On spaces with an arbitrary number of dimensions

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1.

Let \( z_1, z_2, \ldots, z_n \) be \( n \) variables that can take on all real values from \(-\infty\) to \(+\infty\). The \( n \)-fold infinite domain of systems of values of those variables is called an \( n \)-dimensional space and will be denoted by \( S_n \). A system \((z_1^0, z_2^0, \ldots, z_n^0)\) will determine a point \( L_0 \) of that space, and \( z_1^0, z_2^0, \ldots, z_n^0 \) will be called the coordinates of that point.

A system of \( m \) equations will determine a domain of systems of values of \( n-m \) independent variables that will be a space \( S_{n-m} \) of many dimensions that is contained in \( S_n \). A space of only one dimension that forms a simple continuum will be called a line.

Let:

\[
F(z_1, z_2, \ldots, z_n) = 0
\]

be the equation of an \((n-1)\)-dimensional space \( S_{n-1} \). If the function \( F \) is continuous and has just one value for all real values of the coordinate then the space \( S_{n-1} \) will generally separate \( S_n \) into two regions, in one of which one will have \( F < 0 \), and in the other \( F > 0 \). Moreover, one cannot continuously vary the system of values of the coordinates of a point in the first region and pass to a system of values of the coordinates of a point in the other region without passing through a system of values that satisfies equation (1). The two regions will be two \( n \)-dimensional spaces that are bounded by the space \( S_{n-1} \). If one can always pass from the system of values of the coordinates of an arbitrary point in one of the two regions to the system of values of the coordinates of any other point in the same region by a continuous variation without passing through the values of a point in \( S_{n-1} \) then one will say that the region is a connected space.

Let:

\[
\begin{align*}
z_1 &= z_1(u_1, u_2, \ldots, u_{n-1}), \\
z_2 &= z_2(u_1, u_2, \ldots, u_{n-1}), \\
&\vdots \\
z_n &= z_n(u_1, u_2, \ldots, u_{n-1})
\end{align*}
\]

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Let:

\[
\begin{align*}
z_1 &= z_1(u_1, u_2, \ldots, u_{n-1}), \\
z_2 &= z_2(u_1, u_2, \ldots, u_{n-1}), \\
&\vdots \\
z_n &= z_n(u_1, u_2, \ldots, u_{n-1})
\end{align*}
\]
be such continuous, single-valued functions that will satisfy equation (1) identically when they are substituted. The space $S_{n-1}$ can be regarded as the domain of the systems of values of the $n - 1$ real variables $u_1, u_2, \ldots, u_{n-1}$.

It is obvious that $S_{n-1}$ will satisfy the $n - 1$ equations:

$$\frac{dF}{dz_1} \frac{dz_1}{du_m} + \frac{dF}{dz_2} \frac{dz_2}{du_m} + \cdots + \frac{dF}{dz_n} \frac{dz_n}{du_m} = 0,$$

which are obtained by taking $m$ to be the numbers 1, 2, …, $n - 1$, in succession.

Let $A_1, A_2, \ldots, A_n$ denote arbitrary indeterminate quantities and set:

$$\Delta = \begin{vmatrix}
A_1 & \frac{dz_1}{du_1} & \frac{dz_2}{du_1} & \cdots & \frac{dz_n}{du_1} \\
A_2 & \frac{dz_1}{du_2} & \frac{dz_2}{du_2} & \cdots & \frac{dz_n}{du_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_n & \frac{dz_1}{du_n} & \frac{dz_2}{du_n} & \cdots & \frac{dz_n}{du_n}
\end{vmatrix},$$

$$\mu^2 = \sum_{m=1}^{n} \left( \frac{dF}{dz_m} \right)^2,$$

$$M^2 = \sum_{m=1}^{n} \left( \frac{d\Delta}{dz_m} \right)^2.$$

One will then get from equations (3) that:

$$\frac{dF}{dz_m} = \frac{\mu}{M} \frac{d\Delta}{dA_m}.$$

Now let $L$ be a line that is determined by the equations:

$$z_1 = l_1(t), \quad z_2 = l_2(t), \quad \ldots, \quad z_n = l_n(t).$$

If equation (1) is satisfied by only a finite number of real values of $t$ when those values are substituted then the line $L$ will intersect the space $S_{n-1}$ at only a finite number of points. Let $T_0$ be one of those points of intersection that corresponds to $t = t_0$. One will have:
Now consider the two points of \( L \) that correspond to:

\[
t = t_0 + \delta t_0, \\
t = t_0 - \delta t_0,
\]

in which \( \delta t_0 \) is an infinitesimal.

For the first of those values of \( t \), the function \( F \) will become:

\[
\delta F = \delta t_0 \sum_{m=1}^{n} \frac{dF}{dz_m} \frac{dl_m}{dt_0},
\]

and for the second one:

\[
\delta' F = -\delta t_0 \sum_{m=1}^{n} \frac{dF}{dz_m} \frac{dl_m}{dt_0}.
\]

Recalling equations (7), one will get:

\[
\begin{align*}
\delta F &= \frac{\mu D}{M} \delta t_0, \\
\delta' F &= -\frac{\mu D}{M} \delta t_0,
\end{align*}
\]

(9)

in which \( D \) is the determinant \( \Delta \) when the quantities \( dl_m / dt_0 \) are substituted for the \( A_m \).

