, one does not neglect v_{mn} , and one also considers negative-energy states to be intermediate states *m*.

That transition can be produced, on the one hand, in the direct manner by the Coulomb interaction or on the other hand by the mechanism that we discussed in the previous section, namely, through the detour of an intermediate state:

$$m_1 m_2 0 \dots 1_s \dots$$
 (35)

The first process corresponds to the matrix elements:

$$\left(\frac{e_1 e_2}{r_{12}}\right)_{n_1 n_2}^{n_1' n_2'},\tag{36}$$

and from (23), the second process corresponds to:

$$K_{n_{1}n_{2}}^{n_{1}'n_{2}'} = -\frac{c^{2}}{\pi\Omega} \sum_{m_{1},m_{2},s} \frac{Q_{m_{1}m_{2}}^{n_{1}'n_{2}'}(s)Q_{n_{1}n_{2}}^{m_{1}m_{2}}(s)}{\nu_{s}(\nu_{s} + \nu_{m_{1}n_{1}} + \nu_{m_{2}n_{2}})}.$$
(37)

It is reasonable to restrict oneself to the first approximation in the first case and go up to the second in the second case since (36) and (37) are both proportional to $e_1 e_2$.

We would now like to calculate the matrix element (37). We first remark that Q(s) [cf., (18)] consists of two summands Q(1 s) and Q(2 s), each of which depends upon only the coordinates of *one* electron. Since the unperturbed eigenfunctions are products of eigenfunctions of the individual electrons, the matrix elements of Q(s) can correspond to only those transitions that happen to a single electron. In that way, the sum over $m_1 m_2$ will reduce to two terms $m_1 m_2 = n_1 n'_2$ and $m_1 m_2 = n'_1 n_2$. (37) will then become:

$$K_{n_{1}n_{2}}^{n_{1}n_{2}} = -\frac{c^{2}}{\pi\Omega} \sum_{m_{1},m_{2},s} \frac{Q_{n_{1}'n_{1}}(1s)Q_{n_{2}'n_{2}}(2s)}{v_{s}} \left(\frac{1}{v_{s}+v_{n_{1}'n_{1}}} + \frac{1}{v_{s}+v_{n_{2}'n_{2}}}\right).$$
(38)

Q (1 *s*) has the form [cf., (18) and *loc. cit.* (147)]:

$$Q(1 s) = \frac{e_1}{2i} (\gamma_1 A_s) \Big[e^{2\pi i v_s(\alpha_s \mathfrak{r}_1)/c + i\beta_s} - e^{-2\pi i v_s(\alpha_s \mathfrak{r}_1)/c - i\beta_s} \Big].$$
(39)

We write the eigenfunctions of the states n_1 and n'_1 in the form:

$$u_{1} = \frac{a_{1}}{\sqrt{\Omega}} e^{2\pi i (\mathfrak{p}_{1}\mathfrak{r}_{1})/h}, \qquad u_{1}' = \frac{a_{1}'}{\sqrt{\Omega}} e^{2\pi i (\mathfrak{p}_{1}'\mathfrak{r}_{1})/h}, \qquad (40)$$

in which the *a* represent four-component constants that are normalized to unity.

The matrix elements of Q(1 s) will then be non-zero only when:

$$\frac{h\nu_s}{c}\alpha_s = \pm(\mathfrak{p}_1 - \mathfrak{p}_1'). \tag{41}$$

Their values in this case are:

$$\mp \frac{e_1}{2i} \left(\tilde{a}_1' \gamma_1 a_1, A_s \right) e^{\mp i \beta_s} \,. \tag{42}$$

We get a corresponding expression for Q(2 s). Now, since one has to take a mean over the phases β_s , the only terms that will yield non-zero contributions are the ones for which the signs in Q(1 s) and Q(2 s) are opposite. One then has:

$$\mathfrak{p}_1 - \mathfrak{p}_1' = -(\mathfrak{p}_2 - \mathfrak{p}_2') = \pm \frac{h\nu_s}{c} \alpha_s, \qquad \nu_s = \frac{c}{h} |\mathfrak{p}_1 - \mathfrak{p}_1'|, \qquad (43)$$

$$K_{n_{1}n_{2}}^{n_{1}'n_{2}'} = -\frac{e_{1}e_{2}h^{2}}{4\pi\Omega}\sum_{s}\frac{(\tilde{a}_{1}'\gamma_{1}a_{1},A_{s})(\tilde{a}_{2}'\gamma_{2}a_{2},A_{s})}{|\mathfrak{p}_{1}-\mathfrak{p}_{1}'|} \left(\frac{1}{|\mathfrak{p}_{1}-\mathfrak{p}_{1}'|+\frac{(E_{1}'-E_{1})}{c}}+\frac{1}{|\mathfrak{p}_{1}-\mathfrak{p}_{1}'|+\frac{(E_{2}'-E_{2})}{c}}\right). \quad (44)$$