If one now takes the radicals that give positive signs to \( \mu \) and \( M \) and conveniently fixes the ordering of \( z_1, z_2, \ldots, z_m \) while traversing the line \( L \) by continuously increasing \( t \) then one can deduce from equations (9) that if \( D > 0 \) when \( L \) intersects \( S_{n-1} \) at the point \( T_0 \) then one will leave the region in which \( F < 0 \) and enter the one in which \( F > 0 \), and conversely, if \( D < 0 \) then one will leave the region in which \( F > 0 \) and enter the one in which \( F < 0 \).

If one sets:

\[
ds_n^2 = dz_1^2 + dz_2^2 + \cdots + dz_n^2,
\]

and \( ds_n \) is the line element on \( S_n \) (in which case, Riemann called the space \( S_n \) planar) then the line element \( ds_{n-1} \) in the space \( S_{n-1} \) will be given by the formula:

\[
ds_{n-1}^2 = \sum \sum E_{rs} \, du_r \, du_s,
\]

in which:

\[
E_{rs} = \sum \frac{dz_m}{du_r} \frac{dz_m}{du_s},
\]

and from a known property of determinants, one will have:
\[ M^2 = \sum \left( \frac{d\Delta}{dA_m} \right)^2 = \begin{vmatrix} E_{11} & E_{12} & \cdots & E_{1,n-1} \\
E_{12} & E_{22} & \cdots & E_{2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
E_{1,n-1} & E_{2,n-1} & \cdots & E_{n-1,n-1} \end{vmatrix}. \] (10)

If:

\[ dS_n = dz_1 \, dz_2 \ldots \, dz_m \]

is the spatial element of \( S_n \) then the element of \( S_{n-1} \) will be:

\[ dS_{n-1} = M \, du_1 \, du_2 \ldots \, du_{n-1}. \] (11)

Now let:

\[ F_1 (u_1, u_2, \ldots, u_{n-1}) = 0 \]

be the equation of a space \( S_{n-2} \) that is contained in \( S_{n-1} \). If \( F_1 \) is a continuous function and has just one value then \( S_{n-2} \) will generally separate \( S_{n-1} \) into two regions, one of which will have \( F_1 < 0 \) and the other of which will have \( F_1 > 0 \). One can regard \( S_{n-2} \) as the domain of \( n - 2 \) real variables, and one can repeat everything that was said for \( S_{n-2} \). The line element will always be a homogeneous form of degree 2, but the coefficients will have a different form, as well as the coefficient \( M \) by means of which one obtains the spatial element of \( S_{n-2} \). Analogous observations will be true for spaces with a lower number of dimensions.

2.

One says that an \((n - m)\)-dimensional space \( S_{n-m} \) is linearly connected if one can take any two points in it and pass a continuous line that goes from one of those points to the other without leaving \( S_{n-m} \). Say that a space \( S_{n-m} \) is closed if one can divide \( S_n \) into two linearly-connected spaces in such a way that when one is given one of them, one cannot pass a continuous line from any point to another that does not intersect \( S_{n-1} \). One says that a linearly-connected space is closed if it divides a closed space \( S_{n-1} \) into two regions, each of which is linearly connected and is such that one cannot pass a continuous line that is entirely contained in \( S_{n-1} \) from one of them to any point in the other that does not intersect \( S_{n-2} \), and so on.

However, instead of considering just one, now consider an arbitrary number of inequalities:

\[ F_1 < 0, \quad F_2 < 0, \quad \ldots, \quad F_m < 0, \]

which will determine a subset \( R \) of a space \( S_t \) that can be linearly connected. The totality of those \((t - 1)\)-dimensional spaces:

\[ F_1 = 0, \quad F_2 = 0, \quad \ldots, \quad F_m = 0 \]
that bound \( R \) in such a way that one can pass a continuous line from any point of \( R \) to a point outside of \( R \) that does not intersect any of those spaces can be called the \textit{contour} of \( R \).

One says that a space is \textit{finite} if the coordinates of all of its points have finite values.

A space is either finite and linearly connected or closed or it will have a contour.

3.

A finite space has properties that are independent of the magnitude of its dimensions and the form of its elements. Those properties that refer to only the way that its parts are connected were considered by LISTING for ordinary spaces in a treatise that was entitled “Der Census räumlicher Complexe” and were determined for surfaces by RIEMANN.

Other than the linear connections, which are the only ones that present themselves in surfaces, I have observed that one can consider other types of connections in spaces with a number of dimensions that is greater than two.

If any closed \( m \)-dimensional space (with \( m < n \)) in an \( n \)-dimensional space \( R \) that is bounded by one or more \((n - 1)\)-dimensional spaces is the contour of a subset of an \((m + 1)\)-dimensional linearly-connected space that is completely contained in \( R \) then it will be \((m + 1)\)-dimensional connected, and one will simply say that \( R \) has a connection of the \( m^{\text{th}} \) kind that is \textit{simple}. If all of the connections in a space \( R \) are simple then one says that it is \textit{simply connected}. However, if one can imagine a number \( p_m \) of closed \( m \)-dimensional spaces in \( R \) that do not form the contour of a linearly-connected subset of an \((m + 1)\)-dimensional space that is completely contained in \( R \) and is such that any other closed \( m \)-dimensional space forms, by itself or with all or part of the contour, the contour of a linear-connected subset of an \((m + 1)\)-dimensional line that is contained completely in \( R \) then one will say that \( R \) has a \((p_m + 1)^{\text{th}}\)-order connection of the \( m^{\text{th}} \) kind.

\textit{Examples}. – In the ordinary space, the space that is found between two concentric spheres has a second-order connection of the second kind and a simple one of the first kind.

The space inside an annulus has a simple connection of the second kind and a second order connection of the first kind.

Both connections in the space between a sphere and an annulus have second order.

In order to justify the definition that was given for the various kinds of connections, it is necessary to prove that the number \( p_m \) is well-defined for any bounded space \( R \) – i.e., that no matter how one defines the \( m \)-dimensional spaces that possess the stated property, the number of them is always the same. We will base that upon the following lemma, as RIEMANN did in order to prove the corresponding theorem that relates to surfaces:

\textit{If a system of closed} \( m \)-dimensional spaces \( A \), \textit{along with another system} \( C \), \textit{forms the contour of a linearly-connected} \((m + 1)\)-dimensional space \( S_{m+1} \) \textit{that is contained completely with} \( R \), \textit{and if another system of closed} \( m \)-dimensional spaces \( B \), \textit{along with the system} \( C \), \textit{forms the contour of a linearly-connected space} \( S'_{m+1} \) \textit{that is contained completely in} \( R \) \textit{then the system} \( A \), \textit{together with the system} \( B \), \textit{will form the contour of a linearly-connected} \((m + 1)\)-dimensional space that is contained completely in \( R \).
Indeed, the two spaces $S_{m+1}$ and $S'_{m+1}$ will either be on opposite sides of the contour $C$ or on the same side. In the former case, the contour of the space that is composed of $S_{m+1}$ and $S'_{m+1}$ will be the system $A$, along with the system $B$. In the latter case, if one removes $S'_{m+1}$ from $S_{m+1}$ then what will remain will be a space whose contour is the system $A$, along with the system $B$.

If $t$ closed $m$-dimensional spaces $A_1$, $A_2$, ..., $A_t$, by themselves or with any other closed $m$-dimensional space, cannot form the contour of a linearly-connected $(m + 1)$-dimensional space that is contained completely in $R$, and if another system of $t'$ closed $m$-dimensional spaces $B_1$, $B_2$, ..., $B_t$ possesses that property then one will have $t = t'$.

Indeed, suppose that $t' > t$. If $C$ is any closed $m$-dimensional space then either the system $(A_1, A_2, ..., A_t, C)$ or the system $(A_1, A_2, ..., A_t, B_1)$ will form the contour of a linearly-connected $m$-dimensional space that is contained completely in $R$. Therefore, either the system $(A_2, A_3, ..., A_t, C)$ or the system $(A_2, A_3, ..., A_t, B_1)$, together with $A_1$, will form the contour of a linearly-connected $(m + 1)$-dimensional space that is contained completely in $R$, and as a consequence, from the preceding lemma, the system $(A_2, A_3, ..., A_t, C)$, along with the system $(A_2, A_3, ..., A_t, B_1)$ – i.e., the system $(B_1, A_2, A_3, ..., A_t, C)$ – will form the contour of an $(m + 1)$-dimensional space that is contained completely in $R$. Therefore, when the system $(B_1, A_1, A_2, ..., A_t)$ is united with any closed space $C$, it will form the contour of a linearly-connected $(m + 1)$-dimensional space, and if one now continues to substitute one of the spaces $B$ for one of the spaces $A$, in succession, then one will finally have that the system $(B_1, B_2, ..., B_t)$, along with any closed space, and therefore also with $B_{t+1}$, will form the contour of a linearly-connected $(m + 1)$-dimensional space that is contained completely in $R$, but that will contradict what we have supposed when $t' > t$. One likewise proves that one cannot have $t > t'$. Therefore, $t = t'$, which was to be proved.

4.

When one supposes that the connection of a bounded space $R$ is routed along a space with a lower number of dimensions that has its contour on the contour of $R$, one says that one has made a transverse section in $R$.

If one is given an $m$-dimensional space and one separates from it an infinitesimal part with an infinitesimal $(m - 1)$-dimensional space for its contour then one says that one has made a point section.

If a bounded space $R$ can be reduced to another one $R'$ without making any transverse section, and only by means of continuous enlargements and reductions of its parts, then one will say that $R$ can be reduced to $R'$ by continuous transformation.

Two bounded spaces $R$ and $R'$ that can be reduced to each other by means of continuous transformation will have equal orders for their connections of each kind. Now, a point is simply connected, so any space that can be reduced to a point by continuous transformation will be simply connected.
One can always make a space with a contour lose one dimension by continuous transformation.

Indeed, let \( R \) be that space, let \( m \) be its number of dimensions, let \( C \) be its contour, let \( S_m \) be the space that it belongs to, and let \( u_1, u_2, u_3, \ldots, u_m \) denote a system of coordinates in \( S_m \). Imagine an \((m - 1)\)-fold infinite system of lines that occupies all of \( S_m \) continuously. For example, take the lines whose equations are:

\[
\begin{align*}
  u_2 &= a_2, & u_3 &= a_3, & \ldots, & u_{m-1} &= a_{m-1},
\end{align*}
\]

where \( a_2, a_3, \ldots, a_{m-1} \) take all values from \(-\infty\) to \(+\infty\), and consider only that part of the system that contains the lines that meet the contour \( C \) of \( R \). When each of those lines is continued by increasing \( u_1 \) until it meets \( C \), where it enters \( R \) many times and leaves it just as many times, one can approach each entry point indefinitely with the following exit point and thus make \( R \) lose one dimension, as we wished to show.