Due to (43), the sum over *s* consists of only four terms that correspond to the doubled sign in (43) and two mutually-perpendicular directions of polarization. When we sum over *s*, we can then take the mean over all directions of the unit vector A_s that are perpendicular to $\mathfrak{p} - \mathfrak{p}'$ and multiply by 4. If we do that and further observe that due to the law of energy:

$$E_1 - E_1' = E_2' - E_2,$$

then we will find that:

$$K_{n_{1}n_{2}}^{n_{1}'n_{2}'} = -\frac{e_{1}e_{2}h^{2}}{\pi\Omega} \left[(\tilde{a}_{1}'\gamma_{1}a_{1}, \tilde{a}_{2}'\gamma_{2}a_{2}) - \frac{(\tilde{a}_{1}'\gamma_{1}a_{1}, \mathfrak{p}_{1} - \mathfrak{p}_{1}')(\tilde{a}_{2}'\gamma_{2}a_{2}, \mathfrak{p}_{1} - \mathfrak{p}_{1}')}{(\mathfrak{p}_{1} - \mathfrak{p}_{1}')^{2}} \right] \cdot \frac{1}{(\mathfrak{p}_{1} - \mathfrak{p}_{1}')^{2} + \left(\frac{E_{1} - E_{1}'}{c}\right)^{2}}.$$
 (45)

We must still add the matrix element (36) to (45). According to Møller, it has the value:

$$\left(\frac{e_1 e_2}{r_{12}}\right)_{n_1 n_2}^{n'_1 n'_2} = \frac{e_1 e_2 h^2}{\pi \Omega} \frac{(\tilde{a}'_1 a_1)(\tilde{a}'_2 a_2)}{(\mathfrak{p}_1 - \mathfrak{p}'_1)^2}.$$
(46)

The sum is:

$$H_{n_{1}n_{2}}^{n'_{1}n'_{2}} = K_{n_{1}n_{2}}^{n'_{1}n'_{2}} + \left(\frac{e_{1}e_{2}}{r_{12}}\right)_{n_{1}n_{2}}^{n'_{1}n'_{2}}$$

$$= \frac{e_{1}e_{2}h^{2}}{\pi\Omega} \cdot \frac{(\tilde{a}_{1}'a_{1})(\tilde{a}_{2}'a_{2}) - (\tilde{a}_{1}'\gamma_{1}a_{1})(\tilde{a}_{2}'\gamma_{2}a_{2})}{(\mathfrak{p}_{1} - \mathfrak{p}_{1}')^{2} - \left(\frac{E_{1} - E_{1}'}{c}\right)^{2}}$$

$$+ \frac{e_{1}e_{2}h^{2}}{\pi\Omega|\mathfrak{p}_{1} - \mathfrak{p}_{1}'|} \cdot \frac{-\left(\frac{E_{1} - E_{1}'}{c}\right)^{2}(\tilde{a}_{1}'a_{1})(\tilde{a}_{2}'a_{2}) + (\tilde{a}_{1}'\gamma_{1}a_{1},\mathfrak{p}_{1} - \mathfrak{p}_{1}')(\tilde{a}_{2}'\gamma_{2}a_{2},\mathfrak{p}_{1} - \mathfrak{p}_{1}')}{(\mathfrak{p}_{1} - \mathfrak{p}_{1}')^{2} - \left(\frac{E_{1} - E_{1}'}{c}\right)^{2}(\tilde{a}_{1}'a_{1})(\tilde{a}_{2}'a_{2}) + (\tilde{a}_{1}'\gamma_{1}a_{1},\mathfrak{p}_{1} - \mathfrak{p}_{1}')(\tilde{a}_{2}'\gamma_{2}a_{2},\mathfrak{p}_{1} - \mathfrak{p}_{1}')}$$

The first summand on the right-hand side is precisely the **Møller** interaction energy. We must then prove that the second summand will vanish. We next write that summand in a symmetric form by using the energy and impulse theorem:

Factor
$$\cdot \left[+ \frac{E_1 - E_1'}{c} (\tilde{a}_1' a_1) \frac{E_2 - E_2'}{c} (\tilde{a}_2' a_2) - (\tilde{a}_1' \gamma_1 a_1, \mathfrak{p}_1 - \mathfrak{p}_1') (\tilde{a}_2' \gamma_2 a_2, \mathfrak{p}_2 - \mathfrak{p}_2') \right].$$

In order to prove that this vanishes, it suffices to show that one has:

$$(E_{1} - E_{1}')\tilde{a}_{1}'a_{1} = -c(\tilde{a}_{1}'\gamma_{1}a_{1},\mathfrak{p}_{1} - \mathfrak{p}_{1}'), \qquad (49)$$

and a corresponding equation for the second particle. However, (49) follows immediately from the Dirac equation:

$$E_1 a_1 = -c (\mathfrak{p}_1 \gamma_1 a_1) - m_1 c^2 \delta_1 a_1, \qquad (50)$$

and the corresponding equation for \tilde{a}'_1 .

Furthermore, one easily convinces oneself that the **Møller** formula can be derived in precisely the same way when one of the particles is bound.

It is satisfying that no radiation will be emitted in our approximation. One can imagine that it is possible for processes to exist in which the two electrons change their states, and two quanta are emitted. However, in such processes, one must satisfy the law of impulse twice (viz., the impulse of one quantum must be equal to the change in impulse of one electron, and the same thing must be true for the other quantum and the other electron) and the law of energy once. However, that is impossible as long as the no electron has a negative energy in the final state.

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