One can always make a closed space lose one dimension by continuous transformation after one has made a point section.

Indeed, after making a point section, the space will acquire a contour, and therefore, from the previous theorem, it can always lose one dimension by continuous transformation.

If one makes only one point section in a closed \( m \)-dimensional space \( R \) then that will not change the orders of its connections. However, if one makes \( s + 1 \) point sections then the order of the \((m - 1)^{th}\) kind will increase by unity, while the higher orders of connections will not change.

Indeed, let \( \alpha + 1 \) be the order of the connection of the \((m - 1)^{th}\) kind in a closed \( m \)-dimensional space \( R \). One can imagine a system \( A \) of \( \alpha \) closed \((m - 1)\)-dimensional spaces in \( R \) that form the contour of a subset of \( R \), not by themselves, but with any other closed \((m - 1)\)-dimensional space \( C \). Since \( R \) is closed, the system \( A \), together with \( C \), will divide it into two separate regions, \( R' \) and \( R'' \) that both have the same contour – i.e., the system \( A \), along with \( C \). Now if one makes a point section in \( R \) then it will be in one of the two regions; suppose that it is in \( R' \). It is then clear that the system \( A \), along with \( C \), will no longer form the entire contour of \( R' \), but rather, it will always comprise the entire contour of \( R'' \). Therefore, the order of the connection of the \((m - 1)^{th}\) kind of \( R \) will not change under just one point section. However, if one makes two point sections in \( R \) then one can always take \( C \) in such a way that one of those points is in \( R' \) and the other is in \( R'' \), so the system \( A \), along with \( C \), will no longer form the contour of part of \( R \), and it will be necessary to add another closed \((m - 1)\)-dimensional space in order to get the entire contour from part of \( R \). Therefore, one can increase the order of the connection of the \((m - 1)^{th}\) kind of \( R \) with two point sections. One proves analogously that that order will increase by \( 2, 3, \ldots, s \) units with \( 3, 4, \ldots, s + 1 \) point sections, resp.

Now let \( \beta + 1 \) be the order of the connection of the \((m - t - 1)^{th}\) kind of \( R \), where \( 0 < t < m \). One can imagine a system of closed \((m - t - 1)\)-dimensional spaces in \( R \) that does not form the contour of an \((m - t)\)-dimensional space \( T \) that is contained completely in \( R \) by itself. Let as many
point sections as one desires be made in \( R \), but still a finite number of them. With the given contour, one can always arrange that \( T \) does not pass through any of those point sections. Therefore, an arbitrary finite number of point sections will not change the orders of the connections of the kinds that are lower than the \((m - 1)^{th}\).

Since one does not change the orders of the connections of a closed space \( R \) by making just one point section, in order to determine those orders, it will make no difference whether one regards \( R \) as closed or as having an infinitesimal contour. Thus, one can always regard a finite space as being bounded by a contour, and therefore one can always make it lose one dimension by continuous transformation without changing the orders of its connections.

5.

In order to make a finite space \( n \)-dimensional space \( R \) simply connected by means of simply-connected transverse sections, it is necessary and sufficient to make \( p_{n-1} \) linear sections, \( p_{n-2} \) two-dimensional ones, \( p_{n-3} \) three-dimensional ones, …, and \( p_1 \) \((n - 1)\)-dimensional ones, if the orders of its connection of the \( 1^{st} \), \( 2^{nd} \), …, \((n - 1)^{th} \) kind are \( p_1 + 1, p_2 + 1, \ldots, p_{n-1} + 1 \), respectively.

Indeed, let \( p_{n-1} + 1 \) be the order of the connection of the \((n - 1)^{th}\) kind of \( R \). One can imagine a system \( A \) of \((n - 1)\)-dimensional spaces in it that do not form the contour of part of \( R \), but they will form it with any other closed \((n - 1)\)-dimensional space. One will then have more bounded regions, each of which will have a contour that is all or part of the system \( A \) and part of the contour of \( R \). Hence, when one makes that region lose one dimension by a continuous transformation, it will reduce to the system \( A \), which is connected along \((n - 2)\)-dimensional spaces. Thus, \( R \) can reduce to an \((n - 1)\)-dimensional space \( R_1 \) by continuous transformation, which is a space that is composed of \( p_{n-1} \) closed \((n - 1)\)-dimensional spaces \( A \) that are connected with each other by \((n - 2)\)-dimensional spaces, and \( R_1 \) will have the same orders of its connections of the \((n - 2)^{th} \), \((n - 3)^{th} \), …, \(1^{st} \) kind as \( R \). Now, without changing the orders of the connections of \( R_1 \), one can make as many point sections as the number of closed spaces that comprise it – i.e., \( p_{n-1} \). Let \( R'_1 \) be the space \( R_1 \) in which those point sections are made.

If one reduces \( R'_1 \) to \( R \) by continuous transformation then the point sections will acquire one dimension and become continuous lines that go from a point of the contour of \( R \) to another point of the same contour, i.e., they will become transverse linear sections, and therefore the orders of the connections of kinds lower than the \((n - 1)^{th}\) will again remain the same. Hence, one can make only a number \( p_{n-1} \) of transverse sections in \( R \) that do not change the orders of its connections of kinds less than the \((n - 1)^{th}\).

Now each of those \( p_{n-1} \) transverse linear sections will traverse one of the \( p_{n-1} \) closed spaces \( A \), which can at most be arranged in \( R \) such that it will not comprise the contour of a portion of \( R \) by itself, but it will when one adds another one of dimension \( n - 1 \) to it. Therefore, after having made that transverse section, each of the spaces \( A \) will no longer be closed, and thus any closed \((n - 1)\)-dimensional space will become the contour of a portion of \( R \), and the connection of \( R \) of the \((n - 1)^{th} \) kind will be rendered simple.
Therefore, in order to make the connection of the \((n - 1)\)th kind of \(R\) simple by means of simply-connected transverse sections without changing the orders of the connections of the lower kinds, it is necessary and sufficient to make \(p_{n-1}\) transverse linear sections.

The \((n - 1)\)-dimensional space \(R'_1\) to which the space \(R\) in which one makes the \(p_{n-1}\) transverse linear sections is reduced by continuous transformation that has a point section in any of the closed spaces that comprise it and an order of connection of the \((n - 2)\)th kind that is equal to \(p_{n-2}\) can lose one dimension by continuous transformation and reduce to a space \(R_2\) that is composed of \(p_{n-2}\) closed \((n - 2)\)-dimensional spaces that are connected along \((n - 3)\)-dimensional spaces. Now one can make at most \(p_{n-2}\) point sections without changing the orders of the connections of \(R_2\). Let \(R'_2\) denote the space \(R_2\) in which those point sections are made. When one reduces \(R'_2\) to \(R_2\), the point sections of \(R'_2\) will acquire two dimensions and become two-dimensional spaces that have their contours on the contour of \(R\) and will be simply connected since they are reducible to a point by continuous transformation, and will therefore be two-dimensional transverse sections. Let \(R''\) denote the space \(R\) in which one makes the transverse sections in one and two dimensions. The two-dimensional sections will make the connection of the \((n - 2)\)th kind simple. Hence, in order to reduce \(R\) to a space \(R''\) that has connections of the \((n - 1)\)th and \((n - 2)\)th kind that are simple by continuous transformation without changing the orders of the connections of the lower kinds, it is necessary and sufficient to make \(p_{n-1}\) transverse linear sections and \(p_{n-2}\) two-dimensional ones. One then continues to do that for the connections of the lower kinds.

When a finite space \(R\) is reduced to a simply-connected one by means of simply-connected transverse sections, any closed \(m\)-dimensional space that is in \(R\) that is composed of as many closed \(m\)-dimensional spaces as the number of transverse \((n - m)\)-dimensional sections that it meets, the contour of an \((m + 1)\)-dimensional space will be contained completely in \(R\).

Indeed, if \(p_m + 1\) is the order of the connection of the \(m\)th kind of \(R\) then any closed \(m\)-dimensional space \(C\), along with a system of \(p_m\) closed \(m\)-dimensional spaces, will form the contour of an \((m + 1)\)-dimensional space \(S\) that is contained completely in \(R\). Now, each of the spaces \(A\) will be intersected by one and only one transverse \((n - m)\)-dimensional section that makes up part of the one that makes \(R\) simply connected, and therefore since each of those sections has a contour that is on the contour of \(R\), if \(C\) forms the contour of \(S\), along with \(s\) of the spaces \(A\), then it must intersect precisely those \(s\) transverse sections that intersect those \(s\) closed spaces in the system \(A\).

For greater clarity, let us make some applications of that to some ordinary spaces.

The space between two concentric spheres is made simply connected by means of just one transverse linear section that goes from a point of the outer spherical surface to any point of the inner one.

The space inside an annular surface is made simply connected by means of just one transverse surface section that is made along the meridian of the surface.

The space between two annular surfaces is made simply connected by means of just one transverse linear section that goes from a point of the outer annular surface to a point on the inner
one and by means of two transverse surface sections that both go through the linear section, one of which is made along the meridian of the surface and one of them is made along the equator.

The space between a sphere and a ring is made simply connected by means of a transverse linear section that goes from the surface of the sphere to that of the ring, and the other surface section that terminates at the linear section also goes from the surface of the ring to that of the sphere.

6.

Suppose that one is given an \( n \)-dimensional space \( R \) that is bounded by an arbitrary number of closed \((n-1)\)-dimensional ones. Let \( S'_{n-1}, S''_{n-1}, \ldots, S^{(ij)}_{n-1} \) be the ones whose equations are:

\[
F_1 = 0, \quad F_2 = 0, \quad \ldots, \quad F_t = 0,
\]

and let \( R \) be determined from the inequalities:

\[
F_1 < 0, \quad F_2 < 0, \quad \ldots, \quad F_t < 0.
\]

Let \( X_1, X_2, \ldots, X_n \) be \( n \) functions of the points of \( R \) that are finite and continuous. Take under consideration the \( n \)-fold integral:

\[
\Omega_n = \int \left( \frac{dX_1}{dz_1} + \frac{dX_2}{dz_2} + \cdots + \frac{dX_n}{dz_n} \right) dz_1 \, dz_2 \cdots dz_n,
\]

which is extended over all of the space \( R \).

Let even indices distinguish the values of \( X_r \) at the points where the line \( Z_r \) that has the equations:

\[
z_1 = a_1, \quad z_2 = a_2, \quad \ldots, \quad z_{r-1} = a_{r-1}, \quad \ldots, \quad z_n = a_n
\]

(2)

when \( z_r \) increases cross one of the spaces \( S_{n-1} \) and enter into the space \( R \) in which all of the inequalities (1) are satisfied. Let odd indices distinguish the values of \( X_r \) at the points where the line \( Z_r \) crosses one of the spaces \( S_{n-1} \) and leaves the space \( R \). One will have:

\[
\int_{n-1}^{r} \frac{dX_r}{dz_r} \, dz_r = X'_r - X'_r + X''_r - X''_r + \cdots
\]

Thus:

\[
\Omega_n = \sum_{n-1}^{r} \left( X'_r - X'_r + X''_r - X''_r + \cdots \right) dz_1 \, dz_2 \cdots dz_{r-1} \, dz_{r+1} \cdots dz_n.
\]

Now the number of points at which the line \( Z_r \) meets each of the closed spaces \( S_{n-1} \) is even, and \( Z_r \) will enter into \( R \) as many times as it leaves it. For example, the space \( S'_{n-1} \) will be met by \( Z_r \) at the
points 0, 2l₁ + 1, 2l₂, 2l₃ + 1, …, and the part of the integral Ωₙ that refers to the points of \( S'_n \) will be:

\[
Ω'_n = \int_{n-1} \left( X_r^0 - X_r^{(2l_r+1)} + X_r^{(2l_r)} - X_r^{(2l_r+1)_r} + \cdots \right) dz_1 dz_2 \cdots dz_{r-1} dz_{r+1} \cdots dz_n.
\]

Consider \( S'_n \) to be the domain of the \( n-1 \) real variables \( u'_1, u'_2, \ldots, u'_{n-1} \). One has:

\[
dz_1 dz_2 \cdots dz_{r-1} dz_{r+1} \cdots dz_n = \pm \frac{dΔ}{dA_r} du'_1 du'_2 \ldots du'_{n-1},
\]

which will take the + or – sign according to whether \( dΔ / dA_r \) is > 0 or < 0, resp.

Now, as was proved in the first section, it results that one can always take the ordering of the \( z_1, z_2, \ldots, z_m \) in such a way that the sign of \( dΔ / dA_r \) is equal to that of \( dF_1 / dz_r \), if the equation of \( S'_n \) is \( F_1 = 0 \).

Now, \( dF_1 / dz_r < 0 \) at the point of \( S'_n \) where \( Z_r \) enters \( R \), and \( dF_1 / dz_r > 0 \) at the point where it leaves \( R \). One then has:

\[
\int_{n-1} X_r^0 dz_1 dz_2 \cdots dz_{r-1} dz_{r+1} \cdots dz_n = - \int_{n-1} X_r^0 \frac{dΔ}{dA_r} du'_1 du'_2 \ldots du'_{n-1},
\]

\[
\int_{n-1} X_r^{(2l_r+1)} dz_1 dz_2 \cdots dz_{r-1} dz_{r+1} \cdots dz_n = \int_{n-1} X_r^{(2l_r+1)} \frac{dΔ}{dA_r} du'_1 du'_2 \ldots du'_{n-1},
\]

Thus:

\[
Ω'_n = - \int_{n-1} \left( X_r^0 \frac{dΔ}{dA_r} + X_r^{(2l_r+1)} \frac{dΔ}{dA_r} + \cdots \right) du'_1 du'_2 \ldots du'_{n-1},
\]

or

\[
Ω'_n = - \int_{n-1} X_r \frac{dΔ}{dA_r} du'_1 du'_2 \ldots du'_{n-1},
\]

which is extended over all of the space \( S'_n \).

One can make analogous reductions of the other spaces, and one will get:

\[
Ω_n = - \sum_{n-1} \int X_r \frac{dΔ}{dA_r} du'_1 du'_2 \ldots du'_{n-1} - \sum_{n-1} \int X_r \frac{dΔ}{dA_r} du''_1 du''_2 \ldots du''_{n-1} - \cdots
\]

However:
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\[ \sum X_r \frac{d\Delta}{dA_r} = \begin{vmatrix} X_1 \frac{dz_1}{du_1} & \cdots & \frac{dz_1}{du_{n-1}} \\ X_2 \frac{dz_2}{du_1} & \cdots & \frac{dz_2}{du_{n-1}} \\
\vdots & \ddots & \vdots \\
X_n \frac{dz_n}{du_1} & \cdots & \frac{dz_n}{du_{n-1}} \end{vmatrix}. \]

Hence:

\[ \Omega_n = -\sum \int \begin{vmatrix} X_1 \frac{dz_1}{du_1}^{(t)} & \cdots & \frac{dz_1}{du_{n-1}}^{(t)} \\ X_2 \frac{dz_2}{du_1}^{(t)} & \cdots & \frac{dz_2}{du_{n-1}}^{(t)} \\
\vdots & \ddots & \vdots \\
X_n \frac{dz_n}{du_1}^{(t)} & \cdots & \frac{dz_n}{du_{n-1}}^{(t)} \end{vmatrix} du_1^{(t)} du_2^{(t)} \cdots du_{n-1}^{(t)}. \]

If \((z_1^0, z_2^0, \ldots, z_n^0)\) is a point of \(S_{n-1}^{(t)}\) then a line that passes through that point and has the equations:

\[ z_1^0 - z_1^0 = \frac{dF_i}{dz_1^0} \frac{\rho}{\mu}, \]

\[ z_2^0 - z_2^0 = \frac{dF_i}{dz_2^0} \frac{\rho}{\mu}, \]

\[ \cdots \]

\[ z_n^0 - z_n^0 = \frac{dF_i}{dz_n^0} \frac{\rho}{\mu} \]

is called the normal to the space \(S_{n-1}^{(t)}\), and since one has:

\[ \rho^2 = (z_1^0 - z_1^0)^2 + (z_2^0 - z_2^0)^2 + \cdots + (z_n^0 - z_n^0)^2. \]

\(\rho\) is called the distance from the point \((z_1, z_2, \ldots, z_n)\) to \((z_1^0, z_2^0, \ldots, z_n^0)\). If \(\rho\) is infinitesimal and equal to \(dp_t\) then one will have:

\[ \frac{dz_r}{dp_t} = \frac{dF_i}{dz_r} \frac{1}{\mu}. \]

However:
\[
\frac{dF_i}{dz_r} = \frac{d\Delta}{dA_r} \frac{\mu}{M},
\]
so:
\[
\frac{dz_r}{dp_i} = \frac{d\Delta}{dA_r} \frac{1}{M}.
\]
Thus, when one substitutes:
\[
\Omega_n = - \sum_i \int \left( X_1 \frac{dz_1}{dp_i} + X_2 \frac{dz_2}{dp_i} + \cdots + X_n \frac{dz_n}{dp_i} \right) M \, du_1^{(i)} du_2^{(i)} \cdots du_{n-1}^{(i)}.
\]
However, if \( dS_{n-1}^{(i)} \) is the spatial element of \( S_{n-1}^{(i)} \) then one will have:
\[
dS_{n-1}^{(i)} = M \, du_1^{(i)} du_2^{(i)} \cdots du_{n-1}^{(i)},
\]
so:
\[
\Omega_n = - \sum_i \int \sum_r X_r \frac{dz_r}{dp_i} dS_{n-1}^{(i)}.
\]
Therefore, if:
\[
X_r = \frac{dV}{dz_r}
\]
then one will have:
\[
\Omega_n = \int_R \left( \frac{d^2V}{dz_1^2} + \frac{d^2V}{dz_2^2} + \cdots + \frac{d^2V}{dz_n^2} \right) dR = - \sum_i \int \frac{dV}{dp_i} dS_{n-1}^{(i)},
\]
and therefore if:
\[
\frac{d^2V}{dz_1^2} + \frac{d^2V}{dz_2^2} + \cdots + \frac{d^2V}{dz_n^2} = 0
\]
in all of the space \( R \) then one will have:
\[
\sum_i \int \frac{dV}{dp_i} dS_{n-1}^{(i)} = 0.
\]
If the space \( R \) has a simple connection of the \((n - 1)\)th kind then any closed \((n - 1)\)-dimensional space \( C \) that enter into it will form the contour of a portion of \( R \). Thus, if equation (4) is satisfied for all \( R \) and \( V \), along with its first derivatives, are finite and continuous then one will always have:
\[
\int_C \frac{dV}{dp_c} dC = 0.
\]
If the space $R$ has a connection of the $(n-1)^{th}$ kind of order $p_{n-1} + 1$ and one passes $p_{n-1}$ closed $(n-1)$-dimensional spaces $A_1, A_2, \ldots, A_{p_{n-1}}$ through it then any closed space $C$ that is contained in $R$, together with the system $A$, will form the contour of part of $R$, and if $a_1, a_2, \ldots, a_{p_{n-1}}$ are the transverse linear sections that cross the closed spaces $A_1, A_2, \ldots, A_{p_{n-1}}$, respectively, and make the connection of the $(n-1)^{th}$ kind in the space $R$ simple then $C$ will form the contour of a part of $R$ in which the spaces of the system that are crossed by the sections $a$ will meet $C$.

Then set:

$$\int_{A_r} \frac{dV}{dp} \ dA_r = M_r ,$$

and one will have:

$$\int_{C} \frac{dV}{dp} \ dC + \sum M_r = 0 ,$$

in which the sum extends over all values of $r$ that are indices of the transverse sections that meet $C$, and one will have the following theorem:

*If the space $R$ has a connection of the $(n-1)^{th}$ kind of order $p_n + 1$ and that connection can be made simple by transverse linear sections, and if $C$ is a closed $(n-1)$-dimensional space that is contained in $R$ then the integral:

$$\int_{C} \frac{dV}{dp} \ dC$$

will differ from zero by as many moduli of periodicity as the number of transverse linear sections that cross the space $C$.\)*

Since two $(n-1)$-dimensional spaces that have the same contour together form a closed space, one will further have the following theorem:

*If one is given a closed $(n-2)$-dimensional space in an $n$-dimensional space $R$ that has a connection of the $(n-1)^{th}$ kind of order $p_{n-1} + 1$, and the former space can make that connection simple by means of $p_{n-1}$ linear sections then when the previous integral is extended over a space $C$ that has $\Gamma$ for its contour and meets $s$ transverse linear sections, it will differ from the same integral, when it is extended over a space $C'$ that has the same contour and does not meet any section, by moduli of periodicity that relate to those sections, and therefore if the space $R$ has a simple connection of the $(n-1)^{th}$ kind then the integral will always have the same value when it is extended over any space $C$ that is contained in $R$ and has $\Gamma$ for its contour.*
7.

In a closed $n$-dimensional space $R$ that has a connection of the first kind of order $p_1 + 2$, let $s_1$, $s_2$, ..., $s_{p_1}$ be the simply-connected $(n - 1)$-dimensional transverse sections that render the connection of the first kind in $R$ simple. Let $L_1, L_2, ..., L_{p_1}$ be $p_1$ closed lines that cross the sections $s_1, s_2, ..., s_{p_1}$, respectively, and are such that any other closed line $l$, together with the line $L$ that crosses the same sections that the they go through, will form the contour of a two-dimensional space that is contained completely in $R$.

Let:

\[ z_1 = z_1(u), \quad z_2 = z_2(u), \quad \ldots, \quad z_n = z_n(u) \]

be the equations of the line $l$ and take under consideration the integral:

\[ \Omega_1 = \sum \int X_r \, dz_r = \sum \int X_r \frac{dz_r}{du} \, du, \]

which extends over the entire line $l$, in which the $X_r$ are finite and continuous in all of $R$.

Now, if the line $l$ forms part of the contour of $C$ then if the space $C$ is determined by the equations:

\[ z_1 = z_1(v_1, v_2), \quad z_2 = z_2(v_1, v_2), \quad \ldots, \quad z_n = z_n(v_1, v_2), \]

one will have:

\[ \frac{dz_r}{du} \, du = \frac{dz_r}{dv_1} \, dv_1 + \frac{dz_r}{dv_2} \, dv_2, \]

and therefore:

\[ \Omega_1 = \int \sum X_r \frac{dz_r}{dv_1} \, dv_1 + \int \sum X_r \frac{dz_r}{dv_2} \, dv_2. \]

Now, from what was shown in the preceding section:

\[ \iint \left[ \frac{d}{dv_2} \left( \sum X_r \frac{dz_r}{dv_1} \right) - \frac{d}{dv_1} \left( \sum X_r \frac{dz_r}{dv_2} \right) \right] \, dv_1 \, dv_2 = \int \sum X_r \frac{dz_r}{dv_1} \, dv_1 + \int \sum X_r \frac{dz_r}{dv_2} \, dv_2, \]

in which the double integral extends over all of the space $C$, and the simple one extends over the entire system of lines $l, L_1, L_2, \ldots$ that form the contour of $C$.

However, one has:

\[ \iint \left[ \frac{d}{dv_2} \left( \sum X_r \frac{dz_r}{dv_1} \right) - \frac{d}{dv_1} \left( \sum X_r \frac{dz_r}{dv_2} \right) \right] \, dv_1 \, dv_2 \]
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\[
\int \sum \frac{dX_r}{dz_r} \begin{vmatrix} \frac{dz_r}{dv_1} & \frac{dz_r}{dv_2} \\ \frac{dz_r}{dv_1} & \frac{dz_r}{dv_2} \end{vmatrix} dv_1 dv_2 = \int \sum \left( \frac{dX_r}{dz_r} - \frac{dX_r}{dz_r} \right) dv_1 dv_2.
\]

Therefore, the double integral will be zero if the equations:

\[
\frac{dX_r}{dz_r} - \frac{dX_r}{dz_r} = 0
\]

are also verified in all of \( R \). Therefore, if the \( X_r \) satisfy (2) and are finite and continuous in \( R \) then the integral:

\[
\sum \int X_r \, dz_r
\]

will always be zero when it is extended over all of the lines \( l, L_1, L_2, \ldots \) that form the contour of a space \( C \), no matter what the closed line \( l \) is. One gets the following theorem from that:

\[\text{If } R \text{ is an } n\text{-dimensional space that has a connection of the first kind of order } p_1 \text{ and one has simply-connected transverse } (n - 1)\text{-dimensional sections } s_1, s_2, \ldots s_{p_1} \text{ of it that make the connection of the first kind on } R \text{ simple, and } L_1, L_2, \ldots L_{p_1} \text{ are } p_1 \text{ closed lines that meet the sections } s_1, s_2, \ldots s_{p_1}, \text{ respectively, and one sets:}
\]

\[
M_l = \sum_{L_i} \int X_r \, dz_r
\]

\[\text{then the integral:}
\]

\[
\sum \int_{Z_0}^{Z_1} X_r \, dz_r,
\]

\[\text{when extended between two points } Z_0 \text{ and } Z_1 \text{ along a line that meets some sections } s, \text{ will differ from the one that is taken along a line that goes from the point } Z_0 \text{ to the point } Z_1 \text{ without meeting any section } s \text{ by a quantity } M \text{ that relates to the sections } s \text{ that are met, which is taken to be positive or negative according to whether they are met by proceeding in one direction or the other. If the connection of the first kind on the space } R \text{ is simple then the integral will always have the same value when it is taken along any in } R \text{ that goes from } Z_0 \text{ to } Z_1.\]